NONCOMMUTATIVE LOOPS OVER LIE ALGEBRAS

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ABSTRACT. The aim of this paper is to introduce and study Lie algebras over noncommutative rings. For any Lie algebra $\mathfrak{g}$ sitting inside an associative algebra A and any associative algebra $\mathcal{F}$ we introduce and study the $\mathcal{F}$-loop algebra, which is the Lie subalgebra of $\mathcal{F} \otimes A$ generated by $\mathcal{F} \otimes \mathfrak{g}$. In most examples A is the universal enveloping algebra of $\mathfrak{g}$. Our description of the $\mathcal{F}$-loop algebra has a striking resemblance to the commutator expansions of $\mathcal{F}$ used by M. Kapranov in his approach to noncommutative geometry.

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0. INTRODUCTION

The aim of this paper is to introduce and study Lie algebras over noncommutative rings as noncommutative loop Lie algebras. A naive definition of a Lie algebra as a bimodule over a noncommutative algebra $\mathcal{F}$ does not bring any interesting examples beyond $GL_n(\mathcal{F})$ and the corresponding Lie algebra $gl_n(\mathcal{F}) = Mat_n(\mathcal{F}) = \mathcal{F} \otimes gl_n$. Even the special Lie algebra $sl_n(\mathcal{F})$ (which is the set of all matrices in $gl_n(\mathcal{F})$ whose trace belongs to the commutator $[\mathcal{F}, \mathcal{F}]$) is not an $\mathcal{F}$-bimodule. Similarly, the special linear group $SL_n(\mathcal{F})$ is not defined by equations but rather by congruences given by the Dieudonne determinant (see [1]). This is why the approach to classical groups over rings started by J. Dieudonne in [4] and continued by O. T. O'Meara and others (see [5]) does not lead to algebraic groups. Also, unlike in the "commutative case", these methods does not employ rich structure theory of Lie algebras.

Contrary, to this approach, we start this paper by introducing "noncommutative" Lie algebras as *noncommutative loop Lie algebras*. Roughly speaking, for any Lie algebra $\mathfrak{g}$ sitting inside an associative algebra A and any associative algebra $\mathcal{F}$ we define $(\mathfrak{g}, A)(\mathcal{F})$ to be the Lie subalgebra of $\mathcal{F} \otimes A$ generated by $\mathcal{F} \otimes \mathfrak{g}$ (usually we set $A = U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$). In other words, $(\mathfrak{g}, A)(\mathcal{F})$ can be viewed as an $\mathcal{F}$-envelope of $\mathfrak{g}$ in $\mathcal{F} \otimes A$.

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Our main result (Theorem 4.2) is an explicit computation $(\mathfrak{g}, A)(\mathcal{F})$ for all semisimple Lie algebras and all algebras A containing $\mathfrak{g}$. The formula for $(\mathfrak{g}, A)(\mathcal{F})$ has a striking resemblance to the commutator expansions of $\mathcal{F}$ used by M. Kapranov in [6] and then by M. Kontsevich and A. Rosenberg in [7] as an important tool in noncommutative geometry. Our method works for a much larger class of pairs $(\mathfrak{g}, A)$, where A is an associative algebra containing $\mathfrak{g}$. Generalizing Theorem 4.2, we provide a similar computation of the noncommutative loop Lie algebra $(\mathfrak{g}, A)(\mathcal{F})$ for all *perfect pairs* in the sense of Definition 4.1.

Our concluding result is a very compact formula (Theorem 4.8) for $(sl_2(\mathbb{Q}), A)(\mathcal{F})$, which, apparently has a deep physical meaning.

This paper is continuation of our study of algebraic groups over noncommutative rings and their representations started in [2]. In the next paper we will focus on noncommutative semisimple groups, their geometric structure, and representations.

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Throughout the paper, $\mathcal{N}$ will denote the category which objects are algebras over $\mathbb{Q}$ (not necessarily with 1) and arrows are algebra homomorphisms.

1. Commutator expansions

Let $\mathcal{F}$ be an object of $\mathcal{N}$. For $k \geq 0$ define the k-th *commutator space* $\mathcal{F}^{(k)}$ of $\mathcal{F}$ recursively as $\mathcal{F}^{(0)} = \mathcal{F}$, $\mathcal{F}^{(1)} = \mathcal{F}' = [\mathcal{F}, \mathcal{F}]$, $\mathcal{F}^{(2)} = \mathcal{F}'' = [\mathcal{F}, \mathcal{F}']$, …, $\mathcal{F}^{(k)} = [\mathcal{F}, \mathcal{F}^{(k-1)}]$, …, where for any subsets S_1, S_2 of $\mathcal{F}$ the notation $[S_1, S_2]$ stands for the linear span of all commutators $[a, b] = ab - ba$, $a \in S_1$, $b \in S_2$.

The recursive definition defines the canonical map $\pi_k : \mathcal{F}^{\otimes k} \twoheadrightarrow \mathcal{F}^{(k)}$. For any subset S of $\mathcal{F}$ set $S^{(k)} = \pi_k(S)$ for $k > 0$ and $S^{(0)}$ is the $\mathbb{Q}$-linear span of S in $\mathcal{F}$. Using this notation, for any subset $S \subset \mathcal{F}$ define the space $S^{(\bullet)}$ by

$$S^{(\bullet)} = \sum_{k \geq 0} S^{(k)} \tag{1.1}$$

The following result is obvious.

Lemma 1.1. *For any $S \subset \mathcal{F}$ the subspace $S^{(\bullet)}$ is the Lie subalgebra of $\mathcal{F}$ generated by S.*

Following [6] and [7], define the subspaces $I_k^\ell(\mathcal{F})$ by:

$$I_k^\ell(\mathcal{F}) = \sum_{\lambda} \mathcal{F}^{(\lambda_1)} \mathcal{F}^{(\lambda_2)} \cdots \mathcal{F}^{(\lambda_\ell)} ,$$

where the summation goes over all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in (\mathbb{Z}_{\geq 0})^\ell$ such that $\sum_{i=1}^{\ell} \lambda_i = k$. Denote also

$$I_k^{\leq \ell}(\mathcal{F}) := \sum_{1 \leq \ell' \leq \ell} I_k^{\ell'}(\mathcal{F}), I_k(\mathcal{F}) := I_k^{\leq \infty} = \sum_{\ell \geq 1} I_k^\ell(\mathcal{F}) . \tag{1.2}$$

Clearly, $\mathcal{F} I_k^\ell(\mathcal{F}), I_k^\ell(\mathcal{F})\mathcal{F} \subset I_k^{\ell+1}(\mathcal{F})$. Therefore, $I_k(\mathcal{F})$ is a two-sided ideal in $\mathcal{F}$. It is also easy to see that $I_k(\mathcal{F}) = I_k^k(\mathcal{F}) + \mathcal{F} \cdot I_k^k(\mathcal{F})$.

Lemma 1.2. *For each $k, \ell \geq 1$ one has:*
(a) $I_k^\ell(\mathcal{F}) \subset I_{k-1}^\ell(\mathcal{F})$, $I_k^{\leq \ell}(\mathcal{F}) \subset I_{k-1}^{\leq \ell}(\mathcal{F})$.
(b) $[\mathcal{F}, I_{k-1}^\ell(\mathcal{F})] \subset I_k^\ell(\mathcal{F})$, $[\mathcal{F}, I_{k-1}^{\leq \ell}(\mathcal{F})] \subset I_k^{\leq \ell}(\mathcal{F})$.
(c) $I_k^{\leq \ell+1}(\mathcal{F}) = \mathcal{F}[\mathcal{F}, I_{k-1}^{\leq \ell}(\mathcal{F})] + [\mathcal{F}, I_{k-1}^{\leq \ell+1}(\mathcal{F})]$.

Proof. To prove (a) and (b), we need the following obvious recursion for $I_k^\ell(\mathcal{F})$:

$$I_k^\ell(\mathcal{F}) = \sum_{i \geq 0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) \tag{1.3}$$

(with the natural convention that $I_{k'}^{\ell'}(\mathcal{F}) = 0$ if $k' < 0$). Then we prove (a) by induction in ℓ. If $\ell = 1$, the assertion becomes $\mathcal{F}^{(k)} \subset \mathcal{F}^{(k-1)}$. Iterating this inclusion and using the inductive hypothesis, we obtain

$$I_k^\ell(\mathcal{F}) = \sum_{i \geq 0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) = \mathcal{F} I_k^{\ell-1}(\mathcal{F}) + \sum_{i > 0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) \subset$$

$$\subset \mathcal{F} I_{k-1}^{\ell-1}(\mathcal{F}) + \sum_{i > 0} \mathcal{F}^{(i-1)} I_{k-i}^{\ell-1}(\mathcal{F}) = \sum_{i \geq 0} \mathcal{F}^{(i)} I_{k-i-1}^{\ell-1}(\mathcal{F}) = I_{k-1}^\ell(\mathcal{F}) \ .$$

This proves (a).

Prove (b) also by induction in ℓ. If $\ell = 1$, the assertion becomes $[\mathcal{F}, \mathcal{F}^{(k-1)}] \subset \mathcal{F}^{(k)}$, which is obvious. Using the inductive hypothesis, we obtain

$$[\mathcal{F}, I_{k-1}^\ell(\mathcal{F})] = \sum_{i \geq 1} [\mathcal{F}, \mathcal{F}^{(i-1)} I_{k-i}^{\ell-1}(\mathcal{F})] \subset$$

$$\subset \sum_{i \geq 1} [\mathcal{F}, \mathcal{F}^{(i-1)}] I_{k-i}^{\ell-1}(\mathcal{F}) + \mathcal{F}^{(i-1)} [\mathcal{F}, I_{k-i}^{\ell-1}(\mathcal{F})] \subset \sum_{i \geq 0} \mathcal{F}^{(i)} I_{k-i}^{\ell-1}(\mathcal{F}) = I_k^\ell(\mathcal{F}) \ .$$

This proves (b).

Prove (c). Obviously, $I_k^{\leq \ell+1}(\mathcal{F}) \supset \mathcal{F}[\mathcal{F}, I_{k-1}^{\leq \ell}(\mathcal{F})] + [\mathcal{F}, I_{k-1}^{\leq \ell+1}(\mathcal{F})]$ by (b). Therefore, it suffices to prove the opposite inclusion

$$I_k^{\leq \ell+1}(\mathcal{F}) \subset \mathcal{F}[\mathcal{F}, I_{k-1}^{\leq \ell}(\mathcal{F})] + [\mathcal{F}, I_{k-1}^{\leq \ell+1}(\mathcal{F})] \ .$$

We will use the following obvious consequence of (1.3):

$$I_k^{\leq \ell+1}(\mathcal{F}) = \sum_{i \geq 0} \mathcal{F}^{(i)} I_{k-i}^{\leq \ell}(\mathcal{F}) \ .$$

Therefore, it suffices to prove that

$$\mathcal{F}^{(i)} I_{k-i}^{\leq \ell}(\mathcal{F}) \subset \mathcal{F}[\mathcal{F}, I_{k-1}^{\leq \ell}(\mathcal{F})] + [\mathcal{F}, I_{k-1}^{\leq \ell+1}(\mathcal{F})] \tag{1.4}$$

for all $i \geq 0$, $\ell \geq 1$, $k \geq 1$. We prove (1.4) by induction in all pairs (ℓ, i) ordered lexicographically. Indeed, suppose that the assertion is proved for all $(\ell', i') < (\ell, i)$. The base of induction is when $\ell = 1$, $i = 0$. Indeed, $I_k^{\leq 1}(\mathcal{F}) = \mathcal{F}^{(k)}$ for all k and (1.4) becomes $\mathcal{F}\mathcal{F}^{(k)} \subset \mathcal{F}[\mathcal{F}, \mathcal{F}^{(k-1)}] + [\mathcal{F}, I_{k-1}^{\leq 2}(\mathcal{F})]$, which is obviously true since $[\mathcal{F}, \mathcal{F}^{(k-1)}] = \mathcal{F}^{(k)}$.

