FUNCTIONAL EQUATIONS AND QUANTUM SEPARATION OF VARIABLES FOR 3D SPIN MODELS

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ABSTRACT. The recently proposed invariant formulation of the auxiliary linear problem for 3d integrable models provides several new ideas in the solution of the spectral problem of 3d spin models, e.g. the Zamolodchikov-Bazhanov-Baxter model in its vertex formulation. This paper is an announce of the results following from the invariant formulation. Here we formulate the class of 3d spin models which is in fact the appropriately parameterized inhomogeneous Zamolodchikov-Bazhanov-Baxter model, give an expression for the generating function of the complete set of matrices commuting with the transfer matrix of ZBB model (integrals of motion), give the functional equations defining the eigenvalues of the integrals of motion and the transfer matrices, describe explicitly the groupoid of isospectral transformations of the initial system of integrals of motion and finally give an explicit parameterization of a projector to the separated states in the sense of the quantum separation of variables (functional Bethe ansatz).

1. INTRODUCTION

Local invariant formulation of auxiliary linear problem for quantum evolution models in wholly discrete 2+1 dimensional space-time, suggested in [1], provides a powerful method for solution of the corresponding spectral problems. The algebra of observables in the 2+1 quantum evolution models is isomorphic to the local Weyl algebra, which admits the finite dimensional unitary representations when Weyl's parameter q is the primitive root of unity,

$$q = \mathbf{e}^{2\pi i/N} . \tag{1}$$

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In this situation the basic intertwiner, solving the local zero curvature condition, is equivalent to the **R**-matrix of the Zamolodchikov-Bazhanov-Baxter model in its vertex formulation, see [2, 3] for the details. Global operators, acting in the whole Hilbert space, such as an evolution operator or layer-to-layer transfer matrix, are the solutions of some dynamics of the algebra of observables and may be constructed with the help of local **R**-matrices.

Local auxiliary linear problem provides in this case the proof of the exact integrability of ZBB model (in particular, the tetrahedron equation for **R**-matrices is the trivial consequence of the zero curvature condition) and the method of derivation of a set of functional equations for the eigenvalues of ZBB transfer matrices.

N-th powers of the Weyl elements at the root of unity are the centers, which are just the complex parameters for the finite dimensional representations considered. N-th powers of Weyl elements coincide with the modularly dual Weyl pair [4, 5] and therefore have the similar dynamics as the dynamics of primary Weyl generators. ZBB model corresponds to the case of a constant solution of the classical dynamics. But for a finite 3d lattice a complete solution of the classical equations of motion may be derived. In general it gives a parameterization of N-th powers in terms of theta-functions on the Jacobian of a classical spectral curve, depending on a size, form and boundary conditions of 3d lattice. Corresponding 3d spin models belong to a new class of integrable models, and we will discuss them separately.

Simple models like ZBB correspond to the rational limit of the classical spectral curve, and classical equations of motion may still have nontrivial solutions. Corresponding spin counterpart contains modified layer-to-layer transfer matrices and their ordered products, helping one to construct a complete groupoid of isospectral deformations of initial ZBB model. Surprisingly, this groupoid contains special degenerative elements corresponding to projectors to separated states in the sense of the method of quantum separation of variables [6, 7]. Such phenomenon was recently observed for the quantum Toda chain at the root of unity in [8].

This paper is to be considered as the brief announce of all these results. We decided to skip all the proofs for the sake of shortness and so hide completely the method we used. The fragments of the method the reader may find in [1, 8].

Here we first recall the vertex formulation of ZBB model [2, 3], and then step by step we will define the objects and parameterizations necessary for our considerations and simply declare their properties without any kind of proof. Therefore we will formulate a lot of conjectures, but the reader should take into account that all these conjectures are proven, their proof will appear in the forthcoming papers.

2. Formulation of the 3D spin model

2.1. **Cubic lattice.** Let the cubic lattice is formed by the vectors

$$\mathbf{n} \;=\; n_1\mathbf{e}_1 \;+\; n_2\mathbf{e}_2 \;+\; n_3\mathbf{e}_3 \tag{2}$$

with integer n_1, n_2, n_3 and some non-complanar $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Partition function is by the definition the following sum:

$$Z \;=\; \sum_{\{\sigma\}} \prod_{\mathbf{n}} \langle \sigma_{1,\mathbf{n}}, \sigma_{2,\mathbf{n}}, \sigma_{3,\mathbf{n}} | \mathbf{R}_{\mathbf{n}} | \sigma_{1,\mathbf{n}+\mathbf{e}_1}, \sigma_{2,\mathbf{n}+\mathbf{e}_2}, \sigma_{3,\mathbf{n}+\mathbf{e}_3} \rangle \;, \tag{3}$$

where each spin variable $\sigma_{j,\mathbf{n}}$ belongs to a finite set (namely, $\sigma_{j,\mathbf{n}} \in \mathbb{Z}_N$), and the local "Boltzmann weights" are written as matrix elements $\langle \sigma_1, \sigma_2, \sigma_3 | \mathbf{R}_{\mathbf{n}} | \sigma_1', \sigma_2', \sigma_3' \rangle$ of some $N^3 \times N^3$ matrix $\mathbf{R}_{\mathbf{n}}$, whose matrix elements depend in general on $\mathbf{n}$.

If the periodical b.c. are implied in the definition (3), then for the lattice of the size $N_1 \times N_2 \times N_3$ one puts $n_1 \in \mathbb{Z}_{N_1}$, $n_2 \in \mathbb{Z}_{N_2}$, $n_3 \in \mathbb{Z}_{N_3}$.

As a particular case one may consider the homogeneous lattice, when the matrix elements of the Boltzmann weight matrix do not depend on the vertex $\mathbf{n}$: $\mathbf{R}_{\mathbf{n}} = \mathbf{R}$.

