APPROXIMATION BY NODAL SETS

DMITRY JAKOBSON

Abstract. We discuss approximating points on manifolds by nodal sets of eigenfunctions of the Laplacian and explore analogy with approximation by continued fractions.

1. The problem and a simple example

Let M be an n-dimensional compact smooth Riemannian manifold with metric g. Let Δ be the Laplace-Beltrami operator on M, and let $\{\mu_j, \phi_j\}$ denote the spectrum of Δ, where

$$\Delta\phi_j + \mu_j^2\phi_j = 0.$$

In the sequel we shall assume that one of the following two possibilities holds:

i) all μ_j-s are simple (true generically, cf. [U]);
ii) we fix in advance a basis of every eigenspace of Δ.

Weyl's law in dimension n can be formulated as follows:

$$\mu_k \asymp Ck^{1/n}, \tag{1.1}$$

where C depends only on n and the volume of M.

Definition 1.2. Let $\mathcal{N}_\mu := \mathcal{N}(\phi_\mu)$ be the nodal set of the eigenfunction $\Delta\phi_\mu + \mu^2\phi_\mu = 0$.

We would like to study how well points of M can be approximated by $\mathcal{N}_\mu$ as $\mu \to \infty$. Consider first a simple example: eigenfunctions on $M = [0, \pi]$ with the standard metric and (say) with Dirichlet boundary conditions. Then

$$\mu_k = k,\ \phi_k(x) = \sin(kx),\ \mathcal{N}_k = \left\{\frac{\pi j}{k} : 0 \le j \le k\right\}.$$

Accordingly, the set $\mathcal{N}_k$ is $\pi/(2k)$-dense in M.

Interestingly, a similar result holds on any smooth Riemannian manifold (see e.g. [Br]):

Proposition 1.3. *There exists $C > 0$ (which depends only on M, g) such that*

$$B(x, C/\mu) \cap \mathcal{N}(\phi_\mu) \ne \emptyset$$

for any $x \in M$ and $\mu > 0$.

2000 *Mathematics Subject Classification.* Primary: 58J50 Secondary: 11J70, 35P20, 81Q50.
Key words and phrases. eigenfunction, nodal set, continued fractions.
The author was supported by NSERC, NATEQ, Alfred P. Sloan Foundation fellowship and Dawson fellowship. The author would also like to thank IHES and Max Planck Institute for their hospitality.

Here $B(x,r)$ denotes the ball of radius r centered at $x \in M$. Thus $\mathcal{N}_\mu$ is C/μ-dense in M.

To study the rate of approximation by $\mathcal{N}_\mu$ as $\mu \to \infty$ in more detail, consider again the case of $M = [0,\pi]$ where approximating by points in $\mathcal{N}_k$ is equivalent (after rescaling by π) to approximating by rationals with denominator k. It is well-known (see e.g. Khinchin's book [K]) that the distance from any $x \in [0,1]$ to the m-th convergent of its continued fraction expansion is $O(1/m^2)$. However, the *denominator* of the m-th convergent grows exponentially in m for $x \notin \mathbf{Q}$.

Denote by $||x||$ the distance from $x \in \mathbf{R}$ to the nearest integer. The following proposition can be found in [K] and is proved by an easy application of Borel-Cantelli lemma.

Proposition 1.4. *If $\sum_q \psi(q)$ converges, then for Lebesgue-almost all x, there exist only finitely many q such that $||qx|| < \psi(q)$.*

Taking $\psi(q) = C/q^{1+\delta}$ in Proposition 1.4 for any $\delta > 0$, we conclude that

Corollary 1.5. *Given $C, \delta > 0$, for Lebesgue-almost all $x \in [0,1]$ the inequality*

$$|x - p/q| < C/q^{2+\delta}$$

has finitely many integer solutions (p,q).

Equivalently, almost all $x \in M = [0,\pi]$ *cannot* be approximated by points in $\mathcal{N}_k$ to within $C/k^{2+\delta}$ infinitely often. We would like to prove an analogous statement for more general M.