If $\ell \geq 1$, $i > 0$, we obtain, using the Leibniz rule, the following inclusion:

$$\mathcal{F}^{(i)} I_{k-i}^{\leq \ell}(\mathcal{F}) = [\mathcal{F}, \mathcal{F}^{(i-1)}] I_{k-i}^{\leq \ell}(\mathcal{F}) \subset [\mathcal{F}, \mathcal{F}^{(i-1)} I_{k-i}^{\leq \ell}(\mathcal{F})] + \mathcal{F}^{(i-1)} [\mathcal{F}, I_{k-i}^{\leq \ell}(\mathcal{F})] \ .$$

Therefore,

$$\mathcal{F}^{(i)} I_{k-i}^{\leq \ell}(\mathcal{F}) \subset [\mathcal{F}, \mathcal{F}^{(i-1)} I_{k-i}^{\leq \ell}(\mathcal{F})] + \mathcal{F}^{(i-1)} I_{k+1-i}^{\leq \ell}(\mathcal{F})$$

by (b). Finally, using the inductive hypothesis for $(\ell, i-1)$ and taking into account that $\mathcal{F}^{(i-1)} I^{\le \ell}_{k-i}(\mathcal{F}) \subset I^{\le \ell+1}_{k-1}(\mathcal{F})$, and, therefore,

$$[\mathcal{F}, \mathcal{F}^{(i-1)} I^{\le \ell}_{k-i}(\mathcal{F})] \subset [\mathcal{F}, I^{\le \ell+1}_{k-1}(\mathcal{F})] ,$$

we obtain the inclusion (1.4).

If $\ell \ge 2$, $i = 0$, then using the inductive hypothesis for all pairs $(\ell-1, i')$, $i' \ge 0$, we obtain:

$$I^{\le \ell}_k(\mathcal{F}) = \mathcal{F}[\mathcal{F}, I^{\le \ell-1}_{k-1}(\mathcal{F})] + [\mathcal{F}, I^{\le \ell}_{k-1}(\mathcal{F})] .$$

Multiplying by $\mathcal{F}$ on the left and using the distributivity of multiplication of subspaces in $\mathcal{F} \cdot A$, we obtain:

$$\mathcal{F} I^{\le \ell}_k(\mathcal{F}) = \mathcal{F}^2[\mathcal{F}, I^{\le \ell-1}_{k-1}(\mathcal{F})] + \mathcal{F}[\mathcal{F}, I^{\le \ell}_{k-1}(\mathcal{F})] = \mathcal{F}[\mathcal{F}, I^{\le \ell}_{k-1}(\mathcal{F})]$$

because $\mathcal{F}^2 \subset \mathcal{F}$ and $I^{\le \ell-1}_{k-1}(\mathcal{F}) \subset I^{\le \ell}_{k-1}(\mathcal{F})$. This immediately implies (1.4).

Part (c) is proved. The lemma is proved. □

Lemma 1.3. *For any $k', k \ge 0$, and any $\ell, \ell' \ge 1$ one has:*
(a) $I^{\ell}_k(\mathcal{F}) I^{\ell'}_{k'}(\mathcal{F}) \subset I^{\ell+\ell'}_{k+k'}(\mathcal{F})$, $I^{\le \ell}_k(\mathcal{F}) I^{\le \ell'}_{k'}(\mathcal{F}) \subset I^{\le \ell+\ell'}_{k+k'}(\mathcal{F})$.
(b) $[I^{\ell}_k(\mathcal{F}), I^{\ell'}_{k'}(\mathcal{F})] \subset [\mathcal{F}, I^{\ell+\ell'-1}_{k+k'}(\mathcal{F})]$, $[I^{\le \ell}_k(\mathcal{F}), I^{\le \ell'}_{k'}(\mathcal{F})] \subset [\mathcal{F}, I^{\le \ell+\ell'-1}_{k+k'}(\mathcal{F})]$.

Proof. Part (a) follows from the obvious fact that

$$(\mathcal{F}^{(\lambda_1)} \mathcal{F}^{(\lambda_2)} \cdots \mathcal{F}^{(\lambda_{\ell_1})})(\mathcal{F}^{(\mu_1)} \mathcal{F}^{(\mu_2)} \cdots \mathcal{F}^{(\mu_{\ell_2})}) \subset I^{\ell_1+\ell_2}_k(\mathcal{F}) ,$$

where $k = \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell_1} + \mu_1 + \mu_2 + \cdots + \mu_{\ell_2}$.

Prove (b). First, we prove the first inclusion for $\ell = 1$. We proceed by induction in k. The base of induction, $k = 0$, is obvious because $I^1_0(\mathcal{F}) = \mathcal{F}$. Assume that the assertion is proved for all $k_1 < k$, i.e., we have:

$$[\mathcal{F}^{(k_1)}, I^{\ell'}_{k'}(\mathcal{F})] \subset [\mathcal{F}, I^{\ell'}_{k_1+k'}(\mathcal{F})] .$$

Then, using the fact that $\mathcal{F}^{(k)} = [\mathcal{F}, \mathcal{F}^{(k-1)}]$ and the Jacobi identity, we obtain:

$$[\mathcal{F}^{(k)}, I^{\ell'}_{k'}(\mathcal{F})] = [[\mathcal{F}, \mathcal{F}^{(k-1)}], I^{\ell'}_{k'}(\mathcal{F})] \subset$$
$$\subset [\mathcal{F}, [\mathcal{F}^{(k-1)}, I^{\ell'}_{k'}(\mathcal{F})]] + [\mathcal{F}^{(k-1)}, [\mathcal{F}, I^{\ell'}_{k'}(\mathcal{F})]] \subset$$
$$\subset [\mathcal{F}, [\mathcal{F}, I^{\ell'}_{k'+k-1}(\mathcal{F})]] + [\mathcal{F}^{(k-1)}, I^{\ell'}_{k'+1}(\mathcal{F})] \subset [\mathcal{F}, I^{\ell'}_{k'+k}(\mathcal{F})]$$

by the inductive hypothesis and Lemma 1.2(b). This proves the first inclusion of (b) for $\ell = 1$.

Furthermore, we will proceed by induction in ℓ. Now $\ell > 1$, assume that the assertion is proved for all $\ell_1 < \ell$, i.e., we have the inductive hypothesis in the form:

$$[I^{\ell_1}_k(\mathcal{F}), I^{\ell'}_{k'}(\mathcal{F})] \subset [\mathcal{F}, I^{\ell_1+\ell'-1}_{k+k'}(\mathcal{F})]$$

for all $k \ge i \ge 0$.

Using (1.3) and (2.7) with all $a \in \mathcal{F}^{(i)}$, $b \in I^{\ell-1}_{k-i}(\mathcal{F})$, $c \in I^{\ell'}_{k'}(\mathcal{F})$, we obtain for all $i \ge 0$:

$$[\mathcal{F}^{(i)} I^{\ell-1}_{k-i}(\mathcal{F}), I^{\ell'}_{k'}(\mathcal{F})] \subset [\mathcal{F}^{(i)}, I^{\ell-1}_{k-i}(\mathcal{F}) I^{\ell'}_{k'}(\mathcal{F})] + [I^{\ell-1}_{k-i}(\mathcal{F}), \mathcal{F}^{(i)} I^{\ell'}_{k'}(\mathcal{F})] \subset$$
$$[\mathcal{F}^{(i)}, I^{\ell+\ell'-1}_{k+k'-i}(\mathcal{F})] + [I^{\ell-1}_{k-i}(\mathcal{F}), I^{\ell'+1}_{k'+i}(\mathcal{F})] \subset [\mathcal{F}, I^{\ell+\ell'-1}_{k+k'}(\mathcal{F})]$$

by the the already proved (a) and inductive hypothesis. This finishes the proof of the first inclusion of (b). The second inclusion of (b) also follows. □

Generalizing (1.3), for any subset S of $\mathcal{F}$ denote by $I_k^\ell(\mathcal{F},S)$ the image of S under the canonical map $\mathcal{F}^{\otimes k\ell} \twoheadrightarrow I_k^\ell(\mathcal{F})$, i.e.,

$$I_k^\ell(\mathcal{F},S)=\sum_\lambda S^{(\lambda_1)}S^{(\lambda_2)}\cdots S^{(\lambda_\ell)}\ . \tag{1.5}$$

In particular, $I_k^1(\mathcal{F},S)=S^{(k)}$ and $I_0^\ell=S^\ell$.

The following result is obvious.

Lemma 1.4. *Let $\mathcal{F}$ be an object of $\mathcal{N}$ and $S\subset\mathcal{F}$. Then:*
(a) For any $k\geq 0$, $\ell\geq 2$ one has

$$I_k^\ell(\mathcal{F},S)=\sum_{i=0}^k S^{(i)}I_{k-i}^{\ell-1}(\mathcal{F},S)\ .$$

(b) For any $k',k\geq 0$, and any $\ell,\ell'\geq 1$ one has:

$$I_k^\ell(\mathcal{F},S)I_{k'}^{\ell'}(\mathcal{F},S)\subset I_{k+k'}^{\ell+\ell'}(\mathcal{F},S),\ [I_k^\ell(\mathcal{F},S),I_{k'}^{\ell'}(\mathcal{F},S)]\subset I_{k+k'+1}^{\ell+\ell'-1}(\mathcal{F},S)\ .$$

In particular,

$$S^{(i)}I_k^\ell(\mathcal{F},S)\subset I_{k+i}^{\ell+1}(\mathcal{F},S),\ [S^{(i)},I_k^\ell(\mathcal{F},S)]\subset I_{k+i+1}^{\ell}(\mathcal{F},S)\ . \tag{1.6}$$

2. Noncommutative loop Lie algebras

Denote by **LieAct** the category whose objects are triples $(\mathcal{F},\mathcal{L},\alpha)$, where $\mathcal{F}$ is an object of $\mathcal{N}$, $\mathcal{L}$ is a $\mathbb{Q}$-Lie algebra, and $\alpha:\mathcal{F}\otimes\mathcal{L}\to\mathcal{L}$ is an action of $\mathcal{F}$ on $\mathcal{L}$ ($\alpha(f\otimes l)=f(l)$) satisfying:

$$[f_1,f_2](l)=f_1(f_2(l))-f_2(f_1(l)),\ f([l_1,l_2])=[f(l_1),l_2]+[l_1,f(l_2)]$$

for all $f,f_1,f_2\in\mathcal{F}$, $l,l_1,l_2\in\mathcal{L}$ (i.e., α defines a semidirect product $\mathcal{F}\ltimes\mathcal{L}$)); a morphism $(\mathcal{F}_1,\mathcal{L}_1,\alpha_1)\to(\mathcal{F}_2,\mathcal{L}_2,\alpha_2)$ is a pair (φ,ψ), where $\varphi:\mathcal{F}_1\to\mathcal{F}_2$ is a morphism in $\mathcal{N}$ and $\psi:\mathcal{L}_1\to\mathcal{L}_2$ such that the following diagram is commutative

$$\begin{array}{ccc}\mathcal{F}_1\otimes\mathcal{L}_1 & \xrightarrow{\alpha_1} & \mathcal{L}_1\\ {\scriptstyle\varphi\otimes\psi}\downarrow & & \downarrow{\scriptstyle\psi}\\ \mathcal{F}_2\otimes\mathcal{L}_2 & \xrightarrow{\alpha_2} & \mathcal{L}_2\end{array}$$

In particular, this defines a natural projection functor $\pi:\mathbf{LieAct}\to\mathcal{N}$ by $\pi(\mathcal{F},\mathcal{L},\alpha)=\mathcal{F}$ and $\pi(\varphi,\psi)=\varphi$.

Definition 2.1. A *noncommutative loop Lie algebra* (*$\mathcal{N}$-loop Lie algebra*) is a functor $\mathfrak{s}:\mathcal{N}\to\mathbf{LieAct}$ such that $\pi\circ\mathfrak{s}=Id_\mathcal{N}$ (i.e., $\mathfrak{s}$ is a section of π).

