On the Minimal Number of Closed Characteristics on Hypersurfaces Diffeomorphic to a Sphere

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Abstract

It is proved that each contact type hypersurface $\Sigma \subset \mathbb{R}^{2n}$ diffeomorphic to a sphere carries at least n closed characteristics. This generalizes a theorem by Ekeland and Lasry on pinched convex hypersurfaces. The same result is proved for all hypersurfaces of $\mathbb{R}^{2n}$ carrying at least one closed characteristic, i.e. the existence of one implies the existence of n solutions.

1. Introduction

Periodic solutions for Hamiltonian systems have been the object of research since Poincaré, as they are the rare trajectories for which the usually difficult question of long time behaviour and stability can be answered. Moreover, it is well known, that closed geodesics are periodic solutions of fixed energy in the cotangent bundle of the underlying Riemannnian manifold. Both in quasiclassical approximations of quantum mechanics and quantisation of classical mechanics symplectic actions of periodic solutions are important geometrical and physical invariants. The recent discovery of symplectic capacities [EH89], [EH90] is a beautiful illustration of the actuality of the problem, in particular the question whether periodic solutions exist on a given energy hypersurface. The periodic solutions of given period and variable energy, including multiple coverings of such, play a central role in Floer homology of compact symplectic manifolds, in a similar way as critical points of functions for Morse homology. We call *prime* loops the underlying one fold covering of a trajectory, see e.g. [Kl78].

The aim of the paper is to prove the following theorem in the standard symplectic vector space.

Theorem 1: *Let Σ be a closed hypersurface in $(\mathbb{R}^{2n},\omega)$ of class C^2 diffeomorphic to the sphere S^{2n-1}, which carries at least one closed characteristic with positive symplectic area. Then there are at least n geometrically different prime closed characteristics.*

Using Viterbo's theorem for contact type hypersurfaces [V87], on gets:

Theorem 2: *Let Σ be a closed hypersurface in $(\mathbb{R}^{2n},\omega)$ of class C^2 diffeomorphic to the sphere S^{2n-1} of restricted contact type. Then there are at least n geometrically different prime closed characteristics.*

It is worthwile to mention here that the case of hypersurfaces diffeomeorphic to spheres is the case with probably the least number of closed characteristics, due to the idea that hypersurfaces with nontrivial first homotopy group tend to have more solutions, conjecturally at least one for each homotopy class. Moreover, it is well known that generic ellipsoids (i.e. uncoupled nonresonant harmonic oscillators) have exactly n 'modes', as prime periodic solutions are called by physicists. So the result is a statement about the *minimal* number of closed characteristics on hypersurfaces diffeomorphic to S^{2n-1}.

As convex hypersurfaces are of contact type, theorem 1 implies a generalisation of a theorem by Ekeland and Lasry [EL80], who had to assume a pinching condition for the hypersurface $\Sigma = \partial K$: $B(r) \subset K \subset B(R)$, for $R < \sqrt{2}r$, see also [BLMR]. The first result in this direction was proved by Weinstein [W73] and Moser [M76] near equilibra, which has been generalised by Bartsch [B94] just these days.

The question whether periodic solutions exist on *given* energy levels bears several difficulties, see e.g. the monograph [E90], chapter V, where for instance theorem 2 is conjectured for the convex case. One difficulty is due to the fact that the usual variational methods have to be formulated in a space of parametrised loops with a given fixed period, but solutions with fixed energy and fixed period generally do not exist. An other difficulty of the usual approach is that all prime and iterated loops (coverings of the underlying prime loop) are arbitraily close to each other. So one needs a device to separate the prime solutions from the others.

Since Rabinowitz' break through with variational methods in the subject [R78], one often considers a homogenous Hamiltonian having energy levels homothetical to the hypersurface Σ. Then a critical point of the Hamiltonian action functional is a solution which is homothetical to a solution on Σ. This approach has the disadvantage that a multiplicity of solutions on different levels can be homothetical and therefore it is difficult to count the actual number of solutions on Σ.

Ekeland and Lasry used convex homogenous Hamilton functions and the *dual* Hamiltonian action functional, which has the advantage that it is bounded from below and that there is a level of the functional below which all points belong to non iterated loops. In this sublevel set, where the S^1- action is free, it is possible to find an equivariant homeomorphism of the sphere S^{2n-1}, which means that its Borel homology is the one of $\mathbb{C}P^n$, so that one has at least n critical points, see [E90], section V.2., and [B93] for more about the topological methods. One can also show, that after the homothecy, they belong to different prime closed orbits on Σ if the above pinching condition holds. The beautiful dual approach is restricted to convex hypersurfaces, a notion which is not invariant by symplectic maps.

Our approach uses the following ideas. Instead of a space of loops in the euclidean space, we consider the loops space $H^2(S^1, \Sigma)$ of the hypersurface. Therefore, one does not need any growth conditions of Hamitonian functions, more precisely, Hamiltonians are not needed at all; see also [K90] and [K91] for a similar approach to symplectic capacities. We study the characteristic equation which is invariant by the group Diff(S^1) of reparametrisations and a H^2- gradient flow leaving the hypersurface invariant. This is equivalent to study a functional with constraints, where the Lagrange multiplier λ of a critical point x is the euclidean norm $|\dot{x}(t)|$ of the velocitiy of x.

