A brief survey on the theta correspondence

Dipendra Prasad

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

GERMANY

Mehta Research Institute
Allahabad-211 002

India

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Dipendra Prasad
Mehta Research Institute, Allahabad-211 002, INDIA

The following is an expanded version of the notes for a survey lecture on the theta correspondence given at Trichy in January, 1996. The aim of this lecture was to give an idea of this theory and some of the directions in which the theory has been developing. Perhaps some of the material contained in sections 2,3 and 9 is new.

The theory of theta correspondence gives one of the few general methods of constructing automorphic forms of groups over number fields, or admissible representations of groups over local fields. The method has its origin in the classical construction of theta functions which are modular forms–perhaps of half integral weight on the upper half plane. Here is a special case of this:

Let q be a positive definite quadratic form on a lattice $L \subseteq \mathbf{R}^n$, $q : L \to \mathbb{Z}$. Let p be a homogeneous polynomial on $\mathbf{R}^n$ which is harmonic with respect to q, i.e. $\Delta_q p = 0$ where Δ_q denotes the Laplacian with respect to q (Δ_q is the unique homogeneous differential operator of order 2 invariant under the orthogonal group of q). Then,

$$\theta(z) = \sum_{v \in L} p(v) e^{\pi i q(v) z}$$

is a modular form on the upper half plane of weight $\frac{n}{2} + \deg p$. It is a cusp form whenever the polynomial p is of positive degree.

We refer the reader to the original papers of Howe [Ho1], [Ho2], [Ho3], of Weil [We1], [We2], the book of Gelbart [Ge1] for the one-dimensional case, the book of Moeglin-Vigneras-Waldspurger [MVW] for a comprehensive account of the local theory, and the exposition of Gelbart [Ge2] as well as an earlier exposition of the author [P1] for more details on theta correspondence.

The plan of this paper is as follows :

(i) Construction of the Weil Representation.
(ii) K-type of the Weil representation.
(iii) Character of the Weil representation.
(iv) Dual reductive pairs and the local theta correspondence.
(v) Global theta correspondence.
(vi) Towers of theta lifts.
(vii) An example of global theta lift: A theorem of Waldspurger.
(viii) Functoriality of theta correspondence.
(ix) Theta correspondence for unitary groups.
(x) The Siegel-Weil formula.
(xi) Application to cohomology of Shimura Variety.
(xii) Recent generalisations of theta correspondence.

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1 Construction of the Weil Representation

Let k be a field which is not of characteristic 2, and is either a local field or a finite field, and let W be a finite dimensional vector space over k together with a non-degenerate symplectic form:

$$W \times W \to k$$

$$\omega_1, \omega_2 \to < \omega_1, \omega_2 > .$$

Define the Heisenberg group $H(W) = \{(w,t) | w \in W, t \in k\}$ with the law of multiplication :

$$(w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2,\ t_1 + t_2 + \frac{1}{2} < w_1, w_2 >).$$

The Heisenberg group is a central extension of W by k :

$$0 \to k \to H(W) \to W \to 0.$$

The most important property about the representation theory of the Heisenberg group is the following uniqueness theorem due to Stone and von Neumann.

Theorem 1 *For any non-trivial character $\psi : k \to \mathbb{C}$, there exists a unique irreducible representation ρ_ψ of $H(W)$ on which $k \subseteq H(W)$ acts by ψ.*

Now we observe that the symplectic group $\mathrm{Sp}(W)$ operates on $H(W)$ by $g(w,t) = (gw,t)$. By the uniqueness of ρ_ψ, $\mathrm{Sp}(W)$ acts by intertwining operators on ρ_ψ. Namely, there exists $\omega_\psi(g)$, unique up to scalars, such that

$$\rho_\psi(gw,t) = \omega_\psi(g)\rho_\psi(w,t)\omega_\psi(g)^{-1}.$$

The map $g \to \omega_\psi(g)$ is a projective representation of the symplectic group $\mathrm{Sp}(W)$, and gives rise to an ordinary representation of a two fold covering of $\mathrm{Sp}(W)$. We will denote this two fold cover by $\widetilde{\mathrm{Sp}}(W)$, and the representation of it so obtained again by $\omega_\psi(g)$. The group $\widetilde{\mathrm{Sp}}(W)$ is called the metaplectic group, and the representation $\omega_\psi(g)$ the Weil or metaplectic representation. In this paper we will often abuse terminology to call the Weil representation of the metaplectic group $\widetilde{\mathrm{Sp}}(W)$ as simply the Weil representation of the symplectic group $\mathrm{Sp}(W)$.

Remark 1: If W is a symplectic vector space over a separable field extension K of a local field k, then $\mathrm{tr} <,>$ gives a symplectic structure on W with its k vector space structure, to be denoted by $R_{K/k}W$. If ψ is a character on k, then $\psi_K(x) = \psi(\mathrm{tr}x)$ gives a character on K. We have the inclusion of $\mathrm{Sp}(W)$ in $\mathrm{Sp}(R_{K/k}W)$, and it is easy to see that the restriction of the Weil representation of $\mathrm{Sp}(R_{K/k}W)$ for the character ψ of k to $\mathrm{Sp}(W)$ gives the Weil representation of $\mathrm{Sp}(W)$ for the character ψ_K of K. Moreover, if W_1 and W_2 are symplectic vector spaces, then $\mathrm{Sp}(W_1)\times\mathrm{Sp}(W_2)$ is contained in $\mathrm{Sp}(W_1 \oplus W_2)$, and the restriction of the Weil representation of $\mathrm{Sp}(W_1 \oplus W_2)$ to $\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2)$ is the tensor product of the Weil representations of $\mathrm{Sp}(W_1)$ and $\mathrm{Sp}(W_2)$. Obvious as these properties are, the reader will of course notice how special these are to the Weil representation: that their restriction to such small subgroups remains of finite length.

1.1 Explicit realisation of the Weil representation

Let $W = X \oplus Y$ where X and Y are subspaces of W on which the symplectic form is identically zero. The Weil representation of $\mathrm{Sp}(W)$ can be realised on the Schwarz space $\mathcal{S}(X)$ which is the space of locally constant, compactly supported functions on X if k is non-Archimedean, and has the usual definition if k is Archimedean; the action of $\mathrm{Sp}(W)$ on $\mathcal{S}(X)$ is as follows:

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f(X) = |\det A|^{\frac{1}{2}} f({}^tAX)$$

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} f(X) = \psi(\tfrac{{}^tXBX}{2}) f(X)$$

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} f(X) = \gamma \hat{f}(X)$$

where γ is an 8-th root of unity, and $\hat{f}$ denotes the Fourier transform of f, and ψ is a non-trivial additive character of k.

Remark 2: It will be interesting to construct a model of the Weil representation which is defined over a number field. Since all the known models require the additive character ψ in an essential way, it does not seem obvious if it can be done at all. We note that the Weil representation of SL(2) can be defined over a number field because it is sum of its even and odd pieces, both of which are defined over number fields: the even piece because it occurs in an explicit principal series, and the odd piece because it is induced from compact open subgroup.

2 K-type of the Weil representation

The decomposition of a representation of a p-adic group when restricted to a maximal compact subgroup is an important information used in the representation theory of p-adic groups. In this section we carry out this decomposition for the Weil representation of $\mathrm{Sp}(W)$ where W is a symplectic vector space of dimension $2n$ over a non-Archimedean local field k. We let $< .,. >$ denote the symplectic form on W, and let ψ be the character of k of conductor 0 used in the definition of the Weil representation. The investigation in this section is in response to a question of B.H.Gross.

We will assume that the cardinality of the residue field of k is q which is odd so that the metaplectic covering splits over the maximal compact subgroup $\mathrm{Sp}(\mathcal{L})$ of $\mathrm{Sp}(W)$ stabilising a lattice $\mathcal{L}$ on which the symplectic form $< .,. >$ is non-degenerate. The Weil representation of $\mathrm{Sp}(W)$ becomes a representation of $\mathrm{Sp}(\mathcal{L})$. Let $\Gamma(i)$ be the standard filtration on $\Gamma = \mathrm{Sp}(\mathcal{L})$, so that $\Gamma(0) = \mathrm{Sp}(\mathcal{L})$, and $\Gamma(i) = \ker[\mathrm{Sp}(\mathcal{L}) \to \mathrm{Sp}(\mathcal{L}/\pi^i\mathcal{L})]$.

We will prove the following theorem in this section.

Theorem 2 *The Weil representation ω of $\mathrm{Sp}(W)$ when restricted to the compact open subgroup $\mathrm{Sp}(\mathcal{L})$ decomposes as the sum of irreducible representations as follows:*

$$\omega = \omega_0 \oplus \sum_{m \geq 1} (\omega_{2m}^+ \oplus \omega_{2m}^-)$$

where ω_0 is the trivial representation of $\mathrm{Sp}(\mathcal{L})$, and $\omega_{2m}^+, \omega_{2m}^-$ are irreducible representations of $\mathrm{Sp}(\mathcal{L}/\pi^{2m}\mathcal{L})$ of dimension $\frac{1}{2}[q^{2mn} - q^{2(m-1)n}]$ for $m \geq 1$.