In what follows we will suppress the tensor sign in expressions like $\mathcal{F}\otimes A$ and write $\mathcal{F}\cdot A$ instead.

The next construction provides a first example of $\mathcal{N}$-loop Lie algebras. For any objects A and $\mathcal{F}$ of $\mathcal{N}$ one defines an object $\mathfrak{s}_A(\mathcal{F})=(\mathcal{F},\mathcal{F}\cdot A,\alpha)$ of **LieAct**, where the action α is given by

$$f_1(f_2\cdot a)=[f_1,f_2]\cdot a \tag{2.1}$$

for all $f_1,f_2\in\mathcal{F}$, $a\in A$.

The following result is obvious.

Lemma 2.2. *The association $\mathcal{F}\mapsto\mathfrak{s}_A(\mathcal{F})$ defines a noncommutative loop Lie algebra $\mathfrak{s}_A:\mathcal{N}\to\mathbf{LieAct}$.*

Our main object of study will a refinement of the above example. Given an object A of $\mathcal{N}$, and a subspace $\mathfrak{g} \subset A$ such that $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ (i.e., $\mathfrak{g}$ is a Lie subalgebra of A), we say that $(\mathfrak{g}, A)$ is a *compatible pair.* For any compatible pair $(\mathfrak{g}, A)$ and an object $\mathcal{F}$ of $\mathcal{N}$, denote by $(\mathfrak{g}, A)(\mathcal{F})$ the Lie subalgebra of the $\mathcal{F} \cdot A = \mathcal{F} \otimes A$ (under the commutator bracket) generated by $\mathcal{F} \cdot \mathfrak{g}$, that is, $(\mathfrak{g}, A)(\mathcal{F}) = (\mathcal{F} \cdot \mathfrak{g})^{(\bullet)}$ in the notation (1.1).

Proposition 2.3. *For any compatible pair $(\mathfrak{g}, A)$ the association $\mathcal{F} \mapsto (\mathfrak{g}, A)(\mathcal{F})$ defines the noncommutative loop Lie algebra*

$$(\mathfrak{g}, A) : \mathcal{N} \to \mathbf{LieAct} \ .$$

Proof. It suffices to show that any arrow φ in $\mathcal{N}$, i.e., any algebra homomorphism $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ defines a homomorphism of Lie algebras $(\mathfrak{g}, A)(\mathcal{F}_1) \to (\mathfrak{g}, A)(\mathcal{F}_2)$. We need the following obvious fact.

Lemma 2.4. *Let A_1, A_2 be objects of $\mathcal{N}$ and let $\varphi : A_1 \to A_2$ be an algebra homomorphism. Let $S_1 \subset A_1$ and $S_2 \subset A_2$ be two subsets such that $\varphi(S_1) \subset S_2$. Then the restriction of φ to the Lie algebra $S_1^{(\bullet)}$ (in the notation* (1.1)*) is a homomorphism of Lie algebras $S_1^{(\bullet)} \to S_2^{(\bullet)}$.*

Indeed, applying Lemma 2.4 with $A_i = \mathcal{F}_i \cdot A$, $S_i = \mathcal{F}_i \cdot \mathfrak{g}$, $i = 1, 2$, $\varphi = f \otimes id_A : \mathcal{F}_1 \cdot A \to \mathcal{F}_2 \cdot A$, the trivial extension of $\mathcal{F}$, we obtain a Lie algebra homomorphism $(\mathfrak{g}, A)(\mathcal{F}_1) = (\mathcal{F}_1 \cdot \mathfrak{g})^{(\bullet)} \to (\mathcal{F}_2 \cdot \mathfrak{g})^{(\bullet)} = (\mathfrak{g}, A)(\mathcal{F}_2)$.

It remains to construct the action of $\mathcal{F}$ on $\mathcal{L} = (\mathfrak{g}, A)(\mathcal{F}) = (\mathcal{F} \cdot \mathfrak{g})^{(\bullet)}$. Indeed, $S = \mathcal{F} \cdot \mathfrak{g}$ is invariant under the action (2.1) on $\mathcal{F} \cdot A$. Therefore, $\mathcal{L} = (\mathcal{F} \cdot \mathfrak{g})^{(\bullet)}$ is also invariant under this action of $\mathcal{F}$. The proposition is proved. $\square$

If $\mathcal{F}$ is commutative, then $(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g}$ is the $\mathcal{F}$-loop algebra. Therefore, if $\mathcal{F}$ is an arbitrary object of $\mathcal{N}$, the Lie algebra $(\mathfrak{g}, A)(\mathcal{F})$ deserves a name of noncommutative loop Lie algebra *associated with the compatible pair* $(\mathfrak{g}, A)$.

If $A = U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, then we will abbreviate $\mathfrak{g}(\mathcal{F}) := (\mathfrak{g}, U(\mathfrak{g}))(\mathcal{F})$. Another natural choice of A is the algebra $End(V)$, where V is a faithful $\mathfrak{g}$-module. In this case, we will sometimes abbreviate $(\mathfrak{g}, V)(\mathcal{F}) := (\mathfrak{g}, End(V))(\mathcal{F})$.

The following result provides an estimation of $(\mathfrak{g}, A)(\mathcal{F})$ from below. Set

$$\langle \mathfrak{g} \rangle = \sum_{k \geq 1} \mathfrak{g}^k \ , \tag{2.2}$$

i.e., $\langle \mathfrak{g} \rangle$ is the associative subalgebra of A, maybe without unit, generated by $\mathfrak{g}$.

Proposition 2.5. *Let $(\mathfrak{g}, A)$ be a compatible pair and $\mathcal{F}$ be an object of $\mathcal{N}$. Then:*
(a) $\mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \subset (\mathfrak{g}, A)(\mathcal{F})$ and $\mathcal{F}\mathcal{F}^{(k)} \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] \subset (\mathfrak{g}, A)(\mathcal{F})$ for all $k \geq 0$.
(b) If $\mathfrak{g}$ is abelian, i.e., $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = 0$, then

$$(\mathfrak{g}, A)(\mathcal{F}) = \sum_{k \geq 0} \mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \ . \tag{2.3}$$

(c) If $[\mathcal{F}, \mathcal{F}] = \mathcal{F}$ (i.e., $\mathcal{F}$ is perfect as a Lie algebra), then $(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \langle \mathfrak{g} \rangle$.

Proof. Prove (a). We need the following technical result.

Lemma 2.6. *Let $\mathfrak{g}$ be a Lie subalgebra of an associative $\mathbb{Q}$-algebra A. For all $m \geq 2$ denote by $\widetilde{\mathfrak{g}^m}$ the $\mathbb{Q}$-linear span of all g^k, $g \in \mathfrak{g}$. Then for any $m \geq 2$ one has*

$$\widetilde{\mathfrak{g}^m} + (\mathfrak{g}^{m-1} \cap \mathfrak{g}^m) = \mathfrak{g}^m \tag{2.4}$$

Proof. Since $\mathfrak{g}^{i-1}\mathfrak{g}'\mathfrak{g}^{m-i-1} \subset \mathfrak{g}^{m-1}$ for all $i \leq m-1$, we obtain the following congruence for any $c = (c_1, \cdots, c_m) \in (\mathbb{Q}^\times)^m$ and $x = (x_1, \ldots, x_m) \in \mathfrak{g}^{\times m}$:

$$(c_1x_1 + \cdots c_mx_m)^m \equiv \sum_\lambda \binom{m}{\lambda} c^\lambda x^\lambda \mod (\mathfrak{g}^{m-1} \cap \mathfrak{g}^m) \ ,$$

where the summation is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ of m and we abbreviated $c^\lambda = c_1^{\lambda_1} \cdots c_m^{\lambda_m}$ and $x^\lambda = x_1^{\lambda_1} \cdots x_m^{\lambda_m}$. Varying $c = (c_1, \cdots, c_m) \in (\mathbb{Q}^\times)^m$, the above congruence implies that that each monomial x^λ belongs to $\widetilde{\mathfrak{g}^m} + (\mathfrak{g}^{m-1} \cap \mathfrak{g}^m)$. In particular, taking $\lambda = (1, 1, \ldots, 1)$, we obtain $\mathfrak{g}^m \subseteq \widetilde{\mathfrak{g}^m} + (\mathfrak{g}^{m-1} \cap \mathfrak{g}^m)$. Taking into account that $\widetilde{\mathfrak{g}^m} \subseteq \mathfrak{g}^m$, we obtain (2.6). The lemma is proved. □

We also need the following useful identity in $\mathcal{F} \cdot A$:

$$[sE, tF] = st \cdot [E, F] + [s, t] \cdot FE = ts \cdot [E, F] + [s, t] \cdot EF \tag{2.5}$$

for any $s, t \in \mathcal{F}$, $E, F \in A$.

We will prove the first inclusion (a) by induction in k. If $k = 0$, one obviously has $\mathcal{F}^{(0)}\mathfrak{g}^1 = \mathcal{F} \cdot \mathfrak{g} \subset (\mathfrak{g}, A)(\mathcal{F})$. Assume now that $k > 0$. Then for $g \in \mathfrak{g}$ we obtain using (2.5):

$$[\mathcal{F} \cdot g, \mathcal{F}^{(k-1)} \cdot g^k] = [\mathcal{F}, \mathcal{F}^{(k-1)}] \cdot g^{k+1} = \mathcal{F}^{(k)} \cdot g^{k+1}$$

which implies that $\mathcal{F}^{(k)} \cdot \widetilde{\mathfrak{g}^{k+1}} \subset (\mathfrak{g}, A)(\mathcal{F})$ (in the notation of Lemma 2.6). Using Lemma 2.6, we obtain

$$\mathcal{F}^{(k)} \cdot \widetilde{\mathfrak{g}^{k+1}} \equiv \mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \mod \mathcal{F}^{(k)} \cdot (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) \ .$$

Taking into account that $\mathcal{F}^{(k)} \cdot (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) \subset \mathcal{F}^{(k-1)} \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$ by the inductive hypothesis (where we used the inclusion $\mathcal{F}^{(k)} \subset \mathcal{F}^{(k-1)}$), the above implies that $\mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1}$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. This proves the first inclusion of (a). To prove the second inclusion, we compute:

$$[\mathcal{F} \cdot \mathfrak{g}, \mathcal{F}^{(k-1)} \cdot \mathfrak{g}^k] \equiv \mathcal{F}\mathcal{F}^{(k-1)} \cdot [\mathfrak{g}, \mathfrak{g}^k] \mod \mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \ .$$

Therefore, using the already proved inclusion $\mathcal{F}^{(k)} \cdot \mathfrak{g}^{k+1} \subset (\mathfrak{g}, A)(\mathcal{F})$, we see that $\mathcal{F}\mathcal{F}^{(k-1)} \cdot [\mathfrak{g}, \mathfrak{g}^k]$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. This finishes the proof of (a).

Prove (b). Clearly, (a) implies that $(\mathfrak{g}, A)(\mathcal{F})$ contains the right hand side of (2.3). Therefore, it suffices to prove that the latter space is closed under the commutator. Indeed, since $\mathfrak{g}$ is abelian, one has

$$[\mathcal{F}^{(k_1)} \cdot \mathfrak{g}^{k_1+1}, \mathcal{F}^{(k_2)} \cdot \mathfrak{g}^{k_2+1}] = [\mathcal{F}^{(k_1)}, \mathcal{F}^{(k_2)}] \cdot \mathfrak{g}^{k_1+k_2+2} \subset$$
$$\subset \mathcal{F}^{(k_1+k_2+1)} \cdot \mathfrak{g}^{k_1+k_2+2} \subset (\mathfrak{g}, A)(\mathcal{F})$$

because $[\mathcal{F}^{(k_1)}, \mathcal{F}^{(k_2)}] \subset \mathcal{F}^{(k_1+k_2+1)}$. This finishes the proof of (b).

