Biholomorphic automorphisms of Siegel domains in C^4

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1 Results

Let $z = (z^j)$, $j = 1, \ldots, n$, $w = u + iv = (w^j) = (u^j + iv^j)$, $j = 1, \ldots, k$ be coordinates in $\mathbf{C}^n \times \mathbf{C}^k$, $k \leq n$;

$$\langle z, z \rangle =^{\tau} (\langle z, z \rangle^1, \ldots, \langle z, z \rangle^k)$$

a $\mathbf{R}^k$-valued hermitian form and Ω the Siegel domain of second kind, associated with the form $\langle z, z \rangle$, i.e.

$$\Omega = \{(z, w) \in \mathbf{C}^{n+k} : v - \langle z, z \rangle \in V\},$$

where V is the cone $\mathbf{R}conv\{\langle z, z \rangle : z \in \mathbf{C}^n\}$ ($\mathbf{R}conv$ stands for convex hull in $\mathbf{R}^k$).

The quadric $Q = \{(z, w) \in \mathbf{C}^{n+k} : v - \langle z, z \rangle = 0\}$, which is the Shilov boundary of Ω, is presumed to be nondegenerate, i.e.

$i)\langle z, b \rangle^j = 0$ for all z implies $b = 0$
$ii)\langle z, z \rangle^j$ are linearly independent $j = 1, \ldots, k$.

The last condition means that the cone V has nonempty interior.

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Let $f : \Omega \to \Omega$ be a proper holomorphic map. It was shown in [3, 5, 2] that f extends to a holomorphic automorphism of its Shilov boundary Q and this map occurs to be birational in $\mathbf{C}^{n+k}$ with the degree uniformly bounded within the same (n,k), and, conversely, any local C-R diffeomorphism $\Phi : Q \to Q$ extends to a holomorphic automorphism of Ω. This result might be considered as a generalization of the Poincaré-Alexander theorem [4, 1] about the extension of a local CR diffeomorphism of a hyperquadric in $\mathbf{C}^{n+1}$.

Since Q is a homogeneous manifold ($\mathrm{Aut}Q$ acts transitively via the transformations $z \mapsto p + z$, $w \mapsto q + w + 2i\langle z, p\rangle$ with $(p,q) \in Q$) then $\mathrm{Aut}Q \cong Q \times \mathrm{Aut}_0 Q$, where $\mathrm{Aut}_0 Q$ is the isotropy group of a fixed point, say the origin.

Due to the mentioned extension theorem, we may consider $\mathrm{Aut}_0 Q$ as a group of germs of biholomorphic (in our case, birational) transformations $\Phi : (Q, 0) \to (Q, 0)$.

$\mathrm{Aut}_0 Q$ is a finite dimensional Lie group iff Q is nondegenerate (see [6, 2]).

Our goal is to find the explicit description for $\mathrm{Aut}_0 Q$ and hence for $\mathrm{Aut}\Omega$.

In this paper we consider the case $n = 2, k = 2$. This case covers the nonequivalent domains related to the quadrics:

$$\begin{aligned} Q_1 : v^1 &= |z^1|^2 + |z^2|^2 \\ v^2 &= z^1\bar{z}^2 + z^2\bar{z}^1 \end{aligned} \tag{1}$$

$$\begin{aligned} Q_{-1} : v^1 &= |z^1|^2 - |z^2|^2 \\ v^2 &= z^1\bar{z}^2 + z^2\bar{z}^1 \end{aligned} \tag{2}$$

$$\begin{aligned} Q_0 : v^1 &= |z^1|^2 \\ v^2 &= z^1\bar{z}^2 + z^2\bar{z}^1. \end{aligned} \tag{3}$$

All other possible nondegenerate quadrics are isomorphic to one of these types by means of the action of the group $G^{2,2} = \mathrm{GL}(2, \mathbf{C}) \times \mathrm{GL}(2, \mathbf{R})$:

$$(C, \rho)(\langle z, z\rangle) = \rho\langle C^{-1}z, C^{-1}z\rangle.$$

These 3 cases are called hyperbolic, elliptic and parabolic, according to the distribution of the roots of the polynomial invariant

$$P(t) = \det(A^1 + tA^2),$$

where $\langle z, z\rangle^i = \sum_{jk} A^i_{jk} z^j \bar{z}^k$ for $i = 1, 2$. We denote the corresponding Siegel domains by Ω_{-1}, Ω_1, Ω_0.

In suitable coordinates the hyperbolic quadric takes the form $v^1 = |z^1|^2$, $v^2 = |z^2|^2$, thus it is the direct product of two spheres S^3. Due to a theorem of Beloshapka [6] $\mathrm{Aut}\Omega_1 \cong Q_1 \times \mathrm{Aut}_0(Q_1) = Q_1 \times \mathrm{Aut}_0(S^3) \times \mathrm{Aut}_0(S^3)$, and, hence, each element $\Phi \in \mathrm{Aut}(\Omega_1)$ can be represented as a composition $\Phi = \Phi_2 \circ \Phi_1$, where

$$\begin{aligned}
\Phi_1 : z^j &\mapsto \frac{e^{i\phi^j}\lambda^j(z^j + a^j w^j)}{1 - 2i\bar{a}^j z^j - (r^j + i|a^j|^2)w^j} \\
w^j &\mapsto \frac{(\lambda^j)^2 w^j}{1 - 2i\bar{a}^j z^j - (r^j + i|a^j|^2)w^j} \\
\Phi_2 : z &\mapsto p + z \\
w &\mapsto q + w + 2i\langle z, p\rangle, \quad \mathrm{Im}\, q = \langle p, p\rangle
\end{aligned}$$

for $j = 1, 2$, where $\lambda^j > 0, \phi^j, r^j \in \mathbf{R}, a^j \in \mathbf{C}$.