Proof : The proof of this theorem will be carried out in the lattice model of the Weil representation, cf. [MVW], which is most suitable for the study of K-types. We recall that the Weil representation of $\mathrm{Sp}(W)$ in this model is realised on the space $\mathcal{S}_{\mathcal{L},\psi}(W)$ of locally constant compactly supported functions on W such that

$$f(x + a) = \psi(< x, a >) f(x)$$

for all $x \in W$, and $a \in \mathcal{L}$. The action of an element g of $\mathrm{Sp}(\mathcal{L})$ on a function f in $\mathcal{S}_{\mathcal{L},\psi}(W)$ is given by

$$(g \cdot f)(x) = f(gx).$$

Let $\mathcal{S}_m \subset \mathcal{S}_{\mathcal{L},\psi}(W)$ be the subspace of functions on W which are supported on $\pi^{-m}\mathcal{L}$. From the functional equation $f(x + a) = \psi(< x, a >) f(x)$, we find that the functions in $\mathcal{S}_m$ are invariant under translation by $\pi^m\mathcal{L}$, and therefore $\mathcal{S}_m$ can be thought of as a space of functions on $\pi^{-m}\mathcal{L}/\pi^m\mathcal{L}$ with a specified property under translation by $\mathcal{L}/\pi^m\mathcal{L}$. It follows that

$$\dim \mathcal{S}_m = \sharp(\pi^{-m}\mathcal{L}/\mathcal{L}) = q^{2mn}.$$

Define $\mathcal{S}_m^+$ (resp. $\mathcal{S}_m^-$) to be the subspace of $\mathcal{S}_m$ consisting of even (resp. odd) functions. Define ω_{2m}^+ (resp. ω_{2m}^-) to be $\mathcal{S}_m^+/\mathcal{S}_{m-1}^+$ (resp. $\mathcal{S}_m^-/\mathcal{S}_{m-1}^-$). We find that

$$\dim \omega_{2m}^+ = \dim \omega_{2m}^- = \frac{1}{2}[q^{2mn} - q^{2(m-1)n}].$$

Clearly, Γ leaves $\mathcal{S}_m, \mathcal{S}_m^+, \mathcal{S}_m^-$ invariant, and it is easy to see that $\Gamma(2m)$ acts trivially on $\mathcal{S}_m$. Since $\Gamma(m)/\Gamma(2m)$ is an abelian group, it acts by characters which we now find. Let $g = 1+\pi^m\gamma \in \Gamma(m)$, $f \in \mathcal{S}_m$, and $x \in \pi^{-m}\mathcal{L}$, then

$$\begin{aligned}(g \cdot f)(x) &= f(gx) \\ &= f(x + \gamma\pi^m x) \\ &= \psi(< x, \gamma\pi^m x >) f(x) \\ &= \psi(< x, gx >) f(x).\end{aligned}$$

It follows that the action of $\Gamma(m)/\Gamma(2m)$ on $\mathcal{S}_m$ is via the characters $g \to \psi(< x, gx >)f(x)$ where $x \in \pi^{-m}\mathcal{L}$. We denote this character of $\Gamma(m)/\Gamma(2m)$ by ψ_x, so $\psi_x(g) = \psi(< x, gx >)$. It is easy to see that $\psi_x = \psi_y$ if $x \equiv y \bmod \mathcal{L}$, or if $x \equiv -y \bmod \mathcal{L}$. Conversely, if $\psi_x = \psi_y$ then either $x \equiv y \bmod \mathcal{L}$, or $x \equiv -y \bmod \mathcal{L}$. Moreover, $\psi_{Ax}(g) = \psi_x(A^{-1}gA)$, for $A \in \mathrm{Sp}(\mathcal{L})$. It is a well-known fact that $\mathrm{Sp}(\mathcal{L})$ acts transitively on vectors in $\pi^{-m}\mathcal{L}$ which do not belong to $\pi^{-m+1}\mathcal{L}$. Therefore, if ψ_x appears in an irreducible representation of $\mathrm{Sp}(\mathcal{L})$ for one value of x in $\pi^{-m}\mathcal{L}$ not belonging to $\pi^{-m+1}\mathcal{L}$, ψ_y for all values of y in $\pi^{-m}\mathcal{L}$ not in $\pi^{-m+1}\mathcal{L}$ appears. Since $\psi_x = \psi_y$ if and only if $x \equiv y \bmod \mathcal{L}$, or $x \equiv -y \bmod \mathcal{L}$, we find that the dimension of an irreducible representation of $\mathrm{Sp}(\mathcal{L})$ containing ψ_x is divisible by $\frac{1}{2}[q^{2mn} - q^{2(m-1)n}]$. Since ω_{2m}^+, ω_{2m}^- are representations of $\mathrm{Sp}(\mathcal{L})$ of dimension $\frac{1}{2}[q^{2mn} - q^{2(m-1)n}]$, these must therefore be irreducible, proving the decomposition

$$\omega = \omega_0 \oplus \sum_{m \geq 1} (\omega_{2m}^+ \oplus \omega_{2m}^-).$$

3 Character of the Weil representation for SL(2)

In this section we calculate the character of the Weil representation of $SL(2, F)$ for a non-Archimedean local field F. Because of the relation of

the character formula obtained in this case to transfer factors, it would be very interesting to calculate the character of the Weil representation of the general symplectic group. For some work along this direction in the Archimedean case, see [Ad2].

The calculation of the character will be done in 2 steps. The first step consists in identifying those characters of E^1 (the subgroup of norm 1 elements of a quadratic extension E of F) which appear in the Weil representation. The answer is in terms of an epsilon factor. This is a result due to Rogawski [Ro], based on an earlier work of Moen. Next we need to add all the characters of E^1 which appear in the Weil representation to find the character of the Weil representation restricted to E^1. This kind of summation of characters was done by the author (for a different purpose!) in [P2]. We set up some notation now.

Let E be a quadratic field extension of the local field F with $\omega_{E/F}$, the quadratic character of F^* associated to E by local class field theory. The group E^1 of norm 1 elements of E is contained in $SL(2,F)$, and the metaplectic cover of $SL(2,F)$ splits over E^1. However, there is no natural choice for this splitting. For each character γ of E^* whose restriction to F^* is $\omega_{E/F}$, one can construct such a splitting. Let ψ be an additive character of F and δ be an element of E^* with $\mathrm{tr}(\delta) = 0$. This choice of δ gives rise to a symplectic structure on the 2-dimensional F vector space $E : x, y \to \mathrm{tr}(\delta x \bar{y})$. Let $\omega(\psi, \delta)$ denote the Weil representation of $SL(2,F)$ associated to the additive character ψ and this symplectic structure on E. Let $\omega(\gamma, \psi, \delta)$ denote the restriction of $\omega(\psi, \delta)$ to E^1 via the splitting given by γ. The following theorem is due to Rogawski [Ro, Prop. 3.4].

Theorem 3 *Let χ be a character of E^1. Then χ appears in $\omega(\gamma, \psi, \delta)$ if and only if*

$$\epsilon(\gamma\chi_E^{-1}, \psi_E) = \chi(-1)\gamma(2\delta)$$

where $\chi_E(x) = \chi(x/\bar{x}), \psi_E(x) = \psi(\mathrm{tr}\, x)$.

The following theorem is Lemma 3.1 of [P2] proved there only for odd residue characteristic.

Theorem 4 *For the character ψ_0 of E defined by $\psi_0(x) = \psi(tr[-\frac{\delta x}{2}])$,*

we have

$$\sum_{\substack{\epsilon(\chi,\psi_0)=1 \\ \chi|_{F^*}=\omega_{E/F}}} \chi(x) = \epsilon(\omega_{E/F},\psi)\frac{\omega_{E/F}\left(\frac{x-\bar{x}}{\delta-\bar{\delta}}\right)}{\left|\frac{(x-\bar{x})^2}{x\bar{x}}\right|_F^{\frac{1}{2}}}$$

where, as is usual, the summation on the left is by partial sums over all characters of E^* of conductor $\leq n$.

Since $\epsilon(\gamma\chi_E^{-1},\psi_E) = \chi(-1)\gamma(2\delta)\epsilon(\gamma\chi_E^{-1},\psi_0)$, it is easy to see that the above two theorems imply the following.

Theorem 5 *The character of the Weil representation $\omega(\gamma,\psi,\delta)$ at an element $y = x/\bar{x} \in E^1$ is equal to*

$$\epsilon(\omega_{E/F},\psi)\gamma(-x)\frac{\omega_{E/F}\left(\frac{x-\bar{x}}{\delta-\bar{\delta}}\right)}{\left|\frac{(x-\bar{x})^2}{x\bar{x}}\right|_F^{\frac{1}{2}}}.$$

Remark 1: We note that any 2 of the theorems of this section implies the third. Theorem 3 is valid for any residue characteristic, but Theorem 4 is known only for odd residue characteristic, and therefore Theorem 5 is available only in odd residue characteristic.