Moreover, we consider 'comparing homeomorphisms' between S^{2n-1} and Σ, which transport the fibres of the Hopf fibration $S^{2n-1} \to \mathbb{C}P^n$ to Σ. To construct a flow invariant set of loops, the deformations of these homeomorphisms by the flow are considered. The notion of δ-regular k-maximal descending cylinders is introduced as a kind of connecting orbits, in order to rule out homeomorphisms which would collapse within infinite time. Like this, we can imitate the well known Courant-Minmax-principle, with which one finds eigenvalues of linear operators by 'fishing' with subspaces of consecutive dimensions.

Let us fix some notations. Consider the standard symplectic vector space $(\mathbb{R}^{2n}, J, .)$, where $J_z : T_z\mathbb{R}^{2n} \to T_z\mathbb{R}^{2n}$ is the standard integrable almost complex structure, $J_z^2 = -Id, J_z \simeq J$, and "." denotes the scalar product. We use the usual identification $\mathbb{C}^n \to (\mathbb{R}^{2n}, J)$.

Then $\omega(v,w) = \frac{1}{2}Jv.w$ defines the standard exact symplectic form, whose integral θ evaluated over loops in $\mathbb{R}^{2n}$ is the symplectic action functional

$$A : H^2(S^1, \mathbb{R}^{2n}) \to \mathbb{R}, \quad x \mapsto A(x) = \frac{1}{2}\int \big(- J\dot{x}.x\big)_{\mathbb{R}^{2n}}\, dt = \frac{1}{2}\big\langle ax, x\big\rangle_{L^2}$$

measuring the symplectic area of any disc bounded by the trajectory Imx.

Here is a the operator 'conjugate to time'

$$H^2([0,1]) \to H^1([0,1]) \hookrightarrow L^2, \quad x \mapsto -J\frac{dx}{dt} = -J\dot{x}.$$

The kernel of $\omega|_\Sigma$ at any point is one dimensional, since ω is nondegenerate. The following differential inclusion is called 'characteristic equation'of Σ

$$\dot{\gamma}(t) \in \ker \omega|_\Sigma, \tag{C}$$

and its solutions are the 'characteristics'of Σ.

Of course, equation (C) does not fix the parametrisation nor the orientation of the trajectories, which is the main difference to a geometrically equivalent Hamiltonian system. Two parametrisations γ and $\tilde{\gamma}$ are called geometrically different if their images Imγ and Im$\tilde{\gamma}$ are different, i.e. if there is no reparametrisation s such that $\gamma = \tilde{\gamma} \circ s$ or $\tilde{\gamma} = \gamma \circ s$. Specialising now to closed characteristics parametrised by the unit interval, we denote $\mathcal{R} := \{s : [0,1] \overset{H^2}{\to} [0,1] \mid s(0) = 0, s(1) = 1\} \supset \mathrm{Diff}(S^1)$ the set of H^2- reparametrisations of closed curves. Let us emphasize that we do not assume s to be monotone, but that we do assume that they are functions, not merely relations. This means that $\mathcal{R}$ is no group. It is well known, that the symplectic action A is invariant by $\mathcal{R}$, whereas the Hamiltonian action $A(x) - \int G(x(t))dt$ is only S^1- invariant.

Consider the Hopf circles $\mathcal{C} = \{c_\xi : \mathbb{R}/\mathbb{Z} \to S^{2n-1} \mid c_\xi(t) = e^{2\pi J t}\xi, \xi \in S^{2n-1}\}$ solving the initial value problem $\dot{c}_\xi(t) = 2\pi J c_\xi(t),\ c_\xi(0) = \xi$. Fis any point $\xi_0 \in S^{2n-1}$.

Lemma 1: *Assume Σ is of class $C^\ell, \ell \geq 0$. Then given any embedded loop γ of class C^ℓ, parametrised with period 1, there exists a homeomorphism $h : S^{2n-1} \to \Sigma$ of class C^ℓ such that $h(c_{\xi_0}) = \gamma$.*

The proof is standard, see e.g. [H76].

We assumed from now on that the number of prime closed characteristics with positive symplectic action is $N < \infty$; we have to show $N \geq n$. Recall that Viterbo's theorem means that $N \geq 1$ for contact type Σ. Now apply Lemma 1 to these N curves $\gamma_i, i = 1, \ldots, N$, which we assume to be parametrised by arc length, say. The difference quotient of the corresponding diffeomorphism, $\ell = 2$, is bounded from below and above by a constant $\delta_i > 0$ and $\frac{1}{\delta_i} < \infty$ respectively. Set $\delta := \frac{1}{2}\min_i \delta_i > 0$ to be fixed in the sequel.

Let us define $\mathcal{H}_\delta = \{h \in C^2(S^{2n-1}, \Sigma) \mid \frac{1}{\delta}|\xi - \xi'| \geq |h(\xi) - h(\xi')| \geq \delta|\xi - \xi'|\ \forall \xi, \xi' \in S^{2n-1}\}$ and $L_\delta = \{\gamma \in C^2(S^1, \Sigma) \mid |\gamma(t) - \gamma(t')|_{\mathbb{R}^{2n}} \geq \delta \mid q^t - q^{t'} \mid_{\mathbb{C}}$, where $q = e^{2\pi i}$. It is evident that $h(c_\xi) \in L_\delta$ for all ξ and $h \in \mathcal{H}_\delta$.

Assume that the critical actions are arranged in increasing order $0 < a_1 \leq a_2 \leq \ldots \leq a_N \leq B := \max_{\xi \in S^{2n-1}} A(h_N(c_\xi)) + 1$ and repeated according to multiplicity.