Prove (c). Since $\mathcal{F}' = \mathcal{F}$, the already proved part (a) implies that $\mathcal{F} \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$ for all $k \geq 1$, therefore, $\mathcal{F} \cdot \langle \mathfrak{g} \rangle \subseteq (\mathfrak{g}, A)(\mathcal{F})$. But since $\langle \mathfrak{g} \rangle$ is an associative subalgebra of A containing $\mathfrak{g}$, we obtain an opposite inclusion $(\mathfrak{g}, A)(\mathcal{F}) \subseteq \mathcal{F} \cdot \langle \mathfrak{g} \rangle$. This finishes the proof of (c).

The proposition is proved. □

Remark 2.7. Proposition 2.5(c) shows that the case when $[\mathcal{F},\mathcal{F}]=\mathcal{F}$ is not of much interest. This happens, for example, when $\mathcal{F}$ is a Weyl algebra or the quantum torus. In these cases a natural anti-involution on $\mathcal{F}$ can be taken into account. We will discuss it in a separate paper.

Definition 2.8. A pair $(\mathfrak{g},A)$ is of finite type if there exists $m>0$ such that $\mathfrak{g}+\mathfrak{g}^2+\cdots+\mathfrak{g}^m=A$, and we call such minimal m the type of $(\mathfrak{g},A)$. If such m does not exists, we say that $(\mathfrak{g},A)$ is of infinite type.

Note that $(\mathfrak{g},A)$ is of type 1 if and only if $\mathfrak{g}=A$, which, in its turn, implies that $(\mathfrak{g},A)(\mathcal{F})=\mathcal{F}\cdot A$ for all objects $\mathcal{F}$ of $\mathcal{N}$. Note also that if $\langle g\rangle=A$ and A is finite dimensional over $\mathbb{Q}$, then $(\mathfrak{g},A)$ is always of finite type.

Proposition 2.9. *Assume that* $(\mathfrak{g},A)$ *is of type* 2, *i.e.,* $\mathfrak{g}\neq A$ *and* $\mathfrak{g}+\mathfrak{g}^2=A$. *Then*

$$(\mathfrak{g},A)(\mathcal{F})=\mathcal{F}\cdot\mathfrak{g}+\mathcal{F}'\cdot A+\mathcal{F}\mathcal{F}'\cdot[A,A]\ , \tag{2.6}$$

where $\mathcal{F}'=[\mathcal{F},\mathcal{F}]$.

Proof. Proposition 2.5(a) guarantees that

$$\mathcal{F}\cdot\mathcal{F}\cdot\mathfrak{g}+\mathcal{F}'\cdot\mathfrak{g}^2+\mathcal{F}\mathcal{F}'\cdot[\mathfrak{g},\mathfrak{g}^2]\subset(\mathfrak{g},A)(\mathcal{F})\ .$$

Clearly, $\mathcal{F}\cdot\mathfrak{g}+\mathcal{F}'\cdot\mathfrak{g}^2=\mathcal{F}\cdot\mathfrak{g}+\mathcal{F}'\cdot A$ (because $\mathcal{F}'\subset\mathcal{F}$). Let us now prove that $[\mathfrak{g},A]=[A,A]$. Obviously, $[\mathfrak{g},A]\subseteq[A,A]$. The opposite inclusion immediately follows from the following one $[\mathfrak{g}^2,\mathfrak{g}^2]\subseteq[\mathfrak{g},\mathfrak{g}^3]$. That latter inclusion is a direct consequence of the following useful analogue of Jacobi and Leibniz identities in $\mathcal{F}$:

$$[ab,c]+[bc,a]+[ca,b]=0 \tag{2.7}$$

for all $a,b,c\in\mathcal{F}$. Indeed, taking any $a\in\mathfrak{g}$, $b\in\mathfrak{g}$, $c\in\mathfrak{g}^2$, we obtain $[ab,c]\subset[\mathfrak{g},\mathfrak{g}^3]$.

Using the equation $[\mathfrak{g},A]=[A,A]$ we obtain $\mathcal{F}\mathcal{F}'\cdot[A,A]\subset(\mathfrak{g},A)(\mathcal{F})$. This proves that $(\mathfrak{g},A)(\mathcal{F})$ contains the right hand side of (2.6).

To finish the proof, it suffices to show that the latter space is closed under the commutator. Indeed, abbreviating $A'=[A,A]$, we obtain

$$[\mathcal{F}\cdot\mathfrak{g},\mathcal{F}'\cdot A]\subset\mathcal{F}\mathcal{F}'\cdot[\mathfrak{g},A]+[\mathcal{F},\mathcal{F}']\cdot A\mathfrak{g}\subset\mathcal{F}\mathcal{F}'\cdot A'+\mathcal{F}'\cdot A\subset(\mathfrak{g},A)(\mathcal{F})\ ,$$

$$[\mathcal{F}\cdot\mathfrak{g},\mathcal{F}\mathcal{F}'\cdot A']\subset\mathcal{F}^2\mathcal{F}'\cdot[\mathfrak{g},A']+[\mathcal{F},\mathcal{F}\mathcal{F}']\cdot A'\mathfrak{g}\subset\mathcal{F}\mathcal{F}'\cdot A'+\mathcal{F}'\cdot A\subset(\mathfrak{g},A)(\mathcal{F})\ ,$$

$$[\mathcal{F}'\cdot A,\mathcal{F}'\cdot A]\subset(\mathcal{F}')^2\cdot A'+[\mathcal{F}',\mathcal{F}']\cdot A^2\subset\mathcal{F}\mathcal{F}'\cdot A'+\mathcal{F}'\cdot A\subset(\mathfrak{g},A)(\mathcal{F})\ ,$$

$$[\mathcal{F}'\cdot A,\mathcal{F}\mathcal{F}'\cdot A']\subset\mathcal{F}'\mathcal{F}\mathcal{F}'\cdot[A,A']+[\mathcal{F}',\mathcal{F}\mathcal{F}']\cdot A'A\subset(\mathfrak{g},A)(\mathcal{F})$$

because $\mathcal{F}'\mathcal{F}\mathcal{F}'\cdot[A,A']\subset\mathcal{F}\mathcal{F}'\cdot A'\subset(\mathfrak{g},A)(\mathcal{F})$ and $[\mathcal{F}',\mathcal{F}\mathcal{F}']\cdot A'A\subset\mathcal{F}'\cdot A\subset(\mathfrak{g},A)(\mathcal{F})$. Finally,

$$[\mathcal{F}'\mathcal{F}\cdot A',\mathcal{F}\mathcal{F}'\cdot A']\subset(\mathcal{F}'\mathcal{F})^2\cdot[A',A']+[\mathcal{F}'\mathcal{F},\mathcal{F}\mathcal{F}']\cdot(A')^2\subset(\mathfrak{g},A)(\mathcal{F})$$

because $(\mathcal{F}'\mathcal{F})^2\cdot[A',A']\subset\mathcal{F}'\mathcal{F}\cdot A'\subset(\mathfrak{g},A)(\mathcal{F})$ and $[\mathcal{F}'\mathcal{F},\mathcal{F}\mathcal{F}']\cdot[A,A]^2\subset\mathcal{F}'\cdot A\subset(\mathfrak{g},A)(\mathcal{F})$. The proposition is proved. □

For any $\mathbb{Q}$-vector space V and any object $\mathcal{F}$ of $\mathcal{N}$ we abbreviate $sl(V,\mathcal{F}):=(sl(V),End(V))(\mathcal{F})$. We will also naturally make a natural convention $gl(V,\mathcal{F}):=(End(V),End(V))(\mathcal{F})=\mathcal{F}\cdot End(V)$.

Corollary 2.10. *Let V be a finite-dimensional $\mathbb{Q}$-vector space. Then the pair $(\mathfrak{g}, A) = (sl(V), End(V))$ is of type 2. In particular,*

$$sl(V, \mathcal{F}) = \mathcal{F} \cdot sl(V) + \mathcal{F}' \cdot 1 \ .$$

Hence $sl(V, \mathcal{F})$ is the set of all $X \in gl(V, \mathcal{F})$ such that $Tr(X) \in \mathcal{F}' = [\mathcal{F}, \mathcal{F}]$ (where $Tr : gl(V, \mathcal{F}) = \mathcal{F} \cdot End(V) \to \mathcal{F}$ is the trivial extension of the ordinary trace $End(V) \to \mathbb{Q}$).

Proof. Let us prove that the pair $(\mathfrak{g}, A) = (sl(V), gl(V))$ is of type 2, i.e., $sl(V) + sl(V)^2 = gl(V)$. It suffices to show that $1 \in sl(V)^2$. To prove it, choose a basis $e_1, \ldots, e_n$ in V so that $V \cong \mathbb{Q}^n$, $sl(V) \cong sl_n(\mathbb{Q})$ and $A = End(V) \cong M_n(\mathbb{Q})$. Indeed, for any indices $i \neq j$ both E_{ij} and E_{ji} belong to $sl(V)$ and, therefore, $E_{ij}E_{ji} = E_{ii} \in sl(V)^2$. Therefore, $1 = \sum_{i=1}^n E_{ii}$ also belongs to $sl(V)$. Applying Proposition 2.9 and using the obvious fact that $[A, A] = sl(V)$, we obtain

$$sl(V, \mathcal{F}) = \mathcal{F} \cdot sl(V) + \mathcal{F}' \cdot A + \mathcal{F}\mathcal{F}'[A, A] = \mathcal{F} \cdot sl(V) + \mathcal{F}' \cdot 1 \ .$$

This proves the first assertion. The second one follows from the obvious fact that the trace $Tr : \mathcal{F} \cdot End(V) \to \mathcal{F}$ is the projection to the second summand of the direct sum decomposition

$$\mathcal{F} \cdot End(V) = \mathcal{F} \cdot sl(V) + \mathcal{F} \cdot 1 \ .$$

This proves the second assertion. The corollary is proved. $\square$

We can construct more pairs of type 2 as follows. Let V be a $\mathbb{Q}$-vector space and $\Phi : V \times V \to \mathbb{Q}$ be a bilinear form on V. Denote by $o(\Phi)$ the orthogonal Lie algebra of Φ, i.e.,

$$o(\Phi) = \{M \in End(V) : \Phi(M(u), v) + \Phi(u, M(v)) = 0 \ \forall \ u, v \in V\} \ .$$

Denote by $K = K_\Phi \subset V$ the sum of the left and the right kernels of Φ (if Φ is symmetric or skew-symmetric, then K is the left kernel of Φ). Finally, denote by $End(V, K)$ the parabolic subalgebra of $End(V)$ which consists of all $M \in End(V)$ such that $M(K) \subset K$. Clearly, $o(\Phi) \subset End(V, K)$, i.e., $(o(\Phi), End(V, K))$. For and any object $\mathcal{F}$ of $\mathcal{N}$ we abbreviate $o(\Phi, \mathcal{F}) := (o(\Phi), End(V, K))(\mathcal{F})$.

Corollary 2.11. *Let V be a finite-dimensional $\mathbb{Q}$-vector space and Φ be a symmetric or skew-symmetric bilinear form on V. Then $(o(\Phi), End(V, K))$ is of type 2. In particular,*

$$o(\Phi, \mathcal{F}) = \mathcal{F} \cdot o(\Phi) + \mathcal{F}' \cdot 1 + \mathcal{F}' \cdot 1_K + (\mathcal{F}\mathcal{F}' + \mathcal{F}') \cdot sl(V, K) \ . \tag{2.8}$$

Here $sl(V, K)$ is the set of all M in $End(V, K)$ such that $Tr(M) = 0$ and $Tr(M_K) = 0$, where $M_K : K \to K$ is the restriction of M to K and $1_K \in End(V, K)$ is any element such that $\mathbb{Q} \cdot 1 + \mathbb{Q} \cdot 1_K + sl(V, K) = End(V, K)$. If $K = 0$, we set $1_K = 0$.

Proof. First prove that $(\mathfrak{g}, A) = (o(\Phi), End(V, K))$ is of type 2. We complexify the involved objects, i.e., replace both V and K with $\overline{V} = \mathbb{C} \cdot V = \mathbb{C} \otimes V$, $\overline{K} = \mathbb{C} \cdot K$ etc. Using the obvious fact that $\overline{U + U'} = \overline{U} + \overline{U'}$ and $\overline{U \cdot U'} = \overline{U} \cdot \overline{U'}$ for any subspaces of $End(V)$ and $\overline{o(\Phi)} = o(\overline{\Phi})$, we see that it suffices to show that the pair $(o(\overline{\Phi}), End(\overline{V}, \overline{K})$ is of type 2.

