Some results on the eigenfunctions of the quantum trigonometric Calogero-Sutherland model related to the Lie algebra D_4

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Abstract

We express the Hamiltonian of the quantum trigonometric Calogero-Sutherland model related to the Lie algebra D_4 in terms of a set of Weyl-invariant variables, namely, the characters of the fundamental representations of the Lie algebra. This parametrization allows us to solve for the energy eigenfunctions of the theory and to study properties of the system of orthogonal polynomials associated to them such as recurrence relations and generating functions.

1 Introduction

Integrable models play a prominent role in theoretical physics. The reason is not only the direct phenomenological interest of some of them, but also the fact that they often provide some deep insights into the mathematical structure of the theories in which they arise. Sometimes, they even reveal unexpected relations among different physical or mathematical theories. In classical mechanics, integrability not only shows up itself in some of the most important and time-honored problems, such as the Keplerian motion or the Lagrange or Kovalevskaya top. It appears also in a plethora of new hypothetical, highly nontrivial sytems discovered mainly during the three last decades of the past century (see [1, 2] for comprehensives reviews). Among these, the so-called Calogero-Sutherland models form a distinguised class. The first analysis of a system of this kind was performed by Calogero [3] who studied, from the quantum standpoint, the dynamics on the infinite line of a set of particles interacting pairwise by rational plus quadratic potentials, and found that the problem was exactly solvable. Soon afterwards, Sutherland [4] arrived to similar results for the quantum problem on the circle, this time with trigonometric interaction, and Moser [5] showed that the classical version of both models enjoyed integrability in the Liouville sense. The identification of the general scope of these discoveries comes with the work of Olshanetsky and Perelomov [6, 7], who realized that it was possible to associate models of this kind to all the root sytems of the simple Lie algebras, and that all these models were integrable, both in the classical and in the quantum framework [8, 9]. Nowadays, there is a widespread interest in this type of integrable systems, and many mathematical and physical applications for them have been found, see for instance [10].

The eigenfunctions of the Calogero-Sutherland Hamiltonian associated to the root system of a simple Lie algebra L are proportional to some polynomials which form a complete orthogonal system in the quantum Hilbert space. For the specials values $\kappa_\alpha = 1$, where $g_\alpha = \kappa_\alpha(\kappa_\alpha - 1)$ are the coupling constants, they coincide with the irreducible characters of L. For $L = A_n$, these polynomials provide

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natural generalizations to n variables of the classical orthogonal polynomials in one indeterminate. In particular, for the case with a trigonometric potential, one obtains a generalized system of Gegenbauer polynomials. As it was shown in the papers [11, 12, 13], these generalized Gegenbauer polynomials obey a set of recurrence relations which constitute a κ-deformation of the Clebsch-Gordan series of the algebra. The finding of these recurrence relations opened the way to obtain many concrete results on the system of polynomials, as for example explicit expressions, ladder operators or generating functions [14, 15]. The recurrence relations are also the key ingredient to formulate a perturbative approach to most general among the Calogero-Sutherland models, that involving the Weierstrass $\wp$-function as potential [16].

The aim of this paper is to extend some of the results which have been obtained for A_n to the polynomials related to other simple algebras. We think that it is a good idea to begin with a concrete case. We choose to work in the first place the problem associated to D_4 because of the triality symmetry exhibited by this algebra, which will help us in simplifying of the treatment. The organization of the paper is as follows. In Sect. 2, we explain how to express the Calogero-Sutherland Hamiltonian in terms of the fundamental characters of the algebra and how to solve the Schrödinger equation. Then, in Sect. 3, we obtain the main recurrence relations among the polynomials and use them to give algorithms to calculate some subsets of them. Sect. 4 is devoted to find the generating functions for some classes of characters and monomials functions of D_4. More recurrrence relations and some other relevant results are included in Sect. 5, and finally, in Sect. 6, we give some brief conclusions. Also, we offer two appendices. In Appendix A, for the convenience of the reader, we collect some of the basic facts about D_4 which we use in the main text. In Appendix B we list some polynomials, characters and monomial functions.

2 The eigenvalue problem

The Hamiltonian operator for the trigonometric Calogero-Sutherland model related to the root system of a simple Lie algebra of rank r has the form

$$H = \frac{1}{2}(p,p) + \sum_{\alpha\in R^+} \kappa_\alpha(\kappa_\alpha - 1)\sin^{-2}(\alpha, q), \tag{1}$$

where $q = (q_1, \ldots, q_r)$, $p = (p_1, \ldots, p_r)$, $(\ ,\ )$ is the usual euclidean inner product in $\mathbf{R}^r$, R^+ is the set of positive roots of the algebra, and κ_α are constants such that $\kappa_\alpha = \kappa_\beta$ if $\|\alpha\| = \|\beta\|$. In particular, for the case of the algebra D_4 (see Appendix A), this leads to the following Schrödinger equation:

$$\begin{aligned} H\Psi^\kappa \;&=\; E(\kappa)\,\Psi^\kappa, \\ H = -\frac{1}{2}\Delta \;&+\; \kappa(\kappa-1)\left(\sum_{j<k}^{4}\sin^{-2}(q_j - q_k) + \sum_{j<k}^{4}\sin^{-2}(q_j + q_k)\right), \quad \Delta = \sum_{j=1}^{4}\frac{\partial^2}{\partial q_j^2}\,. \end{aligned} \tag{2}$$

The q coordinates are assumed to take values in the $[0,\pi]$ interval, and therefore the equation can be interpreted as describing the dynamics of a system of four particles moving on the circle. Let us notice that there is not translational invariance. We recapitulate some important facts about this model which follow from the general structure of the quantum Calogero-Sutherland models related to Lie algebras [9]. The ground state energy and the (non-normalized) wavefunction are

$$\begin{aligned} E_0(\kappa) \;&=\; 2(\rho,\rho)\,\kappa^2 = 28\,\kappa^2, \\ \Psi_0^\kappa(q) \;&=\; \left\{\prod_{j<k}^{4}\sin(q_j - q_k)\sin(q_j + q_k)\right\}^\kappa, \end{aligned} \tag{3}$$

where ρ is the standard Weyl vector, $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, with the sum extended over all the positive roots of D_4. The excited states depend on a four-tuple of quantum numbers $\mathbf{m} = (m_1, m_2, m_3, m_4)$

$$\begin{aligned} H\,\Psi^\kappa_{\mathbf{m}} &= E_{\mathbf{m}}(\kappa)\,\Psi^\kappa_{\mathbf{m}}, \\ E_{\mathbf{m}}(\kappa) &= 2(\lambda+\kappa\rho, \lambda+\kappa\rho), \end{aligned} \tag{4}$$

where λ is the highest weight of the irreducible representation of D_4 labelled by $\mathbf{m}$, i.e., $\lambda = \sum_{i=1}^4 m_i\lambda_i$, and λ_i are the fundamental weights of D_4. By substitution in (4) of

$$\Psi^\kappa_{\mathbf{m}}(q) = \Psi^\kappa_0(q)\,\Phi^\kappa_{\mathbf{m}}(q), \tag{5}$$

we are led to the eigenvalue problem

$$-\Delta^\kappa\Phi^\kappa_{\mathbf{m}} = \varepsilon_{\mathbf{m}}(\kappa)\,\Phi^\kappa_{\mathbf{m}} \tag{6}$$

with

$$\Delta^\kappa = \frac{1}{2}\Delta + \kappa\sum_{j<k}^{4}\left(\mathrm{ctg}(q_j-q_k)\left(\frac{\partial}{\partial q_j}-\frac{\partial}{\partial q_k}\right) + \mathrm{ctg}(q_j+q_k)\left(\frac{\partial}{\partial q_j}+\frac{\partial}{\partial q_k}\right)\right) \tag{7}$$

and

$$\varepsilon_{\mathbf{m}}(\kappa) = E_{\mathbf{m}}(\kappa) - E_0(\kappa) = 2(\lambda, \lambda+2\kappa\rho). \tag{8}$$

Introducing the inverse Cartan matrix $A^{-1}_{jk} = (\lambda_j, \lambda_k)$, we can give a more explicit expression for $\varepsilon_{\mathbf{m}}(\kappa)$:

$$\begin{aligned} \varepsilon_{\mathbf{m}}(\kappa) &= 2\sum_{j,k=1}^{4} A^{-1}_{jk}\,m_j m_k + 4\kappa\sum_{j,k=1}^{4} A^{-1}_{jk}\,m_j = 2(m_1^2+m_3^2+m_4^2) + 4m_2^2 + 2(m_1m_3+m_1m_4+m_3m_4) \\ &+ 4m_2(m_1+m_3+m_4) + 12\kappa(m_1+m_3+m_4) + 20\kappa m_2. \end{aligned} \tag{9}$$