Idea: Define n minmax-values σ_k of A similar to $\inf_{h \in \mathcal{H}_\delta} \sup_{\xi \in S^{2k-1}} A(h(c_\xi))$. Using an extended gradient like flow on a class of continuous maps $h : S^{2n-1} \to \Sigma$ with mapping degree 1, we show that each σ_k equals to one of the *prime* critical values $a_i = A(\gamma_i)$ of $A|_\Sigma$, Equality of the values σ_k is showed to be impossible if there are only finitely many solutions, therefore they belong to geometrically different loops.

Before we can give the precise definition of σ_k, see §4, we need to construct a gradient like flow on the loop space of the surface (§2) and to show that it yields a deformation lemma (§3). The proof of theorem 1 is completed in §5.

2. The gradient flow

Let the hypersurface $\Sigma = \{G(z) = 1\}$ be given by a function $G : \mathbb{R}^{2n} \to \mathbb{R}, G \in C^2$, such that $\{G(z) < 1\}$ is compact. Then $N(z) = \frac{G'(z)}{|G'(z)|}$ is the outward normal of Σ at the point z.

Note that $H^2(I, \mathbb{R}^{2n})$ is continuously and compactly embedded in $C^1(I, \mathbb{R}^{2n})$ for any interval I, so that $x \in H^2$ is a parametrisation of a differentiable trajectory $\mathrm{Im}x$. Therefore it makes sense to consider the space of H^2-parametrised loops of Σ, $H^2(S^1, \Sigma)$, which is a Hilbert manifold with tangent space $T_x H^2(S^1, \Sigma) = H^2(x^* T\Sigma)$, see e.g. [Kl78] for an analogous situation. For given $x \in H^2(S^1, \Sigma)$, ax and $N(x)$ are elements of $L^2(x^* T\mathbb{R}^{2n}$ and $H^2(x^* T\mathbb{R}^{2n})$ respectively.

The restriction of A to Σ has the L^2-gradient $-L$ with

$$L(x) = -ax + \big(ax.N(x)\big)N(x),$$

which is an element of $L^2(x^* T\Sigma) \subset L^2(x^* T\mathbb{R}^{2n})$. Its zeros, the solutions of the equation

$$ax = \big(ax.N(x)\big)N(x) \tag{$*$}$$

are parametrisations of the characteristics of the hypersurface Σ. We remark that *all* reparametrisations of a characteristic solve $(*)$.

In order to find a regularisation of $L(x)$, consider the extension $T_0 : H^2 \to (H^2)^{*_{L^2}} = H^{-2}$ of $x \mapsto (\frac{d}{dt})^4 x$ and the operator $T = Id + T_0$. The latter relates the H^2- to the L^2- scalar product by

$$\langle u, v\rangle_{H^2} = \langle Tu, v\rangle_{L^2} = \langle u, v\rangle_{L^2} + \langle \ddot{u}, \ddot{v}\rangle_{L^2}.$$

Fourier expansion with respect to the basis $b_{k,i}$ of L^2 given by $b_{k,i}(t) = \exp(2\pi Jkt)\ \varepsilon_i = c_{\varepsilon_i}(kt) =: c_{\varepsilon_i}^{(k)}(t)$, $k \in \mathbb{Z}$, for an orthonormal basis ε_i, $i = 1, \ldots, 2n$ in $\mathbb{R}^{2n}$ yields the equivalent formula for the scalar product in H^2:

$$\langle u, v\rangle_{H^2} = \sum_{k \in \mathbf{Z}} \big(1 + (2\pi k)^4\big)\big(u_k.v_k\big),$$

where $u_k, v_k \in \mathbb{R}^{2n}$ are the Fourier coefficients of u and v.

Remark : $T = Id + T_0$ is invertible, whereas T_0 is only right invertible, because the spectrum $\sigma(T_0) = \{(2\pi k)^4 \mid k \in \mathbb{N}_0\}$ contains 0 but not -1. Nonetheless, observe that we will only use that T is invertible from the right.

The inverse of T is given by

$$K : (H^2)^* \to H^2, \quad Ky = \sum_{k \in \mathbf{Z}} \frac{1}{1 + (2\pi k)^4} \exp(2\pi Jkt)\ y_k$$

which satisfies

$$\|Ky\|^2_{H^2} = \sum_{k\in\mathbf{Z}} \frac{1}{1+(2\pi k)^4} |y_k|^2_{\mathbb{R}^{2n}} \leq \|y\|^2_{L^2}$$
$$\langle u, v\rangle_{L^2} = \langle Ku, v\rangle_{H^2}.$$

The looked for minus- gradient vector field of A is given by

$$X(x) = KL(x) \in H^2(x^*T\mathbb{R}^{2n}),$$

as X satisfies

$$\begin{aligned} DA|_\Sigma(x)(v) &= -\langle L(x), v\rangle_{L^2} = -\langle KL(x), v\rangle_{H^2} \\ &= -\langle X(x), v\rangle_{H^2} = -\langle TX(x), v\rangle_{L^2} = -\langle TKL(x), v\rangle_{L^2}. \end{aligned}$$