Remark 2 : One can use Remark 2 of section 1 together with the character formula for the Weil representation of $SL(2)$ to calculate the character of the Weil representation at many elements of $\mathrm{Sp}(n)$ for general n. This presumes of course that, in the notation of that remark, the restriction of the character of $\mathrm{Sp}(R_{K/k}W)$ to $\mathrm{Sp}(W)$ is the character of the Weil representation of $\mathrm{Sp}(W)$, which can be justified.

4 Dual reductive pairs and the local theta correspondence

Definition: A pair of subgroups (G_1, G_2) in $\mathrm{Sp}(W)$ is called a dual reductive pair if

1. $Z(G_1) = G_2$, and $Z(G_2) = G_1$ where $Z(G_1)$ (resp. $Z(G_2)$) denotes the centraliser of G_1 (resp. G_2) in $\mathrm{Sp}(W)$.

2. G_1 and G_2 are reductive groups, i.e., any G_1 invariant subspace of W has a G_1-invariant complement; similarly for G_2.

We refer to [MVW] and [P1] for a detailed discussion on dual reductive pairs. Here we only note the following examples.

(i) Let V be an orthogonal space (i.e. a finite dimensional vector space over k with a non-degenerate quadratic form), and let W be a symplectic space. Then $V \otimes W$ is a symplectic space in natural way, and we have a map

$$O(V) \times \mathrm{Sp}(W) \to \mathrm{Sp}(V \otimes W).$$

The pair $(O(V), \mathrm{Sp}(W))$ is a dual reductive pair.

(ii) Let K be a quadratic extension of k, V a Hermitian and W a skew-Hermitian space over K. Then the k-vector space $V \otimes_K W$ is naturally a symplectic space under the pairing

$$< v_1 \otimes w_1, v_2 \otimes w_2 >= \mathrm{tr}_{K/k}(< v_1, v_2 >< w_1, w_2 >)$$

and gives rise to a dual reductive pair $(U(V), U(W))$ in $\mathrm{Sp}(V \otimes_K W)$.

For a dual reductive pair (G_1, G_2) in $\mathrm{Sp}(W)$, let $\bar{G}_1$ be the inverse image of G_1 in $\overline{\mathrm{Sp}}(W)$, and $\bar{G}_2$ the inverse image of G_2. The groups $\bar{G}_1$ and $\bar{G}_2$ are known to commute. The Weil representation ω_ψ of $\overline{\mathrm{Sp}}(W)$ can therefore be restricted to $\bar{G}_1 \times \bar{G}_2$. Let π be an irreducible representation of $\bar{G}_1$. Define

$$A(\pi) = \omega_\psi / \cap \mathrm{Ker}\phi : \ \phi \in \mathrm{Hom}_{\bar{G}_1}(\omega_\psi, \pi).$$

$A(\pi)$ is a smooth representation of $\bar{G}_1 \times \bar{G}_2$, and can be written as $A(\pi) = \pi \otimes \theta_0(\pi)$ for a smooth representation $\theta_0(\pi)$ of $\bar{G}_2$.

The following theorem is due to Waldspurger [Wa2] building on the earlier work of Howe.

Theorem 6 *The representation $\theta_0(\pi)$ is of finite length, and, if the residue characteristic of k is not 2, it has a unique irreducible quotient.*

Notation : The unique irreducible quotient of $\theta_0(\pi)$ is called the local theta lift of π, and is denoted by $\theta(\pi)$. We warn the reader that the representation ω_ψ and hence $\theta(\pi)$ depends on the choice of the additive character $\psi : k \to \mathbb{C}$.

Examples:

(i) For $V = K$, a quadratic extension of k thought of as a 2-dimensional vector space over k with the norm form as the quadratic

form, $SO(V) = K^1$, and the theta lift from $O(V)$ to $SL(2)$ can be used to construct representations of $GL(2)$ from characters of K^*. This construction is due to Shalika and Tannaka.

(ii) For $V = D$, the quaternion division algebra over k, together with the reduced norm as the quadratic form, $SO(V)$ is essentially $D^1 \times D^1$, and the theta lift from $SL(2)$ to $D^1 \times D^1$ is related to the Jacquet-Langlands correspondence between discrete series representations of $GL(2)$ and representations of D^*.

Remark : The question about the field of definition of the Weil representation (cf. Remark 2 at the end of section 1) is interesting also because one could use it to give a field of definition to $\theta(\pi)$ in terms of a field of definition for π.

5 Global Theta Correspondence

Let $\mathbf{A}$ be the adele ring of a number field F. Let ψ: $\mathbf{A}/F \to \mathbb{C}$ be a non-trivial character. Let $W = X \oplus Y$ be a symplectic vector space over F with X and Y maximal isotropic subspaces. We have $\mathcal{S}(X(\mathbf{A})) = \otimes_v \mathcal{S}(X(F_v))$, and there is a pojective representation of $\mathrm{Sp}(W)(\mathbf{A})$ on it by taking the tensor product of local representations. This projective representation of $\mathrm{Sp}(W)(\mathbf{A})$ becomes an ordinary representation of a 2-fold cover $\overline{\mathrm{Sp}}(W)(\mathbf{A})$ of $\mathrm{Sp}(W)(\mathbf{A})$.

The following theorem is due to Weil [Wel].

Theorem 7 *The covering* $\overline{\mathrm{Sp}}(W)(\mathbf{A}) \to \mathrm{Sp}(W)(\mathbf{A})$ *splits over* $\mathrm{Sp}(W)(F)$.

Because of this theorem, $\mathrm{Sp}(W)(F)$ operates on $\mathcal{S}(X(\mathbf{A}))$. Define a distribution θ on $\mathcal{S}(X(\mathbf{A}))$ by $\theta(\phi) = \sum_{x \in X(F)} \phi(x)$. This distribution is $\mathrm{Sp}(W)(F)$-invariant. Therefore, the function $g \to \theta(g \cdot \phi) = \theta_\phi(g)$ defines a function $\theta_\phi : \mathrm{Sp}(W)(F)\backslash\overline{\mathrm{Sp}}(W)(\mathbf{A}) \to \mathbb{C}$ These are called theta functions. They are slowly increasing and therefore automorphic. For an appropriate choice of ϕ on $\mathbf{A}$, this gives the adelic analogue of the classical theta function of weight $= \frac{1}{2}$ on the upper half plane given by

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z}.$$

Now let (G_1, G_2) be a dual reductive pair in $\mathrm{Sp}(W)$ which is defined over F. Let π_1 be a cuspidal representation of G_1, realised on a space of

cusp forms on $G_1(F)\backslash G_1(\mathbf{A})$. For a function $\phi \in \mathcal{S}(X(\mathbf{A}))$ and $f \in \pi_1$, define

$$\theta_\phi(f)(g_2) = \int_{G_1(F)\backslash G_1(\mathbf{A})} \theta_\phi(g_1, g_2) f(g_1) dg_1.$$

$\theta_\phi(f)$ is an automorphic form on G_2, and is called the theta lift of f.

Example: The theta function $\theta(z) = \sum_{v \in L} p(v) e^{\pi i q(v)}$ defined in the introduction is such a lift from $O(V)$ to $SL(2)$ where V is the quadratic space $\mathbf{R}^n$ with q as the qudratic form.

Basic questions for the global theta lift are

(i) When is the space generated by $\theta_\phi(f)$ for $\phi \in \mathcal{S}(X(\mathbf{A}))$ and f in the representation space π_1 not identically zero?

(ii) What is the relation of the space spanned by $\theta_\phi(f)$ to local theta correspondence?

As regards (ii) we have the following due to Rallis [Ra1].

Theorem 8 *If $\theta_\phi(f)$ consists of cusp forms, then the space generated by them is irreducible and is $\otimes\theta_v(\pi_v^\vee)$ where $\pi_v^\vee$ denotes the contragredient of π_v.*

Rallis has a theory about when $\theta_\phi(f)$ is cuspidal in terms of "towers of theta lifts", which we review next; this theory also partially answers part (i) of the above question (see Theorem 5(ii) below).

6 Towers of theta lifts

Let V be an even dimensional quadratic space over a number field which does not represent any zero. Let $\mathbf{H}$ be the two dimensional hyperbolic quadratic space (with quadratic form XY). The "towers of theta lifts" consists in looking at theta lifts from $\mathrm{Sp}(W)$ to various $O(V + n\mathbf{H})$. The following theorem is due to S. Rallis [Ra1].