The main problem is to solve equation (6). As it has been shown for the case of the algebra A_n [11, 12, 13], the best way to do that is to use a set of independent variables which are invariant under the Weyl symmetry of the Hamiltonian, namely, the characters of the four fundamental representations of the algebra D_4. Unfortunately, the expression of these characters in terms of the q-variables (which play the role of coordinates on the maximal torus of D_4) is not very simple. Denoting the character of the irreducible representation of maximal weight λ_j as z_j, we find

$$\begin{aligned} z_1 &= \sum_{j=1}^{4} x_j + \sum_{j=1}^{4} x_j^{-1} \\ z_2 &= \sum_{i<j}^{4} x_i x_j + \sum_{i<j}^{4}(x_i x_j)^{-1} + \sum_{i,j}^{4} x_i^{-1}x_j \\ z_3 &= \bar{x}\sum_{i=1}^{4}\frac{1}{x_i} + \frac{1}{\bar{x}}\sum_{i=1}^{4} x_i, \\ z_4 &= \bar{x} + \frac{1}{\bar{x}} + \frac{1}{\bar{x}}\sum_{i<j}^{4} x_i x_j \end{aligned}$$

where $x_j = e^{2iq_j}$, and $\bar{x} = \sqrt{x_1x_2x_3x_4}$. These expressions make the direct change of variables from q_i to z_k quite cumbersome. We refrain from trying that approach, and choose an indirect route which has the added advantage of being also applicable to other algebras in which the expressions for the characters are even more involved. We can infer from (7) the structure of Δ^κ when written in the z-variables:

$$\Delta^\kappa = \sum_{j,k=1}^{4} a_{jk}(z_i)\, \partial_{z_j}\, \partial_{z_k} + \sum_{j=1}^{4} \left[b_j^{(0)}(z_i) + \kappa b_j^{(1)}(z_i) \right] \partial_{z_j}. \tag{10}$$

On the other hand, as it is well known [17], the $\Phi^\kappa_{\mathbf{m}}$ are polynomials which, with some precise partial ordering for the monomials to be described later, start as follows:

$$\Phi^\kappa_{\mathbf{m}}(z_i) = P^\kappa_{\mathbf{m}}(z_i) = z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4} + \cdots. \tag{11}$$

Therefore, making use of (9), we conclude that

$$\begin{aligned} a_{jk}(z_i) &= 2\, A^{-1}_{jk}\, z_j z_k + \text{lower order terms}, \\ b_j^{(r)}(z_i) &= c_j^{(r)} z_j + d_j^{(r)}, \qquad r = 0, 1. \end{aligned} \tag{12}$$

Now, to obtain the full expressions for these coefficients, we rely on the fact that, for $\kappa = 1$, the $P^\kappa_{\mathbf{m}}$ polynomial gives the character of the irreducible representation of D_4 with maximal weight $\sum_{i=1}^4 m_i \lambda_i$, while for $\kappa = 0$ the same polynomial is the corresponding symmetric monomial function [9]. Both, characters and monomial functions, can be computed by using the information available in the literature (see, for instance, the "Reference Chapter" of [18]). In fact, the following short list of polynomials

$$\begin{aligned} P^{(1)}_{2,0,0,0}(z) &= z_1^2 - z_2 - 1, \\ P^{(1)}_{1,1,0,0}(z) &= z_1 z_2 - z_3 z_4, \\ P^{(1)}_{1,0,1,0}(z) &= z_1 z_3 - z_4, \\ P^{(1)}_{0,2,0,0}(z) &= z_2^2 - z_1 z_3 z_4 + z_2, \\ P^{(0)}_{2,0,0,0}(z) &= z_1^2 - 2 z_2 \end{aligned}$$

is all we need to obtain Δ^κ. By substituting these polynomials in (6) and using (9), (10), (12) and the triality symmetry (that here implies that the final expression for Δ^κ should be invariant under permutations of the indices 1,3,4), we get enough simple linear algebraic equations to fix all the coefficients. We give here only the final result:

$$\begin{aligned} \frac{1}{2}\Delta^\kappa &= \left(z_1^2 - 2 z_2 - 8\right) \partial^2_{z_1} + \left[2\, z_2^2 - 4\left(z_1^2 + z_3^2 + z_4^2\right) - 2\, z_1\, z_3\, z_4 + 8\, z_2 \right] \partial^2_{z_2} + \left(z_3^2 - 2 z_2 - 8\right) \partial^2_{z_3} \\ &+ \left(z_4^2 - 2 z_2 - 8\right) \partial^2_{z_4} + (2 z_1 z_2 - 6 z_3 z_4 - 8 z_1)\, \partial_{z_1}\, \partial_{z_2} + (z_1\, z_3 - 8\, z_4)\, \partial_{z_1}\, \partial_{z_3} \\ &+ (z_1 z_4 - 8 z_3)\, \partial_{z_1}\, \partial_{z_4} + (2 z_2 z_3 - 6 z_1 z_4 - 8 z_3)\, \partial_{z_2}\, \partial_{z_3} + (2 z_2 z_4 - 6 z_1 z_3 - 8 z_4)\, \partial_{z_2}\, \partial_{z_4} \\ &+ (z_3 z_4 - 8 z_1)\, \partial_{z_3}\, \partial_{z_4} + (6\kappa + 1)\, z_1\, \partial_{z_1} + [2(5\kappa + 1)\, z_2 + 8(\kappa - 1)]\, \partial_{z_2} + (6\kappa + 1)\, z_3\, \partial_{z_3} \\ &+ (6\kappa + 1)\, z_4\, \partial_{z_4}. \end{aligned} \tag{13}$$

Once the explicit expression for the operator Δ^κ in the z variables is given, the Schrödinger equation can be solved iteratively. By direct application of Δ^κ to $z^{\mathbf{m}} \equiv z_1^{m_1}\, z_2^{m_2}\, z_2^{m_3}\, z_4^{m_4}$, we find

$$\begin{aligned} \Delta^\kappa z^{\mathbf{m}} &= \varepsilon_{\mathbf{m}}(\kappa)\, z^{\mathbf{m}} - \sum_{i=1}^{4} a^i_{\mathbf{m}}\, z^{\mathbf{m}-\alpha_i} - \sum_{j \in I} b^j_{\mathbf{m}}\, z^{\mathbf{m}-(\alpha_2+\alpha_j)} - \sum_{ij \in T} c^{ij}_{\mathbf{m}}\, z^{\mathbf{m}-(\alpha_2+\alpha_i+\alpha_j)} \\ &- 2\, a^2_{\mathbf{m}} \sum_{ij \in T} z^{\mathbf{m}-(2\alpha_2+\alpha_i+\alpha_j)} - d_{\mathbf{m}}\, z^{\mathbf{m}-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)} \\ &- 4 \sum_{j \in I} a^j_{\mathbf{m}}\, z^{\mathbf{m}-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4+\alpha_j)}, \end{aligned} \tag{14}$$

where the sets of indices are $I=\{1,3,4\}$ and $T=\{13,14,34\}$, and

$$\begin{aligned} a^i_{\mathbf{m}} &= 4\,m_i(m_i-1), \qquad\qquad b^j_{\mathbf{m}}=12m_2m_j,\\ c^{ij}_{\mathbf{m}} &= 16\,m_i\,m_j, \qquad\qquad d_{\mathbf{m}}(\kappa)=16\,m_2\left(2-m_2-\kappa+\sum_{j\in I}m_j\right). \end{aligned}$$

All monomials in $\Delta^\kappa z^{\mathbf{m}}$ take the form $z^{\mathbf{m}-\mu}$ with μ as a positive root. Thus, the polynomial $P^\kappa_{\mathbf{m}}$ has the form

$$P^\kappa_{\mathbf{m}}(z)=\sum_{\mu\in Q^+(\mathbf{m})} c_\mu z^{\mathbf{m}-\mu}, \tag{15}$$

where we choose the normalization $c_0=1$ and, if Q^+ is the cone of positive roots,

$$Q^+(\mathbf{m})=\{\mu\in Q^+ \mid z^{\mathbf{m}-\mu} \text{ is well defined if } z_1\,z_2\,z_3\,z_4=0\}\,. \tag{16}$$