Furthermore, without loss of generality we consider the case when $K = 0$, i.e., the form Φ is non-degenerate. If Φ is symmetric, one can choose a basis of $\overline{V}$ so that $\overline{V} \cong \mathbb{C}^n$, and $\overline{\Phi}$ is the standard dot product on $\mathbb{C}^n$. In this case $o(\overline{\Phi})$ is $o_n(\mathbb{C})$, the Lie algebra of orthogonal matrices, which is generated by all elements

$E_{ij} - E_{ji}$ where E_{ij} is the corresponding elementary matrix. Using the identity $(E_{ij} - E_{ji})^2 = -(E_{ii} + E_{jj})$ for $i \neq j$, we see that $o_n(\mathbb{C})^2$ contains all diagonal matrices. Furthermore, if i, j, k are pairwise distinct indices then $(E_{ij} - E_{ji})(E_{jk} - E_{kj}) = E_{ik}$. Thus we have shown that $o_n(\mathbb{C})^2 = M_n(\mathbb{C}) = End(\overline{V}, \overline{K})$. Therefore, $o_n(\mathbb{Q})^2 = M_n(\mathbb{Q}) = End(V, K)$. This proves the assertion for the symmetric Φ.

If Φ is skew-symmetric and non-degenerate, then $n = 2m$ and one can choose a basis of $\overline{V}$ such that V is identified with $\mathbb{C}^n$ and $o(\overline{\Phi})$ is identifies the symplectic Lie algebra $sp_{2m}(\mathbb{C})$.

Recall that a basis in $sp_{2m}(\mathbb{C})$ can be chosen as follows. It consists of elements $E_{ij} - E_{j+m,i+m}$ for $i, j \leq m$ and $E_{ik} + E_{i+m,\ell-m}$ for $i \leq m, \ell > m$. Using the identity $(E_{i,i+m} + E_{i+m,i})^2 = E_{ii} + E_{i+m,i+m}$ and the fact that $(E_{ii} - E_{i+m,i+m}) \in sp_{2m}(\mathbb{Q})$, we see that all diagonal matrices belong $sp_{2m}(\mathbb{C}) + sp_{2m}(\mathbb{C})^2$.

Also, the identity $(E_{ii} - E_{i+m,i+m})(E_{ij} - E_{j+m,i+m}) = E_{ij}$ for $i \neq j$ implies that $E_{ij} \in sp_{2m}(\mathbb{C})$ for all $i, j \leq m$. Similarly, one can prove that $E_{ij} \in sp_{2m}(\mathbb{C})$ for $i, j \geq m$.

Furthermore, the identity $(E_{ii} - E_{i+m,i+m})(E_{i\ell} + E_{i+m,\ell-m}) = E_{i\ell} - E_{i+m,\ell-m}$ implies that $sp_{2m}(\mathbb{C}) + sp_{2m}(\mathbb{C})^2$ contains all E_{ik} for $i \leq m, k > m$ and for $i > m, k \leq m$. Thus we have shown that $sp_{2m}(\mathbb{C}) + sp_{2m}(\mathbb{C})^2 = M_n(\mathbb{C}) = End(\overline{V}, \overline{K})$. Therefore, $sp_{2m}(\mathbb{Q}) + sp_{2m}(\mathbb{Q})^2 = M_n(\mathbb{Q}) = End(V, K)$. This proves the assertion for the skew-symmetric Φ.

Prove (2.8) now. We abbreviate $A = End(V, K)$. Obviously, $[A, A] = sl(V, K)$ and, if $K \neq \{0\}$, then $\mathbb{Q} \cdot 1 + sl(V, K)$ is of codimension 1 in A, i.e, 1_K always exists. Therefore, applying Proposition 2.9, we obtain

$$o(\Phi, \mathcal{F}) = \mathcal{F} \cdot o(\Phi) + \mathcal{F}' \cdot A + \mathcal{F}\mathcal{F}'[A, A] = \mathcal{F} \cdot o(\Phi) + \mathcal{F}' \cdot 1 + \mathcal{F}' \cdot 1_K + (\mathcal{F}\mathcal{F}' + \mathcal{F}') \cdot sl(V, K)\ .$$

This finishes the roof of Corollary 2.11. □

3. Upper bounds of loop Lie algebras

For any compatible pair $(\mathfrak{g}, A)$ define two subspaces $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ and $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ of $\mathcal{F} \cdot A$ by:

$$\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k \geq 1} I_k(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] + [\mathcal{F}, I_{k-1}(\mathcal{F})] \cdot \mathfrak{g}^{k+1}\ , \tag{3.1}$$

where $I_k(\mathcal{F})$ is defined in (1.3); and

$$\overline{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum I_{k_1}^{\ell_1+1} I_{k_2}^{\ell_2+1} \cdot [J_{\ell_1}^{k_1+1}, J_{\ell_2}^{k_2+1}] + [I_{k_1}^{\ell_1+1}, I_{k_2}^{\ell_2+1}] \cdot J_{\ell_2}^{k_2+1} J_{\ell_1}^{k_1+1}, \tag{3.2}$$

where the summation is over all $k_1, k_2 \geq 0$, $\ell_1, \ell_2 \geq 0$, and we abbreviated $I_k^\ell := I_k^\ell(\mathcal{F})$, $J_k^\ell := I_k^\ell(A, \mathfrak{g})$ in the notation (1.5).

We will refer to $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ as the *upper bound* of $(\mathfrak{g}, A)(\mathcal{F})$ and to $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ as *refined upper bound* of $(\mathfrak{g}, A)(\mathcal{F})$.

It is easy to see that the assignments $\mathcal{F} \mapsto \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ and $\mathcal{F} \mapsto \overline{(\mathfrak{g}, A)}(\mathcal{F})$ are functors $\widetilde{(\mathfrak{g}, A)}$ and $\overline{(\mathfrak{g}, A)}$ from $\mathcal{N}$ to the category $Vect_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces.

The following result is obvious.

Lemma 3.1. *If $(\mathfrak{g}, A)$ is a compatible pair of type m (see Definition 2.8), then*

$$\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k=1}^{m-1} I_k(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] + [\mathcal{F}, I_{k-1}(\mathcal{F})] \cdot \mathfrak{g}^{k+1}\ . \tag{3.3}$$

The following is the main result of this section, which explains this terminology and proves that both $\widetilde{(\mathfrak{g},A)}$ and $\overline{(\mathfrak{g},A)}(\mathcal{F})$ are noncommutative loop Lie algebras $\mathcal{N} \to$ **LieAct**.

Theorem 3.2. *For any compatible pair $(\mathfrak{g},A)$ and any object $\mathcal{F}$ of $\mathcal{N}$ one has:*
(a) The subspace $\widetilde{(\mathfrak{g},A)}(\mathcal{F})$ is a Lie subalgebra of $\mathcal{F}\cdot A$.
(b) The subspace $\overline{(\mathfrak{g},A)}(\mathcal{F})$ is a Lie subalgebra of $\mathcal{F}\cdot A$.
(c) $(\mathfrak{g},A)(\mathcal{F}) \subseteq \overline{(\mathfrak{g},A)}(\mathcal{F}) \subseteq \widetilde{(\mathfrak{g},A)}(\mathcal{F})$.

Proof. Prove (a). Using (2.5), we obtain

$$[\mathcal{F}\cdot\mathfrak{g},\mathcal{F}\cdot\mathfrak{g}] \subset \mathcal{F}^2\cdot[\mathfrak{g},\mathfrak{g}] + [\mathcal{F},\mathcal{F}]\cdot\mathfrak{g}^2 \subset \widetilde{(\mathfrak{g},A)}(\mathcal{F})$$

because $\mathcal{F}^2 \subset \mathcal{F}$, $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$, and $I_0(\mathcal{F}) = \mathcal{F}$. Furthermore,

$$[\mathcal{F}\cdot\mathfrak{g}, I_k(\mathcal{F})\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]] \subset \mathcal{F}I_k(\mathcal{F})\cdot[\mathfrak{g},[\mathfrak{g},\mathfrak{g}^{k+1}]] + [\mathcal{F},I_k(\mathcal{F})]\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]\mathfrak{g} \subset \widetilde{(\mathfrak{g},A)}(\mathcal{F})$$

because $\mathcal{F}I_k(\mathcal{F}) \subset I_k(\mathcal{F})$, $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}^{k+1}]] \subset [\mathfrak{g},\mathfrak{g}^{k+1}]$, and $[\mathfrak{g},\mathfrak{g}^{k+1}]\mathfrak{g} \subset \mathfrak{g}^{k+2}$. Finally, abbreviating $J_k := [\mathcal{F}, I_{k-1}(\mathcal{F})]$, we obtain:

$$[\mathcal{F}\cdot\mathfrak{g}, J_k\cdot\mathfrak{g}^{k+1}] \subset \mathcal{F}\cdot J_k\cdot[\mathfrak{g},\mathfrak{g}^{k+1}] + [\mathcal{F},J_k]\cdot\mathfrak{g}^{k+2} \subset \widetilde{(\mathfrak{g},A)}(\mathcal{F})$$

because, taking into the account that $J_k \subset I_{k-1}(\mathcal{F})$ since $I_{k-1}(\mathcal{F})$ is a two-sided ideal in $\mathcal{F}$, we have $\mathcal{F}J_k \subset \mathcal{F}I_{k-1}(\mathcal{F}) \subset I_{k-1}(\mathcal{F})$ and, taking into account that $J_k \subset I_k(\mathcal{F})$ by Lemma 1.2(b) taken with $\ell = \infty$, we have $[\mathcal{F},J_k] \subset [\mathcal{F},I_k(\mathcal{F})]$.

Furthermore, we need the following obvious consequence of (2.7).

Lemma 3.3. *For any Lie subalgebra $\mathfrak{g}$ of A one has*

$$[\mathfrak{g}^{k+1},\mathfrak{g}^{m+1}] \subset [\mathfrak{g},\mathfrak{g}^{k+m-1}]$$

for any $k,m \geq 1$.

Therefore, for any $k,m \geq 1$ one has:

$$[I_k(\mathcal{F})\cdot\mathfrak{g}^{k+1}, I_m(\mathcal{F})\cdot\mathfrak{g}^{m+1}] \subset$$

$$\subset I_k(\mathcal{F})I_m(\mathcal{F})\cdot[\mathfrak{g}^{k+1},\mathfrak{g}^{m+1}] + [I_k(\mathcal{F}),I_m(\mathcal{F})]\cdot\mathfrak{g}^{k+m+2} \subset \widetilde{(\mathfrak{g},A)}(\mathcal{F})$$

because $I_k(\mathcal{F})I_m(\mathcal{F}) = I_{k+m}(\mathcal{F})$ by Lemma 1.3(a), $[\mathfrak{g}^{k+1},\mathfrak{g}^{m+1}] \subset [\mathfrak{g},\mathfrak{g}^{k+m-1}]$ by Lemma 3.3, and $[I_k(\mathcal{F}),I_m(\mathcal{F})] \subset [\mathcal{F},I_{k+m-1}(\mathcal{F})]$ by Lemma 1.3(b) taken with $\ell = \infty$. Therefore, taking into account that

$$[\mathbf{I}_k,\mathbf{I}_m] \subset [I_k(\mathcal{F})\cdot\mathfrak{g}^{k+1}, I_m(\mathcal{F})\cdot\mathfrak{g}^{m+1}] \subset \widetilde{(\mathfrak{g},A)}(\mathcal{F})$$

for $\mathbf{I}_r$ stands for any of the spaces $I_r(\mathcal{F})\cdot[\mathfrak{g},\mathfrak{g}^{r+1}], [\mathcal{F},I_{r-1}(\mathcal{F})]\cdot\mathfrak{g}^{r+1}$ we finish the proof of (a).