Theorem 9 *(i) Let π be a cuspidal automorphic representation of $\mathrm{Sp}(W)$, and $\theta_n(\pi)$ its theta lift to $O(V + n\mathbf{H})$. Let n_0 be the smallest integer ≥ 0 such that $\theta_{n_0}(\pi) \neq 0$. Then $\theta_{n_0}(\pi)$ is cuspidal, and $\theta_n(\pi)$, $n \geq n_0$, are non-zero, and never cuspidal for $n > n_0$.*

(ii) $\theta_n(\pi) \neq 0$ for $n \geq \dim W$.

There is an analogous statement in which W is varying in the tower $\mathbf{H}, 2\mathbf{H}, \ldots$ ($\mathbf{H}$ now hyperbolic symplectic), or for dual reductive pairs consisting of unitary groups.

Remark: Let I_j be the space of cusp forms on $\mathrm{Sp}(W)$ whose theta lifts to $O(V + j\mathbf{H})$ is non-zero but is zero to $O(V + i\mathbf{H})$, $i < j$. Then the theorem above gives an interesting decomposition of the space S of cusp forms on $\mathrm{Sp}(W)$ as

$$S = I_0 \oplus I_1 \oplus \ldots I_{2n}$$

where $\dim W = 2n$. The cusp forms in different I_j's behave quite differently.

The theory of towers of theta lifts works in the local case also when one replaces the word cuspidal by supercuspidal, cf. the work of Kudla in [Ku]. This work of Kudla in [Ku] determines the principal series in which $\theta_n(\pi)$ lies as a subquotient in terms of $\theta_{n_0}(\pi)$, and is based on the calculation of the Jacquet functor of the Weil representation of $\mathrm{Sp}(V \otimes W)$ with respect to the unipotent radical of maximal parabolics of $O(V)$ and $\mathrm{Sp}(W)$.

Recently Harris, Kudla and Sweet have used these towers of theta lifts to construct a very interesting infinite family of supercuspidals on Unitary groups of dimension tending to infinity from one supercuspidal. We review this work briefly.

Let V and W be two Hermitian spaces over a quadratic extension K of a local field k. Let δ be a non-zero element of K whose trace to k is 0. Multiplication by δ turns a Hermitian space into a skew-Hermitian space, and therefore $(U(V), U(W))$ form a dual reductive pair in $\mathrm{Sp}(V \otimes_K W)$. By [HKS] the metaplectic cover of $\mathrm{Sp}(V \otimes_K W)$ splits over $U(V) \times U(W)$, the splitting depending on choice of characters χ_1, χ_2 of K^* such that $\chi_1|_{k^*} = \omega_{K/k}^{\dim W}$, and $\chi_2|_{k^*} = \omega_{K/k}^{\dim V}$. Here $\omega_{K/k}$ is the quadratic character of k^* associated to the quadratic extension K of k by local class field theory.

Note that there are two isomorphism classes of Hermitian or skew-Hermitian spaces of a given dimension depending on their discriminant over a non-archimedean local field.

The following theorem is due to Harris, Kudla and Sweet [HKS]. Here we fix the splitting of the metaplectic cover of $Sp(V \otimes_K W)$ over $(U(V), U(W))$ for the choice of the characters χ_1, χ_2 of K^* with $\chi_1 = \chi_2 = \chi$.

Theorem 10 *(i) Suppose that π is a supercuspidal representation of $U(V)$. Then the theta lift of π to $U(W)$ is non-zero for exactly one Hermitian space W with* $\dim(W) = \dim(V)$.

(ii) Given a Hermitian space W with $\dim(W) = \dim(V)$, *the theta lift of π to $U(W)$ is non-zero if and only if*

$$\omega_{K/k}(\mathrm{disc}V) = \epsilon(BC(\pi) \otimes \chi, \psi_K)\omega_\pi(-1)\chi(\delta^{-n})\omega_{K/k}(\mathrm{disc}W),$$

where $\epsilon(BC(\pi)\otimes\chi, \psi_K)$ is defined and studied via the "doubling-method" in [HKS].

This theorem implies that starting with a supercuspidal representation π on $U(V)$, one can construct a supercuspidal representation on $U(W \oplus n\mathbf{H})$ for $n > 0$, dim W = dim V, and the space W has the property that the theta lift of π to $U(W)$ is zero. Repeating the process one gets an infinite family of supercuspidal representations on unitary groups of increasing dimensions. This theorem of Harris, Kudla and Sweet also generalises Theorem 3 due to Rogawski.

7 An example of global theta lift: A theorem of Waldspurger

The theorem of Waldspurger [Wa1] completely describes the global theta correspondence between $PGL(2)$ and $\overline{SL}(2)$, and is proto-type of theorems expected in general.

Theorem 11 *(i) For a cuspidal automorphic representation π of $PGL(2)$, $\theta_\pi \neq 0$ on $\overline{SL}(2)$ if and only if*

$$L(\pi, \frac{1}{2}) \neq 0.$$

(ii) For an automorphic representation π of $\overline{SL}(2)$, $\theta_\pi \neq 0$ if and only if π has a ψ-Whittaker model.

(iii) For an automorphic representation π on $PGL(2)$, $\otimes_v \theta_v(\pi_v)$ is automorphic if and only if the sign in the functional equation for the L-function of π, $\varepsilon(\pi, \frac{1}{2}) = 1$.

Remark : For generalisations of Waldspurger's theorem above, see the book of Rallis [Ra3], and also a recent paper of Furusawa [Fu].

8 Functoriality of theta correspondence

The basic question about local theta correspondence is to classify those representations π_1 of G_1 for which $\theta(\pi_1) \neq 0$, and then to understand $\theta(\pi_1)$ which is a representation of G_2 in terms of the representation π_1 of G_1. Here, for simplicity, we assume that the metaplectic cover of the symplectic group splits over $G_1 \times G_2$, and that we have fixed such a splitting to regard the Weil representation of the metaplectic group as a representation of $G_1 \times G_2$, to define the theta correspondence between representations of G_1 and G_2.

We recall that according to conjectures of Langlands, representations of a reductive group G over a local field k are parametrised by certain homomorphisms of the Weil-Deligne group W'_k of k into a complex group associated to G, called the L-group of G and denoted by ${}^LG$:

$$\phi : W'_k \to {}^LG.$$

This conjecture is known in the Archimedean case and in some other cases. The conjecture implies in particular that if there is a map between L-groups ${}^LG_1 \to {}^LG_2$, then there is a way of associating representations of G_2 to representations of G_1. Correspondence between representations of two groups arising in this way is said to be "functorial".

We refer to the work of Rallis [Ra2] in this regard which proves the functoriality of the theta correspondence for spherical representations. When the groups G_1 and G_2 are of "similar" size then again the theta lifting seems functorial. There are some conjectures and evidences on this by Adams in the Archimedean case [Ad1], which are refined and extended by this author [P3]. However theta correspondence in general does not respect functoriality; see [Ad1], and [HKS].

We also refer to the recent work of Gelbart, Rogawski and Soudry [GRS] for rather complete information about theta lifting from $U(2)$ to $U(3)$ both locally and globally and its relation to functoriality.

9 Theta correspondence for unitary groups

In this section we propose two conjectures, one local and the other global, about theta correspondence between two unitary groups $U(V)$ and $U(W)$ where V and W are Hermitian spaces of dimension n over a

quadratic extension K of a local or a global field k. The local conjecture is motivated by the work [HKS] recalled in section 6 which we refine using the Vogan parametrisation.

If the local field is non-Archimedean, then there are precisely two isomorphism classes of Hermitian spaces, and if $k = \mathbf{R}$, then there are $n+1$ isomorphism classes represented by $U(n,0), U(n-1,1), \cdots, U(1,n-1), U(0,n)$. We will denote the L-group of $U(V)$ by ${}^LU(n)$ which is a semi-direct product of $GL(n,\mathbf{C})$ by the Weil group of k which acts via its quotient $\mathrm{Gal}(K/k)$ by $g \to \Phi_n {}^t g^{-1} \Phi_n^{-1}$ where Φ_n is the $n \times n$ matrix whose only non-zero entries are at the places $(i, n+1-i), 1 \leq i \leq n$ where it takes the value $(-1)^{i+1}$.

Let V and W be Hermitian spaces of the same dimension over a quadratic extension K of a local field k. We fix ψ to be a non-trivial additive character of k, and ψ_K the additive character of K obtained by composing ψ with the trace map from K to k. In this section we fix χ to be a character of K^* whose restriction to k^* is $\omega_{K/k}$, the quadratic character of k^* associated to K if n is odd, and we let χ to be the trivial character of K^* if n is even. Let $\delta \in K^*$ be fixed with $\mathrm{tr}(\delta) = 0$. Multiplication by δ turns the Hermitian space W into a skew-Hermitian space, and therefore we get a symplectic structure on $V \otimes_K W$. The character χ is used to define a splitting of the metaplectic covering of $\mathrm{Sp}(V \otimes_K W)$ over $U(V) \times U(W)$, cf. [HKS]. The Weil representation of $\mathrm{Sp}(V \otimes_K W)$ associated to the character ψ thus gives rise to a representation of $U(V) \times U(W)$. When working globally, we take global analogues of ψ and χ.