The vector field $X(x)$ along x is indeed tangent to the hypersurface, i.e. it can be understood as an H^2-section $S^1 \to x^*T\Sigma$. To see this, observe that for all sections $\xi \in L^2(x^*T\Sigma)$ which are L^2-orthogonal to $N(x)$, the regularised section $K\xi \in H^2(x^*T\mathbb{R}^{2n})$ is H^2-orthogonal to $\lambda N(x)$:

$$\langle K\xi, \lambda N(x)\rangle_{H^2} = \langle TK\xi, \lambda N(x)\rangle_{L^2} = \langle \xi, \lambda N(x)\rangle_{L^2} = 0.$$

for all scaling functions $\lambda \in C^2(S^1, \mathbb{R})$ considered as test functions in $H^2(S^1, \mathbb{R})$. This is equivalent to

$$\big(N(x(t)).K\xi(t)\big)_{\mathbb{R}^{2n}} = 0 \quad \forall t.$$

In order to show that L and X are differentiable vector fields in their respective spaces, we need an estimate of $|D_vN(x)|_{\mathbb{R}^{2n}}$:

As we suppose $G \in C^2$, one can consider the 'shape operator'of Σ at the point x:

$$S_x : T_x\Sigma \to T_x\Sigma, \quad v \mapsto D_vN = \frac{G''}{|G'|}v - \frac{G'}{|G'|^3}\big(G'.G''v\big),$$

whose operator norm is equal to the biggest absolute value $|\lambda_{\max}(x)|$ of the principal curvatures of Σ.

With $\kappa := \max_{x\in\Sigma}|\lambda_{\max}(x)|$ one gets a uniform estimate $|D_vN(x)|\,|S_xv| \leq \|S_x\|\,|v| = |\lambda_{\max}(x)|\,|v| \leq \kappa\,|v|$. □

The L^2- gradient vector field $L : H^2 \to H^1 \hookrightarrow L^2$ is formally differentiated by

$$\begin{aligned} DL(x)(v) = -av &+ \big(av.N(x)\big)_{\mathbb{R}^{2n}} N(x) + \big(ax.D_vN(x)\big)_{\mathbb{R}^{2n}} N(x) \\ &+ \big(ax.N(x)\big)_{\mathbb{R}^{2n}} D_vN(x). \end{aligned}$$

This expression is a bounded linear map in $v \in H^1$ and a fortiori in $v \in H^2$:

$$\|DL(x)(v)\|_{L^2} \leq 2\,\|\dot v\|_{L^2} + 2\kappa\,\|\dot x\|_{L^2}\,\|v\|_{L^2} \leq 2\,\|v\|_{H^2}\big(1 + \kappa\,\|\dot x\|_{L^2}\big).$$

So L is differentiable as map $H^2 \to L^2$, from where it follows by the boundedness of K, that KL is differentiable as map $H^2 \to H^2$.

Consider moreover a smooth monotone function $\rho : \mathbb{R} \to [0,1], \rho(a) = 0$ for $a \leq \frac{1}{2}a_1, \rho(a) = 1$ for $a \geq \frac{3}{4}a_1$. Then $\rho \circ A : H^2 \to \mathbb{R}$ is $\mathcal{C}^1$, from where $V(x) := \frac{\rho(A(x))}{\|X(x)\|_{H^2}} X(x) \in H^2(x^*T\Sigma)$.

Now one can apply the local uniqueness and existence theorem for first order ordinary differential equations to get a local flow line $\varphi^s(x), s \in]-\varepsilon(x), \varepsilon(x)[$, $\varepsilon(x) > 0$ of the regularised and normed minus- gradient flow equation of $A|_\Sigma$:

$$\frac{d}{ds}\varphi^s(x) = V(\varphi^s(x)) = \frac{\rho(A(\varphi^s(x)))}{\|\varphi^s(x)\|_{H^2}} X(\varphi^s(x))$$
$$\varphi^0(x) = x.$$

But the flow exists globally, for all $s \in \mathbb{R}$, because $\|V(x)\|_{H^2}$ is bounded by 2:

Assume there is a time s^+ for which $\varphi^{s^+}(x)$ is not defined, but defined for all $s < s^+$. Then for any sequence $s_n < s^+, s_n \to s$, one gets

$$\|\varphi^{s_m}(x) - \varphi^{s_n}(x)\|_{H^2} \leq \int_{s_n}^{s_m} \left\| \frac{d}{ds}\varphi^{s'}(x) \right\|_{H^2} ds' \leq 2\,|s_n - s_m|,$$

i.e. $\varphi^{s_n}(x)$ is a Cauchy sequence whence has a limit point in H^2 which we denote by $\varphi^{s^+}(x)$. By further application of the local existence theorem, the flow line can be extended to an open interval around s^+ and whence to $\mathbb{R}_+$ and similarly to negative s.

Consider the images of the Hopf circles $\mathcal{C} = \{c_\xi : \mathbb{R}/\mathbb{Z} \to S^{2n-1} \mid c_\xi(t) \doteq e^{2\pi Jt}\xi, \xi \in S^{2n-1}\}$ by a smooth map $h : S^{2n-1} \to \Sigma$.

The map $\mathbb{R} \times S^{2n-1} \to \Sigma$ given by $\xi \mapsto \varphi^s(h(c_\xi))(0)$ is a continous homotopy belonging to h, as the following maps are all differentiable:

$$\xi \mapsto c_\xi \mapsto h(c_\xi) \mapsto \varphi^s\big(h(c_\xi)\big) \overset{\iota}{\mapsto} \varphi^s\big(h(c_\xi)\big) \mapsto \varphi^s\big(h(c_\xi)\big)(0),$$

where the second last map is the Sobolev inclusion $\iota : H^2(S^1, \Sigma) \hookrightarrow \mathcal{C}^1$.