For the formulation of the main result, concerning the domains Ω_{-1} and Ω_0 it is convenient to introduce the following notation:

Let $\tau : \mathbf{C}^2 \to gl(2, \mathbf{C})$ be the lifting of the form:

$$\tau : z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \mapsto Z = \begin{pmatrix} z^1 & \delta z^2 \\ z^2 & z^1 \end{pmatrix},$$

$\delta = 0, -1$.

Let $a = (a^1, a^2) \in \mathbf{C}^2, r = (r^1, r^2) \in \mathbf{R}^2$,

$$\begin{aligned}
\Delta &= \mathrm{id}\ - 2i\tau(z)\tau(a) - (\tau(r) + i\tau(a)\overline{\tau(a)})\tau(w) = \\
&= \mathrm{id}\ - 2i \begin{pmatrix} \langle z, a\rangle^1 & \delta\langle z, a\rangle^2 \\ \langle z, a\rangle^2 & \langle z, a\rangle^1 \end{pmatrix} \\
&\quad - \begin{pmatrix} \langle w, r - i\langle a, a\rangle\rangle^1 & \delta\langle w, r - i\langle a, a\rangle\rangle^2 \\ \langle w, r - i\langle a, a\rangle\rangle^2 & \langle w, r - i\langle a, a\rangle\rangle^1 \end{pmatrix}
\end{aligned}$$

$\delta = 0, -1$.

We prove the following

Theorem 1 *Each element $\Phi_\delta \in \mathrm{Aut}(\Omega_\delta)$, where $\delta = 0, -1$ admits a representation of the form $\Phi_\delta = \Phi_\delta^3 \circ \Phi_\delta^2 \circ \Phi_\delta^1$:*

$$\begin{aligned} \Phi_\delta^1 : z &\mapsto \Delta^{-1} \begin{pmatrix} z^1 + \langle w, \bar{a} \rangle^1 \\ z^2 + \langle w, \bar{a} \rangle^2 \end{pmatrix} \\ w &\mapsto \Delta^{-1} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \Phi_\delta^2 : \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} &\mapsto C \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \\ \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} &\mapsto \rho \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \end{aligned}$$

where $C \in \mathrm{GL}(2, \mathbf{C}), \rho \in \mathrm{GL}(2, \mathbf{R})$ of the form

$$C = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad \rho = C\bar{C}$$

with $A, B \in \mathbf{C}$ in the elliptic case and

$$C = e^{i\phi} \begin{pmatrix} \lambda & 0 \\ \zeta & \mu \end{pmatrix}, \quad \rho = \begin{pmatrix} \lambda^2 & 0 \\ 2\mathrm{Re}\,\zeta\lambda & \lambda\mu \end{pmatrix}$$

where $\phi, \lambda, \mu \in \mathbf{R}$ and $\zeta \in \mathbf{C}$ in the parabolic case.

$$\begin{aligned} \Phi_\delta^3 : z &\mapsto p + z \\ w &\mapsto q + w + 2i\langle z, p \rangle, \end{aligned}$$

with $(p, q) \in Q_\delta$.

Remark. Without using the matrix notation the transformations take in the parabolic case the form

$$
\begin{aligned}
\Phi_0^1 : z^1 &\mapsto \frac{z^1 + \langle w, \bar{a}\rangle^1}{1 - 2i\langle z, a\rangle^1 - \langle w, r - i\langle a, a\rangle\rangle^1} \\
z^2 &\mapsto \frac{z^2 + \langle w, \bar{a}\rangle^2}{1 - 2i\langle z, a\rangle^1 - \langle w, r - i\langle a, a\rangle\rangle^1} \\
&\quad + \frac{(z^1 + \langle w, \bar{a}\rangle^1)(2i\langle z, a\rangle^2 + \langle w, r - i\langle a, a\rangle\rangle^2)}{(1 - 2i\langle z, a\rangle^1 - \langle w, r - i\langle a, a\rangle\rangle^1)^2} \\
w^1 &\mapsto \frac{w^1}{1 - 2i\langle z, a\rangle^1 - \langle w, r - i\langle a, a\rangle\rangle^1} \\
w^2 &\mapsto \frac{w^2}{1 - 2i\langle z, a\rangle^1 - \langle w, r - i\langle a, a\rangle\rangle^1} \\
&\quad + \frac{w^1(2i\langle z, a\rangle^2 + \langle w, r - i\langle a, a\rangle\rangle^2)}{(1 - 2i\langle z, a\rangle^1 - \langle w, r - i\langle a, a\rangle\rangle^1)^2} \\
\Phi_0^2 : z &\mapsto e^{i\phi + \lambda} \begin{pmatrix} e^{\mu} & 0 \\ \zeta & e^{-\mu} \end{pmatrix} z \\
w &\mapsto e^{2\lambda} \begin{pmatrix} e^{2\mu} & 0 \\ 2e^{\mu} Re\zeta & 1 \end{pmatrix}, \\
\Phi_0^3 : z &\mapsto p + z \\
w &\mapsto q + w + 2i\langle z, p\rangle,
\end{aligned}
$$

where $(p, q) \in Q_0, r^1, r^2, \lambda, \phi, \mu \in \mathbf{R}, a^1, a^2, \zeta \in \mathbf{C}$.