We will assume the conjecture of Langlands and its refinement due to Vogan [Vo] about parametrization of representations of reductive groups over local fields. Very briefly put, let G be a quasi-split reductive group over a local field k with ${}^LG$ its L-group which is a semi-direct product of $G^\vee$ with the Weil group of k. Fix a non-degenerate character of the unipotent radical of a Borel subgroup of G. Then according to Vogan, the isomorphism classes of pairs (ϕ, μ) where ϕ is a parameter for G, and μ is an irreducible representation of $\mathcal{S}_\phi$, the group of connected components of the centraliser of ϕ in $G^\vee$, is in one-to-one correspondence with the set of irreducible admissible representations of pure inner forms of G. For G, the quasi-split unitary group $U(V_0)$, the pure inner forms of $U(V_0)$ are in one-to-one correspondence with the isomorphism classes of Hermitian spaces of dimension $n = \dim(V_0)$. The additive character ψ_δ of the trace zero elements of K defined by $\psi_\delta(x) = \psi(\delta x)$ can be used to fix a non-degenerate character on the

unipotent radical of a Borel subgroup of $U(V_0)$. In the case of a non-Archimedean local field, the discriminant of V, denoted disc$V$, which is an element of $k^*/\mathrm{Nm}K^*$ determines the isomorphism class of V. A Vogan parameter (ϕ, μ) corresponds to a representation of $U(V)$ such that

$$\mu(-\mathbf{1}) = \omega_{K/k}\left(\frac{\mathrm{discV}}{\mathrm{discV}_0}\right) \qquad 9.0.1.$$

where $-\mathbf{1}$ is the element of $\mathcal{S}_\phi$ represented by the negative of the identity matrix in $GL(n, \mathbf{C})$.

9.1 Component group for parameters of unitary group

Let ϕ be a parameter for unitary group $U(V)$ over a local field k defined in terms of a Hermitian form on an n-dimensional vector space V over the quadratic extension K of k. The restriction of ϕ to the Weil group W_K of K is a group homomorphism $\phi_K : W_K \to GL(n, \mathbf{C})$. Assume that ϕ_K decomposes as a direct sum of irreducible representations $\phi_K = \sum n_i\phi_i$ where n_i is the multiplicity of the irreducible representation ϕ_i. The representation ϕ_K has the property that $\bar{\phi}_K \cong \phi_K^*$ where ϕ_K^* denotes the dual of ϕ_K, and $\bar{\phi}_K$ is the representation of W_K obtained from ϕ_K by conjugation by an element of W_k which does not belong to W_K. It follows that the set of irreducible representations ϕ_i which appear in the decomposition of ϕ_K is stable under $\phi_i \to \bar{\phi}_i^*$. The group of connected components of the centraliser of ϕ in $GL(n, \mathbf{C})$ is a product of $\mathbf{Z}/2$'s, the product being indexed by those irreducible representations ϕ_i which have the property that $\phi_i \cong \bar{\phi}_i^*$ and either n_i is odd, or if n_i is even, then the dimension of ϕ_i is even. We do not give detailed proof of this here, except to point out that these connected components arise because of the maps,

$$^LU(n) \times O(2m+1, \mathbf{C}) \to {}^LU(2mn+n),$$
$$^LU(2n+1) \times \mathrm{Sp}(2m, \mathbf{C}) \to {}^LU(4mn+2m),$$
$$^LU(2n) \times O(2m, \mathbf{C}) \to {}^LU(4mn).$$

These maps arise essentially because Φ_n is symmetric if n is odd, and is skew symmetric if n is even, and the fact that under a suitable ordering of basis, $\Phi_n \otimes \Phi_m = \Phi_{nm}$ if either n or m is odd, and $\Phi_{2n} \otimes \Psi_{2m} = \Phi_{4mn}$ where Ψ_{2m} is the $2m \times 2m$ matrix whose only non-zero entries are at the places $(i, 2m+1-i), 1 \leq i \leq 2m$ where it takes the value 1.

9.2 The character χ_ϕ and the local conjecture

For representations ϕ_i of W_K with the property $\bar{\phi}_i = \phi_i^*$, $\det \phi_i$ which is a character of K^* is trivial on those elements of k^* which are norms from K^*. Recall that we have fixed $\delta \in K^*$ with $\mathrm{tr}(\delta) = 0$. Since $\delta^2 = -\mathrm{Nm}(\delta)$, $\det \phi_i(\delta^2) = \det \phi_i(-1)$. We will now be using epsilon factors associated to representations of the Weil-Deligne group for which we refer to the paper of Tate [Ta]. Since $\epsilon(\phi_i, \psi_K) = \epsilon(\bar{\phi}_i, \psi_K)$, and $\epsilon(\phi_i, \psi_K) \cdot \epsilon(\phi_i^*, \psi_K) = \det \phi_i(-1)$, we find that

$$\epsilon(\phi_i, \psi_K)^2 = \det \phi_i(-1) = \det \phi_i(\delta^2),$$

and therefore

$$\epsilon(\phi_i, \psi_K) \cdot \det \phi_i(\delta^{-1}) = \pm 1.$$

To a parameter ϕ of the unitary group $U(V)$, we will now define a character χ_ϕ on the group of connected components S_ϕ of the centraliser of ϕ in $GL(n, \mathbf{C})$. For the decomposition of ϕ restricted to K, $\phi_K = \sum n_i \phi_i$, as we have seen, the group of connected components is parametrised by those ϕ_i for which either n_i is odd, or if n_i is even, $\dim \phi_i$ is even. Define χ_ϕ on the $\mathbf{Z}/2$ associated to ϕ_i to be the character of $\mathbf{Z}/2$ taking the non-trivial element of $\mathbf{Z}/2$ to

$$\epsilon(\phi_i \otimes \chi, \psi_K) \cdot \det(\phi_i \otimes \chi)(\delta^{-1}).$$

Remark 9.2.1: From the definition of χ_ϕ given above, it is easy to see that

$$\chi_\phi(-\mathbf{1}) = \epsilon(\phi_K \otimes \chi, \psi_K) \cdot \det(\phi_K \otimes \chi)(\delta^{-1}),$$

where $-\mathbf{1}$ is the element of S_ϕ represented by the negative of the identity matrix in $GL(n, \mathbf{C})$.

Conjecture 1: If an irreducible admissible representation $\pi_V \otimes \pi_W$ of $U(V) \times U(W)$ appears as a quotient in the Weil representation of $\mathrm{Sp}(V \otimes_K W)$ restricted to $U(V) \times U(W)$, then

(i) The Langlands parameters associated to π_V and π_W are the same; call it ϕ.

(ii) The theta correspondence between $U(V)$ and $U(W)$ as V and W vary over the isomorphism classes of Hermitian spaces of dimension n define a bijection between the irreducible admissible representations of pure inner forms of $U(V)$ belonging to one Vogan L-packet to itself.

(iii) The characters χ_V and χ_W associated to the representations π_V and π_W are related by

$$\chi_V = \chi_W \cdot \chi_\phi$$

where χ_ϕ is the character of the component group $\mathcal{S}_\phi$ defined earlier.

Remark 9.2.2: If for an irreducible admissible representation π of $U(V)$, the theta lift of π to $U(W)$ is non-zero for a Hermitian space W with $\dim V = \dim W$, then from the above conjecture together with Remark 9.2.1 above, and equation 9.0.1, the discriminants of V and W are related by

$$\omega_{K/k}(\mathrm{disc}V) = \epsilon(\phi_K \otimes \chi, \psi_K) \det(\phi_K \otimes \chi)(\delta^{-1})\omega_{K/k}(\mathrm{disc}W),$$

where ϕ_K is the parameter of π base changed to K.

Remark 9.2.3: One can formulate a conjecture analogous to Conjecture 1 in the case when $\dim V$ and $\dim W$ differ by 1. Since this will entail introduction of alot more notation, we dont do it here, except to say that if $\dim W = \dim V + 1$, then for an irreducible admissible representation π on $U(V)$, with Vogan parameter (ϕ, μ), the character on the group of connected component of the parameter ϕ' associated to the theta lift of π to $U(W)$ is the various ways in which $\mu\chi_\phi$ can be extended to $\mathcal{S}_{\phi'}$ under the natural map from $\mathcal{S}_\phi$ to $\mathcal{S}_{\phi'}$.