Prove (b). For any subsets X and Y of an associative algebra A and $\varepsilon \in \{0,1\}$ denote

$$X \bullet_\varepsilon Y := \begin{cases} X\cdot Y & \text{if } \varepsilon = 0 \\ [X,Y] & \text{if } \varepsilon = 1 \end{cases}$$

We need the following result.

Lemma 3.4. *Let Γ be an abelian group and let A and $\mathcal{F}$ be objects of $\mathcal{N}$. Assume that $A_\alpha \subset \mathcal{F}$ and $B_\alpha \subset A$ are two families labeled by Γ such that*

$$A_\alpha \bullet_\varepsilon A_\beta \subseteq A_{\alpha+\beta+\varepsilon\cdot v},\ B_\beta \bullet_\varepsilon B_\alpha \subseteq B_{\alpha+\beta-\varepsilon\cdot v} \tag{3.4}$$

for all $\alpha,\beta \in \Gamma$, $\varepsilon \in \{0,1\}$, where v is a fixed element of Γ. Then for any $\alpha_0 \in \Gamma$ the subspace

$$\mathfrak{h} = A_{\alpha_0} \cdot B_{\alpha_0+v} + \sum_{\alpha,\beta\in\Gamma,\varepsilon\in\{0,1\}} (A_\alpha \bullet_{1-\varepsilon} A_\beta)\cdot(B_{\beta+v} \bullet_\varepsilon B_{\alpha+v})$$

is a Lie subalgebra of $\mathcal{F}\cdot A = \mathcal{F}\otimes A$.

Proof. The equation (2.5) implies that

$$[A\cdot B, A'\cdot B'] \subset (A \bullet_{1-\delta} A')\cdot(B' \bullet_\delta B)$$

for each $\delta \in \{0,1\}$. Therefore,

(i) Taking $A = A_\alpha \bullet_{1-\varepsilon} A_\beta$, $B = B_{\beta+v} \bullet_\varepsilon B_{\alpha+v}$, $A' = A_{\alpha'} \bullet_{1-\varepsilon'} A_{\beta'}$, $B' = B_{\beta'+v} \bullet_{\varepsilon'} B_{\alpha'+v}$, and taking into the account that $A \subseteq A_{\alpha''}$, $A' \subseteq A_{\beta''}$, $B \subseteq B_{\alpha''+v}$, and $B' \subseteq B_{\beta''+v}$ by (3.4), where $\alpha'' = \alpha+\beta+(1-\varepsilon)\cdot v$ and $\beta'' = \alpha'+\beta'+(1-\varepsilon')\cdot v$, we obtain for each $\delta\in\{0,1\}$:

$$[A\cdot B, A'\cdot B'] \subset (A_{\alpha''} \bullet_{1-\delta} A_{\beta''})\cdot(B_{\beta''+v} \bullet_\delta B_{\alpha''+v}) \subset \mathfrak{h}\ .$$

(ii) Taking $A = A_{\alpha_0}$, $B = B_{\alpha_0+v}$, $A' = A_{\alpha'} \bullet_{1-\varepsilon'} A_{\beta'}$, $B' = B_{\beta'+v} \bullet_{\varepsilon'} B_{\alpha'+v}$, and taking into the account that $A' \subseteq A_{\beta''}$ and $B' \subseteq B_{\beta''+v}$ by (3.4), where $\beta'' = \alpha'+\beta'+(1-\varepsilon')\cdot v$, we obtain for each $\delta\in\{0,1\}$:

$$[A\cdot B, A'\cdot B'] \subset (A_{\alpha_0} \bullet_{1-\delta} A_{\beta''})\cdot(B_{\beta''+v} \bullet_\delta B_{\alpha_0+v}) \subset \mathfrak{h}\ .$$

(ii) Taking $A = A' = A_{\alpha_0}$, $B = B' = B_{\alpha_0+v}$, we obtain for each $\delta\in\{0,1\}$:

$$[A\cdot B, A'\cdot B'] \subset (A_{\alpha_0} \bullet_{1-\delta} A_{\alpha_0})\cdot(B_{\alpha_0+v} \bullet_\delta B_{\alpha_0+v}) \subset \mathfrak{h}\ .$$

The lemma is proved. □

Taking in Lemma 3.4: $\Gamma = \mathbb{Z}^2$, $\alpha = (k,\ell+1)\in\mathbb{Z}^2$, $v = (1,-1)$,

$$A_\alpha = \begin{cases} I_k^{\ell+1}(\mathcal{F}) & \text{if } k,\ell\ge 0\\ 0 & \text{otherwise}\end{cases},\ B_{\alpha+v} = \begin{cases} I_\ell^{k+1}(A,\mathfrak{g}) & \text{if } k,\ell\ge 0\\ 0 & \text{otherwise}\end{cases},$$

Lemma 1.4 implies that (3.4) holds for all $\alpha,\beta\in\mathbb{Z}^2$, $\varepsilon\in\{0,1\}$. Therefore, applying Lemma 3.4 with $\alpha_0 = (0,1)$, we finish the proof of the assertion that $\overline{(\mathfrak{g},A)}(\mathcal{F})$ is a Lie subalgebra of $\mathcal{F}\cdot A$. This finishes the proof of (b).

Prove (c). The first inclusion $(\mathfrak{g},A)(\mathcal{F}) \subset \overline{(\mathfrak{g},A)}(\mathcal{F})$ is obvious because $\mathcal{F}\cdot\mathfrak{g} \subset \overline{(\mathfrak{g},A)}(\mathcal{F})$ and $\overline{(\mathfrak{g},A)}(\mathcal{F})$ is a Lie subalgebra of $\mathcal{F}\cdot A$.

Let us prove the second inclusion $(\mathfrak{g},A)(\mathcal{F}) \subset \overline{(\mathfrak{g},A)}(\mathcal{F})$ of (c).

Rewrite the result of Lemma 1.3(b) (with $\ell_1 = \ell_2 = \infty$) in the form of (3.4) as:

$$I_{k_1}^{\ell_1+1}(\mathcal{F}) \bullet_{1-\varepsilon} I_{k_2}^{\ell_2+1}(\mathcal{F}) \subset I_{k_1}(\mathcal{F}) \bullet_{1-\varepsilon} I_{k_2}(\mathcal{F}) \subset \begin{cases} I_{k_1+k_2}(\mathcal{F}) & \text{if } \varepsilon = 1\\ [\mathcal{F}, I_{k_1+k_2-1}(\mathcal{F})] & \text{if } \varepsilon = 0\end{cases}.$$

Using the obvious inclusion $J_k^{\ell+1} = I_\ell^{k+1}(A,\mathfrak{g}) \subset \mathfrak{g}^{k+1}$ for all $k,\ell\ge 0$ and Lemma 3.3, we obtain

$$J_{\ell_2}^{k_2+1} \bullet_\varepsilon J_{\ell_1}^{k_1+1} \subset \mathfrak{g}^{k_2+1} \bullet_\varepsilon \mathfrak{g}^{k_1+1} \subset \begin{cases} \mathfrak{g}^{k_1+k_2+2} & \text{if } \varepsilon = 0\\ [\mathfrak{g}, \mathfrak{g}^{k_1+k_2+1}] & \text{if } \varepsilon = 1\end{cases}$$

for all $k_1, k_2, \ell_1, \ell_2 \geq 0$, $\varepsilon \in \{0,1\}$. Therefore, we obtain the inclusion:

$$(I_{k_1}^{\ell_1+1} \bullet_{1-\varepsilon} I_{k_2}^{\ell_2+1}) \cdot (J_{\ell_1}^{k_2+1} \bullet_{\varepsilon} J_{\ell_1}^{k_1+1}) \subset$$
$$\subset \begin{cases} I_{k_1+k_2}(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{k_1+k_2+1}] & \text{if } \varepsilon = 1 \\ [\mathcal{F}, I_{k_1+k_2-1}(\mathcal{F})] \cdot \mathfrak{g}^{k_1+k_2+2} & \text{if } \varepsilon = 0 \end{cases} \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F}) \ .$$

This proves the inclusion $\overline{(\mathfrak{g}, A)}(\mathcal{F}) \subset \widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ and finishes the proof of (c).

Therefore, Theorem 3.2 is proved. □

Now we will refine Theorem 3.2 by introducing a natural filtration on each involved Lie algebra and proving the "filtered" version of the theorem.

For any compatible pair $(\mathfrak{g}, A)$, an object $\mathcal{F}$ of $\mathcal{N}$, and each $m \geq 1$ we define the subspaces $\mathcal{F} \cdot \langle \mathfrak{g} \rangle_m$, $(\mathfrak{g}, A)_m(\mathcal{F})$, $\widetilde{(\mathfrak{g}, A)}_m(\mathcal{F})$ and $\overline{(\mathfrak{g}, A)}_m(\mathcal{F})$ of $\mathcal{F} \cdot A$ by:

$$\mathcal{F} \cdot \langle \mathfrak{g} \rangle_m = \sum_{1 \leq k \leq m} \mathcal{F} \cdot \mathfrak{g}^k$$

$$(\mathfrak{g}, A)_m(\mathcal{F}) = \sum_{0 \leq k < m} (\mathcal{F} \cdot \mathfrak{g})^{(k)} \tag{3.5}$$

$$\widetilde{(\mathfrak{g}, A)}_m(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{1 \leq k < m} I_k^{\leq m-k}(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] + [\mathcal{F}, I_{k-1}^{\leq m-k}(\mathcal{F})] \cdot \mathfrak{g}^{k+1} \ , \tag{3.6}$$

where $I_k^{\leq \ell}(\mathcal{F})$ is defined in (1.3) and

(3.7)
$$\overline{(\mathfrak{g}, A)}_m(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum I_{k_1}^{\ell_1+1} I_{k_2}^{\ell_2+1} \cdot [J_{\ell_1}^{k_1+1}, J_{\ell_2}^{k_2+1}] + [I_{k_1}^{\ell_1+1}, I_{k_2}^{\ell_2+1}] \cdot J_{\ell_2}^{k_2+1} J_{\ell_1}^{k_1+1},$$

where the summation is over all $k_1, k_2 \geq 0$, $\ell_1, \ell_2 \geq 0$ such that $k_1+k_2+\ell_1+\ell_2+2 \leq m$, and we abbreviated $I_k^\ell := I_k^\ell(\mathcal{F})$, $J_k^\ell := I_k^\ell(A, \mathfrak{g})$ in the notation (1.5).

Recall that a Lie algebra $\mathfrak{h} = (\mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \ldots)$ is called a *filtered Lie algebra* if $[\mathfrak{h}_{k_1}, \mathfrak{h}_{k_2}] \subset \mathfrak{h}_{k_1+k_2}$ for all $k_1, k_2 \geq 0$.

Taking into account that $[\mathfrak{g}^{k_1+1}, \mathfrak{g}^{k_2+1}] \subset \mathfrak{g}^{k_1+k_2+1}$, we see that $\mathfrak{h}_m = \mathcal{F} \cdot \langle \mathfrak{g} \rangle_m$, $m \geq 0$ defines an increasing filtration on the Lie algebra $\mathcal{F} \cdot \langle \mathfrak{g} \rangle$ (where $\langle \mathfrak{g} \rangle$ is as in (2.2).

The following result is a filtered version of Theorem 3.2.

Theorem 3.5. *For any compatible pair* $(\mathfrak{g}, A)$ *and an object* $\mathcal{F}$ *of* $\mathcal{N}$ *one has:*
(a) The subspace $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$ *is a filtered Lie subalgebra of* $\mathcal{F} \cdot \langle \mathfrak{g} \rangle$*.*
(b) The subspace $\overline{(\mathfrak{g}, A)}(\mathcal{F})$ *is a filtered Lie subalgebra of* $\mathcal{F} \cdot \langle \mathfrak{g} \rangle$*.*
(c) A chain of inclusions of filtered Lie algebras:

$$(\mathfrak{g}, A)(\mathcal{F}) \subseteq \overline{(\mathfrak{g}, A)}(\mathcal{F}) \subseteq \widetilde{(\mathfrak{g}, A)}(\mathcal{F}) \ .$$

The proof of Theorem 3.5 is almost identical to that of Theorem 3.2.