In the elliptic case it is convenient to use coordinates, where Q_{-1} has the form $v^1 = \text{Im } z^1\bar{z}^2, v^2 = \text{Re } z^1\bar{z}^2$. Then the corresponding transformations are

$$
\begin{aligned}
\Psi^1 : z^1 &\mapsto \frac{z^1 - a^1(w^1 - iw^2)}{1 - \bar{a}^2 z^1 + a^1\bar{a}^2(w^1 - iw^2)} \\
z^2 &\mapsto \frac{z^2 + a^2(w^1 + iw^2)}{1 - \bar{a}^1 z^2 - \bar{a}^1 a^2(w^1 + iw^2)} \\
w^1 &\mapsto \frac{1}{2}\frac{w^1 - iw^2}{1 - \bar{a}^2 z^1 + a^1\bar{a}^2(w^1 - iw^2)} + \\
&\quad + \frac{1}{2}\frac{w^1 + iw^2}{1 - \bar{a}^1 z^2 - \bar{a}^1 a^2(w^1 + iw^2)}
\end{aligned}
$$

$$
\begin{aligned}
w^2 &\mapsto \frac{i}{2}\frac{w^1 - iw^2}{1 - \bar{a}^2 z^1 + a^1\bar{a}^2(w^1 - iw^2)} \\
&\quad -\frac{i}{2}\frac{w^1 + iw^2}{1 - \bar{a}^1 z^2 - \bar{a}^1 a^2(w^1 + iw^2)} \\
\Psi^2 : z &\mapsto \begin{pmatrix} \frac{1}{1-r(w^1-iw^2)} & 0 \\ 0 & \frac{1}{1-\bar{r}(w^1+iw^2)} \end{pmatrix} z \\
w &\mapsto \begin{pmatrix} \frac{1}{2}\frac{w^1-iw^2}{1-r(w^1-iw^2)} + \frac{1}{2}\frac{w^1+iw^2}{1-\bar{r}(w^1+iw^2)} \\ \frac{i}{2}\frac{w^1-iw^2}{1-r(w^1-iw^2)} - \frac{i}{2}\frac{w^1+iw^2}{1-\bar{r}(w^1+iw^2)} \end{pmatrix} \\
\Phi^2_{-1} : z &\mapsto e^{\lambda+i\phi}\begin{pmatrix} e^{\mu+i\theta} & 0 \\ 0 & e^{-\mu-i\theta} \end{pmatrix} z \\
w &\mapsto e^{2\lambda}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} w, \\
\Phi^3_{-1} : z &\mapsto p + z \\
w &\mapsto q + w + 2i\langle z, p\rangle,
\end{aligned}
$$

where $(p,q) \in Q_{-1}, a^1, a^2, r \in \mathbf{C}, \lambda, \mu, \theta, \phi \in \mathbf{R}$. $\Phi^1_{-1} = \Psi^2 \circ \Psi^1$.

This theorem leads to the notion of biholomorphically invariant 2 dimensional "chains", analogous to the Chern-Moser chains on a hypersurface which are in the case of hyperquadrics the intersections with complex lines.

A k dimensional surface Γ on $Q \subset \mathbf{C}^{n+k}$ is called a chain if there exists $\Phi \in \mathrm{Aut}Q$ such that

$$\Phi(\Gamma) = \Gamma_0 = \{(z,w) \in \mathbf{C}^{n+k} : v = 0, z = 0\}.$$

The k-plane Γ_0 is called standard chain.

Corollary 1 *Chains passing throught the origin are 2 dimensional real-analytic surfaces of the form*

$$
\begin{aligned}
z &= \Delta^{-1}\langle a, u\rangle \\
w &= \Delta^{-1} u
\end{aligned}
$$

where Δ is as in Theorem 1, and $u = (u^1, u^2)$ with $u^1, u^2 \in \mathbf{R}$.

The explicit formulas for the automorphisms provide an obvious extension of the automorphisms.

Corollary 2 *Any local CR diffeomorphism of a nondegenerate quadric in* $\mathbf{C}^4$ *extends to a birational map of* $\mathbf{C}^4$ *which degree does not exceed 2.*

This Corollary provides the general theorem of Henkin and Tumanov [3, 5] with precise estimate of the degree.

In Section 5 we give a linear representation of the automorphism groups in $\mathbf{C}^6$.

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2 Motivations from the case of a hypersphere $S \subset \mathbf{C}^2$

We show that the projective transformations which compose the isotropy group of a hypershere can be obtained by "gradual normalization" of the equation of S.

In the next section we will see that the considerations concerning the hypersphere can be applied to the codimension 2 case by means of some matrix substitution. Using this result we can define some 2 dimensional submanifolds at the outlined quadrics Q being analogous to the Chern Moser chains at the oulined quadrics and obtain the transformations Φ_δ^1.

Let $z, w = u + iv$ - coordinates in $\mathbf{C}^2$. We consider $S = S \subset \mathbf{C}^2$ given by $v = |z|^2$.

The isotropy group $I_0(S)$ consists of projective transformations and its Lie algebra $\mathcal{I}(S)$ is composed with vector fields:

$$((\mu + i\phi)z + aw + 2i\bar{a}z^2 + rwz)\frac{\partial}{\partial z} + (2\mu w + 2i\bar{a}zw + rw^2)\frac{\partial}{\partial w},$$

where $\mu, \phi, r \in \mathbf{R}, a \in \mathbf{C}$.

Let $\Phi \in I_0(S)$. Without loss of generality we may assume that the tangent map to Φ restricted to $T_0^{\mathbf{C}}S$ is identical, since any automorphism is a composition of such a Φ and a linear transformation of the form $z \mapsto \lambda e^{i\phi}z, w \mapsto \lambda^2 w$.

We will represent Φ as composition of several maps which are not automorphisms of S.