The above-mentioned partial ordering of monomials is given simply by the height of μ, i. e. $z^{\mathbf{m}-\mu_1} > z^{\mathbf{m}-\mu_2}$ if $ht(\mu_1)<ht(\mu_2)$. From (14), the coefficients c_μ obey the iterative formula

$$c_\mu=\frac{N_\mu}{\varepsilon_{\mathbf{m}-\mu}(\kappa)-\varepsilon_{\mathbf{m}}(\kappa)} \tag{17}$$

with

$$\begin{aligned} N_\mu &= \sum_{i=1}^4 a^i_{\mathbf{m}-(\mu-\alpha_i)}\,c_{\mu-\alpha_i}+\sum_{j\in I} b^j_{\mathbf{m}-(\mu-\alpha_2-\alpha_j)}\,c_{\mu-(\alpha_2+\alpha_j)}+\sum_{ij\in T} c^{ij}_{\mathbf{m}-(\mu-\alpha_2-\alpha_i-\alpha_j)}\,c_{\mu-(\alpha_2+\alpha_i+\alpha_j)}\\ &+ 2\sum_{ij\in T} a^2_{\mathbf{m}-(\mu-2\alpha_2-\alpha_i-\alpha_j)}\,c_{\mu-(2\alpha_2+\alpha_i+\alpha_j)}+d_{\mathbf{m}-(\mu-\alpha_1-2\alpha_2-\alpha_3-\alpha_4)}\,c_{\mu-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}\\ &+ 4\sum_{j\in I} a^j_{\mathbf{m}-(\mu-\alpha_1-2\alpha_2-\alpha_3-\alpha_4-\alpha_j)}\,c_{\mu-(\alpha_1+2\alpha_2+\alpha_3+\alpha_4+\alpha_j)}. \end{aligned}$$

Along with the explicit expressions for the roots given in Appendix A, it is suitable for the implementation on a symbolic computer program. A list of polynomials obtained through the use of this formula is offered in Appendix B.

3 The structure of the recurrence relations

As it is well known, all the systems of orthogonal polynomials in one indeterminate z, such that $P_m(z)=z^m+\cdots$ satisfy a recursive formula $z\,P_m(z)=a_m\,P_{m+1}(z)+b_m\,P_m(z)+c_m\,P_{m-1}(z)$. In particular, the orthogonal polynomials associated to the trigonometric Calogero-Sutherland model for the case of two particles and Lie algebra A_1 are the classical Gegenbauer polynomials, whose recursive formula is known to be

$$z\,P^\kappa_m(z)=P^\kappa_{m+1}(z)+\frac{m(m-1+2\kappa)}{(m-1+\kappa)(m+\kappa)}\,P^\kappa_{m-1}(z).$$

This formula is reminiscent of the Clebsch-Gordan series for A_1. In fact, for $\kappa=1$ it reduces exactly to this Clebsch-Gordan series: the polynomials are the characters of A_1 and the coefficents are equal to one. Immediately the question arises about the existence of analogous recurrence relations, i.e., with the structure of κ-deformations of the corresponding Clebsch-Gordan series, for the polynomials related to Calogero-Sutherland models associated to other simple Lie algebras. As it was shown in [11], the answer turns out to be in the afirmative for all root systems, but to obtain the expressions for the deformed coefficients it is necessary to proceed through a case-by-case analysis. Once the coefficients are known, many applications are possible. The aim of this section is to fix the structure of the basic recurrence relations for the case of D_4 and to give a simple illustration of their use.

We want to study the formulas for $z_i P^{\kappa}_{\mathbf{m}}(z)$, $i = 1, 2, 3, 4$. Therefore, as $P^{(1)}_{\mathbf{m}}(z) = z_i$ for $m_j = (\delta_{ji})$, and the recursive formulas are deformations of the Clebsch-Gordan series, we need to know the weights of the irreducible representations whose integral dominant weights are λ_1, λ_2, λ_3 and λ_4. For the case of λ_1, λ_3 and λ_4, these representations have dimension eight. On the other hand, if we act on the highest weight with the Weyl group in the way explained in the Appendix A, we obtain eight different weights. Thus, these representations include only one orbit of the Weyl group and we are done. For the case of λ_2, the representation has dimension 28 and the orbit of the Weyl group containing λ_2 has only 24 elements. But $\lambda_2 = \alpha^+_{12}$, the highest root, and thus this representation is the adjoint one and includes a second orbit: the Cartan subalgebra, with four elements of weight zero. Let us summarize.

- Weights in z_1: $\pm\lambda_1$, $\pm(\lambda_1 - \lambda_2)$, $\pm(\lambda_2 - \lambda_3 - \lambda_4)$, $\pm(\lambda_3 - \lambda_4)$.
- Weights in z_2: $\pm\lambda_2$, $\pm(\lambda_2 - 2\lambda_j)$, $\pm(2\lambda_2 - \lambda_1 - \lambda_3 - \lambda_4)$, $\pm(\lambda_2 + \lambda_i - \lambda_j - \lambda_k)$, $\pm(\lambda_i + \lambda_j - \lambda_k)$, $\pm(\lambda_2 - \lambda_1 - \lambda_3 - \lambda_4)$, 0, with $i, j, k \in I$.
- Weights in z_3: $\pm\lambda_3$, $\pm(\lambda_3 - \lambda_2)$, $\pm(\lambda_2 - \lambda_1 - \lambda_4)$, $\pm(\lambda_1 - \lambda_4)$.
- Weights in z_4: $\pm\lambda_4$, $\pm(\lambda_4 - \lambda_2)$, $\pm(\lambda_2 - \lambda_1 - \lambda_3)$, $\pm(\lambda_1 - \lambda_3)$.

With these weights, the structure of the recurrence relations results to be as follows:

$$\begin{aligned} z_1\, P^{\kappa}_{m_1,m_2,m_3,m_4}(z) &= P^{\kappa}_{m_1+1,m_2,m_3,m_4}(z) + a^1_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1-1,m_2,m_3,m_4}(z) + b^1_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1+1,m_2-1,m_3,m_4}(z) \\ &+ c^1_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1-1,m_2+1,m_3,m_4}(z) + d^1_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2+1,m_3-1,m_4-1}(z) \\ &+ e^1_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2-1,m_3+1,m_4+1}(z) + f^1_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2,m_3+1,m_4-1}(z) \\ &+ g^1_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2,m_3-1,m_4+1}(z) \end{aligned}$$

$$\begin{aligned} z_2\, P^{\kappa}_{m_1,m_2,m_3,m_4}(z) &= P^{\kappa}_{m_1,m_2+1,m_3,m_4}(z) + A_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2-1,m_3,m_4}(z) + B_{\mathbf{m}}(\kappa)^{1\pm}\, P^{\kappa}_{m_1\pm 2,m_2\mp 1,m_3,m_4}(z) \\ &+ B_{\mathbf{m}}(\kappa)^{3\pm}\, P^{\kappa}_{m_1,m_2\mp 1,m_3\pm 2,m_4}(z) + B_{\mathbf{m}}(\kappa)^{4\pm}\, P^{\kappa}_{m_1,m_2\mp 1,m_3,m_4\pm 2}(z) \\ &+ C_{\mathbf{m}}(\kappa)^{\pm}\, P^{\kappa}_{m_1\mp 1,m_2\pm 2,m_3\mp 1,m_4\mp 1}(z) + D_{\mathbf{m}}(\kappa)^{1\pm}\, P^{\kappa}_{m_1\pm 1,m_2\pm 1,m_3\mp 1,m_4\mp 1}(z) \\ &+ D_{\mathbf{m}}(\kappa)^{3\pm}\, P^{\kappa}_{m_1\mp 1,m_2\pm 1,m_3\pm 1,m_4\mp 1}(z) + D_{\mathbf{m}}(\kappa)^{4\pm}\, P^{\kappa}_{m_1\mp 1,m_2\pm 1,m_3\mp 1,m_4\pm 1}(z) \\ &+ E_{\mathbf{m}}(\kappa)^{1\pm}\, P^{\kappa}_{m_1\mp 1,m_2,m_3\pm 1,m_4\pm 1}(z) + E_{\mathbf{m}}(\kappa)^{3\pm}\, P^{\kappa}_{m_1\pm 1,m_2,m_3\mp 1,m_4\pm 1}(z) \\ &+ E_{\mathbf{m}}(\kappa)^{4\pm}\, P^{\kappa}_{m_1\pm 1,m_2,m_3\pm 1,m_4\mp 1}(z) + F_{\mathbf{m}}(\kappa)^{\pm}\, P^{\kappa}_{m_1\pm 1,m_2\mp 1,m_3\pm 1,m_4\pm 1}(z) \\ &+ G_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2,m_3,m_4}(z), \end{aligned}$$

$$\begin{aligned} z_3\, P^{\kappa}_{m_1,m_2,m_3,m_4}(z) &= P^{\kappa}_{m_1,m_2,m_3+1,m_4}(z) + a^3_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2,m_3-1,m_4}(z) + b^3_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2-1,m_3+1,m_4}(z) \\ &+ c^3_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2+1,m_3-1,m_4}(z) + d^3_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1-1,m_2+1,m_3,m_4-1}(z) \\ &+ e^3_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1+1,m_2-1,m_3,m_4+1}(z) + f^3_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1+1,m_2,m_3,m_4-1}(z) \\ &+ g^3_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1-1,m_2,m_3,m_4+1}(z), \end{aligned}$$