The "monodromy" and "transfer" matrices may be defined for the cubic lattice. We choose the first direction as the "time", i.e. consider the layer of $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$ with n_1 frozen, so that $\sigma_{1,\mathbf{n}} = \sigma_{1:n_2,n_3}$ and $\sigma_{1,\mathbf{n}+\mathbf{e}_1} = \sigma'_{1:n_2,n_3}$ Then by definition, the "monodromy" layer-to-layer matrix is defined by

$$\begin{aligned} &\langle \sigma_1, \sigma_2, \sigma_3 | \widehat{\mathbf{Q}}_{n_1} | \sigma_1', \sigma_2', \sigma_3' \rangle \;= \\ &= \sum_{\sigma_2,\sigma_3} \prod_{n_2,n_3} \langle \sigma_{1:n_2,n_3}, \sigma_{2:n_2,n_3}, \sigma_{3:n_2,n_3} | \mathbf{R}_{\mathbf{n}} | \sigma'_{1:n_2,n_3}, \sigma_{2:n_2+1,n_3}, \sigma_{3:n_2,n_3+1} \rangle \;, \end{aligned} \tag{4}$$

where the matrix elements of $\widehat{\mathbf{Q}}_{n_1}$ are taken between the sets

$$\sigma_1 = \;\{\sigma_{1:n_2,n_3}\},\;\; \sigma_2 = \{\sigma_{2:0,n_3}\},\;\; \sigma_3 = \{\sigma_{3:n_2,0}\},\;\; n_2 = 0, ..., N_1 - 1,\;\; n_3 = 0, ..., N_3 - 1 \tag{5}$$

and

(6)

$$\sigma'_1 = \{\sigma'_{1:n_2,n_3}\}, \ \ \sigma'_2 = \{\sigma_{2:N_2,n_3}\}, \ \ \sigma'_3 = \{\sigma_{3:n_2,N_3}\}, \ \ n_2 = 0, ..., N_2 - 1, \ \ n_3 = 0, ..., N_3 - 1 \ ,$$

and the summation is taken with respect to all "internal" $\sigma_{2:n_2,n_3}, \sigma_{3:n_2,n_3}$, $0 < n_2 < N_2$, $0 < n_3 < N_3$. The "layer-to-layer" transfer matrix is the result of the tracing of the monodromy matrix:

$$\langle \sigma_1 | \mathbf{Q}_{n_1} | \sigma'_1 \rangle \ = \ \sum_{\sigma_2, \sigma_3} \langle \sigma_1, \sigma_2, \sigma_3 | \widehat{\mathbf{Q}}_{n_1} | \sigma'_1, \sigma_2, \sigma_3 \rangle \ , \tag{7}$$

where the summation is taken over all $\sigma_2 = \{\sigma_{2:0,n_3} = \sigma_{2:N_2,n_3}\}$ and $\sigma_3 = \{\sigma_{3:n_2,0} = \sigma_{3:n_2,N_3}\}$. Due to this periodicity we will always imply below

$$n_2 \ \in \ \mathbb{Z}_{N_2} \ , \quad n_3 \ \in \ \mathbb{Z}_{N_3} \tag{8}$$

The partition function is

$$Z = \mathrm{Tr}\ \mathbf{Z}_1 \ , \tag{9}$$

where $\mathbf{Z}_1$ is the partition function for the cubic lattice, open in the "first" direction:

$$\mathbf{Z}_1 \ = \ \mathbf{Q}_0 \cdot \mathbf{Q}_1 \cdot \ldots \mathbf{Q}_{N_1 - 1} \ . \tag{10}$$

Integrability of simple 3d spin model is based on the existence of the commutative set of the transfer matrices:

$$[\mathbf{Q}, \mathbf{Q}'] \ = \ 0 \ , \tag{11}$$

where $\mathbf{Q}$ is formed by the weights $\mathbf{R}_{n_2,n_3}$ and $\mathbf{Q}'$ is formed by analogous but different weights $\mathbf{R}'_{n_2,n_3}$. The commutativity is provided by the Tetrahedron equation for $\mathbf{R}$-s. The model formulated is integrable if all $\mathbf{Q}_{n_1}$ in (10) belong to the same commutative family.

Complicated 3d spin models are governed by the so called "modified tetrahedron equation", leading at first to the modified exchange relations for the layer-to-layer transfer matrices, but finally the integrability means the existence of the commutative family of $\mathbf{Z}_1$-matrices (10).

The main example of the 3d spin model is the Zamolodchikov-Bazhanov-Baxter model. Below the explicit form of its vertex weight is given.

2.2. **Fermat curve and the function W.** Let p be a point on the Fermat curve $\mathcal{F}$

$$p \stackrel{def}{=} (x,y) \in \mathcal{F} \quad \Leftrightarrow \quad x^N + y^N = 1\ . \tag{12}$$

$W_p(n)$, $p \in \mathcal{F}$, $n \in \mathbb{Z}_N$, we define as follows:

$$\frac{W_p(n)}{W_p(n-1)} \;=\; \frac{y}{1\,-\,x\,q^n}\ ,\quad W_p(0)\;=\;1\ . \tag{13}$$

The root of unity q is given by (1).

Function $W_p(n)$, being "q - gamma function", "q - exponent" and "q - dilogarithm" at the root of unity, has a lot of remarkable properties, see e.g. the Appendix of [3] for an introduction into "q - root of unity - hyper-geometry".