S contains a straight line $l : v = 0, z = 0$. We denote by $\gamma_\Phi : z = p(t), w = q(t)$ - a Φ-"characteristic" curve on S, the so called "chain", that is mapped to l by Φ.

We take some "natural" parameter on γ and look for the first map Ψ_1 being defined in some neighbourhood of the origin in $\mathbf{C}^2$ in the form

$$
\begin{aligned}
\Psi_1^{-1} : z &\mapsto z + p(w) + 2iT(z,w), \\
w &\mapsto q(w) + 2ig(z,w), \\
&\quad T = O(z^2), g = O(z),
\end{aligned}
$$

choosing T and g to eliminate as many as possible terms in the new equation of S, using all given functional freedom in T and g, and leaving one parameter $a = \frac{dp}{du}\Big|_0$ free.

Then we find Ψ_2, having the form

$$
\begin{aligned}
z &\mapsto e^{i\theta(w)}\sqrt{h'(w)}z \\
w &\mapsto h(w), \\
&\quad h(u), \theta(u) \in \mathbf{R}, h'(0) = 1,
\end{aligned}
\tag{4}
$$

where we use all the freedom in θ and h to eliminate some other terms in the equation , leaving $\eta = \frac{d^2h}{(dw)^2}\Big|_0$ free. It is convenient to represent Ψ_2 in the form $\Phi_2 \circ \Psi_2'$, where Φ_2 is some automorphism of S and for Ψ_2' $\eta = 0$.

We set $\Phi_1 := \Psi_2' \circ \Psi_1$

Thus, we have found a unique $\tilde{\Phi} = \Phi_2 \circ \Phi_1$ with the prescribed set of parameters (a, η), that "kills" a number of "observable" terms of the equation. Since there exists an automorphism $\Phi \in I_0(S)$ with the same parameters (a, η), it follows that $\tilde{\Phi} = \Phi$.

We emphasize that we "kill" only some terms of the power series, which are easy to observe, until we "eat up" all the possible functional freedom in the transformation and do not pay attention to the other terms.

The precise computation gives us the following:

Consider $S : v = |z|^2$ and apply the transformation

$$
\begin{aligned}
z &\mapsto z + p(w) + 2iT(z,w), \\
w &\mapsto q(w) + 2ig(z,w).
\end{aligned}
$$

We choose T, g, p, q such that in the new equation

$$v = |z|^2 + \sum_{k,l} F_{kl}(z, \bar{z}, u)$$

(F_{kl} are polynomials in $(z, \bar{z})$ of degree (k, l) with coefficients analytic in u.) the terms $F_{10}, F_{k0}, F_{k1}, F_{32}, k = 2, 3, \ldots$ vanish.

We choose the parameter on $\gamma : z = p(u), w = q(u)$ so that

$$\frac{dq}{du} = 1 + 2i\frac{dp}{du}\bar{p} \tag{5}$$

The vanishing of $F_{k0}, k = 1, 2, \ldots$ implies for T and g:

$$
\begin{aligned}
g_1 &= z\bar{p}(w), \\
g_k &= 2iT_k\bar{p}(w), k = 2, 3, \ldots
\end{aligned}
$$

where g_k and T_k are polynomials in z of degree k with coefficients being analytic functions of w, such that

$$
\begin{aligned}
g &= \sum_{k=1}^{\infty} g_k, \\
T &= \sum_{k=2}^{\infty} T_k,
\end{aligned}
\tag{6}
$$

We introduce the operator D acting as follows

$$D(F(z,w)) = iv\frac{\partial F}{\partial w}(u)\bigg|_{v=|z|^2}.$$

We will denote the derivative $\frac{\partial\omega}{\partial u}(u)$ by ω'

The vanishing of $F_{k1}, k = 2, 3, \ldots$ implies then:

$$
\begin{aligned}
Dg_1 &= 2iT_2\bar{z} - i|z|^2 z\bar{p}' \\
Dg_k &= 2iT_{k+1}\bar{z} + 2i(-i|z|^2)T_k\bar{p}' + 2i(i|z|^2)T_k{}'\bar{p}, k = 2, 3, \ldots
\end{aligned} \tag{7}
$$

On the other hand, deriving (6), we obtain

$$
\begin{aligned}
Dg_1 &= zi|z|^2 z\bar{p}' \\
Dg_k &= 2i(i|z|^2)T_k\bar{p}' + 2i(i|z|^2)T_k{}'\bar{p}, k = 2, 3, \ldots
\end{aligned} \tag{8}
$$

It follows immediately,

$$T_k = (2i)^{k-2} z^k (\bar{p}'(w))^{k-1}, k = 2, 3, \ldots,$$

and, hence, Ψ_1^{-1} takes the form

$$
\begin{aligned}
z &\mapsto p(w) + \frac{z}{1 - 2i\bar{p}'(w)z} \\
w &\mapsto q(w) + \frac{2i\bar{p}(w)z}{1 - 2i\bar{p}'(w)z}
\end{aligned} \tag{9}
$$

Now we are going to compute the term F_{22}:

$$
\begin{aligned}
F_{22} &= -\frac{1}{2i}q''\frac{(i|z|^2)^2}{2} + \frac{1}{2i}q''\frac{(-i|z|^2)^2}{2} + 4T_2\bar{T}_2 \\
&\quad + p''\bar{p}\frac{(i|z|^2)^2}{2} + p\bar{p}''\frac{(-i|z|^2)^2}{2} + \bar{p}'p'(-i|z|^2)(i|z|^2) \\
&= 6|z|^4|p'|^2
\end{aligned}
$$