We denote $\varphi^s\big(h(c_\xi)\big)$ by $\Phi^s(h)(c_\xi)$, and call $\Phi^s(h)$ the *sphere flow* of h.

Remark : In order to get a flow which is equivariant by parameter transformations from $\mathcal{R}$ (not only monotone ones) one can study the vector field

$$\tilde{L}(x)(t) = \frac{L(x)(t)}{\sigma_x(t)\,|\dot{x}(t)|},$$

where $\sigma_x : [0,1] \to \{0,1\}$ is the sign function, defined to be 1 if $\dot{x}(t)$ vanishes or is directed in positive direction with respect to a given orientation of Imx and -1 if $\dot{x}(t)$ points in negative direction. Then it is easy to see that for any $s \in \mathcal{R}$, $\tilde{L}(x \circ s)(t) = \tilde{L}(x)(s(t))$ a.e., i.e. $\tilde{L}(x)$ invariant by $\mathcal{R}$. Moreover, $\tilde{L}$ is bounded by 2, differentiable as map $\tilde{L} : H^1 \to L^2$ and satisfies $\tilde{L}(x)(t) = 0 \iff x$ solves $(*)$.

3. Deformation Lemma

In order to prove that minmax-values are critical, one usually assumes that they are not and shows by a gradient-like deformation that this leads to a contradiction.

Lemma 2: *Assume $\alpha \in [\frac{a_1}{2}, B]$ is not critical. Then there is ε_0, such that for any $h \in \mathcal{H}_\delta$ and for any $s_0 \geq 0$, there is $\Delta(h, s_0) > 0$ such that $\forall \varepsilon \leq \varepsilon_0$ $\forall s \in [s_0, s_0+1]$ we get: $\forall \xi$ with $A(\varphi^s(h(c_\xi))) \in [\alpha - 2\varepsilon, \alpha + 2\varepsilon]$ one has $\|X(\varphi^s(h(c_\xi)))\|_{H^2} \geq \Delta(h, s_0)$.*

Proof:

Define the set $U_\varepsilon := U_{\varepsilon,h,s_0}(\alpha) = \{(s,\xi) \in [s_0, s_0+1] \times S^{2n-1} \mid A(\varphi^s(h(c_\xi))) \in [\alpha - 2\varepsilon, \alpha + 2\varepsilon]\}$, which is compact. Assume that there is an h which allows a sequence $(s_n, \xi_n) \in U_\varepsilon$ such that

$$\|X(\varphi^{s_n}(h(c_{\xi_n})))\|_{H^2} \to 0 \text{ and } A(\varphi^{s_n}(h(c_{\xi_n}))) \to \alpha.$$

Because of the compactness of $U_\varepsilon(h)$, there is a limit point (s_*, ξ_*). The corresponding loop $\varphi^{s_*}(h(c_{\xi_*})) \in H^2$ satisfies, thanks to the continuity of $X(\varphi^s(h(c_\xi)))$ and $A(\varphi^s(h(c_\xi)))$ in the variables $(s,\xi) \in \mathbb{R} \times S^{2n-1}$:

$$X(\varphi^{s_*}(h(c_{\xi_*}))) = 0 \text{ and } A(\varphi^{s_*}(h(c_{\xi_*}))) = \alpha.$$

This means

$$X(x) = \rho(x)\frac{KL(x)}{\|x\|} = \frac{KL(x)}{\|x\|} = 0 \ \Rightarrow L(x) = 0,$$

i.e. α is critical, which is a contradiction. □

Observe that ε_0 is independent of h.

Remark : A lemma like this is usually proved using the Palais-Smale condition. It is worthwile to observe that the only compactness we use here is the trivial compactness of $[s_0, s_0+1] \times S^{2n-1}$.

Deformation Lemma: *Assume $\alpha \in [\frac{a_1}{2}, B]$ is not critical. Consider ε_0 and any $0 < \varepsilon < \varepsilon_0$ as in Lemma 2. Then for any $h \in \mathcal{H}_\delta$ $\forall s_0 \geq 0$ $\forall \xi \in S^{2n-1}$ the following implication holds*

$$A(\varphi^{s_0}(h(c_\xi))) \leq \alpha + \varepsilon \implies A(\varphi^{s_0+1}(h(c_\xi))) \leq \alpha - \varepsilon\ .$$

Proof:

With lemma 2, $\forall h \in \mathcal{H}_\delta$, $\forall s_0 \geq 0$, there is $\Delta(h, s_0) > 0$, such that for the pairs $(s,\xi) \in [s_0, s_0+1] \times S^{2n-1}$ satisfying $A(\varphi^{s_0}(h(c_\xi))) \in [\alpha - 2\varepsilon, \alpha + 2\varepsilon]$ one has the lower bound $\|X(\varphi^s(h(c_\xi)))\|_{H^2} \geq \Delta(h, s_0)$. We have $A(\varphi^s(h(x))) \leq A(\varphi^{s_0}(h(x)))$ $\forall s \geq s_0$, more precisely

$$\begin{aligned} A(\varphi^{s_0}(h(c_\xi))) - A(\varphi^{s}(h(c_\xi))) &= \int_s^{s_0} \langle KL(\varphi^{s'}(h(c_\xi))), \frac{d\varphi^{s'}}{ds}(\varphi^{s'}(h(c_\xi)))\rangle_{H^2} ds' \\ &= -\int_s^{s_0} \left\|KL(\varphi^{s'}(h(c_\xi)))\right\|^2_{H^2} ds' \\ &\geq \Delta^2(s - s_0) \end{aligned}$$

for all ξ satisfying $A\big(\varphi^{s'}(h(c_\xi))\big) \geq \alpha - 2\varepsilon \; \forall s' \leq s$. For all other ξ, $A\big(\varphi^s(h(c_\xi))\big) \leq \alpha - 2\varepsilon$.