9.3 Theta lifting for unitary groups: The Archimedean case

We verify conjecture 1 for the discrete series representations of the unitary group $U(p,q)$ here. The discrete series representations of $U(p,q)$ are parametrized by $p+q$ tuples of distinct numbers $(\lambda_1, \lambda_2, \cdots, \lambda_{p+q})$ where all the λ_i are integers if $p+q$ is odd, and all the λ_i are half-integers (i.e., belong to $\frac{1}{2}\mathbf{Z}$ but not to $\mathbf{Z}$) if n is even. We can further assume that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p, \quad \text{and}$$
$$\lambda_{p+1} > \lambda_{p+2} > \cdots > \lambda_{p+q}.$$

Assume that

$$\lambda_1 > \lambda_2 > \cdots \lambda_a > 0 > \lambda_{a+1} > \cdots > \lambda_p,$$

$$\lambda_{p+1} > \cdots > \lambda_{p+b} > 0 > \lambda_{p+b+1} > \cdots \lambda_{p+q},$$

then it follows from the results of J.-S. Li [Li1] that the local theta lift of π_λ to $U(r,s)$ for $r+s=p+q$, is non-zero if and only if

$$r = a - b + q,$$

$$s = b - a + p,$$

and in this case the local theta lift is again a discrete series whose Harish-Chandra parameter is

$$(\lambda_1, \cdots, \lambda_a, \lambda_{p+b+1}, \cdots, \lambda_{p+q}, \lambda_{p+1}, \cdots, \lambda_{p+b}, \lambda_{a+1}, \cdots, \lambda_p).$$

Actually as Li is concerned with local theta lifts of discrete series representations of $U(p,q)$ to general unitary groups, he expresses his results in results in terms of Vogan-Zuckermann $A_q(\lambda)$ which even in the case of same size unitary groups that we are considering, seem much more complicated because of various shifts involved.

Given this result of J.-S. Li, conjecture 1 can now be easily checked for discrete series representations of $U(p,q)$, but before we can do that, we need to review the Vogan parametrization of representations of real reductive groups.

Let G be a connected real reductive group, and ${}^LG$ its L-group which is a semi-direct product of its connected component $G^\vee$ by the Weil group $W_{\mathbf{R}}$ of $\mathbf{R}$. Let $\phi : W_{\mathbf{R}} \to {}^LG$ be a discrete series parameter. The centraliser $\mathcal{S}_\phi$ of ϕ in $G^\vee$ is a finite abelian group isomorphic to $(\mathbf{Z}/2)^r$ where r is the rank of G over $\mathbf{C}$. The pure inner forms of $G = U(p,q)$ are $U(p+q,0), U(p+q-1,1), \cdots, U(0,p+q)$. We will fix a quasi split pure inner form of G together with a non-degenerate character on the unipotent radical of a Borel subgroup. According to Vogan, given this data, every character of $(\mathbf{Z}/2)^{p+q}$ determines in a bijective manner, a pure inner form of G and a discrete series representation on it with parameter ϕ. Starting with a pure inner form and a discrete series representation on it, one constructs the character of $(\mathbf{Z}/2)^{p+q}$ as follows. Let T be a compact Cartan subgroup of G, and $\chi : T \to S^1$ a regular character, i.e., $< \chi, \alpha > \neq 0$ for every root α of G. This defines a system of positive roots by declaring a root to be positive if and only if $< \chi, \alpha >> 0$. Define a root to be compact if the corresponding root space is contained in the unique maximal compact subgroup of G containing T, and non-compact otherwise.

The torus T in G with associated +ve root system gives rise to a torus $T^\vee$, and a Borel subgroup $B^\vee$ in $G^\vee$, and for each root α of T, we have the coroot $\alpha^\vee : \mathbf{G}_m \to T^\vee$. A coroot $\alpha^\vee$ is called compact if and only if the root α is compact. Now given a discrete series parameter $\phi : W_{\mathbf{R}} \to {}^L G$, we can assume that its restriction $\phi_{\mathbf{C}^*}$ to $\mathbf{C}^*$ lands inside $T^\vee$. The discrete series representations with Harish-Chandra parameter $\chi : T \to S^1$, has $\phi_{\mathbf{C}^*}$ given by $\phi_{\mathbf{C}^*}(z) = (z/\bar{z})^\chi$ where χ belongs to $X^*(T) \cong X_*(T^\vee) \otimes \mathbf{C}$.

The centraliser of $\phi : W_{\mathbf{R}} \to {}^L G$ in $G^\vee$ is precisely the elements of order ≤ 2, $T^\vee[2]$, in $T^\vee(\mathbf{C})$. For each positive simple root α, we have $\alpha^\vee : \mathbf{G}_m \to T^\vee$. Let $a_\alpha = \alpha^\vee(-1) \in T^\vee[2]$. Corresponding to the discrete series on G with Harish-Chandra parameter χ, define the character $\chi : T^\vee[2] \to \mathbf{C}^*$ by demanding $\chi(a_\alpha) = -1$ if α is a compact simple root. In the case of unitary groups, the a_α's generate a subgroup of index 2 of $T^\vee[2]$, so one needs one bit more of information to fix χ (which distinguishes $U(p,q)$ from $U(q,p)$), but which we dont describe here.

We will not verify conjecture 1 for general discrete series representations, but we will do it in a particular case; the general case is very similar. We will take the discrete series representation on $U(p,q)$ with Harish-Chandra parameter

$$\Lambda = \lambda_1 > \cdots > \lambda_p > \lambda_{p+1} > \cdots > \lambda_{p+a} > 0 > \lambda_{p+a+1} > \cdots > \lambda_{p+q}.$$

According to Vogan, this representation of $U(p,q)$ defines a character χ_A on the component group $(\mathbf{Z}/2)^{p+q}$ which for the standard basis $\{e_i\}$ of $(\mathbf{Z}/2)^{p+q}$ has the property that:

$$\chi_A(e_i e_{i+1}) = -1 \qquad \text{for} \quad i = 1, ..., p+q-1, i \neq p.$$

By the result of J.-S. Li recalled above, the lift of this discrete series representation on $U(p,q)$ is non-zero for $U(p+q-a, a)$, and the Harish-Chandra parameter for the theta lift is:

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p > \lambda_{p+a+1} > \cdots > \lambda_{p+q},$$
$$\lambda_{p+1} > \lambda_{p+2} > \cdots > \lambda_{p+a}.$$

The corresponding character χ_B on the component group $(\mathbf{Z}/2)^{p+q}$ has the property that:

$$\chi_B(e_i e_{i+1}) = -1, \qquad \text{for} \quad i = 1, ..., p+q-1, i \neq p, i \neq p+a.$$

For $m \in \frac{1}{2}\mathbf{Z}$, let χ_m be the character of $\mathbf{C}^*$ given by $z \to (z/\bar{z})^m$, or $re^{i\theta} \to e^{2im\theta}$. We have [Ta,3.2.5]

$$\epsilon(\chi_m) = i^{-2m}, \quad \text{if} \quad m > 0, \quad \text{and}$$

$$\epsilon(\chi_m) = i^{2m}, \quad \text{if} \quad m < 0.$$

It follows that for $m \in \frac{1}{2}\mathbf{Z} - \mathbf{Z}$,

$$\chi_m(i)^{-1}\epsilon(\chi_m) = -1 \quad \text{if} \quad m > 0$$

and

$$\chi_m(i)^{-1}\epsilon(\chi_m) = 1 \quad \text{if} \quad m < 0.$$

If n is odd, then the character χ fixed at the beginning of this section (to split $\mathrm{Sp}(V \otimes_K W)$ over $U(V) \times U(W)$) can be taken to be $\chi_{\frac{1}{2}}$, and if n is even, then as always, χ is taken to be the trivial character. It follows that for any value of n, $\Lambda \otimes \chi$ is a tuple consisting of elements in $\frac{1}{2}\mathbf{Z} - \mathbf{Z}$. Therefore the character χ_Λ associated to the parameter Λ is

$$\chi_\Lambda(e_i) = 1, \quad \text{if} \quad i \leq p + a,$$

$$\chi_\Lambda(e_i) = -1, \quad \text{if} \quad i > p + a.$$

Clealy, $\chi_B = \chi_A \cdot \chi_\Lambda$ on the index 2 subgroup of $(\mathbf{Z}/2)^{p+q}$ generated by $e_i e_{i+1}$. If n is odd, then $-I = (1, \cdots, 1) \in (\mathbf{Z}/2)^{p+q}$ gives a complement to this index 2 subgroup. Since χ_A corresponds to a representation of $U(p,q)$, and χ_B to a representation of $U(p+q-a,a)$, from 9.0.1, $\chi_B(-I) = \chi_A(-I)(-1)^{q+a}$, verifying $\chi_B = \chi_A\chi_\Lambda$. If n is even, our result has this ambiguity by 2.

9.4 Global theta correspondence for unitary groups

Conjecture 2: Let V be an n-dimensional Hermitian space over a quadratic extension K of a number field k, and let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $U(V)$. Then,

(i) There exists an n-dimensional Hermitian space W over K such that the local theta lift is non-zero at each place of k if and only if the global epsilon factor $\epsilon(BC(\pi) \otimes \chi, \psi_K) = 1$, where $BC(\pi)$ denotes the base change of π to $GL(n,K)$.