Using the vanishing of F_{23} we determine the function p. Therefore, we have to compute F_{23}:

$$
\begin{aligned}
F_{23} &= -\bar{g}_1'(-iF_{22}) - \bar{g}_1''\frac{(-i|z|^2)^2}{2} + p'\bar{z}iF_{22} + \\
&\quad p''\bar{z}\frac{(i|z|^2)^2}{2} + 4T_2\bar{T}_3 - 2iz\bar{T}'(-i|z|^2) \\
&= -2|z|^4(p''\bar{z} - 2i\bar{z}|p'|^2 p')
\end{aligned}
$$

Hence, p satisfies the equation

$$p'' = 2i|p'|^2 p',$$

with $p(0) = 0$, and, therefore,

$$p = \frac{e^{2i\theta}}{2iR}(e^{2iR^2 u} - 1),$$

where $R > 0$ and $\theta \in \mathbf{R}$.
From condition (5) and $q(0) = 0$, we obtain

$$q = \frac{1}{2iR^2}(e^{2iR^2 u} - 1).$$

Now we will try to eliminate the term F_{22} by means of some transformation of the form

$$\begin{aligned} z^* &= U(w)z \\ w^* &= w, \end{aligned} \tag{10}$$

where $|U(u)| = 1$.
The vanishing condition of F_{33} gives the equation

$$iU\bar{U}' - iU'\bar{U} = -6|p'|^2.$$

Solving this equation we obtain

$$U = e^{-3iR^2 u}.$$

After the transformation (10) the term F_{33} takes the following form:

$$F_{33} = -\frac{2}{3}R^4|z|^6.$$

By means of a transformation

$$\begin{aligned} z^* &= \sqrt{h'(w)}z \\ w^* &= h(w), \end{aligned} \tag{11}$$

we shall eliminate F_{33}. Therefore h has to satisfy the equation

$$h''' - \frac{3}{2}\frac{(h'')^2}{h'} - 2R^4 h' = 0$$

Using the substitution $h' = \frac{1}{s^2}$ one finds a particular solution

$$h' = \frac{1}{\cos^2(R^2 u)}$$

It follows that

$$h = \frac{\tan(R^2 u)}{R^2}.$$

Hence,

$$\begin{aligned} z^* &= \frac{z}{\cos(R^2 w)} \\ w^* &= \frac{\tan(R^2 w)}{R^2}. \end{aligned} \tag{12}$$

Since we have chosen only a particular solution for eliminating F_{33} we look now for transformations of the form (11) preserving F_{33}.

Then $h = h_0(w)$ has to satisfy the homogenious equation

$$h''' - \frac{3}{2}\frac{(h'')^2}{h'} = 0. \tag{13}$$

Using the substitution $h' = \frac{1}{s^2}$ once more, we find

$$h_0' = \frac{1}{(\gamma - ru)^2}$$

We have required that $h'(0) = 1$. Therefore

$$h_0' = \frac{1}{(1 - ru)^2}$$

and,

$$h_0 = \frac{u}{1 - ru}$$

The parameter $\eta = 2r$ and the "homogenious reparametrization" takes the form

$$\begin{aligned} z^* &= \frac{z}{1-rw} \\ w^* &= \frac{w}{1-rw}. \end{aligned} \tag{14}$$

Now we can write down the explicit maps Φ_1 and Φ_2:

$$\begin{aligned} (\Phi_1)^{-1} : z &= \frac{\tilde{z} + a\tilde{w}}{1 - i|a|^2\tilde{w} - 2i\bar{a}\tilde{z}} \\ w &= \frac{\tilde{w}}{1 - i|a|^2\tilde{w} - 2i\bar{a}\tilde{z}} \end{aligned}$$

with $a = Re^{2i\theta}$.

$$\begin{aligned} \Phi_2 : z^* &= \frac{z}{1-rw} \\ w^* &= \frac{w}{1-rw}. \end{aligned}$$

The chain in the parameter obtained by transformation (12) is then

$$\begin{aligned} z &= \frac{a}{1 - i|a|^2 u} \\ w &= \frac{1}{1 - i|a|^2 u} \end{aligned}$$

It is seen to be the intersection of the quadric with the complex plane $z = aw$.

3 Reduction to the hypersphere case

Let $\mathcal{A}$ be the algebra of complex 2×2 matrices. For any real δ there are commutative subalgebras $\mathcal{A}_\delta$ consisting of matrices of the form:

$$\begin{pmatrix} \zeta^1 & \delta\zeta^2 \\ \zeta^2 & \zeta^1 \end{pmatrix}$$

Since δ is real, the following conjugation is correctly defined:

$$\overline{\begin{pmatrix} \zeta^1 & \delta\zeta^2 \\ \zeta^2 & \zeta^1 \end{pmatrix}} = \begin{pmatrix} \bar{\zeta}^1 & \delta\bar{\zeta}^2 \\ \bar{\zeta}^2 & \bar{\zeta}^1 \end{pmatrix}$$

Notice that any of these algebras with real structure is equivalent to one of the following: $\mathcal{A}_{-1}, \mathcal{A}_0, \mathcal{A}_1$.

Now we consider the equation defining the hypersphere in the algebras $\mathcal{A}_{-1}, \mathcal{A}_0, \mathcal{A}_1$. We obtain

$$\begin{pmatrix} v^1 & \delta v^2 \\ v^2 & v^1 \end{pmatrix} = \begin{pmatrix} |z^1|^2 + \delta|z^2|^2 & \delta(z^1\bar{z}^2 + z^2\bar{z}^1) \\ z^1\bar{z}^2 + z^2\bar{z}^1 & |z^1|^2 + \delta|z^2|^2 \end{pmatrix} \tag{15}$$

These equations define Q_1, Q_0, Q_{-1} for $\delta = 1, 0, -1$ respectively.