If we now choose ε' small enough such that $\Delta(h, s_0) \geq \sqrt{\varepsilon'} > 0$ and $s = s_0 + 1$, then we get in both cases $A\big(\varphi^{s_0+1}(h(c_\xi))\big) \leq A\big(\varphi^{s_0}(h(c_\xi))\big) - \Delta^2 \leq \alpha - \varepsilon'$, from where the claim follows. □

4. Construction of σ_k

Consider the filtration by standard spheres in $\mathbb{C}^k \cong \mathbb{R}^{2k}, k = 1, \ldots, n$, with origin in ξ_0, namely

$$\xi_0 \in S^1 \subset S^3 \subset \ldots \subset S^{2n-1}.$$

For each $h \in \mathcal{C}(S^{2n-1}, \Sigma)$ with mapping degree 1, this induces a filtration on Σ for which we choose non unique maximal points $\mu_k(h) \in H^2(S^1, \Sigma)$:

$$A\big(\mu_k(h)\big) = \max_{\xi \in S^{2k-1}} A\big(h(c_\xi)\big).$$

We have showed in section 2 that there exists a regularised extended minus- gradient flow Φ^s on S^{2n-1}- families of loops, which is induced from the regularised flow φ^s on loops and which diminishes the symplectic action $A(\gamma) = \int_\gamma \theta$ of each circle which is an image of a Hopf circle (unless it is critical for the restriction $A|_\Sigma$).

Now think of a flow line $\{\Phi^s(h)(S^{2n-1}) \mid s \in \mathbb{R}_+\}$ starting from $h \in \mathcal{H}_\delta$ as a deformation of circle fibrations on Σ; For each k, we have again a maximal point for any s, which we denote by $\mu_k^s(h)$:

$$A\big(\mu_k^s(h)\big) = \max_{\xi \in S^{2k-1}} A\big(\Phi^s(h)(c_\xi)\big),$$

which is attained at some ξ^s satisfying $\mu^s(h) = \Phi^s(h)(c_{\xi^s})$.

Definition 1: *Given $h \in \mathcal{H}_\delta$, every piecewise continous family $\mu_k^s(h), s \in \mathbb{R}$ satisfying*
(i) $A\big(\mu_k^s(h)\big) = \max_{\xi \in S^{2k-1}} A\big(\Phi^s(h)(c_\xi)\big)$ (k-maximality)
(ii) $\exists M < \infty$ such that $\forall s \geq M$, $\mu_k^s \in L_\delta$ (δ-regularity)
is called a **δ-regular k-maximal descending cylinder for** *h.*

We observe that $\mu^s, s \geq 0$ is comparable to 'half'a connecting orbit as used in Floer theory, but far simpler to find since one does not have to solve any non linear partial differential equation. It is called 'descending'cylinder as its action descends although it does not follow the steepest descent: In general $\varphi^s(\mu^0) \neq \mu^s$. Moreover φ^s does not have the flow property on the cylinders μ^s as $\varphi^1(\mu^s) \neq \mu^{s+1}$.

Definition 2: *Let $\mathcal{I}_k \subset \mathcal{H}_\delta$ be the set of initial homeomorphisms which realize one of the closed characteristics γ_i (i.e. $\exists \xi \in S^{2k-1}, i \in \{1, \ldots, N\}$ such that $h(c_\xi) = \gamma_i$), from which $A(\mu_k^s(h)) \geq a_1 > 0 \; \forall s \geq 0$,*
and which are provided with a δ-regular k-maximal descending cylinder $\mu_k^s(h)$.
Let σ_k be defined the following values

$$\sigma_k = \inf_{s \in \mathbb{R}} \inf_{h \in \mathcal{H}_\delta} \sup_{\xi \in S^{2k-1}} A\big(\Phi^s(h)(c_\xi)\big) \geq a_1 > 0.$$

Remark : In contrast to an earlier version of this paper, we do not need ask that $\Phi^s(h)$ is invertible for $s \geq M$.

Lemma 4: *For any $\mu, \gamma \in C^1, \delta_0 > 0$, there is an $\varepsilon > 0$ small enough such that one gets:*

$$\left.\begin{array}{l} |\mu - \gamma|_{C^1} < \varepsilon \\ |\gamma(t) - \gamma(t')|_{\mathbf{R}^{2n}} \geq \delta_0 |q^t - q^{t'}|_{\mathbf{C}} \ \forall t, t' \end{array}\right\} \Longrightarrow |\mu(t) - \mu(t')| \geq \frac{\delta_0}{2} |q^t - q^{t'}| \ \forall t, t'.$$

In other words, one gets $\gamma \in L_{2\delta} \Longrightarrow \mu \in L_\delta$ for μ C^1- close enough to γ.

Proof:

Let us recall that we use the parametrisation of the circle S^1 by $q^t := e^{2\pi i t}$.