(ii) Suppose that W is an n-dimensional Hermitian space over K such that for each place v of k, the theta lift, call it $\theta(\pi_v)$, of π_v is non-zero. Then the global representation $\otimes_v \theta(\pi_v)$ is a cuspidal automorphic representation of $U(W)$.

(iii) If W is as in (ii), then the global theta lift of π to $U(W)$ is non-zero if and only if $L(BC(\pi) \otimes \chi, \frac{1}{2}) \neq 0$.

Remark 9.4.1: Part (i) of the above conjecture follows from the local conjecture, specially the remark 9.2.2 following from Conjecture 1. When the number field K splits at a place v of k, then the corresponding local unitary group is just $GL(n, k_v)$, and we are granting ourselves that the duality correspondence for the pair $(GL(n, k_v), GL(n, k_v))$ is just the identity map. One also needs to use the classification theorem of Hermitian forms over a global field due to Landherr (or, the Hasse principle in modern language).

Remark 9.4.2: For $n = 1$, part (i) of the above conjecture is a theorem due to Rogawski, cf [Ro]; part (ii) is a tautology; part (iii) is due to Gelbart and Rogawski [GR, cor.5.2.2]. For $n = 2$, part (iii) of the above conjecture is proved by M. Harris [Ha, Thm. 4.5] in most cases, but part (ii) seems open. Since the unitary group in 2 variables is closely connected to $GL(2)$, part (ii) of the above conjecture should be a consequence of Jacquet-Langlands correspondence together with the multiplicity formula of Labesse-Langlands, but which this author has not verified.

Remark 9.4.3: The reader will of course not have failed to notice the analogy of conjecture 2 to the theorem of Waldspurger recalled in section 7. We would like to point out that in Waldspurger's theorem, there was no local condition as the local theta lift from $PGL(2)$ to $\widetilde{SL}(2)$ is always non-zero.

Remark 9.4.4: In the general theory of automorphic forms, there is a conjecture due to J.Arthur [Ar] building on the work of Labesse-Langlands in the context of $SL(2)$, which answers when a representation of the adele group which is associated to a parameter ϕ is automorphic. This criterion depends on, again conjectural, pairing

$$<,>: \quad \mathcal{S}_\phi \times \Pi \to \mathbf{Z}/2,$$

where $\mathcal{S}_\phi$ is the group of connected components of a global parameter ϕ, and Π is the set of representations of the adele group $G(\mathbf{A})$ whose all the local components belong to the L-packet determined by ϕ (which for simplicity we take to be a tempered parameter). For each representation τ in Π, $\chi_\tau(s) =< s, \tau >$ defines a character on $\mathcal{S}_\phi$. Suppose that π is a cuspidal automorphic representation belonging to Π which has

a Whittaker model. In such a circumstance, by Vogan there are local pairings

$$<,>_v: \quad \mathcal{S}_{\phi_v} \times \Pi_v \to \mathbf{Z}/2.$$

It seems natural to expect that these local pairings defined by Vogan give rise to the global pairing in the sense that for $\tau = \otimes_v \tau_v$, a representation of $G(\mathbf{A})$, $< s, \tau >= \prod_v < s_v, \tau_v >_v$ for $s \in \mathcal{S}_\phi$ and s_v its image in $\mathcal{S}_{\phi_v}$. Now the conjecture of Arthur is that a global representation τ of $G(\mathbf{A})$ is automorphic if and only if the character χ_τ is trivial. Now we will show how this conjecture implies part (ii) of conjecture 2. So, we start with a tempered, cuspidal automorphic representation $\pi_V = \otimes_v \pi_v$ of $U(V)$, such that $\pi_W = \otimes_v \theta(\pi_v)$ is a non-zero representation of $U(W)(\mathbf{A})$. By Arthur, we are given that the character χ_{π_V} is the trivial character, and we need to verify that χ_{π_W} is also the trivial character. Since the global component group $\mathcal{S}_\phi$ is $\mathbf{Z}/2$ generated by $\pm$Id in $GL(n, \mathbf{C})$, from Remark 9.2.1, this will be so if and only if $\epsilon(BC(\pi) \otimes \chi, \psi_K) = 1$. But since π_V has a non-zero theta lift to $U(W)$, by part (i) of the above conjecture, $\epsilon(BC(\pi) \otimes \chi, \psi_K)$ is indeed 1.

10 The Siegel-Weil Formula

The aim of this section is to describe the Siegel-Weil formula which is at the heart of many applications of theta correspondence. It relates the integral of a theta function to an Eisenstein series. The simplest case of the Siegel-Weil formula is the identity

$$\sum \frac{\theta_Q(z)}{\sharp(\mathrm{Aut}Q)} = E_n(z)$$

where Q is the set of integral positive definite even unimodular quadratic forms in $2n$ variables, and $E_n(Z)$ is the Eisenstein series of weight n for $SL_2(\mathbf{Z})$.

We will be working with a number field F in this section. Let $G = \mathrm{Sp}(W)$ be the symplectic group of rank n, and $H = O(V)$ be the orthogonal group of a quadratic space V over F which we assume has even dimension $= 2m$. Then G and H form a dual reductive pair inside $\mathrm{Sp}(V \otimes W)$. Since the dimension of V is assumed to be even, the metaplectic cover of $\mathrm{Sp}(V \otimes W)$ splits over $G \times H$ both locally and globally. Weil representation of $\mathrm{Sp}(V \otimes W)$ therefore gives rise to a representation ω of $G(\mathbf{A}) \times H(\mathbf{A})$ on the Schwartz space $\mathcal{S}(V(\mathbf{A})^n)$.

For $\phi \in \mathcal{S}(V(\mathbf{A})^n)$, define the theta function as usual by

$$\theta(g,h;\phi) = \sum_{x \in V(F)^n} \omega(g)\phi(h^{-1}x).$$

It is a theorem of Weil [We2] that the integral

$$I(g,\phi) = \int_{H(F)\backslash H(\mathbf{A})} \theta(g,h;\phi)dh,$$

converges if either V is an-isotropic (i.e., V does not represent any zero over F), or if r is the dimension of the maximal isotropic subspace of V, then $m > n + r + 1$.

For $s \in \mathbf{C}$, let $\Phi(g,s)$ be the function on $G(\mathbf{A})$ defined by

$$\Phi(g,s) = (\omega(g)\phi)(0)|a(g)|^{s-s_0}.$$

Here if $g = pk$ where $p \in P(\mathbf{A})$ for P the Siegel parabolic with $GL(n)$ as its Levi subgroup embedded as $\begin{pmatrix} X & 0 \\ 0 & {}^tX^{-1} \end{pmatrix}$ inside $\mathrm{Sp}(2n)$, and $k \in K = \prod_v K_v$ a standard maximal compact subgroup of $\mathrm{Sp}(n)(\mathbf{A})$, then $a(g) = \det X$.

Let $(.,.)$ denote the Hilbert symbol of the number field F. For the quadratic space V with discriminant $d(V)$, let χ_V denote the character of the idele class group of F defined by $\chi_V(x) = (x, (-1)^m d(V))$. Let $I_n(s,\chi_V)$ denote the principal series representation of $\mathrm{Sp}(n)(\mathbf{A})$ induced from the character $g \to |a(g)|^s \chi_V(a(g))$ of the Siegel parabolic subgroup. It is easy to see that the map sending ϕ to $\Phi(g,s_0)$ defines a $G(\mathbf{A})$ intertwining map from $\mathcal{S}(V(\mathbf{A})^n)$ to $I_n(s_0,\chi_V)$. Define now the Eisenstein series by

$$E(g,s,\Phi) = \sum_{\gamma \in P(k)\backslash G(k)} \Phi(\gamma g, s).$$

The above series is absolutely convergent for $Re(s) > \frac{n+1}{2}$, and has analytic continuation to all of complex plane; it is known to be holomorphic at $s_0 = \frac{m-n-1}{2}$.

The identity expressed in the following theorem is called the Siegel-Weil formula.

Theorem 12 *Assume that either V is an-isotropic over F, or if r is the dimension of the maximal isotropic subspace of V, then $m >$*

$n+r+1$ so that the integral defining $I(g,\phi)$ is absolutely convergent. Let $s_0 = \frac{m-n-1}{2}$, then with the notation as above, we have

$$E(g, s_0, \Phi) = \kappa I(g, \phi)$$

where $\kappa = 2$ if $m \leq n+1$, and 1 otherwise.

This theorem was proved by Weil in [We2] when $m > 2n+2$, i.e., the case in which the Eisenstein series involved converged absolutely; the general case of the Siegel-Weil formula, including the holomorphicity of the Eisenstein series at s_0 is due to Kudla and Rallis [KR1], [KR2]. Weil proved it for general dual reductive pairs in [We2], and not just for the $(O(n), \mathrm{Sp}(m))$ case considered here. The extension of this formula to other dual reductive pairs where the Eisenstein series does not converge absolutely is not yet complete.