2.3. **R-matrix.** Define $N^3 \times N^3$ matrix $\mathbf{R}$ as the matrix function of four Fermat points p_1, p_2, p_3, p_4 as follows [3]:

$$\begin{aligned} &\langle \sigma_1, \sigma_2, \sigma_3 | \mathbf{R} | \sigma_1', \sigma_2', \sigma_3' \rangle \;\stackrel{def}{=}\; R_{\sigma_1,\sigma_2,\sigma_3}^{\sigma_1',\sigma_2',\sigma_3'} \;= \\ &=\; \delta_{\sigma_2+\sigma_3,\sigma_2'+\sigma_3'}\; q^{(\sigma_1'-\sigma_1)\,\sigma_3'}\, \frac{W_{p_1}(\sigma_2-\sigma_1)\,W_{p_2}(\sigma_2'-\sigma_1')}{W_{p_3}(\sigma_2'-\sigma_1)\,W_{p_4}(\sigma_2-\sigma_1')}\ , \end{aligned} \tag{14}$$

where x-coordinates of four Fermat curve points are identically related by

$$x_1\; x_2 \;=\; q\;\; x_3\; x_4\ . \tag{15}$$

$\mathbf{R}$-matrix (14) is considered as the universal building block for 3d spin integrable models. In the expressions for the "monodromy" or "transfer" matrices (4,7), as well as in the expression for the partition function (3), the dependence on $\mathbf{n}$ means simply

$$\mathbf{R}_{\mathbf{n}} \;=\; \mathbf{R}(p_{1,\mathbf{n}}, p_{2,\mathbf{n}}, p_{3,\mathbf{n}}, p_{4,\mathbf{n}})\ . \tag{16}$$

It is known for the homogeneous case, $p_{k,\mathbf{n}} = p_k$, two transfer matrices $\mathbf{Q}$ and $\mathbf{Q}'$, defined via

$$p_1, p_2, p_3, p_4 \;\mapsto\; \text{eq. } (14) : \mathbf{R} \;\mapsto \text{eqs. } (4,7) :\; \mathbf{Q} \tag{17}$$

and $p_1', p_2', p_3', p_4' \;\mapsto\; \mathbf{R}' \;\mapsto\; \mathbf{Q}'$, commute if $x_3/x_1 = x_3'/x_1'$ [3].

2.4. **Inhomogeneous ZBB model.** Define the symbol $e(X,Y)$, $X,Y\in\mathbb{C}$ as

$$e(X,Y)^N \;=\; X-Y\;. \tag{18}$$

Conjecture 1. *The inhomogeneous ZBB model corresponds to the following parameterization of* $\mathbf{R_n}$ *in (3) in the sense of (16):*

$$\begin{aligned}
x_{1,\mathbf{n}} &= q^{-1}\frac{e(X_{n_1},Z'_{n_3})e(Y_{n_2},Z_{n_3})}{e(X_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\,, & y_{1,\mathbf{n}} &= \frac{e(X_{n_1},Y_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\,,\\
x_{2,\mathbf{n}} &= \frac{e(X'_{n_1},Z'_{n_3})e(Y'_{n_2},Z_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\,, & y_{2,\mathbf{n}} &= \frac{e(X'_{n_1},Y'_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\,,\\
x_{3,\mathbf{n}} &= q^{-1}\frac{e(X_{n_1},Z'_{n_3})e(Y'_{n_2},Z_{n_3})}{e(X_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\,, & y_{3,\mathbf{n}} &= q\frac{e(X_{n_1},Y'_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\,,\\
x_{4,\mathbf{n}} &= q^{-1}\frac{e(X'_{n_1},Z'_{n_3})e(Y_{n_2},Z_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\,, & y_{4,\mathbf{n}} &= \frac{e(X'_{n_1},Y_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\,.
\end{aligned} \tag{19}$$

Here the pair of complex parameters X_{n_1},X'_{n_1} corresponds to n_1-th layer in the "first" direction, Y_{n_2},Y'_{n_2} corresponds to n_2-th layer in the "second" direction and so on. So this pairs may be considered as the *free* spectral parameters of the model, they are related to but do not coincide with Zamolodchikov's dihedral angles.

2.5. **Eigenvalue and eigenvector problem.**

Conjecture 2. *Any two transfer matrices* $\mathbf{Q}_{n_1}$ *and* $\mathbf{Q}_{m_1}$*, defined by (4,7) with the parameterization (19), commute if they have the same set of* $Y_{n_2},Y'_{n_2},Z_{n_3},Z'_{n_3}$ *(i.e. the same* $e(Y^{\#}_{n_2},Z^{\#}_{n_3})$*) and arbitrary* X_{n_1},X'_{n_1} *and* X_{m_1},X'_{m_1}*.*

It means that they have the same set of eigenvectors $|\Psi_t\rangle$,

$$\mathbf{Q}_{n_1}|\Psi_t\rangle \;=\; |\Psi_t\rangle Q_{n_1,t}\;, \tag{20}$$

where the symbol t stands as a formal labeling of eigenvectors (i.e. a multi-index), and $Q_{n_1,t}$ is the eigenvalue of $\mathbf{Q}_{n_1}$ for this eigenvector. $|\Psi_t\rangle$ depends on the set of $Y_{n_2},Y'_{n_2},Z_{n_3},Z'_{n_3}$ and do not depend on X_{n_1},X'_{n_1}. Together with the right eigenvectors one may use the left ones, defining the spectral decomposition of $\mathbf{Q}_{n_1}$:

$$\mathbf{Q}_{n_1} \;=\; \sum_t\,|\Psi_t\rangle\,Q_{n_1,t}\,\langle\Psi_t|\;. \tag{21}$$

The general problem is to find out the eigenvalues and eigenvectors of $\mathbf{Q}_{n_1}$. To solve this problem explicitly is quite hopeless.

In this paper we will give the solution for more simple problems. At the first, we will define the structure of t, i.e. we will derive the complete set of independent matrices, commuting with $\mathbf{Q}_{n_1}$ (i.e. the "integrals of motion").

Next we will give a set of equations for the spectrum of the integrals of motion as well the equations connecting the spectrum of $\mathbf{Q}_{n_1}$ and the spectrum of the integrals of motion (such type equations are called usually the functional equations for the spectrum).