Substituting in the automorphisms Φ_1 and Φ_2 of the hypersphere $\xi = z, w, a, r$ by

$$\begin{pmatrix} \xi^1 & \delta\xi^2 \\ \xi^2 & \xi^1 \end{pmatrix}$$

we obtain automorphism•groups of real dimension 6.

Now we can write down the equations of the chains being analogous to the Moser Chern chains. These chains are the 2 dimensional surfaces which can be mapped by an automorphism to the plane $v = 0, z = 0$.

They have the form

$$\begin{aligned} z &= (1 - ia\bar{a}u)^{-1}au \\ w &= (1 - ia\bar{a}u)^{-1}u. \end{aligned}$$

It follows that they are the intersections of the quadric with special complex 2-planes

$$z = aw,$$

where $a \in \mathcal{A}_\delta$, the "matrix lines".

In the next section we will compute the groups of linear automorphisms. Adding these groups we get automorphism groups of dimension 10 in the elliptic and 11 in the parabolic case. It follows then from a result of Beloshapka (see [6]) that these are the complete automorphism groups.

Another way to verify that the transformations obtained above are all automorphisms Φ with

$$\left.\frac{\partial \Phi}{\partial z}\right|_{T_0 \mathbf{C}_Q} = \mathrm{id}$$

is the following:

We show that any holomorphic map

$$\begin{aligned} F : z &\mapsto z + p(w) + 2i \sum_{k=2}^{\infty} T_k(z,w), \\ w &\mapsto q(w) + 2i \sum_{k=1}^{\infty} g_k(z,w), \end{aligned}$$

with the property that the equation of the image of Q via F does not contain terms of degree $(1,0), (k,0), (k,1)$ for $k > 1, \ldots$ has the the form:

$$\begin{aligned} F : z^* &= \sum_{n=0}^{\infty} A_n(w) z^n \\ w^* &= \sum_{n=0}^{\infty} B_n(w) z^n, \end{aligned}$$

where z, w, z^*, w^* are matrices of the given form and $A_n(w), B_n(w)$ can be represented as

$$
\begin{aligned}
A_n &= \sum_{n=0}^{\infty} A_{n,m} w^m \\
B_n &= \sum_{n=0}^{\infty} B_{n,m} w^m .
\end{aligned}
$$

Then all considerations from the hypersphere case can be formally applied to the quadrics Q, with one exception concerning the "reparametrization" map Φ_3 in the parabolic case. We will return to this question at the end of the following section.

We need the following

Lemma 1 *Let $G : \mathbf{C}^2 \to \mathbf{C}^2$ be a holomorphic map defined in some neighbourhood of the origin with the property:*

$$
\begin{aligned}
\frac{\partial G^1}{\partial w^1} &= \frac{\partial G^2}{\partial w^2} \\
\frac{\partial G^1}{\partial w^2} &= \delta \frac{\partial G^2}{\partial w^1}
\end{aligned}
\tag{16}
$$

then

$$
\begin{pmatrix} G^1 & \delta G^2 \\ G^2 & G^1 \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} a_n & \delta b_n \\ b_n & a_n \end{pmatrix} \begin{pmatrix} w^1 & \delta w^2 \\ w^2 & w^1 \end{pmatrix}^n
$$

.

Proof. We prove that

$$
\sum_{k+l=m} \frac{\partial^n}{(\partial w^1)^k (\partial w^2)^l} \begin{pmatrix} G^1 & \delta G^2 \\ G^2 & G^1 \end{pmatrix} \frac{(w^1)^k (w^2)^l}{k!l!} =
$$

$$
\frac{1}{n!} \begin{pmatrix} \frac{\partial^n G^1}{\partial (w^1)^n} & \delta \frac{\partial^n G^2}{\partial (w^1)^n} \\ \frac{\partial^n G^2}{\partial (w^1)^n} & \frac{\partial^n G^1}{\partial (w^1)^n} \end{pmatrix} \begin{pmatrix} w^1 & \delta w^2 \\ w^2 & w^1 \end{pmatrix}^n .
$$

It can be verified, by induction, that

$$\begin{pmatrix} w^1 & \delta w^2 \\ w^2 & w^1 \end{pmatrix}^n = \frac{1}{2\sqrt{\delta}}((w^1+\sqrt{\delta}w^2)^n - (w^1-\sqrt{\delta}w^2)^n)\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}$$
$$+\frac{1}{2}((w^1+\sqrt{\delta}w^2)^n + (w^1-\sqrt{\delta}w^2)^n)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

It follows immediately from (16) that

$$\begin{aligned} \frac{\partial^n G^1}{(\partial w^1)^k(\partial w^2)^l} &= \sqrt{\delta}^l \frac{\partial^n G^1}{(\partial w^1)^n} \\ \frac{\partial^n G^2}{(\partial w^1)^k(\partial w^2)^l} &= \frac{\partial^n G^2}{(\partial w^1)^n}, \end{aligned}$$

if l is even, and

$$\begin{aligned} \frac{\partial^n G^1}{(\partial w^1)^k(\partial w^2)^l} &= \sqrt{\delta}^{l+1} \frac{\partial^n G^2}{(\partial w^1)^n} \\ \frac{\partial^n G^2}{(\partial w^1)^k(\partial w^2)^l} &= \frac{\partial^n G^1}{(\partial w^1)^n}, \end{aligned}$$

if l is odd. These identities prove the lemma. □

Now we deduce that F has the desired form.