$$\begin{aligned} z_4\, P^{\kappa}_{m_1,m_2,m_3,m_4}(z) &= P^{\kappa}_{m_1,m_2,m_3,m_4+1}(z) + a^4_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2,m_3,m_4-1}(z) + b^4_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2-1,m_3,m_4+1}(z) \\ &+ c^4_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1,m_2+1,m_3,m_4-1}(z) + d^4_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1-1,m_2+1,m_3-1,m_4}(z) \\ &+ e^4_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1+1,m_2-1,m_3+1,m_4}(z) + f^4_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1-1,m_2,m_3+1,m_4}(z) \\ &+ g^4_{\mathbf{m}}(\kappa)\, P^{\kappa}_{m_1+1,m_2,m_3-1,m_4}(z), \end{aligned}$$

where $B_{\mathbf{m}}(\kappa)^{1\pm}\, P^{\kappa}_{m_1\pm 2, m_2\mp 1, m_3, m_4}(z)$ means $B_{\mathbf{m}}(\kappa)^{1+}\, P^{\kappa}_{m_1+2, m_2-1, m_3, m_4}(z) + B_{\mathbf{m}}(\kappa)^{1-}\, P^{\kappa}_{m_1-2, m_2+1, m_3, m_4}(z)$, etc, and it is understood that all polynomials involving negative quantum numbers are zero. The recurrence relations reflect triality in the fact that not all the coefficients appearing in these formulas are independent. There are coincidences upon permutations of the quantum numbers, for instance

$$a^1_{m_1,m_2,m_3,m_4} = a^3_{m_3,m_2,m_1,m_4} = a^4_{m_4,m_2,m_3,m_1}, \tag{18}$$

and similarly for $b^j_{\mathbf{m}}, c^j_{\mathbf{m}}, d^j_{\mathbf{m}}, e^j_{\mathbf{m}}, f^j_{\mathbf{m}}, g^j_{\mathbf{m}}$. In the same fashion, we have also

$$B^{1\pm}_{m_1,m_2,m_3,m_4} = B^{3\pm}_{m_3,m_2,m_1,m_4} = B^{4\pm}_{m_4,m_2,m_3,m_1} \tag{19}$$

and similarly for $D^{j\pm}_{\mathbf{m}}, E^{j\pm}_{\mathbf{m}}$.

As an example, let us consider a simple case in which only one of the quantum numbers is nonvanishing, namely,

$$z_1\, P^{\kappa}_{m,0,0,0}(z) = P^{\kappa}_{m+1,0,0,0}(z) + a_m(\kappa)\, P^{\kappa}_{m-1,0,0,0}(z) + c_m(\kappa)\, P^{\kappa}_{m-1,1,0,0}(z), \tag{20}$$

where we write $a_m(\kappa) = a^1_{m,0,0,0}(\kappa)$ and $c_m(\kappa) = c^1_{m,0,0,0}(\kappa)$. Using formulae

$$\begin{aligned}
P^{\kappa}_{m,0,0,0}(z) &= z_1^m - \frac{m(m-1)\left[4\kappa^2 + 4(m-2)\kappa + (m-1)(m-2)\right]}{(m-1+\kappa)(m-1+3\kappa)(m-2+\kappa)}\, z_1^{m-2} - \frac{m(m-1)}{m-1+\kappa} z_1^{m-2}\, z_2 + \cdots \\
P^{\kappa}_{m,1,0,0}(z) &= z_1^m\, z_2 + \frac{4\,\kappa(\kappa-1)(m-2+2\kappa)}{(m+1+5\kappa)(m+2\kappa)(m-1+\kappa)}\, z_1^m + \cdots,
\end{aligned}$$

we obtain the coeffficients in (20)

$$\begin{aligned}
a_m(\kappa) &= \frac{m(m+2\kappa)(m-1+4\kappa)(m-1+6\kappa)}{(m-1+\kappa)(m-1+3\kappa)(m+3\kappa)(m+5\kappa)}, \\
c_m(\kappa) &= \frac{m(m-1+2\kappa)}{(m+\kappa)(m-1+\kappa)}.
\end{aligned}$$

As a byproduct of triality, we can also write other two recurrence relations with the same coefficients:

$$\begin{aligned}
z_3 P^{\kappa}_{0,0,m,0}(z) &= P^{\kappa}_{0,0,m+1,0}(z) + a_m(\kappa)\, P^{\kappa}_{0,0,m-1,0}(z) + c_m(\kappa)\, P^{\kappa}_{0,1,m-1,0}(z) \\
z_4\, P^{\kappa}_{0,0,0,m}(z) &= P^{\kappa}_{0,0,0,m+1}(z) + a_m(\kappa)\, P^{\kappa}_{0,0,0,m-1}(z) + c_m(\kappa)\, P^{\kappa}_{0,1,0,m-1}(z).
\end{aligned} \tag{21}$$

The first of these recurrence relations can be used to devise an algorithm for the calculation of the polynomials of the form $P^{\kappa}_{m,0,0,0}(z)$ and $P^{\kappa}_{m,1,0,0}(z)$. By multiplying (20) by the differential operator $\Delta^{\kappa} - \varepsilon_{m-1,1,0,0}(\kappa)$, the term involving $P^{\kappa}_{m-1,1,0,0}$ cancels. Using the explicit expressions (9), (13), we find

$$\begin{aligned}
P^{\kappa}_{m+1,0,0,0} &= \frac{1}{4(m+\kappa)}\left[\Delta^{\kappa}, z_1\right] P^{\kappa}_{m,0,0,0}(z) - \frac{1+4\kappa}{2(m+\kappa)}\, z_1 P^{\kappa}_{m,0,0,0}(z) \\
&+ \frac{m(m+2\kappa)(m-1+4\kappa)(m-1+6\kappa)}{(m-1+\kappa)(m-1+3\kappa)(m+\kappa)(m+3\kappa)}\, P^{\kappa}_{m-1,0,0,0}(z),
\end{aligned}$$

where, from (13),

$$\begin{aligned}
\left[\Delta^{\kappa}, z_1\right] &= 4\left(z_1^2 - 2z_2 - 8\right)\partial_{z_1} + 2\left(z_1 z_3 - 8z_4\right)\partial_{z_3} + 2\left(z_1 z_4 - 8z_3\right)\partial_{z_4} \\
&+ 4\left(z_1 z_2 - 3z_3 z_4 - 4z_1\right)\partial_{z_2} + 2(6\kappa+1)\, z_1.
\end{aligned}$$

where, from (13),

$$\begin{aligned}
\left[\Delta^{\kappa}, z_1\right] &= 4\left(z_1^2 - 2z_2 - 8\right)\partial_{z_1} + 2\left(z_1 z_3 - 8z_4\right)\partial_{z_3} + 2\left(z_1 z_4 - 8z_3\right)\partial_{z_4} \\
&+ 4\left(z_2 z_2 - 3z_3 z_4 - 4z_1\right)\partial_{z_2} + 2(6\kappa+1)\, z_1.
\end{aligned}$$

Once the polynomials $P^{\kappa}_{m,0,0,0}(z)$ are known, the recurrence relation (20) provides a formula for each $P^{\kappa}_{m,1,0,0}(z)$:

$$c_{m+1}(\kappa)\, P^{\kappa}_{m,1,0,0}(z) = z_1\, P^{\kappa}_{m+1,0,0,0}(z) - P^{\kappa}_{m+2,0,0,0}(z) - a_{m+1}(\kappa)\, P^{\kappa}_{m,0,0,0}(z). \tag{22}$$

4 Some generating functions

We present in this section the generating functions for some characters and symmetric monomial functions. Let us consider first the case of the monomial functions with only one non-vanishing quantum number in the form $P^{(0)}_{m,0,0,0}(z)$. The generating function for this subset is

$$F_0(t,z) = \sum_{m=0}^{\infty} t^m\, P^{(0)}_{m,0,0,0}(z). \tag{23}$$

In terms of the x variables, the general expression for these monomial functions is

$$P^{(0)}_{m,0,0,0}(x) = \sum_{j=1}^{4} \left(x_j^m + x_j^{-m} \right), \tag{24}$$

and, in particular, we define $P^{(0)}_{0,0,0,0}(z) = 8$. In these variables, the computation of $F_0(t,x)$ only requires to sum the geometric series:

$$F_0(t,x) = \sum_{j=1}^{4} \left(\frac{1}{1 - t x_j} + \frac{1}{1 - \frac{t}{x_j}} \right). \tag{25}$$