Finally we will construct (explicitly) a projector to a "separated state" $|\Phi_\alpha\rangle$ such that

$$\langle\Psi_t|\Phi_\alpha\rangle \;=\; \prod_{n_1} Q_{n_1,t}(\alpha) \tag{22}$$

for some special values of X_{n_1} and some special value of N_1. α here is a multi-index, defined by some particular choices of N-th roots of e-symbol in the definition of $Q_{n_1,t}$. Equation (22) is called usually the quantum separation of variables.

3. Exact integrability

3.1. **A sample Q-matrix.** We will investigate one separate $\mathbf{Q}$-matrix from the set of $\mathbf{Q}_{n_1}$-matrixes. For this we will omit for a moment the index n_1. $\widehat{\mathbf{Q}}$ as well as $\mathbf{Q}$ are given by (4,7) with $\mathbf{R}_\mathbf{n} = \mathbf{R}_{n_2,n_3}$ parameterized by (19) with $X_{n_1} = X$, $X'_{n_1} = X'$.

Besides the usual transfer matrix $\mathbf{Q}$ we need a set of degenerative transfer matrices $\mathbf{Q}(\alpha,\beta)$ defined by

$$\langle\sigma_1|\mathbf{Q}(\alpha,\beta)|\sigma'_1\rangle \;=\; \sum_{\substack{\sigma_2\,:\,\sum_{n_3}\sigma_{2:n_3}=-\beta\\ \sigma_3\,:\,\sum_{n_2}\sigma_{3:n_2}=\alpha}} \langle\sigma_1,\sigma_2,\sigma_3|\widehat{\mathbf{Q}}|\sigma'_1,\sigma_2,\sigma_3\rangle\;, \tag{23}$$

so that

$$\mathbf{Q} \;=\; \sum_{\alpha,\beta}\mathbf{Q}(\alpha,\beta)\;. \tag{24}$$

see (4,7).

Conjecture 3. *Matrices $\mathbf{Q}(\alpha,\beta)$ also form the commutative family with respect to X, X' and α,β.*

3.2. **Local Weyl algebra.** The structure of $\mathbf{Q}$, defined by its matrix elements between the vectors

$$\langle\sigma_1| = \bigotimes_{n_2,n_3} \langle\sigma_{1:n_2,n_3}| \quad\text{and}\quad |\sigma_1'\rangle = \bigotimes_{n_2,n_3} |\sigma'_{1:n_2,n_3}\rangle \tag{25}$$

belonging to the vector spaces $\mathcal{V}^*$ and $\mathcal{V}$ correspondingly, such that each σ-variable is $\mathbb{Z}_N$-invariant, implies that we may define the algebra if observables as the direct product of simple Weyl algebra at the root of unity.

Namely, let the simple Weyl algebra is formed by the $N\times N$ matrices $\mathbf{x}$ and $\mathbf{z}$, defined in the basis $|\sigma\rangle = |\sigma \mod N\rangle$, $\langle\sigma|\sigma'\rangle = \delta_{\sigma,\sigma'}$,

$$\mathbf{x}|\sigma\rangle = |\sigma\rangle q^{\sigma}\;, \quad \mathbf{z}|\sigma\rangle = |\sigma+1\rangle\;, \quad \mathbf{x}\,\mathbf{z} \;=\; q\;\mathbf{z}\mathbf{x}\;. \tag{26}$$

The corresponding components in the tensor product of $\Delta = N_2N_3$ copies of the simple Weyl algebra we will denote as

$$\mathbf{x}_{1:n_2,n_3} \;=\; 1\otimes 1\otimes \ldots\otimes \underbrace{\mathbf{x}}_{\substack{(n_2,n_3)-th\\ place}} \otimes \ldots\;, \quad \mathbf{z}_{1:n_2,n_3} \;=\; 1\otimes 1\otimes \ldots\otimes \underbrace{\mathbf{z}}_{\substack{(n_2,n_3)-th\\ place}} \otimes \ldots\;, \tag{27}$$

so that the structure of the indices in (25) matches exactly the structure of observables in (27).

3.3. **Matrix ℓ and the determinant.** Define now the operator-valued $\Delta\times\Delta$ matrix ℓ (the auxiliary linear problem coefficients matrix) as follows:

$$\begin{aligned}
&\ell_{n_2+1,n_3|n_2,n_3} \;=\; a_{n_2}\;, \quad \ell_{n_2,n_3+1|n_2,n_3} \;=\; b_{n_3}\,\mathbf{v}_{1:n_2,n_3}\;,\\
&\ell_{n_2,n_3|n_2,n_3} \;=\; -\,\mathbf{w}_{1:n_2,n_3}\;, \quad \ell_{n_2+1,n_3+1|n_2,n_3} \;=\; -\,a_{n_2}b_{n_3}\,\mathbf{u}_{1:n_2,n_3}\;,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
&\mathbf{v}_{1:n_2,n_3} \;=\; \frac{e(Y_{n_2},Z'_{n_3})}{e(Y'_{n_2},Z_{n_3})}\,\mathbf{x}_{1:n_2,n_3}\mathbf{z}_{1:n_2,n_3}\;,\\
&\mathbf{w}_{1:n_2,n_3} \;=\; \frac{e(Y_{n_2},Z_{n_3})}{e(Y'_{n_2},Z_{n_3})}\,\mathbf{z}_{1:n_2,n_3}\;, \quad \mathbf{u}_{1:n_2,n_3} \;=\; \frac{e(Y'_{n_2},Z'_{n_3})}{e(Y'_{n_2},Z_{n_3})}\,\mathbf{x}_{1:n_2,n_3}\;.
\end{aligned} \tag{29}$$

Here as usual $n_2\in\mathbb{Z}_{N_2}$, $n_3\in\mathbb{Z}_{N_3}$. a_{n_2} and b_{n_3} are extra parameters. Since the elements of different columns of the matrix ℓ commute, the formal determinant of ℓ is well defined.