(a) large times:

$$\begin{aligned} \delta_0 |q^t - q^{t'}| &\leq |\gamma(t) - \gamma(t')| \leq |\gamma(t) - \mu(t)| + |\mu(t) - \mu(t')| + |\mu(t') - \gamma(t')| \\ &\leq 2\varepsilon + |\mu(t) - \mu(t')| \end{aligned}$$

$$|\mu(t) - \mu(t')| \geq \delta_0 |q^t - q^{t'}| - 2\varepsilon \geq \frac{\delta_0}{2} |q^t - q^{t'}|,$$

where the last inequality holds if $2\varepsilon < \frac{\delta_0}{2} |q^t - q^{t'}|$, i.e. if $|q^t - q^{t'}| > \frac{4\varepsilon}{\delta_0}$.

(b) small times:

One obtains $|\dot\gamma(t)| \geq 2\pi\delta_0$, moreover $|\dot\mu(t)| \geq |\dot\gamma(t)| - \varepsilon \geq 2\pi\delta_0 - \varepsilon$. As the difference quotient approximates $\dot\mu(t)$, using the estimate $2\pi |t - t'| \geq |q^t - q^{t'}| \geq 4 |t - t'|$ for $|t - t'| \leq \frac{1}{2}$, we get, for ε small enough, that $|q^t - q^{t'}| \leq \frac{4\varepsilon}{\delta_0}$ implies

$$\frac{|\mu(t) - \mu(t')|}{|q^t - q^{t'}|} = \frac{|\mu(t) - \mu(t')|\,|t - t'|}{|t - t'|\,|q^t - q^{t'}|} \geq (|\dot\mu(t)| - \varepsilon) \frac{1}{2\pi} \geq \frac{2\pi\delta_0 - 2\varepsilon}{2\pi} = \delta_0 - \frac{\varepsilon}{\pi} \geq \frac{\delta_0}{2}.$$

□

Lemma 5: $\mathcal{I}_n \neq \emptyset$.

Proof:

Consider $h := h_N$ such that $\gamma_N = h_N(c_{\xi_0})$ and $A(\gamma_N) = a_N$. The interval $]a_N, B]$, for $B := \max_{\xi \in S^{2n-1}} A(h(c_\xi)) + 1$, say, does not contain any prime critical values but at most a finite number of multiples of such. With lemma 2, the flow Φ^s diminishes the action of all non critical loops by a positive amount.

Now assume that $\mu^s(h)$, defined by $A(\mu^s(h)) = \max_{\xi \in S^{2n-1}} A(\varphi^s(h(c_\xi)))$, is stopped by one of the iterate closed characteristics, $\gamma_i^{(j)}$, say. It means that $\|\mu^s(h) - \gamma_i^{(j)}\|_{H^2} \to 0, s \to \infty$. Since every critical point x of A has Morse index ∞, i.e. the dimension of directions at x with negative slope is infinite, it is possible to choose a small perturbation $\tilde h$ of h with $\gamma_N = \tilde h(c_{\xi_0})$ and $\max A(\tilde h(c_\xi)) \leq B$, with the property that the flow avoids $\gamma_i^{(j)}$.

Applying this argument at most a finite number of times, we end with the situation that $\mu^s(h)$ is not stopped at any iterate but at the fixed point of φ^s: $\varphi^s(h(c_{\xi_0})) = h(c_{\xi_0}) = \gamma_N$.

This means $\mu^s(h) \stackrel{H^2}{\to} \gamma_N$, from which there exists $M(\varepsilon) < \infty$, such that $\|\gamma_N - \mu^s(h)\|_{C^1} < \varepsilon \quad \forall s \geq M(\varepsilon)$. As a consequence of Lemma 4, $\mu^s(h) \in L_\delta \ \forall s \geq M$, i.e. $\mu^s(h)$ is δ- regular and therefore $h = h_N \in \mathcal{I}_n$. □

In order to normalize the situation, we look for a holomorphic rotation R^s such that $c_{\xi^s} = R^s c_{\xi_0}$, the maximal action is attained at the parameter ξ_0.

With the standard identification $\mathbb{C}^n \to (\mathbb{R}^{2n}, J)$ one gets $GL(n, \mathbb{C}) \cong \{R \in GL(2n, \mathbb{R}) \mid RJ = JR\}$. Then

$$O(2n) \cap GL(n, \mathbb{C}) \cong \{R \in GL(2n, \mathbb{R}) \mid RJ = JR, R^T R = Id\} \cong U(n).$$

Using $RJ = JR$ and the uniqueness of the initial value problem

$$\frac{d}{dt} Rc_\xi = R\dot{c}_\xi = 2\pi RJc_\xi = 2\pi JRc_\xi \text{ and } Rc_\xi(0) = R\xi$$

we get $Rc_\xi = c_{R\xi}$.

Now one can consider a path of holomorphic rotations $R^s \in U(n)$ with $R^s\xi_0 = \xi^s, R^s J\xi_0 = J\xi^s$. Then $\mu_k^s = \Phi^s(h) \circ R^s c_{\xi_0}$ is the searched for normalisation to ξ_0 of one of the maximal points.