From the condition that the terms $(0,k)$ in the new equation of Q vanish we derive

$$\begin{aligned} g_1(z,w) &= z\bar{p}(w) \\ g_k(z,w) &= T_k(z,w)\bar{p}(w) \text{ for } k>1. \end{aligned}$$

Here g_k, T_k and p are matrix-valued functions of the form

$$\begin{pmatrix} \xi^1 & \delta\xi^2 \\ \xi^2 & \xi^1 \end{pmatrix}$$

with $\xi = g_k, T_k, p$.

The vanishing of terms $(1,k)$ gives

$$T_k(z,w)\bar{z} = 2T_{k-1}D\bar{p}(w) \text{ for } k > 2. \tag{17}$$

It follows from the holomorphy of F that

$$\begin{aligned}\frac{\partial p^1}{\partial u^1} &= \frac{\partial p^2}{\partial u^2}\\ \frac{\partial p^1}{\partial u^2} &= \delta\frac{\partial p^2}{\partial u^1}.\end{aligned}$$

In fact, (17) is equivalent to

$$T_k(z,w)\bar{z} = 2iT_{k-1}\bar{p}' z\bar{z} + 2iT_{k-1}R(z^1\bar{z}^2 + z^2\bar{z}^1)$$

where

$$\bar{p}' = \begin{pmatrix} \frac{\partial\bar{p}^1}{\partial u^1} & \delta\frac{\partial\bar{p}^2}{\partial u^1} \\ \frac{\partial\bar{p}^2}{\partial u^1} & \frac{\partial\bar{p}^1}{\partial u^1} \end{pmatrix},$$

and,

$$R = \begin{pmatrix} \frac{\partial\bar{p}^1}{\partial u^2} - \delta\frac{\partial\bar{p}^2}{\partial u^1} & \delta\left(\frac{\partial\bar{p}^2}{\partial u^2} - \frac{\partial\bar{p}^1}{\partial u^1}\right) \\ \frac{\partial\bar{p}^2}{\partial u^2} - \frac{\partial\bar{p}^1}{\partial u^1} & \frac{\partial\bar{p}^1}{\partial u^2} - \delta\frac{\partial\bar{p}^2}{\partial u^1} \end{pmatrix}.$$

The holomorphy of T_k implies that

$$\frac{\partial}{\partial\bar{z}^i}R(z^1\bar{z}^2 + z^2\bar{z}^1)(\bar{z})^{-1} = 0$$

for $i = 1, 2$.
We get

$$R\begin{pmatrix} z^1\bar{z}^2 & \delta|z^2|^2 \\ |z^2|^2 & z^1\bar{z}^2 \end{pmatrix} = R\begin{pmatrix} |z^1|^2 & \delta\bar{z}^1 z^2 \\ \bar{z}^1 z^2 & |z^2|^2 \end{pmatrix} = 0$$

and, hence, $R = 0$.
Then (17) takes the form

$$T_k(z,w) = 2iT_{k-1}\bar{p}'(w)z.$$

It follows from Lemma 1 that p can be represented as a power series of matrices. This implies immediately that F has the desired form.

4 The linear automorphisms

We look for linear transformations of the form:

$$\begin{aligned} z &\mapsto Cz \\ w &\mapsto \rho w \end{aligned}$$

with $C \in \mathrm{GL}(2, \mathbf{C})$ and $\rho \in \mathrm{GL}(2, \mathbf{R})$ preserving the forms $\mathrm{Re}\, w^1 = |z^1|^2 + \delta|z^2|^2$ and $\mathrm{Re}\, w^2 = z^1\bar{z}^2 + z^2\bar{z}^1$.

The elements $X \in gl(2, \mathbf{C})$, $s \in gl(2, \mathbf{R})$ of the corresponding Lie algebra satisfy the following conditions:

$$\begin{aligned} \bar{x}_{21} + x_{21} &= s_{12} \\ \bar{x}_{12} + x_{12} &= \delta s_{21} \\ \bar{x}_{11} + x_{22} &= \bar{x}_{22} + x_{11} = s_{22} \\ \bar{x}_{11} + x_{11} &= s_{11} \\ \delta\bar{x}_{22} + \delta x_{22} &= \delta s_{11} \\ \bar{x}_{12} + \delta x_{21} &= \delta\bar{x}_{21} + x_{12} = s_{12}. \end{aligned}$$

If $\delta = 0$ it follows

$$X = i\phi \mathrm{id} + \begin{pmatrix} \lambda + \mu & 0 \\ \xi + i\eta & \lambda - \mu \end{pmatrix}.$$

If $\delta \neq 0$ it follows

$$X = i\phi \mathrm{id} + \begin{pmatrix} \lambda & \delta(\xi + i\eta) \\ \xi + i\eta & \lambda \end{pmatrix}.$$

Applying the exponential map we obtain the form of the matrices C:

$$C = e^{\lambda + i\phi} \begin{pmatrix} e^{\mu} & 0 \\ \zeta & e^{-\mu} \end{pmatrix}.$$

if $\delta = 0$. Here $\lambda, \mu, \phi, \xi, \eta \in \mathbf{R}, \zeta \in \mathbf{C}$.