Parameters a_{n_2} and b_{n_3} enters this determinant only via

$$A = \prod_{n_2} a_{n_2}\,, \quad B = \prod_{n_3} b_{n_3}\,. \tag{30}$$

The determinant is a polynomial with respect to A, B. Further we will need two normalizations of it:

$$\mathbf{J}(A,B) = \det ||\ell|| \prod_{n_2,n_3} (-a_{n_2} b_{n_3} \mathbf{u}_{1:n_2,n_3})^{-1} \tag{31}$$

and

$$\widetilde{\mathbf{J}}(A,B) = \det ||\ell|| \prod_{n_2,n_3} (-\mathbf{w}_{1:n_2,n_3})^{-1} \tag{32}$$

Matrices $\mathbf{J}$ and $\widetilde{\mathbf{J}}$ in the special points play the important role:

$$a_{n_2} = \frac{e(X,Y_{n_2})}{e(X,Y'_{n_2})}\,,\quad b_{n_3} = \frac{e(X,Z_{n_3})}{e(X,Z'_{n_3})} \Rightarrow \mathbf{J}(A,B) \stackrel{def}{=} \mathbf{J}(X)\,, \tag{33}$$

and

$$a_{n_2} = \frac{e(X',Y_{n_2})}{e(X',Y'_{n_2})}\,,\quad b_{n_3} = \frac{e(X',Z_{n_3})}{e(X',Z'_{n_3})} \Rightarrow \widetilde{\mathbf{J}}(A,B) \stackrel{def}{=} \widetilde{\mathbf{J}}(X')\,, \tag{34}$$

3.4. **Integrals of motion.** $\mathbf{J}$ defined by (31) is the polynomial with respect to A^{-1} and B^{-1}:

$$\mathbf{J}(A,B) = \sum_{m_2=0}^{N_2} \sum_{m_3=0}^{N_3} A^{-m_3} B^{-m_2}\, \mathbf{J}_{m_2,m_3}\,, \quad \mathbf{J}_{0,0} = 1\,. \tag{35}$$

Conjecture 4. *If* $\mathbf{Q}$ *is the transfer matrix prepared with the help of the same* $e(Y^{\#}_{n_2}, Z^{\#}_{n_3})$ *as* $\mathbf{J}$*, then*

$$\mathbf{J} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{J} \tag{36}$$

for any X, X' *entering* $\mathbf{Q}$ *and any* A, B *entering* $\mathbf{J}$.

Conjecture 5. *Coefficients* $\mathbf{J}_{m_2,m_3}$ *form the following algebra:*

$$\mathbf{J}_{m_2,m_3} \mathbf{J}_{m'_2,m'_3} = q^{m'_2 m_3 - m'_3 m_2}\, \mathbf{J}_{m'_2,m'_3} \mathbf{J}_{m_2,m_3}\,. \tag{37}$$

Note, $\widetilde{\mathbf{J}}$ is equivalent to $\mathbf{J}$ due to $\widetilde{\mathbf{J}} \sim \mathbf{J}\mathbf{J}_{N_2,N_3}^{-1}$.

N_2+N_3 remarkable simple matrices belong to the system of $\mathbf{J}_{m_2,m_3}$:

$$\mathbf{U}_{n_2} \;=\; \prod_{n_3} \mathbf{x}_{1:n_2,n_3}\;, \quad \mathbf{W}_{n_3} \;=\; \prod_{n_2} \mathbf{z}_{1:n_2,n_3}\;. \tag{38}$$

Since $\mathbf{U}_{n_2}\mathbf{W}_{n_3} = q\mathbf{W}_{n_3}\mathbf{U}_{n_2}$, they contain one "mass center". Let

$$\mathbf{J}_{m_2,m_3} \;=\; \mathbf{W}_0^{m_3}\mathbf{U}_0^{-m_2}\,\mathbf{t}_{m_2,m_3}\;. \tag{39}$$

Conjecture 6. *The set $\mathbf{t}_{m_2,m_3}$ is commutative, and each $\mathbf{t}_{m_2,m_3}$ commute with $\mathbf{U}_0$ and $\mathbf{W}_0$.*

Conjecture 7. *Matrices $\mathbf{t}_{m_2,m_3}$ for the perimeter of Newton's polygon $0\le m_2\le N_2$, $0\le m_3\le N_3$ are polynomials with respect to $\mathbf{U}_0^{-1}\mathbf{U}_{n_2}$ and $\mathbf{W}_0^{-1}\mathbf{W}_{n_3}$.*

Conjecture 8. *$g=(N_2-1)(N_3-1)$ nontrivial matrices $\mathbf{t}_{m_2,m_3}$ inside the Newton's polygon are algebraically independent.*

Conjecture 9. *g matrices from inside the Newton's polygon, N_2-1 matrices $\mathbf{U}_0^{-1}\mathbf{U}_{n_2}$, N_3-1 matrices $\mathbf{W}_0^{-1}\mathbf{W}_{n_3}$ and any non-degenerative combination of $\mathbf{U}_0,\mathbf{W}_0$ is the complete set of commutative matrices.*

Additionally one has

Conjecture 10.

$$\mathbf{t}_{m_2,m_3}\,\mathbf{Q}(\alpha,\beta) \;=\; \mathbf{Q}(\alpha,\beta)\,\mathbf{t}_{m_2,m_3}\;, \tag{40}$$

and moreover, $\mathbf{t}_{m_2,m_3}$ and $\mathbf{Q}(\alpha,\beta)$ may be diagonalized simultaneously.