Lemma 6: *$\mathcal{I}_k \subset \mathcal{I}_{k-1}$ up to $R \in U(k)$. Therefore $\sigma_{k-1} \leq \sigma_k$ for all $k = 2, \ldots, n$.*

Proof:

Pick $h \in \mathcal{I}_k$ and one of its δ-regular k-maximal descending cylinders $\mu_k^s(h)$. There is a piecewise continous curve $\xi^s = R^s c_{\xi_0}, R^s \in U(k)$ with $\mu_k^s(h) = \Phi^s(h)(c_{\xi^s})$. ξ^s has a convergent subsequence ξ^{s_i} whose limit point is denoted by $\xi^* = R^*\xi_0 = R\xi_0$. Then choose $h \circ R$ as initial homeomorphism and, because $R \in U(k)$, one gets the same maximal curve as for h: $\mu_k^s(h \circ R) = \mu_k^s(h)$. It satisfies

$$\|\mu_k^s(h) - \Phi^s(h \circ R)(c_{\xi_0})\|_{H^2} \to 0$$

But this means, using $\xi_0 \in S^{2k-3} \subset S^{2n-1}$ and the continuity of A on $H^2(S^1, \Sigma)$, that the maximum of A on S^{2k-3} approaches $A(\mu_k^s(h))$ for s big enough. Therefore, there is $\mu_{k-1}^s(h)$ realising the maximum of A on S^{2k-3} for s big enough, which approaches $\mu_k^s(h)$ for a subsequence:

$$\left\|\mu_k^{s_i}(h) - \mu_{k-1}^{s_i}(h)\right\|_{H^2} \to 0.$$

By lemma 4, $\mu_k^s(h)$ is δ- regular and $\left\|\mu_{k-1}^s(h) - \Phi^s(h \circ R)(c_{\xi_0})\right\| \to 0$. Therefore $h \in \mathcal{I}_{k-1}$. □

5. Proof of Theorem 1

By construction of σ_k, for any $\varepsilon > 0$ there are $h \in \mathcal{I}_k$ and $s_0 \geq 0$ such that

$$A\big(\varphi^{s_0}(h(c_\xi))\big) < \sigma_k + \varepsilon \quad \forall \xi \in S^{2k-1}.$$

From this, we prove by contradiction that σ_k is critical and that $\sigma_k \neq \sigma_{k+1}$ for all $k = 1 \ldots n$, which yields that there are n geometrically different closed characteristics.

(a) Assume first that σ_k for one k is not critical, then by lemma 2 there exists ε_0, such that we can apply the deformation lemma to $\alpha = \sigma_k$ and $\varepsilon \leq \varepsilon_0$:

$$A\big(\varphi^s(h(c_\xi))\big) < \alpha - \varepsilon \quad \forall \xi \in S^{2k-1} \quad \forall s \geq s_0 + 1,$$

which is the contradiction to $\sigma_k = \min_{s\in\mathbb{R}} \max_{\xi\in S^{2k-1}} A\big(\varphi^s(h(c_\xi))\big)$ we aimed for.

(b) Assume now that $\sigma_k = \sigma_{k+1}$. We pick $h \in \mathcal{I}_{k+1}$ and $s_0 \geq 0$ such that $A\big(\varphi^{s_0}(h(c_\xi))\big) < \sigma_k + \varepsilon \ \forall \xi \in S^{2k+1}$. One can assume that all loops $\varphi^{s_0}(h(c_\xi))$ are parametrised proportional to arc length.

Consider $\xi_k \in S^{2k-1}$ such that $\mu_k^{s_0}(h) = \varphi^{s_0}(h(c_{\xi_k}))$. By definition one has

$$\sigma_k \leq A\big(\mu_k^{s_0}(h)\big) \leq A\big(\mu_{k+1}^{s_0}(h)\big) < \sigma_k + \varepsilon,$$

from where

$$\sigma_k \leq A\big(\varphi^{s_0}(h(c_\xi))\big) < \sigma_k + \varepsilon \quad \forall \xi \in D := S^{2k+1} \setminus S^{2k-1},$$

in particular for an S^1- invariant round 3-sphere such that $S^3 \cap \bar{D} = \{c_{\xi_k}(t) \mid t \in S^1\}$. The existence of such a sphere is easily seen in coordinates, after a suitable rotation $R \in U(k+1)$.

Distinguish two cases:

(1) All loops $\varphi^s(h(c_\xi))$, $\xi \in D = S^{2k+1} \setminus S^{2k-1}$, converge to a unique point, i.e. to the point to which μ_{k+1}^s converges:

$$\varphi^s(h(c_\xi)) \stackrel{H^2}{\to} \mu_{k+1}^s \stackrel{H^2}{\to} \tilde{\gamma} \in L_\delta, \ s \to \infty, \quad \forall \xi \in D.$$

Therefore, one has for any $\rho > 0$ an $s_0 \geq 0$ such that

$$\tilde{h}(\xi) := \varphi^{s_0}(h(c_\xi)) \in B_\rho(\tilde{\gamma}) := \{x \in H^2 \mid \|x - \tilde{\gamma}\|_{H^2} < \rho\}$$

Then for an ε small enough, by lemma 4, $B_\rho(\tilde{\gamma}) \subset L_\delta$, which means that $\tilde{h}$ sends S^3 equivariantly in the set L_δ with free S^1-action. Any S^1- invariant functional has at least two critical points in a set with free S^1- action containing the image of a 3- sphere of an S^1- equivariant map. Therefore A has two critical points in $\mathrm{clos}(B_\rho(\tilde{\gamma}))$, for all ρ, which contradicts the finiteness of the number N of prime closed characteristics through the fact that they have to be isolated.

(2) In the other case, there must be a family of geometrically different critical loops to which $\varphi^{s_m}(h(c_\xi))$, $\xi \in D$ converge, so we have again a contradiction to the finiteness of N. □

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