2. Estimate for real-analytic manifolds

To characterize the *rate* of approximation by nodal sets, we make the following definition:

Definition 2.1. Given $b > 0$ *(exponent)*, and $C > 0$ *(constant)*, let $M(b,C)$ to be the set of all $x \in M$ such that there exists an infinite sequence $\mu_k \to \infty$ for which

$$B\left(x, \frac{C}{\mu_k^b}\right) \cap \mathcal{N}(\phi_{\mu_k}) \neq \emptyset.$$

For example, Proposition 1.3 implies that $M(1,C) = M$ for some $C > 0$. Also, Corollary 1.5 implies that for $M = [0,\pi]$, we have $\mathrm{meas}(M(2+\delta, C)) = 0\ \forall C, \delta > 0$. Here meas denotes the Lebesgue measure.

We now state an analogue for general M:

Theorem 2.2. *Let M be a real-analytic manifold of dimension n. Then for any $C > 0, \delta > 0$,*

$$\mathrm{vol}(M(n+1+\delta, C)) = 0,$$

where vol *denotes the Riemannian volume on M.*

Equivalently, almost all $x \in M$ *cannot* be approximated by points in $\mathcal{N}_\mu$ to within $C/\mu^{n+1+\delta}$ infinitely often.

The proof of Theorem 2.2 uses the following estimate proved by Donnelly and Fefferman proved in [DF]:

Proposition 2.3. *Let M be real-analytic manifold of dimension n. Then there exist $0 < C_1 < C_2$ independent of $\mu > 0$ such that*

$$C_1\mu \le \mathrm{meas}_{n-1}(\mathcal{N}_\mu) \le C_2\mu, \tag{2.4}$$

where meas_{n-1} *is the* $(n-1)$*-dimensional measure on* M *induced by the metric.*

One can regard this result as a generalization of the fact that the k-th eigenfunction of a Sturm-Liouville problem has $(k-1)$ zeros.

Proof of Theorem 2.2. The proof proceeds similarly to the proof of Corollary 1.5. Fix $C > 0, \delta > 0$. Let A_j be the neighborhood of $\mathcal{N}(\phi_j)$ of radius $C/\mu_j^{n+1+\delta}$. As $\mu_j \to \infty$, we have

$$\mathrm{vol}(A_j) \le C_3/\mu_j^{n+\delta}$$

by (2.4). Accordingly,

$$\mathrm{vol}\left(\cup_j A_j\right) \le C_3 \sum_j \mu_j^{-n-\delta}. \tag{2.5}$$

To finish the proof, it suffices by Borel-Cantelli Lemma to show that $\sum_j \mu_j^{-n-\delta}$ converges. This follows from the asymptotics $\mu_k \sim k^{1/n}$ of (1.1) as $k \to \infty$. □

Remark 2.6. In the proof of Theorem 2.2 we have only used the *upper bound* in (2.4). For general smooth manifolds, such a bound is unknown but is conjectured to hold by Yau. Yau's conjecture would imply Theorem 2.2 for such M. In dimension two, one can show that $\mathrm{vol}(M(7/2+\delta, C)) = 0$ for all $C, \delta > 0$ using the estimate $\mathrm{meas}_1(\mathcal{N}_\mu) \le C\mu^{3/2}$ of [DF2].

Remark 2.7. By results of [J-L], Theorem 2.2 continues to hold for *level sets* of eigenfunctions (since the level set of an eigenfunction is a nodal set of a linear combination of that eigenfunction with a constant eigenfunction). Question: which level sets are C/μ-dense?

3. Discussion

For a given M it seems interesting to find

$$E(M) := \sup\{b : \mathrm{vol}(M(b, C)) > 0 \ for\ some\ C > 0\}.$$

Theorem 2.2 implies that real-analytic n-dimensional manifolds, $E(M) \le n + 1$.

In dimension one, it follows from the theory of continued fractions that $E(M) = 2$ for $M = [0, \pi]$. In fact, $M(2, \pi) = M$ while $\mathrm{vol}(M(2+\delta, C)) = 0 \forall \delta > 0$.