The change to the original z variables can be done by the inspection of the coefficients of the powers of t in both the numerator and denominator of this rational expression, with the result

$$F_0(t,z) = \frac{N_0(t,z)}{D(t,z)}, \tag{26}$$

where

$$\begin{aligned} N_0(t,z) &= 8 - 7 z_1\, t + 6 z_2\, t^2 - 5(z_3 z_4 - z_1)\, t^3 + 4(z_3^2 + z_4^2 - 2 z_2 - 2)\, t^4 - 3(z_3 z_4 - z_1)\, t^5 \\ &+ 2 z_2\, t^6 - z_1\, t^7, \\ D(t,z) &= 1 - z_1\, t + z_2\, t^2 - (z_3 z_4 - z_1)\, t^3 + (z_3^2 + z_4^2 - 2 z_2 - 2)\, t^4 - (z_3 z_4 - z_1)\, t^5 \\ &+ z_2\, t^6 - z_1\, t^7 + t^8. \end{aligned} \tag{27}$$

There is an alternative approach. As the monomial functions are eigenfunctions of $\Delta^{(0)}$ with eigenvalues $\varepsilon_{m,0,0,0}(0) = 2m^2$, we have

$$\frac{1}{2}\, \Delta^{(0)}\, F_0(t,z) = \sum_{m=0}^{\infty} m^2 t^m\, P^{(0)}_{m,0,0,0}(z),$$

and, therefore, we can write a differential equation for $F_0(t,z)$:

$$\left[\frac{1}{2} \Delta^{(0)} - (t\, \partial_t)^2 \right] F_0(t,z) = 0, \qquad F_0(0,z) = 8. \tag{28}$$

One can verify by substitution that (26) satisfies this equation. When $F_0(t,z)$ is known, we can easily obtain the generating function

$$G_0(t,z) = \sum_{m=0}^{\infty} t^m\, P^{(0)}_{m,1,0,0}(z) \tag{29}$$

by only recalling (20), which for $\kappa = 0$ is simply

$$z_1 P^{(0)}_{m,0,0,0}(z) = P^{(0)}_{m+1,0,0,0}(z) + P^{(0)}_{m-1,0,0,0}(z) + P^{(0)}_{m-1,1,0,0}(z). \tag{30}$$

This gives

$$G_0(t,z) = \frac{M_0(t,z)}{D(t,z)} \tag{31}$$

with

$$\begin{aligned}
M_0(t,z) &= z_2 - 4 + (6\,z_1 - 3\,z_3 z_4)\,t \\
&+ (-8 - 2\,z_1^2 - 10\,z_2 - z_2^2 + 4\,z_3^2 + 2\,z_1 z_3 z_4 + 4\,z_4^2)\,t^2 \\
&+ (10\,z_1 + 5\,z_1 z_2 - 3\,z_1 z_3^2 - 4\,z_3 z_4 + z_2 z_3 z_4 - 3\,z_1 z_4^2)\,t^3 \\
&+ (8\,z_2 - 4\,z_1^2 + 2\,z_2^2 - z_2 z_3^2 + 4\,z_1 z_3 z_4 - z_2 z_4^2)\,t^4 \\
&+ (-6\,z_1 - 6\,z_1 z_2 - z_3 z_4 + z_2 z_3 z_4)\,t^5 + (8 + 6 z_1^2 + 2\,z_2 - z_2^2)\,t^6 \\
&+ (-10\,z_1 + z_1 z_2)\,t^7 + (4 - z_2)\,t^8 .
\end{aligned}$$

The computation of the generating functions for the characters $P^{(1)}_{m,0,0,0}$ and $P^{(1)}_{m,1,0,0}$ goes through similar arguments. In this case, the eigenvalues are $\varepsilon_{m,0,0,0}(1) = 2m^2 + 12m$. Hence,

$$F_1(t,z) = \sum_{m=0}^{\infty} t^m P^{(1)}_{m,0,0,0}(z)\,, \qquad P^{(1)}_{0,0,0,0}(z) \equiv 1 \tag{32}$$

is the solution of the equation

$$\left[\frac{1}{2}\Delta^{(1)} - (t\,\partial_t)^2 - 6t\,\partial_t\right] F_1(t,z) = 0, \qquad F_1(0,z) = 1. \tag{33}$$

The Weyl character formula implies that the denominator of $F_1(t,z)$ should be the same $D(t,z)$ found before. Thus, we try an Ansatz

$$F_1(t,z) = \frac{N_1(t,z)}{D(t,z)} \tag{34}$$

and obtain the simple answer

$$N_1(t,z) = 1 - t^2. \tag{35}$$

Applying the recurrence relation (20)) we obtain the generating function $G_1(t,z)$ for the characters $P^{(1)}_{m,1,0,0}$:

$$G_1(t,z) = \frac{1}{D(t,z)}\left\{z_2 - z_3 z_4 t + (z_3^2 + z_4^2 - 2z_2 - 1)\,t^2 - (z_3 z_4 - z_1)\,t^3 + z_2\,t^4 - z_1\,t^5 + t^6\right\}. \tag{36}$$

5 More recurrence relations and other results

In this Section, we give the remaining recurrence relations involving the product of a fundamental character times a polynomial with only one non-vanishing quantum number. We also comment the existence of some peculiar values for κ for which the polynomials associated to some special excited states are proportional to integer powers of the fundamental state wavefunction.

To obtain the mentioned recurrence relations, it is necessary to compute the coefficients of a limited number of terms of the polynomials involved. Once the form of these terms is known, we can obtain the coefficients in the recurrence relations solving a system of linear algebraic equations. We do not give here the full expressions for the coefficients of the required terms, because some of them are too long, and only list them:

$$\begin{array}{rcl}
P^\kappa_{1,0,m,0}(z) &=& z_1 z_3^m + A\, z_3^{m-1} z_4 + \cdots, \\
P^\kappa_{0,m,0,0}(z) &=& z_2^m + B\, z_2^{m-1} + C\, z_2^{m-2} + D\, z_1 z_2^{m-2} z_3 z_4 + E\, z_1 z_2^{m-3} z_3 z_4 \\
&+& F\,(z_1^2 z_2^{m-2} + z_2^{m-2} z_3^2 + z_2^{m-2} z_4^2) + \cdots, \\
P^\kappa_{1,m,0,0}(z) &=& z_1 z_2^m + G\, z_1 z_2^{m-1} z_4 + H\, z_2^{m-1} z_3 z_4 + \cdots, \\
P^\kappa_{0,m,1,1}(z) &=& z_2^m z_3 z_4 + I\, z_1 z_2^m + \cdots, \\
P^\kappa_{m,0,0,0}(z) &=& z_1^m + J\, z_1^{m-2} + K\, z_1^{m-2} z_2 + \cdots, \\
P^\kappa_{m,1,0,0}(z) &=& z_1^m z_2 + L\, z_1^{m-2} z_2 + N\, z_1^{m-1} z_3 z_4 + M\, z_1^m + \cdots, \\
P^\kappa_{m,0,1,1}(z) &=& z_1^m z_3 z_4 + N\, z_1^{m-1} z_2 + O\, z_1^{m+1} + \cdots, \\
P^\kappa_{1,m,1,1}(z) &=& z_1 z_2^m z_3 z_4 + P\, z_2^m + Q\, z_1 z_2^{m-1} z_3 z_4 + R\,(z_1^2 z_2^m + z_2^m z_3^2 + z_2^m z_4^2) + S\, z_2^{m+1} + \cdots, \\
P^\kappa_{2,m,0,0}(z) &=& z_1^2 z_2^m + T\, z_2^m + U\, z_1 z_2^{m-1} z_3 z_4 + W\, z_2^{m+1} + \cdots.
\end{array}$$

The use of the quantities denoted A to W in the previous formulas in the general structure of the recurrence relations give the following results:

- Formulae of type $z_1 P^\kappa_{0,0,m,0}(z)$:

$$\begin{array}{rcl}
z_1\, P^\kappa_{0,0,m,0}(z) &=& P^\kappa_{1,0,m,0}(z) + b_m(\kappa) P_{0,0,m-1,1}(z) \\
z_1\, P^\kappa_{0,0,0,m}(z) &=& P^\kappa_{1,0,0,m}(z) + b_m(\kappa) P_{0,0,1,m-1}(z) \\
z_3\, P^\kappa_{m,0,0,0}(z) &=& P^\kappa_{m,0,1,0}(z) + b_m(\kappa) P_{m-1,0,0,1}(z) \\
z_3\, P^\kappa_{0,0,0,m}(z) &=& P^\kappa_{0,0,1,m}(z) + b_m(\kappa) P_{1,0,0,m-1}(z) \\
z_4\, P^\kappa_{m,0,0,0}(z) &=& P^\kappa_{m,0,0,1}(z) + b_m(\kappa) P_{m-1,0,1,0}(z) \\
z_4\, P^\kappa_{0,0,m,0}(z) &=& P^\kappa_{0,0,m,1}(z) + b_m(\kappa) P_{1,0,m-1,0}(z)
\end{array}$$

with

$$b_m(\kappa) = \frac{m(m-1+4\kappa)}{(m-1+\kappa)(m+3\kappa)}\ .$$

- Formulae of type $z_1\, P^\kappa_{0,m,0,0}(z)$:

$$\begin{array}{rcl}
z_1\, P^\kappa_{0,m,0,0}(z) &=& P^\kappa_{1,m,0,0}(z) + d_m(\kappa) P_{1,m-1,0,0}(z) + e_m(\kappa) P^\kappa_{0,m-1,1,1}(z) \\
z_3\, P^\kappa_{0,m,0,0}(z) &=& P^\kappa_{0,m,1,0}(z) + d_m(\kappa) P_{0,m-1,1,0}(z) + e_m(\kappa) P^\kappa_{1,m-1,0,1}(z) \\
z_4\, P^\kappa_{0,m,0,0}(z) &=& P^\kappa_{0,m,0,1}(z) + d_m(\kappa) P_{0,m-1,0,1}(z) + e_m(\kappa) P^\kappa_{1,m-1,1,0}(z)
\end{array}$$

with

$$\begin{array}{rcl}
d_m(\kappa) &=& \dfrac{2m(m+\kappa)(m-1+3\kappa)(m-1+4\kappa)(2m-1+6\kappa)}{(m-1+\kappa)(m-1+2\kappa)(m+3\kappa)(2m-1+5\kappa)(2m+5\kappa)}, \\
e_m(\kappa) &=& \dfrac{m(m-1+3\kappa)}{(m-1+\kappa)(m+2\kappa)}\ .
\end{array}$$

- Formulae of type $z_2\, P^{\kappa}_{m,0,0,0}(z)$:

$$\begin{array}{rcl} z_2\, P^{\kappa}_{m,0,0,0}(z) &=& P^{\kappa}_{m,1,0,0}(z) + f_m(\kappa)\, P_{m-2,1,0,0}(z) + g_m(\kappa)\, P^{\kappa}_{m-1,0,1,1}(z) + h_m(\kappa)\, P^{\kappa}_{m,0,0,0}(z) \\ z_2\, P^{\kappa}_{0,0,m,0}(z) &=& P^{\kappa}_{0,1,m,0}(z) + f_m(\kappa)\, P_{0,1,m-2,0}(z) + g_m(\kappa) P^{\kappa}_{1,0,m-1,1}(z) + h_m(\kappa)\, P^{\kappa}_{0,0,m,0}(z) \\ z_2\, P^{\kappa}_{0,0,0,m}(z) &=& P^{\kappa}_{0,1,0,m}(z) + f_m(\kappa)\, P_{0,1,0,m-2}(z) + g_m(\kappa)\, P^{\kappa}_{1,0,1,m-1}(z) + h_m(\kappa)\, P^{\kappa}_{0,0,0,m}(z) \end{array}$$

with

$$\begin{array}{rcl} f_m(\kappa) &=& \dfrac{m(m-1)(m-2+2\kappa)(m+2\kappa)(m-1+4\kappa)(m-1+5\kappa)}{(m-2+\kappa)(m-1+\kappa)^2(m-1+3\kappa)(m+3\kappa)(m+4\kappa)}, \\ g_m(\kappa) &=& \dfrac{m(m-1+3\kappa)}{(m-1+\kappa)(m+2\kappa)}, \\ h_m(\kappa) &=& \dfrac{4\left[-3\kappa^3+5\kappa^2+(6m-1)\kappa+(m^2-1)\right]}{(m-1+\kappa)(1+3\kappa)(m+1+5\kappa)} \;. \end{array}$$

- Formula for $z_2\, P^{\kappa}_{0,m,0,0}(z)$:

$$\begin{array}{rcl} z_2\, P^{\kappa}_{0,m,0,0}(z) &=& P^{\kappa}_{0,m+1,0,0}(z) + k_m(\kappa)\, P_{0,m-1,0,0}(z) + p_m(\kappa)\, P^{\kappa}_{1,m-1,1,1}(z) + q_m(\kappa)\, P^{\kappa}_{1,m-2,1,1}(z) \\ &+& r_m(\kappa)\, \left[P^{\kappa}_{2,m-1,0,0}(z) + P^{\kappa}_{0,m-1,2,0}(z) + P^{\kappa}_{0,m-1,0,2}(z)\right] + s_m(\kappa)\, P^{\kappa}_{0,m,0,0}(z) \end{array}$$

with

$$\begin{array}{rcl} k_m(\kappa) &=& \dfrac{4m(m+\kappa)^2(m+2\kappa)(m-1+3\kappa)(m-1+4\kappa)^2(2m-1+4\kappa)(m-1+5\kappa)(2m-1+6\kappa)}{(m-1+\kappa)(m-1+2\kappa)^2(m+3\kappa)^2(m+4\kappa)(2m-2+5\kappa)(2m-1+5\kappa)^2(2m+5\kappa)}, \\ p_m(\kappa) &=& \dfrac{m(m-1+2\kappa)}{(m-1+\kappa)(m+\kappa)}, \\ q_m(\kappa) &=& \dfrac{2m(m-1)(m+\kappa)^2(m-2+2\kappa)(m-1+3\kappa)^3(2m-1+6\kappa)}{(m-2+\kappa)(m-1+\kappa)^2(m-1+2\kappa)^2(m+2\kappa)^2(2m-1+5\kappa)(2m+5\kappa)}, \\ r_m(\kappa) &=& \dfrac{m(m+\kappa)(m-1+3\kappa)(m-1+4\kappa)}{(m-1+\kappa)(m-1+2\kappa)(m+2\kappa)(m+3\kappa)}, \\ s_m(\kappa) &=& \dfrac{-4t_m(\kappa)}{(\kappa+1)(m-1+\kappa)(m+1+4\kappa)(2m-1+5\kappa)(2m+1+5\kappa)}, \\ t_m(\kappa) &=& (-1+5m^2-4m^4)+(2+25m-7m^2-40m^3+2m^4)\kappa+(20-35m-123m^2+20m^3)\kappa^2, \\ &+& (-22-115m+63m^2)\kappa^3+(-19+65m)\kappa^4+20\kappa^5. \end{array}$$

Finally, we mention that for $\kappa = -\frac{1}{2}(n-1)$, $n \in \mathbf{N}$, the polynomials associated to the dominant weight which is n times the Weyl vector ρ are proportional to a power of the ground state wavefunction, namely

$$P^{-\frac{1}{2}(n-1)}_{n\rho} = (-1)^n\, 2^{12n} \left\{ \prod_{j<k}^{4} \sin(q_j-q_k)\sin(q_j+q_k) \right\}^n .$$

This formula can be verified quite easily by direct application of $\Delta^{-\frac{1}{2}(n-1)}$ in the form (7) to the right-hand side: one finds that the Schrödinger equation (6) with the appropriate eigenvalue is satisfied. The most convenient way to fix the proportionality constant is by performing an analytic continuation to complex q_i and considering the region $x_i \in \mathbf{R}$ and $x_1 \gg x_2 \gg x_3 \gg x_4 \gg 0$. Then, the polynomials are dominated by the leading order term, $P^{-\frac{1}{2}(n-1)}_{n\rho} \simeq z_1^n z_2^n z_3^n z_4^n$, and, on the other hand, using the formulas for the fundamental characters displayed in Section 2. one finds $z_1z_2z_3z_4 \simeq x_1^3x_2^2x_3$ and $\prod_{j<k}^4 \sin(q_j-q_k)\sin(q_j+q_k) \simeq -2^{-12}x_1^3x_2^2x_3$. This gives the proportionality constant written above.

6 Conclusions

In this paper, we have shown how to solve the Schrödinger equation for the trigonometric Calogero-Sutherland model related to the Lie algebra D_4 and we have explored some properties of the energy eigenfunctions. The main point is that the use of a Weyl-invariant set of variables, the characters of the fundamental representations, leads to a formulation of the Schrödinger equation by means of a second order differential operator which is simple enough to make feasible a recursive method for the treatment of the spectral problem. The eigenfunctions provide a complete system of orthogonal polynomials in four variables, and these polynomials obey recurrence relations which are extensions of the Clebsch-Gordan series of the algebra. The structure of some of these recurrence relations has been fixed and, for particular cases, the coefficients involved have been computed. Also, some generating functions for the polynomials with parameter $\kappa = 1$ and $\kappa = 0$ have been obtained. These generating functions can give some hints about the form of the generating function for general κ, see [20].