4. Perfect pairs and achievable upper bounds

Below we lay out some sufficient conditions on the compatible pair $(\mathfrak{g}, A)$ which guarantee that the upper bounds are achievable.

Definition 4.1. We say that a compatible pair $(\mathfrak{g}, A)$ is *perfect* if

$$[\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} + (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) = \mathfrak{g}^{k+1}$$

for all $k \geq 2$.

We say that a compatible pair $(\mathfrak{g}, A)$ is split semisimple if $\mathfrak{g}$ is a split semisimple (over $\mathbb{Q}$) Lie subalgebra of A.

The following is the main result of this section.

Main Theorem 4.2. *Let $(\mathfrak{g}, A)$ be a compatible pair. Then*
(a) If $(\mathfrak{g}, A)$ is perfect, then for any object $\mathcal{F}$ of $\mathcal{N}$ one has

$$(\mathfrak{g}, A)(\mathcal{F}) = \widetilde{(\mathfrak{g}, A)}(\mathcal{F}) \ ,$$

*i.e., the noncommutative loop Lie algebras $(\mathfrak{g}, A), \widetilde{(\mathfrak{g}, A)}\mathcal{N} \to$ **LieAct** are equal.*
(b) If $(\mathfrak{g}, A)$ is split semisimple, then $(\mathfrak{g}, A)$ is perfect.
(c) If $(\mathfrak{g}, A)$ is split semisimple, then for any object $\mathcal{F}$ of $\mathcal{N}$ one has

$$(\mathfrak{g}, A)(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k \geq 2} I_{k-1}(\mathcal{F}) \cdot (\mathfrak{g}^k)_+ + [\mathcal{F}, I_{k-2}(\mathcal{F})] \cdot Z_k(\mathfrak{g}) \ , \tag{4.1}$$

where $(\mathfrak{g}^k)_+ = [\mathfrak{g}, \mathfrak{g}^k]$ is the "centerless" part of $\mathfrak{g}^k$, $Z_k(\mathfrak{g}) = Z(\langle\mathfrak{g}\rangle) \cap \mathfrak{g}^k$, and $Z(\langle\mathfrak{g}\rangle)$ is the center of $\langle\mathfrak{g}\rangle = \sum_{k \geq 0} \mathfrak{g}^k$.

Proof. Prove (a). We need the following assertion regarding the lower bound for $(\mathfrak{g}, A)(\mathcal{F})$.

Proposition 4.3. *Let $\mathfrak{g}$ be a Lie subalgebra of A and $\mathcal{F}$ be any of $\mathcal{N}$.*
(a) Assume that for some $k \geq 2$ one has

$$I \cdot [\mathfrak{g}, \mathfrak{g}^k] \subset (\mathfrak{g}, A)(\mathcal{F})$$

where I is a left ideal in $\mathcal{F}$. Then:

$$[\mathcal{F}, I] \cdot [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} \subset (\mathfrak{g}, A)(\mathcal{F}) \ . \tag{4.2}$$

(b) Assume that for some $k \geq 2$ one has

$$J \cdot \mathfrak{g}^k \subset (\mathfrak{g}, A)(\mathcal{F})$$

where J is a subset of $\mathcal{F}$ such that $[\mathcal{F}, J] \subset J$. Then:

$$[\mathcal{F}, J] \cdot \mathfrak{g}^{k+1} + (\mathcal{F}J + J) \cdot [\mathfrak{g}, \mathfrak{g}^k] \subset (\mathfrak{g}, A)(\mathcal{F}) \tag{4.3}$$

Proof. Prove (a). Indeed,

$$[\mathcal{F} \cdot \mathfrak{g}, I \cdot [\mathfrak{g}, \mathfrak{g}^k]] \equiv [\mathcal{F}, I] \cdot [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} \mod \mathcal{F}I \cdot [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^k]] \ .$$

Since $\mathcal{F}I \subset \mathcal{F}$ and $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^k]] \subset [\mathfrak{g}, \mathfrak{g}^k]$, and, therefore, $\mathcal{F}I \cdot [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^k]] \subset I \cdot [\mathfrak{g}, \mathfrak{g}^k] \subset (\mathfrak{g}, A)(\mathcal{F})$, the above congruence implies that $[\mathcal{F}, I] \cdot [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g}$ also belongs to $(\mathfrak{g}, A)(\mathcal{F})$. This proves (a).

Prove (b). For any $g \in \mathfrak{g}$ we obtain:

$$[\mathcal{F} \cdot g, J \cdot g^k] = [\mathcal{F}, J] \cdot g^{k+1}$$

which implies that $[\mathcal{F}, J] \cdot \widetilde{\mathfrak{g}^{k+1}} \subset (\mathfrak{g}, A)(\mathcal{F})$ (in the notation of Lemma 2.6). Using Lemma 2.6, we obtain

$$[\mathcal{F}, J] \cdot \widetilde{\mathfrak{g}^{k+1}} \equiv [\mathcal{F}, J] \cdot \mathfrak{g}^{k+1} \mod [\mathcal{F}, J] \cdot (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) \ .$$

Taking into account that $[\mathcal{F},J]\cdot(\mathfrak{g}^k\cap\mathfrak{g}^{k+1})\subset[\mathcal{F},J]\cdot\mathfrak{g}^k\subset(\mathfrak{g},A)(\mathcal{F})$, the above implies that $[\mathcal{F},J]\cdot\mathfrak{g}^{k+1}$ also belongs to $(\mathfrak{g},A)(\mathcal{F})$. Furthermore,

$$[\mathcal{F}\cdot\mathfrak{g},J\cdot\mathfrak{g}^k]\equiv\mathcal{F}J\cdot[\mathfrak{g},\mathfrak{g}^k]\mod[\mathcal{F},J]\cdot\mathfrak{g}^{k+1}\ .$$

Therefore, using the already proved inclusion $[\mathcal{F},J]\cdot\mathfrak{g}^{k+1}\subset(\mathfrak{g},A)(\mathcal{F})$, we see that $\mathcal{F}J\cdot[\mathfrak{g},\mathfrak{g}^k]$ also belongs to $(\mathfrak{g},A)(\mathcal{F})$. Finally, using the fact that $[\mathfrak{g},\mathfrak{g}^k]\subset\mathfrak{g}^k$, we obtain $J\cdot[\mathfrak{g},\mathfrak{g}^k]\subset J\cdot\mathfrak{g}^k\subset(\mathfrak{g},A)(\mathcal{F})$. This proves (b).

Proposition 4.3 is proved. $\square$

Now we are ready to finish the proof of Theorem 4.2(a). In view of Theorem 3.2(c), it suffices to prove that $\widetilde{(\mathfrak{g},A)}(\mathcal{F})\subset(\mathfrak{g},A)(\mathcal{F})$, that is,

$$I_k(\mathcal{F})\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]\subset(\mathfrak{g},A)(\mathcal{F}),\ [\mathcal{F},I_k(\mathcal{F})]\cdot\mathfrak{g}^{k+2}\subset(\mathfrak{g},A)(\mathcal{F})\tag{4.4}$$

for $k\geq 0$.

We will prove (4.4) by induction in k. First, verify the base of induction at $k=0$. Obviously, $I_0(\mathcal{F})\cdot[\mathfrak{g},\mathfrak{g}]\subset(\mathfrak{g},A)(\mathcal{F})=\mathcal{F}\cdot[\mathfrak{g},\mathfrak{g}]\subset(\mathfrak{g},A)(\mathcal{F})$. Furthermore, Proposition 4.3(b) taken with $k=1$, $J=\mathcal{F}$ implies that $[\mathcal{F},\mathcal{F}]\cdot\mathfrak{g}^2\subset(\mathfrak{g},A)(\mathcal{F})$.

Now assume that $k>0$. Using a part of the inductive hypothesis in the form $[\mathcal{F},I_{k-1}(\mathcal{F})]\cdot\mathfrak{g}^{k+1}\subset(\mathfrak{g},A)(\mathcal{F})$ and applying Proposition 4.3(b) with $J=[\mathcal{F},I_{k-1}(\mathcal{F})]$, we obtain $(\mathcal{F}J+J)\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]\subset(\mathfrak{g},A)(\mathcal{F})$. In its turn, Lemma 1.2(b) taken with $\ell=\infty$ implies that $\mathcal{F}J+J=I_k(\mathcal{F})$. Therefore, we obtain

$$I_k(\mathcal{F})\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]\subset(\mathfrak{g},A)(\mathcal{F})\ ,$$

which is the first inclusion of (4.4). To prove the second inclusion (4.4), we will use Proposition 4.3(a) with $I=I_k(\mathcal{F})$:

$$(\mathfrak{g},A)(\mathcal{F})\supset[\mathcal{F},I_k(\mathcal{F})]\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]\mathfrak{g}\ .$$

On the other hand, using the perfectness of the pair $(\mathfrak{g},A)$, we obtain:

$$[\mathcal{F},I_k(\mathcal{F})]\cdot[\mathfrak{g},\mathfrak{g}^{k+1}]\mathfrak{g}\equiv[\mathcal{F},I_k(\mathcal{F})]\cdot\mathfrak{g}^{k+2}\mod[\mathcal{F},I_k(\mathcal{F})]\cdot\mathfrak{g}^{k+1}\ .$$

But Lemma 1.2(a) taken with $\ell=\infty$ implies that $I_k(\mathcal{F})\subset I_{k-1}(\mathcal{F})$, therefore,

$$[\mathcal{F},I_k(\mathcal{F})]\cdot(\mathfrak{g}^{k+1}\cap\mathfrak{g}^{k+2})\subset[\mathcal{F},I_k(\mathcal{F})]\cdot\mathfrak{g}^{k+1}\subset[\mathcal{F},I_{k-1}(\mathcal{F})]\cdot\mathfrak{g}^{k+1}\subset(\mathfrak{g},A)(\mathcal{F})$$

by the inductive hypothesis. This gives the second inclusion of (4.4). Therefore, Theorem 4.2(a) is proved.

Prove (b) now. We will first show that each the pair $(\mathfrak{g},A)$ is perfect, whenever $\mathfrak{g}\subset A$ is split semisimple (over $\mathbb{Q}$).

We need the following simple technical result.

Lemma 4.4. *Let $(\mathfrak{g},A)$ be a compatible pair. Assume that $h_0\in\mathfrak{g}$ is such that ad h is diagonalizable over $\mathbb{Q}$, i.e.,*

$$\mathfrak{g}=\bigoplus_{r\in\mathbb{Q}}\mathfrak{g}_r\ ,\tag{4.5}$$

where each $\mathfrak{g}_k$ is the eigenspace of the operator ad $h_0:\mathfrak{g}\to\mathfrak{g}$ of the rational eigenvalue r. Then for each $k\geq 1$ and each $\mathbf{r}=(r_1,\dots,r_{k+1})\in\mathbb{Q}^{k+1}\setminus\{0\}$ the subspace $\mathfrak{g}_{r_1}\cdots\mathfrak{g}_{r_{k+1}}$ of $\mathfrak{g}^k$ belongs to $[\mathfrak{g},\mathfrak{g}^k]\mathfrak{g}+(\mathfrak{g}^k\cap\mathfrak{g}^{k+1})$.

Proof. Clearly, under the adjoint action of h_0 on $\mathfrak{g}^k$ each vector of $x \in \mathfrak{g}_{r_1}\cdots\mathfrak{g}_{r_k}$ satisfies $[h_0, x] = (r_1 + \cdots + r_k)x$. Therefore, for any $(r_1, \ldots, r_k) \in \mathbb{Q}^k$ such that $r_1 + \cdots + r_k \neq 0$ the subspace $\mathfrak{g}_{r_1}\cdots\mathfrak{g}_{r_k}$ belongs to $[\mathfrak{g}, \mathfrak{g}^k]$. Clearly,

$$\mathfrak{g}_{r_1}\cdots\mathfrak{g}_{r_{k+1}} \equiv \mathfrak{g}_{r_{\sigma(1)}}\cdots\mathfrak{g}_{r_{\sigma(k)}}\mathfrak{g}_{r_{\sigma(k+1)}} \mod \mathfrak{g}^k \cap \mathfrak{g}^{k+1}$$

for any permutation $\sigma \in S_{k+1}$.