Remark : The local case of the map $\phi \to \Phi$ defined by

$$\Phi(g, s_0) = (\omega(g)\phi)(0),$$

is also very important. By a theorem of Rallis [Ra1], this induces an injection from $\mathcal{S}(V^n)_{O(V)}$ into $I_n(s_0, \chi_V)$. (Where for a representation X of a group E, X_E denotes the maximal quotient of X on which E acts trivially.) The method of Weil representation gives an important method for the study of the Jordan-Hölder series of $I_n(s_0, \chi_V)$. For instance, Kudla and Rallis [KR3] prove that if $m \leq n+1$ then the image $R_n(V)$ of $\mathcal{S}(V^n)$ in $I_n(s_0, \chi_V)$ is irreducible, and if V_1 and V_2 are the two distinct quadratic spaces with the same discriminant (if $m = 2$, and the discriminant $= -1$, then there is only one quadratic space), then $R_n(V_1)$ and $R_n(V_2)$ are distinct irreducible submodules of $I_n(s_0, \chi)$ ($\chi = \chi_{V_1} = \chi_{V_2}$) such that $I_n(s, \chi)/[R_n(V_1) + R_n(V_2)]$ is also irreducible, and non-zero if and only if $m < n+1$. If $n+1 < m \leq 2n$ (if $\chi = 1$, we exclude the value $m = 2n$; see [KR3] for this case), then $R_n(V_1)$ and $R_n(V_2)$ are maximal submodules of $I_n(s_0, \chi)$, and $R_n(V_1) \cap R_n(V_2)$ is irreducible. These results on Jordan-Holder series in turn find important application to theta liftings as in the work of [HKS] in the unitary case.

11 Application of theta correspondence to cohomology of Shimura variety

A Shimura variety is a topological space of the form $\Gamma\backslash G/K$ where G is a real Lie group, K a maximal compact subgroup of G, and Γ an arithmetic subgroup of G (An example : $SL_2(\mathbb{Z})\backslash SL(2,\mathbb{R})/SO(2)$).

According to a theorem of Matsushima, if $\Gamma\backslash G/K$ is compact, then

$$H^i(\Gamma\backslash G/K, \mathbb{C}) = \oplus_\pi H^i(\mathcal{G}, K, \pi)^{m(\pi)}$$

where $L^2(G\backslash\Gamma) = \sum m(\pi)\pi$.

Therefore the calculation of the cohomology of a Shimura variety is equivalent to finding those automorphic representations whose component at infinity have non-trivial $(\mathcal{G}, K)$ cohomology.

The construction of cohomological automorphic representations of G depends on realising G as a member of a dual reductive pair (G, G') with G' small, and lifting automorphic representation of G' which are discrete series at infinity. Cuspidal automorphic representations of a group with prescribed discrete series component at infinity are known to exist in plenty according to a theorem of Savin, and then the problem splits into two parts:

(i) Realisation of cohomological representations as theta lifts at infinity.

(ii) Determination of the condition under which the global theta lift is non-zero.

For (i), one knows by Vogan and Zuckermann [V-W] an explicit description of cohomological representations. The theta lifts are calculated via the method of Jacquet functors, [Li1].

For (ii), observe that $\theta_\phi(f) \neq 0$ if and only if the inner product $< \theta_\phi(f), \theta_\phi(f) > \neq 0$. There is a formula for this inner product by Rallis [Ra3] involving special values of L-functions. Therefore we know it to be non-zero when the special value is at a point of absolute convergence. Actually, there are finitely many "bad" factors which might contribute a zero and which one needs to deal with also; see [Li2], and [KR4] in a greater generality. The formula of Rallis generalises Waldspurger's result mentioned before, and is proved by expressing an integral of theta functions by an Eisenstein series by the Siegel-Weil formula, and then unfolding the integral.

Non-vanishing theorems about cohomology of Shimura variety via the method of theta correspondence were first proved by Kazhdan for

$SU(n,1)$, and the most recent results are due to J.-S. Li. Here is an example from the work of Li ([Li2], corollary 1.3).

Theorem 13 *Let $G = SO(p,q)$, Γ an arithmetic subgroup of G constructed by a skew-Hermitian form over a quaternion division algebra over a totally real field such that D is split at all but one real place. Then for $p \geq q$, $p+q = 2n \geq 8$, and for Γ deep enough, the qth Betti number of Γ is non-zero.*

12 Recent generalisations of theta correspondence

There has been much activity recently in constructing analogues of the Weil representation, and understanding the associated theta correspondence.

The characteristic property of the Weil representation taken for the purposes of this generalisation is the fact that the Weil representation is a rather small representation, in fact the smallest representation after the trivial representation. We will make this concept precise, but one can get some idea about it from the dimension formula for the Weil representation over finite field $\mathbf{F}_q$. The dimension of the Weil representation of $\mathrm{Sp}(2n, \mathbf{F}_q)$ is q^n, and it splits into two irreducible representation of dimension $\frac{q^n+1}{2}$ and $\frac{q^n-1}{2}$, whereas the dimension of a generic representation of $\mathrm{Sp}(2n, \mathbf{F}_q)$ is of the order of q^{n^2}.

We now make precise the concept of smallness of a representation. For an irreducible representation π of G, define a distribution Θ_π on the Lie algebra $\mathcal{G}$ of G as follows :

$$\Theta_\pi(f) = \mathrm{tr}\left(\int_{\mathcal{G}} f(X)\pi(expX)dX\right)$$

Theorem 14 (Harish-Chandra) *For each nilpotent orbit θ, there exists a complex number C_θ such that for all functions f with small support around the origin of $\mathcal{G}$,*

$$\Theta_\pi(f) = \sum C_\theta \int \hat{f}\mu_\theta,$$

the summation is over the (finite) set of nilpotent orbits, and where μ_θ is a G-invariant measure on the nilpotent orbit θ suitably normalised.

Definition:

(1) Wave front set $WF(\pi) = \cup_{C_\theta \neq 0} \bar{\theta}$.

(2) Gelfand-Kirillov dimension of π, denoted $\dim d(\pi)$ is $\max_{C_\theta \neq 0} \frac{1}{2} \dim \theta$.

Proposition 1 *Let $G(n)$ be the principal congruence subgroup of G of level n. Then for an irreducible admissible representation V of G,*

$$\lim_{n \to \infty} \frac{\dim V^{G(n)}}{q^{nd(\pi)}} = 1.$$

Definition : A representation π is called minimal if $WF(\pi) = \bar{\theta}_{min}$ where θ_{min} is a non-trivial minimal nilpotent orbit. (Such a nilpotent orbit is unique over the algebraic closure.)

Minimal representations have been constructed by many people starting with the work of Kazhdan, by Kazhdan, Savin, Gross-Wallach among others. Here is an example. Consider the Satake parameter corresponding to:

$$F^* \xrightarrow{\psi} SL(2, \mathbb{C}) \xrightarrow{\phi_0} {}^L G$$

where $\psi : x \to \begin{pmatrix} |x|^{\frac{1}{2}} & 0 \\ 0 & |x|^{-\frac{1}{2}} \end{pmatrix}$, and ϕ_0 corresponds to the subregular unipotent orbit. The following theorem is due to G. Savin [Sa].

Theorem 15 *Let G be a simple, simply connected, simply laced split group over a local field F. Then the spherical representation π with the above as Satake parameter is minimal.*

One can classify dual reductive pairs for general groups, cf. [Ru], and in each case when minimal representations are constructed, one would like to prove if the analogue of the Howe duality conjecture is true. Next one would like to prove that the local minimal representations are part of global automorphic representations, and if so, this gives a construction of global automorphic representation on one member of a dual reductive pair in terms of the other just as in the case of the Weil representation. This is a very active area of research at the moment. Here is an example from the work of Gross and Savin[G-S].

The pair $(G_2, PG\mathrm{Sp}(6))$ is a dual reductive pair in E_7. There exists a form of G_2 over $\mathbf{Q}$ which is compact at infinity, and split at all the finite places. The pair $(G_2, PG\mathrm{Sp}(6))$ sits inside a form of E_7

which is of rank 3 at ∞ and split at all finite places. There is an automorphic form on E_7 constructed by H. Kim [Ki] which is minimal at all places. The theta lifts from automorphic form on G_2 to $PG\mathrm{Sp}(6)$ using this is expected to correspond to the mapping of L-groups given by $G_2(\mathbb{C}) \to \mathrm{Spin7}(\mathbb{C})$, as proved in [G-S] for many representations. The interest of this work of Gross and Savin is that they are able to lift automorphic forms from G_2 to $PG\mathrm{Sp}(6)$ using this "exceptional" theta correspondence, and the automorphic form they get on $PG\mathrm{Sp}(6)$ is of holomorphic kind, and so presumably has a motive associated to it, whose Galois group is G_2 (and not smaller as they take care to lift only those which have a Steinberg at a finite place).

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