If $\delta \neq 0$:

$$C = \begin{pmatrix} A & \delta B \\ B & A \end{pmatrix},$$

where $A, B \in \mathbf{C}$ such that $A^2 - \delta B^2 \neq 0$.
The corresponding matrices ρ have the form:
in the parabolic case, $\delta = 0$

$$\rho = e^{2\lambda} \begin{pmatrix} e^{2\mu} & 0 \\ 2e^{\mu}\mathrm{Re}\ \zeta & 1 \end{pmatrix},$$

and in the case $\delta \neq 0$

$$\rho = \begin{pmatrix} |A|^2 + \delta|B|^2 & \delta(A\bar{B} + b\bar{A}) \\ A\bar{B} + B\bar{A} & |A|^2 + \delta|B|^2 \end{pmatrix}.$$

Notice, that a 4 dimensional group of linear (C, ρ) transformations can be obtained from the linear group in the hypersphere case substituting numbers by matrices of given form. In the hyperbolic and elliptic cases this is the whole linear group, but in the parabolic case an additional parameter appears. Therefore, we have to show that this parameter does not give any additional freedom in the "reparametrization" map Φ_3.

Let $\tilde{\Phi}$ be an automorphism of the form

$$\begin{array}{rcl} z^1 & \mapsto & \lambda(w)z^1 \\ z^2 & \mapsto & \xi(w)z^1 + \mu(w)z^2 \\ w^1 & \mapsto & h^1(w) \\ w^2 & \mapsto & h^2(w) \end{array}$$

with

$$\begin{array}{ll} \frac{\partial h^1}{\partial w^1} = \lambda^2(w) & \frac{\partial h^1}{\partial w^2} = 0 \\ \frac{\partial h^2}{\partial w^1} = 2\lambda(w)\xi(w) & \frac{\partial h^2}{\partial w^2} = \lambda(w)\mu(w) \end{array}.$$

It follows that h^1 satisfies the one dimensional equation (13), and

$$h^1 = \frac{w^1}{1 - rw^1}, \quad \lambda(w) = \frac{1}{1 - rw^1}$$

for some $r \in \mathbf{R}$. Therefore, we may suppose, without loss of generality, that $\lambda = 1$ and $h^1 = w^1$. After applying $\tilde{\Phi}$ in the equation of Q_0 appears the term $\frac{\partial \mu}{\partial u^1}|z^1|^2 \mathrm{Im}\,(z^1\bar{z}^2) + 2\frac{\partial \mu}{\partial u^2}\mathrm{Re}\,(z^1\bar{z}^2)\mathrm{Im}\,(z^1\bar{z}^2)$. Hence, μ must be a constant.

5 Linear representation of Aut$Q$ in $\mathbf{C}^6$

Let $\mathcal{A}_\delta^3$ be the $\mathcal{A}_\delta$ module of triples $(\Theta_0, \Theta_1, \Theta_2)$ with $\Theta_i \in \mathcal{A}_\delta$. By $\mathcal{A}_\delta^*$ we denote the ring of invertible elements of $\mathcal{A}_\delta$ and by $\hat{\mathcal{A}}_\delta^3$ the factor space under the natural action of $\mathcal{A}_\delta^*$. $\hat{\mathcal{A}}_\delta^3$ is a compact manifold which can be considered as a compactification of $\mathbf{C}^4 = \mathcal{A}_\delta^2$ by the embedding

$$(z, w) \mapsto (\mathrm{id}\ , z, w).$$

Now, any automorphism of Q_1, Q_0, Q_{-1} can be represented as a linear transformation of $\mathbf{C}^6$ in the following way:

Let Q_1, Q_0, Q_{-1} be given in the form (15). Then the automorphisms can be written as a composition of

$$\begin{aligned} z &\mapsto (z + aw)(\mathrm{id}\ - 2i\bar{a}z - (r + ia\bar{a})w)^{-1} \\ w &\mapsto w(\mathrm{id}\ - 2i\bar{a}z - (r + ia\bar{a})w)^{-1}, \end{aligned}$$

where $a, r \in \mathcal{A}_\delta$, with $r = \bar{r}$, and a linear (C, ρ) transformation.

The first map induces the following linear transformation in $\mathcal{A}_\delta^3$:

$$\begin{aligned} \Theta_0 &\mapsto \Theta_0 - 2i\bar{a}\Theta_1 - (r + ia\bar{a})\Theta_2 \\ \Theta_1 &\mapsto \Theta_1 + a\Theta_2 \\ \Theta_2 &\mapsto \Theta_2. \end{aligned}$$

Hence,

$$\begin{pmatrix} \Theta_0^1 \\ \Theta_0^2 \end{pmatrix} \mapsto \begin{pmatrix} \Theta_0^1 \\ \Theta_0^2 \end{pmatrix} - 2i\bar{a}\begin{pmatrix} \Theta_1^1 \\ \Theta_1^2 \end{pmatrix} - (r + ia\bar{a})\begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} \Theta_1^1 \\ \Theta_1^2 \end{pmatrix} &\mapsto \begin{pmatrix} \Theta_1^1 \\ \Theta_1^2 \end{pmatrix} + a \begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix} \\
\begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix} &\mapsto \begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix}.
\end{aligned}$$

Together with the linear transformation we obtain

$$\begin{aligned}
\begin{pmatrix} \Theta_0^1 \\ \Theta_0^2 \end{pmatrix} &\mapsto \begin{pmatrix} \Theta_0^1 \\ \Theta_0^2 \end{pmatrix} - 2i\bar{a} \begin{pmatrix} \Theta_1^1 \\ \Theta_1^2 \end{pmatrix} - (r + ia\bar{a}) \begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix} \\
\begin{pmatrix} \Theta_1^1 \\ \Theta_1^2 \end{pmatrix} &\mapsto C \begin{pmatrix} \Theta_1^1 \\ \Theta_1^2 \end{pmatrix} + Ca \begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix} \\
\begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix} &\mapsto \rho \begin{pmatrix} \Theta_2^1 \\ \Theta_2^2 \end{pmatrix}.
\end{aligned}$$

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