It is also easy to see that for any $\mathbf{r} = (r_1, \ldots, r_{k+1}) \in \mathbb{Q}^{k+1} \setminus \{0\}$ there exists a permutation $\sigma \in S_{k+1}$ such that $r_{\sigma(1)} + \cdots + r_{\sigma(k)} \neq 0$ and, therefore,

$$\mathfrak{g}_{r_1}\cdots\mathfrak{g}_{r_{k+1}} \subset (\mathfrak{g}_{r_{\sigma(1)}}\cdots\mathfrak{g}_{r_{\sigma(k)}})\mathfrak{g}_{r_{\sigma(k+1)}} + \mathfrak{g}^k \cap \mathfrak{g}^{k+1} \in [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} + (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) \ .$$

The lemma is proved. □

Clearly, if $\mathfrak{g}$ is a split semisimple (over $\mathbb{Q}$) Lie algebra, then it contains a (regular) semisimple element h_0 such which satisfies Lemma 4.4 and, moreover, such that in terms of (4.5) one has $\mathfrak{g}_0 = \mathfrak{h}$ is the Cartan (aka maximal toral) subalgebra of $\mathfrak{g}$, and

$$\mathfrak{n}_- = \bigoplus_{r<0} \mathfrak{g}_r, \mathfrak{n}_+ = \bigoplus_{r>0} \mathfrak{g}_r$$

are opposite maximal nilpotent subalgebras normalized by $\mathfrak{h}$.

Therefore, in view of Lemma 4.4, it suffices to prove the following result.

Lemma 4.5. *If $\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a split semisimple Lie algbra, then*

$$\mathfrak{h}^{k+1} \subset [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} + (\mathfrak{g}^k \cap \mathfrak{g}^{k+1}) \ .$$

Proof. Let us consider the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha\neq 0} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \mathbb{Q}\cdot E_\alpha$ for each root α. For each positive root α (i.e., such that $r = \alpha(h) > 0$, i.e, $E_{-\alpha} \in \mathfrak{n}_-$, $E_\alpha \in \mathfrak{n}_+$) we rescale E_α in such a way so that the triple $(E_{-\alpha}, H_\alpha = [E_\alpha, F_\alpha], E_\alpha)$ is an sl_2-triple.

For each positive root α we obtain the following congruence:

$$[E_\alpha, \mathfrak{h}^{k-1}E_{-\alpha}]\mathfrak{h} \equiv \mathfrak{h}^{k-1}H_\alpha\mathfrak{h} \mod [E_\alpha, \mathfrak{h}^{k-1}]E_{-\alpha}\mathfrak{h}$$

Note that

$$[E_\alpha, \mathfrak{h}^{k-1}]E_{-\alpha}\mathfrak{h} \subset \sum_{i=0}^{k-2} \mathfrak{h}^i E_\alpha \mathfrak{h}^{k-2-i}E_{-\alpha}\mathfrak{h} \subset [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} + (\mathfrak{g}^k \cap \mathfrak{g}^{k+1})$$

by Lemma 4.4. Therefore, $\mathfrak{h}^{k-1}H_\alpha\mathfrak{h} \subset [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} + (\mathfrak{g}^k \cap \mathfrak{g}^{k+1})$ for each positive root α. Since all H_α span $\mathfrak{h}$, this proves that $\mathfrak{h}^{k+1} \subset [\mathfrak{g}, \mathfrak{g}^k]\mathfrak{g} + (\mathfrak{g}^k \cap \mathfrak{g}^{k+1})$. The lemma is proved. □

Therefore, we have proved that each split semisimple pair $(\mathfrak{g}, A)$ is perfect. Based on this and on Theorem 4.2(a), in order to finish the proof of Theorem 4.2(b), it suffices to show that

$$\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F}\cdot\mathfrak{g} + \sum_{k\geq 2} I_{k-1}(\mathcal{F})\cdot(\mathfrak{g}^k)_+ + [\mathcal{F}, I_{k-2}(\mathcal{F})]\cdot Z_k(\mathfrak{g}) \ . \tag{4.6}$$

We need the following simple fact regarding split semisimple pairs $(\mathfrak{g}, A)$.

Lemma 4.6. *For any split semisimple pair $(\mathfrak{g}, A)$ one has the following decomposition of the $\mathfrak{g}$-module $\mathfrak{g}^k$, $k \geq 2$:*

$$\mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^k] + Z_k(\mathfrak{g}),\ \ [\mathfrak{g}, \mathfrak{g}^k] \cap Z_k(\mathfrak{g}) = \{0\} \ ,$$

where $Z_k(\mathfrak{g}) = Z(\langle\mathfrak{g}\rangle) \cap \mathfrak{g}^k$, and $Z(\langle\mathfrak{g}\rangle)$ is the center of $\langle\mathfrak{g}\rangle = \sum_{k\geq 0} \mathfrak{g}^k$.

Proof. Clearly, $\mathfrak{g}^k$ is a semisimple finite-dimensional $\mathfrak{g}$-module (under the adjoint action). Therefore, it uniquely decomposes into isotypic components one of which, the component of invariants, is $Z_k(\mathfrak{g})$. Denote the sum of all non-invariant isotypic components by $(\mathfrak{g}^k)_+$. By definition, $\mathfrak{g}^k = (\mathfrak{g}^k)_+ + Z_k(\mathfrak{g})$ and $(\mathfrak{g}^k)_+ \cap Z_k(\mathfrak{g}) = 0$. It remains to prove that $(\mathfrak{g}^k)_+ = [\mathfrak{g}, \mathfrak{g}^k]$. Indeed, $[\mathfrak{g}, \mathfrak{g}^k] \subseteq (\mathfrak{g}^k)_+$. On the other hand, since each non-trivial simple $\mathfrak{g}$-submodule $V \subset \mathfrak{g}^k$ is faithful, i.e., $[\mathfrak{g}, V] = V$. Therefore, $[\mathfrak{g}, \mathfrak{g}^k]$ contains all non-invariant isotypic components, i.e., $[\mathfrak{g}, \mathfrak{g}^k] \subset (\mathfrak{g}^k)_+$. The obtained double inclusion implies that $(\mathfrak{g}^k)_+ = [\mathfrak{g}, \mathfrak{g}^k]$. The lemma is proved. $\square$

Furthermore, using Lemma 4.6 and the definition (3.1) of $\widetilde{(\mathfrak{g}, A)}(\mathcal{F})$, we obtain

$$\widetilde{(\mathfrak{g}, A)}(\mathcal{F}) = \mathcal{F} \cdot \mathfrak{g} + \sum_{k \geq 1} I_k(\mathcal{F}) \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] + [\mathcal{F}, I_{k-1}(\mathcal{F})] \cdot \mathfrak{g}^{k+1}$$

$$= \mathcal{F} \cdot \mathfrak{g} + \sum_{k \geq 1} (I_k(\mathcal{F}) + [\mathcal{F}, I_{k-1}(\mathcal{F})]) \cdot [\mathfrak{g}, \mathfrak{g}^{k+1}] + [\mathcal{F}, I_{k-1}(\mathcal{F})] \cdot Z_{k+1}(\mathfrak{g}) \ ,$$

which, after taking into account that $[\mathcal{F}, I_{k-1}(\mathcal{F})] \subset I_k(\mathcal{F})$ (and shifting the index of summation), becomes the right hand side of (4.6). This finishes the proof of Theorem 4.2(b).

Therefore, Theorem 4.2 is proved. $\square$

Remark 4.7. Based on the arguments of Lemmas 4.4 and 4.5, we can generalize Theorem 4.2(b) to Kac-Moody Lie algebras.

The following result is a corollary of Theorem 4.2.

Theorem 4.8. *Let A be a $\mathbb{Q}$-algebra with unit containing $sl_2(\mathbb{Q})$ as a Lie subalgebra. Then*

$$(sl_2(\mathbb{Q}), A)(\mathcal{F}) = \mathcal{F} \cdot sl_2(\mathbb{Q}) + [\mathcal{F}, Z_1(A, \mathcal{F})] \cdot 1 + \sum_{i \geq 1} Z_i(A, \mathcal{F}) \cdot V_i \ , \tag{4.7}$$

where

$$Z_i(A, \mathcal{F}) = \sum_{j \geq 0} I_{i+2j-1}(\mathcal{F}) \cdot \Delta^j \ ,$$

$\Delta = 2EF + 2FE + H^2$ is the Casimir element and V_i is the $sl_2(\mathbb{Q})$-submodule of A generated by E^i. In particular, if A is finite dimensional over $\mathbb{Q}$ with unit, then there exists $m \geq 1$ such that $E^{m+1} = 0$, $\Delta \in \mathbb{Q} \cdot 1$, and

$$(sl_2(\mathbb{Q}), A)(\mathcal{F}) = \mathcal{F} \cdot sl_2(\mathbb{Q}) + [\mathcal{F}, \mathcal{F}] \cdot 1 + \sum_{1 \leq i \leq m} I_{i-1}(\mathcal{F}) \cdot V_i \ , \tag{4.8}$$

where V_i is the $sl_2(\mathbb{Q})$-submodule of A generated by E^i.

Proof. Prove (4.7). Clearly, each $\mathfrak{g}^k$ is a finite-dimensional $sl_2(\mathbb{Q})$-module generated by the highest weight vectors $\Delta^j E^i$, $i, j \geq 0$, $i + 2j \leq k$. That is, in notation of (4.1), one has

$$(\mathfrak{g}^k)_+ = \sum_{i>0, j \geq 0, i+2j \leq k} \Delta^j \cdot V_i \ ,$$

where the sum is direct (but some summands may be zero) and

$$V_i = \sum_{r=-i}^{i} \mathbb{Q} \cdot (ad\ F)^{i+r}(E^i)$$

is the corresponding simple $sl_2(\mathbb{Q})$-module; and

$$Z_k(\mathfrak{g}) = \sum_{1\le j\le k/2} \mathbb{Q}\cdot\Delta^j ,$$

where the sum is direct. Therefore, taking into account that $I_k(\mathcal{F})\subset I_{k-1}(\mathcal{F})$, the equation (4.1) simplifies to

$$(sl_2(\mathbb{Q}),A)(\mathcal{F}) = \mathcal{F}\cdot\mathfrak{g} + \sum_{i>0,j\ge 0} I_{i+2j-1}(\mathcal{F})\cdot\Delta^j V_i + \sum_{j\ge 1}[\mathcal{F}, I_{2j-2}(\mathcal{F})]\cdot\Delta^j$$
$$= \mathcal{F}\cdot sl_2(\mathbb{Q}) + [\mathcal{F}, Z_1(A,\mathcal{F})]\cdot 1 + \sum_{i\ge 1} Z_i(A,\mathcal{F})\cdot V_i .$$

This finishes the proof of (4.7).

Prove (4.8). Indeed, now $(\mathfrak{g},A)$ is of finite type, say, m, therefore using (3.3) and the fact that $Z_i(A,\mathcal{F}) = I_{i-1}(\mathcal{F})$ (because $I_k(\mathcal{F})\subset I_{k-1}(\mathcal{F})$ and Δ is a scalar), we obtain from already proved (4.7):

$$(sl_2(\mathbb{Q}),A)(\mathcal{F}) = \mathcal{F}\cdot sl_2(\mathbb{Q}) + [\mathcal{F}, Z_1(A,\mathcal{F})]\cdot 1 + \sum_{i\ge 1} Z_i(A,\mathcal{F})\cdot V_i$$
$$= \mathcal{F}\cdot sl_2(\mathbb{Q}) + [\mathcal{F},\mathcal{F}]\cdot 1 + \sum_{1\le i\le m} I_{i-1}(\mathcal{F})\cdot V_i .$$

Theorem 4.8 is proved. □

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