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Appendix A: Summary of results on the Lie algebra D_4

In this appendix, we review some standard facts about the root and weight systems of the Lie algebra D_4 that the reader could find useful to follow the main text. More extensive and sound treatments of these topics can be found in many excellent textbooks, see for instance [18], [19].

The most convenient explicit representation of D_4 is

$$D_4 = \left\{ \left(\begin{array}{cc} m & b \\ c & -m^t \end{array} \right) \mid m,\, b,\, c \text{ real } 4\times 4 \text{ matrices and } b^t = -b,\ c^t = -c \right\}.$$

This gives $\dim D_4 = 28$. One can choose the following linear basis:

$$\begin{array}{rcll} M_{jk} & = & E_{j,k} - E_{4+j,4+k}, & j,k = 1,2,3,4 \\ B_{jk} & = & E_{j,4+k} - E_{k,4+j}, & j,k = 1,2,3,4, \quad j<k \\ C_{jk} & = & E_{4+j,k} - E_{4+k,j}, & j,k = 1,2,3,4, \quad j<k \end{array}$$

with $(E_{i,j})_{kl} = \delta_{ik}\delta_{jl}$. The Cartan subalgebra is

$$H = \left\{ h = \sum_{i=1}^{4} c_i M_{ii} \mid c_i \in \mathbf{R} \right\}$$

and this confirms that the rank of D_4 is four. The matrix commutators

$$\begin{array}{rcl} [h, M_{jk}] & = & (c_j - c_k) M_{jk}, \\ [h, B_{jk}] & = & (c_j + c_k) B_{jk}, \\ [h, C_{jk}] & = & -(c_j + c_k) C_{jk} \end{array}$$

allow us to classify the 24 roots in two groups

$$\begin{array}{rcll} \alpha_{jk}(h) & = & c_j - c_k, & j \neq k, \\ \alpha^{\pm}_{jk}(h) & = & \pm(c_j + c_k), & j<k. \end{array}$$

One can extract the following basis of simple roots

$$\alpha_1 \equiv \begin{pmatrix} 1, & -1, & 0, & 0 \end{pmatrix} = \alpha_{12}, \qquad \alpha_2 \equiv \begin{pmatrix} 0, & 1, & -1, & 0 \end{pmatrix} = \alpha_{23},$$
$$\alpha_3 \equiv \begin{pmatrix} 0, & 0, & 1, & -1 \end{pmatrix} = \alpha_{34}, \qquad \alpha_4 \equiv \begin{pmatrix} 0, & 0, & 1, & 1 \end{pmatrix} = \alpha^{+}_{34},$$

where we have given the decomposition of these roots in the basis of H^* dual to diag(M_{ii}), i=1,2,3,4. The euclidean relations among the simple roots are

$$\begin{aligned} (\alpha_i, \alpha_i) &= 2, & i &= 1,2,3,4, \\ (\alpha_2, \alpha_i) &= -1, & i &= 1,3,4, \\ (\alpha_i, \alpha_j) &= 0, & i &= 1,3,4. \end{aligned}$$

Thus, the Cartan matrix reads

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

The positive roots are $\alpha_{ij}, \alpha^{+}_{ij}, i < j$, and they can be classified by heights as indicated in the table. The

Height	Positive roots
1	$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$
2	$\alpha_{13} = \alpha_1 + \alpha_2,\ \alpha_{24} = \alpha_2 + \alpha_3,\ \alpha^{+}_{24} = \alpha_2 + \alpha_4$
3	$\alpha_{14} = \alpha_1 + \alpha_2 + \alpha_3,\ \alpha^{+}_{14} = \alpha_1 + \alpha_2 + \alpha_4,\ \alpha^{+}_{23} = \alpha_2 + \alpha_3 + \alpha_4$
4	$\alpha^{+}_{13} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$
5	$\alpha^{+}_{12} = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$

Table 1: Heights of positive roots.

Weyl group is easy to describe. The Weyl reflection on the hyperplane in H^* orthogonal to the root α is $s_\alpha(v) = v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha$. Applying this formula to $\alpha_{ij}, \alpha^{\pm}_{ij}$, one readily finds that the most general Weyl reflection consists in a permutation of the components of v in the e_i basis plus an even number of changes of the signs of these components. This gives $|W| = 192$ for the order of the Weyl group. The fundamental weights λ_k can be obtained from the equation $\alpha_i = \sum_{j=1}^{4} A_{ji}\lambda_j$. They are

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) = \frac{1}{2}\,(\ 2, \ \ 0, \ \ 0, \ \ 0\ ), \\ \lambda_2 &= \frac{1}{2}(2\alpha_1 + 4\alpha_2 + 2\alpha_3 + 2\alpha_4) = \frac{1}{2}\,(\ 2, \ \ 2, \ \ 0, \ \ 0\ ), \\ \lambda_3 &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4) = \frac{1}{2}\,(\ 1, \ \ 1, \ \ 1, \ \ -1\ ), \\ \lambda_4 &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4) = \frac{1}{2}\,(\ 1, \ \ 1, \ \ 1, \ \ 1\ ), \end{aligned}$$

and the geometry of the weight system is summarized by the relations

$$\| \lambda_1 \| = \| \lambda_3 \| = \| \lambda_4 \| = 1, \qquad \| \lambda_2 \| = \sqrt{2},$$
$$(\lambda_i, \lambda_2) = 1, \quad i = 1,3,4, \qquad (\lambda_i, \lambda_j) = \frac{1}{2}, \quad i,j = 1,3,4.$$

The Weyl vector is

$$\rho = \frac{1}{2}\sum_{\alpha\in R^+} = \sum_{j=1}^{4}\lambda_j = \alpha = 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 3\alpha_4 = (\ 3,\quad 2,\quad 1,\quad 0\ ),$$

and the Weyl formula for dimensions applied to the irreducible representation associated to the integral dominant weight $\mathbf{m} = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4$ gives

$$\dim r(\mathbf{m}) = \prod_{\alpha\in R^+}\frac{(\alpha, \mathbf{m}+\rho)}{(\alpha,\rho)} = \frac{P}{1440}$$

with

$$P = \prod_{i=1}^{4}(m_i+1)\ \prod_{j}(m_2+m_j+2)\ \prod_{j<k}(m_2+m_j+m_k+3)\ (m_1+m_2+m_3+m_4)\ (m_1+2m_2+m_3+m_4)$$

where the indices j, k take the values $1, 3, 4$. In particular, for the fundamental representations, one finds:

$$\begin{array}{ll} \dim r(\lambda_1) = 8, & \dim r(\lambda_2) = 28, \\ \dim r(\lambda_3) = 8, & \dim r(\lambda_4) = 8. \end{array}$$

Appendix B: Some polynomials, characters and monomial functions

We list here all the polynomials, characters and monomial functions with total degree lower or equal to three up to triality.

Polynomials

$$\begin{aligned}
P^{\kappa}_{1,0,0,0}(z) &= z_1 \\
P^{\kappa}_{0,1,0,0}(z) &= z_2 + \frac{4(\kappa-1)}{5\kappa+1}, \\
P^{\kappa}_{2,0,0,0}(z) &= z_1^2 - \frac{2}{1+\kappa}\,z_2 - \frac{8\kappa}{(1+\kappa)(1+3\kappa)} \\
P^{\kappa}_{0,2,0,0}(z) &= z_2^2 - \frac{2}{1+\kappa}\,z_1z_3z_4 - \frac{2(-1+\kappa)}{(1+\kappa)(1+2\kappa)}\,(z_1^2+z_3^2+z_4^2) + \frac{4(-3+5\kappa+6\kappa^2+4\kappa^3)}{(1+\kappa)(1+2\kappa)(3+5\kappa)}\,z_2 + \\
&+ \frac{16(-1+\kappa)(3+10\kappa+3\kappa^2+2\kappa^3)}{(1+\kappa)(1+2\kappa)(2+5\kappa)(3+5\kappa)} \\
P^{\kappa}_{1,1,0,0}(z) &= z_1z_2 - \frac{3}{1+2\kappa}\,z_3z_4 + \frac{4(-1+\kappa)(-1+2\kappa)}{(1+2\kappa)(2+5\kappa)}\,z_1 \\
P^{\kappa}_{1,0,1,0}(z) &= z_1z_3 - \frac{4}{1+3\kappa}\,z_4 \\
P^{\kappa}_{3,0,0,0}(z) &= z_1^3 - \frac{6}{2+\kappa}\,z_1z_2 + \frac{6}{(1+\kappa)(2+\kappa)}\,z_3z_4 - \frac{12(1+2\kappa+2\kappa^2)}{(1+\kappa)(2+\kappa)(2+3\kappa)}\,z_1 \\
P^{\kappa}_{0,3,0,0}(z) &= z_2^3 - \frac{6}{2+\kappa}\,z_1z_2z_3z_4 + \frac{6}{(1+\kappa)(2+\kappa)}\,(z_1^2z_3^2+z_1^2z_4^2+z_3^2z_4^2) - \frac{3(2+\kappa+\kappa^2)}{(1+\kappa)^2(2+\kappa)}\,(z_1^2z_2+z_2z_3^2+z_2z_4^2) \\
&+ \frac{6(10+17\kappa+21\kappa^2+10\kappa^3+2\kappa^4)}{5(1+\kappa)^3(2+\kappa)}\,z_2^2 - \frac{3(30+53\kappa+4\kappa^2-15\kappa^3+8\kappa^4)}{5(1+\kappa)^4(2+\kappa)}\,z_1z_3z_4
\end{aligned}$$