The "mass center" simply governs the phases of A,B:

$$\mathbf{U}_0\,\mathbf{J}(A,B) \;=\; \mathbf{J}(q^{-1}A,B)\,\mathbf{U}_0\;,\quad \mathbf{W}_0\,\mathbf{J}(A,B) \;=\; \mathbf{J}(A,q^{-1}B)\,\mathbf{W}_0\;, \tag{41}$$

and analogously

$$\mathbf{U}_0\,\mathbf{Q}(\alpha,\beta) \;=\; \mathbf{Q}(\alpha-1,\beta)\,\mathbf{U}_0\;,\quad \mathbf{W}_0\,\mathbf{Q}(\alpha,\beta) \;=\; \mathbf{Q}(\alpha,\beta-1)\,\mathbf{W}_0\;. \tag{42}$$

Conjecture 11. *$\mathbf{Q}(\alpha,\beta)$ and $\mathbf{t}_{m_2,m_3}$ satisfy the following system:*

$$\mathbf{J}(X)\;\mathbf{Q}(0,0) \;=\; \mathbf{Q}(0,0)\;\widetilde{\mathbf{J}}(X') \;=\; 0\;, \tag{43}$$

where $\mathbf{J}(X)$ and $\widetilde{\mathbf{J}}(X')$ are defined by (33,34).

Conjecture 12. *In the basis of common eigenvectors of* $\mathbf{Q}(\alpha,\beta)$ *and* $\mathbf{t}_{m_2,m_3}$ *(43) is* $2N$ *equations in the matrix block* $\mathbf{U}_0$*,* $\mathbf{W}_0$*, and this system of functional equations allows one (in principle) to find the spectra of* $\mathbf{t}$ *and* $\mathbf{Q}$*.*

Each of equations (43) is exactly Baxter's $t-Q$ equation for $N_2=2$ or $N_3=2$ and generic integer N. For $N_2>2$ and $N_3>2$ (43) are equivalent to all the chain of nested Bethe ansatz system.

4. Isospectrality and separation

4.1. Modified Q-operators.

Conjecture 13. *The system of two algebraic equations for two variables* $P,P'\in\mathbf{C}$

$$\prod_{n_2=0}^{N_2-1}\frac{P'-Y'_{n_2}}{P-Y'_{n_2}}=\prod_{n_2=0}^{N_2-1}\frac{P'-Y_{n_2}}{P-Y_{n_2}}\,,\quad\prod_{n_3=0}^{N_3-1}\frac{P'-Z'_{n_3}}{P-Z'_{n_3}}=\prod_{n_3=0}^{N_3-1}\frac{P'-Z_{n_3}}{P-Z_{n_3}}\tag{44}$$

has $g=(N_2-1)(N_3-1)$ *inequivalent solutions.*

Let us enumerate these solutions as P_k, P'_k, $k=0,...,g-1$.

For some N_1 define the matrix $\mathbf{Z}_1$ by (10) with the following local parameterization of $\mathbf{R_n}$:

(45)

$$x_{1,\mathbf{n}}=q^{-1}\frac{e(X_{n_1},Z'_{n_3})e(Y_{n_2},Z_{n_3})}{e(X_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\frac{\tau_{1,\mathbf{n}}\tau_{2,\mathbf{n}+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}}}\,,\qquad y_{1,\mathbf{n}}=\frac{e(X_{n_1},Y_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}}}\,,$$

$$x_{2,\mathbf{n}}=\frac{e(X'_{n_1},Z'_{n_3})e(Y'_{n_2},Z_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}\tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}}\,,\quad y_{2,\mathbf{n}}=\frac{e(X'_{n_1},Y'_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_2}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}}\,,$$

$$x_{3,\mathbf{n}}=q^{-1}\frac{e(X_{n_1},Z'_{n_3})e(Y'_{n_2},Z_{n_3})}{e(X_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\frac{\tau_{1,\mathbf{n}+\mathbf{e}_2}\tau_{2,\mathbf{n}+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}}}\,,\quad y_{3,\mathbf{n}}=q\frac{e(X_{n_1},Y'_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X_{n_1},Z_{n_3})e(Y'_{n_2},Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{1,\mathbf{n}+\mathbf{e}_2+\mathbf{e}_3}\tau_{2,\mathbf{n}}}\,,$$

$$x_{4,\mathbf{n}}=q^{-1}\frac{e(X'_{n_1},Z'_{n_3})e(Y_{n_2},Z_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\frac{\tau_{1,\mathbf{n}}\tau_{2,\mathbf{n}+\mathbf{e}_1+\mathbf{e}_3}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}}\,,\quad y_{4,\mathbf{n}}=\frac{e(X'_{n_1},Y_{n_2})e(Z_{n_3},Z'_{n_3})}{e(X'_{n_1},Z_{n_3})e(Y_{n_2},Z'_{n_3})}\frac{\lambda_{\mathbf{n}}\tau_{3,\mathbf{n}+\mathbf{e}_1}}{\tau_{1,\mathbf{n}+\mathbf{e}_3}\tau_{2,\mathbf{n}+\mathbf{e}_1}}\,,$$

where

$$\tau^N_{1:\mathbf{n}}=H^{(g)}\left(\left\{\frac{I_k(\mathbf{n})}{\sigma_k(X_{n_1})}\right\}_{k=0}^{g-1}\right)\,,\quad\tau^N_{2:\mathbf{n}}=H^{(g)}\left(\left\{\frac{I_k(\mathbf{n})}{\sigma_k(Y_{n_2})}\right\}_{k=0}^{g-1}\right)\,,$$