The same result likely holds for *separable systems.*[1] In such systems one can separate variables and choose a basis of eigenfunctions that (in appropriate coordinates) have the form $\phi(x_1, \ldots, x_n) = \prod \psi_j(x_j)$, where ψ_j are solutions of 2nd order differential equations. Accordingly, $\mathcal{N}(\phi)$ forms a "grid" of hyperpsurfaces determined by zeros of ψ_j-s, and approximation by $\mathcal{N}(\phi)$ reduces to a series of one-dimensional problems. The proof of the inequality $E(M) \le 2$ seems straightforward, while the proof of the equality looks more difficult.

For manifolds with ergodic geodesic flows (e.g. in negative curvature), eigenfunction behavior has been studied using *random wave model* [Be]. In addition, *percolation model* [BS] has been used to study the statistics of nodal domains in

[1]Examples include surfaces of revolution, Liouville tori and *quantum completely integrable* systems [TZ].

chaotic systems.[2] In the opinion of the author, it would be difficult to use these models directly to predict the "best possible" rate of approximation by nodal sets. The reason is that these models descibe a *single* eigenfunction on a scale of C/μ (several wavelengths). However (as shown by the example of $M = [0, \pi]$) for a given $x \in M$ the values of μ giving the best approximation of x by $\mathcal{N}(\phi_\mu)$ can grow exponentially. It thus seems difficult to take into account simultaneous behavior of all eigenfunctions in such a large energy range. However, one can probably expect that $E(M) > 2$ for such manifolds (in contrast to the integrable case), due to irregularity of nodal lines for such systems.

It also seems interesting to study "level sets" $M(b)$ for the approximation exponent b, e.g. defined by

$$M(b) := \cup_C M(b, C) \setminus (\cup_{a<b} \cup_C M(a, C)) .$$

Acknowledgements: The author would like to thank E. Bogomolny, Y. Fyodorov, J. Marklof, U. Smilansky and J. Toth for stimulating discussions about this problem. The author would also like to thank the organizers of the workshop on wavefunctions at the university of Bristol in september 2005 for their hospitality.

References

[Be] M. Berry. Focusing and twinkling: critical exponents from catastrophes in non-Gaussian random short waves. *J. Phys. A* **10** (1977), 2061–2081.

[BS] E. Bogomolny and C. Schmit. Percolation model for nodal domains of chaotic wave functions. *Phys. Rev. Lett.* **88** (2002), 114102.

[Br] J. Brüning. Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators. *Math. Z.* **158** (1978), no. 1, 15–21.

[DF] H. Donnelly and C. Fefferman. Nodal sets of eigenfunctions of Riemannian manifolds. *Invent. Math.* **93** (1988), 161–183.

[DF2] H. Donnelly and C. Fefferman. Nodal sets for eigenfunctions of the Laplacian on surfaces. *J. Amer. Math. Soc.* **3** (1990), 333–353.

[FGS] G. Foltin, S. Gnutzmann and U. Smilansky. The morphology of nodal lines—random waves versus percolation. *J. Phys. A* **37** (2004), no. 47, 11363–11371.

[J-L] D. Jerison and G. Lebeau. Nodal sets of sums of eigenfunctions. In *Harmonic analysis and partial differential equations* (Chicago, IL, 1996), 223–239, Chicago Lectures in Math., U. Chicago Press, 1999.

[K] A. Khinchin. *Continued fractions.* Dover, Mineola, NY, 1997. *Tsepnye drobi* (Russian), 4th ed. Nauka, Moscow, 1978.

[TZ] J. Toth and S. Zelditch. Riemannian manifolds with uniformly bounded eigenfunctions. *Duke Math. J.* **111** (2002), 97–132.

[U] K. Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.* **98** (1976), 1059–1078.

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 2K6, Canada.

E-mail address: `jakobson@math.mcgill.ca`

[2]We refer the reader to [FGS] and references therein for a nice discussion about applicability of those models for studying various questions about eigenfunctions of chaotic systems.