$$
\begin{aligned}
&- \frac{12\kappa(8+10\kappa+\kappa^2+\kappa^3)}{5(1+\kappa)^4(2+\kappa)}(z_1^2+z_3^2+z_4^2) + \frac{12(30+119\kappa+159\kappa^2+124\kappa^3+80\kappa^4+24\kappa^5+4\kappa^6)}{5(1+\kappa)^4(2+\kappa)(4+5\kappa)} z_2 \\
&+ \frac{16(-30+103\kappa+440\kappa^2+359\kappa^3+98\kappa^4+86\kappa^5+20\kappa^6+4\kappa^7)}{5(1+\kappa)^4(2+\kappa)(3+5\kappa)(4+5\kappa)}
\end{aligned}
$$

$$
\begin{aligned}
P^\kappa_{2,1,0,0}(z) &= z_1^2 z_2 - \frac{2}{1+\kappa} z_2^2 - \frac{1+3\kappa}{(1+\kappa)^2} z_1 z_3 z_4 + \frac{4(-1+\kappa)\kappa^2}{(1+\kappa)^2(3+5\kappa)} z_1^2 + \frac{4}{(1+\kappa)^2}(z_3^2+z_4^2) \\
&- \frac{4(9+27\kappa+28\kappa^2+16\kappa^3)}{(1+\kappa)^2(2+3\kappa)(3+5\kappa)} z_2 - \frac{16(3+5\kappa+2\kappa^3)}{(1+\kappa)^2(2+3\kappa)(3+5\kappa)} \\
P^\kappa_{1,2,0,0}(z) &= z_1 z_2^2 - \frac{2}{1+\kappa} z_1^2 z_3 z_4 - \frac{1+3\kappa}{(1+\kappa)^2} z_2 z_3 z_4 - \frac{2(-1+\kappa)}{(1+\kappa)(1+2\kappa)} z_1^3 + \frac{5-\kappa}{(1+\kappa)^2}(z_1 z_3^2 + z_1 z_4^2) \\
&+ \frac{4(-1+\kappa)(9+19\kappa+10\kappa^2+4\kappa^3)}{(1+\kappa)^2(1+2\kappa)(4+5\kappa)} z_1 z_2 - \frac{4(-1+\kappa)(-5+2\kappa)(1+3\kappa)}{(1+\kappa)^2(1+2\kappa)(4+5\kappa)} z_3 z_4 \\
&+ \frac{8(-9-57\kappa-72\kappa^2+28\kappa^3-2\kappa^4+4\kappa^5)}{(1+\kappa)^2(1+2\kappa)(3+5\kappa)(4+5\kappa)} z_1 \\
P^\kappa_{1,1,1,0}(z) &= z_1 z_2 z_3 - \frac{3}{1+2\kappa}(z_1^2 z_4 + z_3^2 z_4) - \frac{8(-1+\kappa)}{(1+2\kappa)(2+3\kappa)} z_2 z_4 + \frac{4(12+23\kappa-11\kappa^2+6\kappa^3)}{(1+2\kappa)(2+3\kappa)(3+5\kappa)} z_1 z_3 \\
&- \frac{8(3-22\kappa+4\kappa^2)}{(1+2\kappa)(2+3\kappa)(3+5\kappa)} z_4 \\
P^\kappa_{1,0,1,1}(z) &= z_1 z_3 z_4 - \frac{4}{1+3\kappa}(z_1^2+z_3^2+z_4^2) + \frac{12}{(1+2\kappa)(1+3\kappa)} z_2 + \frac{16(1+5\kappa)}{(1+2\kappa)(1+3\kappa)^2}
\end{aligned}
$$

Characters

$$
\begin{aligned}
P^{(1)}_{1,0,0,0}(z) &= z_1 \\
P^{(1)}_{0,1,0,0}(z) &= z_2 \\
P^{(1)}_{2,0,0,0}(z) &= z_1^2 - z_2 - 1 \\
P^{(1)}_{0,2,0,0}(z) &= z_2^2 + z_2 - z_1 z_3 z_4 \\
P^{(1)}_{1,1,0,0}(z) &= z_1 z_2 - z_3 z_4 \\
P^{(1)}_{1,0,1,0}(z) &= z_1 z_3 - z_4 \\
P^{(1)}_{3,0,0,0}(z) &= z_1^3 - 2 z_1 z_2 + z_3 z_4 - 2 z_1 \\
P^{(1)}_{0,3,0,0}(z) &= z_2^3 + 3 z_2^2 + 3 z_2 - 2 z_1 z_2 z_3 z_4 + z_1^2 z_3^2 + z_1^2 z_4^2 + z_3^2 z_4^2 \\
&- (z_1^2 + z_3^2 + z_4^2) z_2 - z_1 z_3 z_4 - z_1^2 - z_3^2 - z_4^2 + 1 \\
P^{(1)}_{2,1,0,0}(z) &= z_1^2 z_2 - z_2^2 - z_1 z_3 z_4 + z_3^2 + z_4^2 - 2 z_2 - 1 \\
P^{(1)}_{1,2,0,0}(z) &= z_1 z_2^2 - z_1^2 z_3 z_4 - z_2 z_3 z_4 + z_1(z_3^2 + z_4^2) - z_1 \\
P^{(1)}_{1,1,1,0}(z) &= z_1 z_2 z_3 + z_1 z_3 - (z_1^2 + z_3^2) z_4 + z_4 \\
P^{(1)}_{1,0,1,1}(z) &= z_1 z_3 z_4 - z_1^2 - z_3^2 - z_4^2 + z_2 + 2
\end{aligned}
$$

Monomial functions

$$
P^{(0)}_{1,0,0,0}(z) = z_1
$$

$$
\begin{aligned}
P^{(0)}_{0,1,0,0}(z) &= z_2 - 4\\
P^{(0)}_{2,0,0,0}(z) &= z_1^2 - 2z_2\\
P^{(0)}_{0,2,0,0}(z) &= z_2^2 - 2\,z_1z_3z_4 + 2\,z_1^2 + 2\,z_3^2 + 2\,z_4^2 - 4\,z_2 - 8\\
P^{(0)}_{1,1,0,0}(z) &= z_1z_2 - 3z_3z_4 + 2z_1\\
P^{(0)}_{1,0,1,0}(z) &= z_1z_3 - 4z_4\\
P^{(0)}_{3,0,0,0}(z) &= z_1^3 - 3z_1z_2 + 3z_3z_4 - 3z_1\\
P^{(0)}_{0,3,0,0}(z) &= z_2^3 + 6z_2^2 + 9z_2 - 3z_1z_2z_3z_4 + 3z_1^2z_3^2 + 3z_1^2z_4^2 + 3z_3^2z_4^2\\
&- 3(z_1^2 + z_3^2 + z_4^2)z_2 - 9z_1z_3z_4 - 4\\
P^{(0)}_{2,1,0,0}(z) &= z_1^2z_2 - 2z_2^2 - z_1z_3z_4 + 4z_3^2 + 4z_4^2 - 6z_2 - 8\\
P^{(0)}_{1,2,0,0}(z) &= z_1z_2^2 - 2z_1^2z_3z_4 - z_2z_3z_4 + 2z_1^3 + 5z_1(z_3^2 + z_4^2) - 9z_1z_2 - 5z_3z_4 - 6z_1\\
P^{(0)}_{1,1,1,0}(z) &= z_1z_2z_3 + 8z_1z_3 - 3(z_1^2 + z_3^2)z_4 + 4z_2z_4 - 4z_4\\
P^{(0)}_{1,0,1,1}(z) &= z_1z_3z_4 - 4z_1^2 - 4z_3^2 - 4z_4^2 + 12z_2 + 16
\end{aligned}
$$

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