(46)

$$\tau^N_{3:\mathbf{n}}=H^{(g)}\left(\left\{\frac{I_k(\mathbf{n})}{\sigma_k(Z_{n_3})}\right\}_{k=0}^{g-1}\right)\,,\quad\lambda_{\mathbf{n}}=H^{(g)}\left(\left\{\frac{I_k(\mathbf{n})\sigma_k(Z'_{n_3})}{\sigma_k(X_{n_1})\sigma_k(Y_{n_2})}\right\}_{k=0}^{g-1}\right)\,,$$

with

$$H^{(g)}(\{f_k\}_{k=0}^{g-1}) \;=\; \frac{\det|P_j^i \,-\, f_j\,P_j'^i|_{i,j=0}^{g-1}}{\prod_{i>j}(P_i \,-\, P_j)}\;,\quad\text{and}\quad \sigma_k(X) \;=\; \frac{P_k'-X}{P_k-X}\;, \tag{47}$$

and $I_k(\mathbf{n})$ defined as follows:

$$I_k(\mathbf{n}) \;=\; f_k\;\left(\prod_{j_1=0}^{n_1-1}\frac{\sigma_k(X'_{j_1})}{\sigma_k(X_{j_1})}\right)\left(\prod_{j_2=0}^{n_2-1}\frac{\sigma_k(Y'_{j_2})}{\sigma_k(Y_{j_2})}\right)\left(\prod_{j_3=0}^{n_3-1}\frac{\sigma_k(Z'_{j_3})}{\sigma_k(Z_{j_3})}\right)\;. \tag{48}$$

g coefficients f_k are generic complex numbers. Obviously this newly defined $\mathbf{Z}_1$ coincides with the previous definition if all $f_k = 0$. But now one has no commutativity of $\mathbf{Q}_{n_1}$. Instead there exists an exchange relation which is equivalent to the following proposition:

Conjecture 14. *The newly defined $\mathbf{Z}_1$ is the symmetrical matrix function with respect to any permutation of the pairs (X_{n_1}, X'_{n_1}), $n_1 = 0, ..., N_1 - 1$.*

In the newly defined $\mathbf{Z}_1$ any such permutation changes $\tau_{j,\mathbf{n}}$ and so it changes significantly the set of $\mathbf{R}_\mathbf{n}$ and so the set of $\mathbf{Q}_{n_1}$.

Corresponding determinants should be defined by the same formulae (28,31) with

(49)

$$\mathbf{v}_{1:n_2,n_3}(n_1) \;=\; \frac{e(Y_{n_2}, Z'_{n_3})}{e(Y'_{n_2}, Z_{n_3})}\,\frac{\tau_{2,\mathbf{n}}\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}\tau_{3,\mathbf{n}}}\,\mathbf{x}_{1:n_2,n_3}\mathbf{z}_{1:n_2,n_3}\;,$$

$$\mathbf{w}_{1:n_2,n_3}(n_1) \;=\; \frac{e(Y_{n_2}, Z_{n_3})}{e(Y'_{n_2}, Z_{n_3})}\,\frac{\tau_{3,\mathbf{n}+\mathbf{e}_2}}{\tau_{3,\mathbf{n}}}\,\mathbf{z}_{1:n_2,n_3}\;,\quad \mathbf{u}_{1:n_2,n_3}(n_1) \;=\; \frac{e(Y'_{n_2}, Z'_{n_3})}{e(Y'_{n_2}, Z_{n_3})}\,\frac{\tau_{2,\mathbf{n}}}{\tau_{2,\mathbf{n}+\mathbf{e}_3}}\,\mathbf{x}_{1:n_2,n_3}\;.$$

Now the determinant depends on n_1 via the dependence of $\tau_{j,\mathbf{n}}$, and we may denote it as $\mathbf{J}_{n_1}(A, B)$.

Conjecture 15. *For given ordering of (X_{n_1}, X'_{n_1}) we have*

$$\mathbf{J}_{n_1}(A, B)\;\mathbf{Q}_{n_1} \;=\; \mathbf{Q}_{n_1}\;\mathbf{J}_{n_1+1}(A, B)\;. \tag{50}$$

4.2. **Isospectrality.** Let now

$$N_1 \;=\; g \;=\; (N_2 - 1)(N_3 - 1)\;. \tag{51}$$

Change the normalization of the coefficients f_k,

$$f_k \;=\; f'_k\,\sigma_k(X_k) \tag{52}$$

and consider the limit when

$$X_k \;\mapsto\; P'_k\;, \quad f'_k \text{ being generic complex numdes.} \tag{53}$$

Then in the set of arguments of $\tau_{2,\mathbf{n}}$ and $\tau_{3,\mathbf{n}}$ the number of nonzero amplitudes $I_k(\mathbf{n})$ is exactly n_1. In particular

$$\mathbf{J}_0(A,B) \;\equiv\; \mathbf{J}(A,B)\;, \tag{54}$$

where $\mathbf{J}$ is our initial τ-independent determinant (31). Applying (50) g times, one obtains

$$\mathbf{J}(A,B)\,\mathbf{Z}_1 \;=\; \mathbf{Z}_1\,\mathbf{J}_g(A,B)\;, \tag{55}$$

giving the following

Conjecture 16. *$\mathbb{C}^g$-parametric family of operators $\mathbf{J}_g(A,B)$ is isospectral to $\mathbf{J}(A,B)$, and newly constructed $\mathbf{Z}_1$ is the similarity operator, non-degenerative for generic $\vec{f}' \in \mathbb{C}^g$.*

Let $|\Psi_t\rangle$ be the set of eigenvectors for initial $\mathbf{t}_{m_2,m_3}$, consisting $\mathbf{J}$, and $|\Psi_t^{(g)}\rangle$ be the set of eigenvectors for the final $\mathbf{t}^{(g)}_{m_2,m_3}$, consisting $\mathbf{J}_g$. Then one has

Conjecture 17.

$$\mathbf{Z}_1 \;=\; \sum_t |\Psi_t\rangle Q_{0,t}Q_{1,t}...Q_{g-1,t}\langle\Psi_t^{(g)}|\;. \tag{56}$$

4.3. **Separation.** After (52) and (53) consider further the limit when

$$f'_k \;\mapsto\; 1\;. \tag{57}$$

In this limit we may construct all the modified $\mathbf{Q}_{n_1}$-operators, but now we are interesting in the modified operators

$$\mathbf{Q}_{n_1}(\alpha) \;=\; \sum_\beta\; \mathbf{Q}_{n_1}(\alpha,\beta)\;. \tag{58}$$

Due to the summation over β, $\mathbf{W}_0\mathbf{Q}_{n_1}(\alpha) \;=\; \mathbf{Q}_{n_1}(\alpha)\mathbf{W}_0$. Let in this limit

$$\mathbf{Z}_1(\vec{\alpha}) \;=\; \mathbf{Q}_0(\alpha_0)\mathbf{Q}_1(\alpha_1)\ldots\mathbf{Q}_{g-1}(\alpha_{g-1}) \;\text{ with }\; X_k\mapsto P'_k,\; f'_k\mapsto 1\;. \tag{59}$$

Quantum separation of variables is based on

Conjecture 18. *In the limit (57) the set of final eigenvectors degenerates, and matrix* $\mathbf{Z}_1(\alpha)$ *has the following decomposition:*

$$\mathbf{Z}_1(\vec{\alpha}) \;=\; \sum_t |\Psi_t\rangle Q_{0,t}(\alpha_0)\dots Q_{g-1,t}(\alpha_{g-1})\langle\chi_{t'}|\;, \tag{60}$$

where t' *is the subset of* t *corresponding to* N_2+N_3-1 *commutative operators* $\mathbf{U}_0^{-1}\mathbf{U}_{n_2}$ *and* $\mathbf{W}_{n_3}$.

Note, $\langle\chi|$ depends on X'_{n_1} and therefore has a significant freedom in the definition. In (60) one may make the summation with respect $t\setminus t'$:

$$|\Phi_{\vec{\alpha},t'}\rangle \;=\; \sum_{t\setminus t'} |\Psi_t\rangle\, Q_{0,t}(\alpha_0)\dots Q_{g-1,t}(\alpha_{g-1})\;. \tag{61}$$

The basis of vectors $|\Phi_{\vec{\alpha},t'}\rangle$ is called the separated basis. Now we will try to explain its origin.

Turn back to the matrix $\ell_{s|v}$ (28,29). Let $\mathbf{A}_{v|s}$ be its matrix of algebraic supplements, defined by

$$\sum_{v\in\mathbb{Z}_{N_2}\times\mathbb{Z}_{N_3}} \ell_{s|v}\,\mathbf{A}_{v|s'} \;=\; \delta_{s|s'}\,\mathbf{J}(A,B)\;, \tag{62}$$

where $\mathbf{J}$ is given by (31).

"Usual" way to derive the quantum separation of variables [6, 7] is to construct a rather complicated procedure that may be described symbolically as

$$\text{“quantum resultant”}_{\text{with respect to } B}(\mathbf{J}(A,B),\mathbf{A}_{s|v}(A,B)) = \widetilde{\mathbf{b}}_v(A)\,\mathbf{b}_s(A)\;, \tag{63}$$

where $\mathbf{b}_s(A)$ is a g-th power polynomial with respect to A. (61) are the eigenvectors of $\mathbf{b}_s(A)$ for $s=(0,0)$. Fortunately one may get rid of the consideration of the "quantum resultant", choosing the following direct way.

Let $\mathcal{V}_s(A,B)$ be the subspace of the complete vector space $\mathcal{V}$ defined by

$$|\Phi\rangle\in\mathcal{V}_s(A,B) \;\Leftrightarrow\; \mathbf{J}(A,B)\,|\Phi\rangle = 0 \;\text{ and }\; \mathbf{A}_{v|s}\,|\Phi\rangle = 0\;\;\forall v\;. \tag{64}$$

Conjecture 19. $\dim\mathcal{V}_s(A,B) = N^{\Delta-2}$ *(*$\Delta = N_2N_3$*) if*

$$A = A_k \overset{def}{=} q^{\alpha_k}\prod_{n_2}\frac{e(P'_k,Y_{n_2})}{e(P'_k,Y'_{n_2})}\;,\quad B = B_k \overset{del}{=} q^{\beta_k}\prod_{n_3}\frac{e(P'_k,Z_{n_3})}{e(P'_k,Z'_{n_3})}\;, \tag{65}$$

for any $k=0,\dots,g-1$.

Let further

$$\mathcal{V}_s(k,\alpha_k) = \bigcup_{n\in\mathbb{Z}_N} \mathcal{V}_s(A_k, q^n B_k)\;, \quad \mathcal{V}_s(\vec{\alpha}) = \bigcap_{k=0}^{g-1} \mathcal{V}_s(k,\alpha_k)\;. \tag{66}$$

Obviously, due to the construction, $\dim \mathcal{V}_s(\vec{\alpha}) = N^{\Delta-g}$, and $\mathbf{b}_s(A_k)\cdot\mathcal{V}_s(\vec{\alpha}) = 0$.

Our approach to the quantum separation of variables is based technically on the following main

Conjecture 20. *The first modified operator $\mathbf{Q}_0(\alpha_0,\beta_0)$ in the sequence (59) obeys in the limit (53)*

$$\mathbf{J}(A_0,B_0)\cdot\mathbf{Q}_0(\alpha_0,\beta_0) \;=\; \mathbf{A}_{v|s}(A_0,B_0)\cdot\mathbf{Q}_0(\alpha_0,\beta_0) \;=\; 0 \quad \forall v \tag{67}$$

for $s=(n_2,n_3)$ if $\tau_{1,0\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3}=0$.

Therefore $\mathbf{Q}_0(\alpha_0)$ is the projector to $\mathcal{V}_s(0,\alpha_0)$ and due to the conjecture 14 on the page 12, eq. (59) gives the projector to $\mathcal{V}_s(\vec{\alpha})$. The limit (57) corresponds to the choice $s=(0,0)$.

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