Structure and Rigidity in Hyperbolic Groups I

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Abstract

We introduce a filtration of hyperbolic groups according to their possible actions on real trees. Using this filtration and results from the theory of (small) group actions on real trees we study the structure of hyperbolic groups and their automorphism group.

In [Gr] M. Gromov has introduced hyperbolic groups and show how geometric notions, tools and results, mostly from the theory of negatively curved manifolds, can be adapted to obtain deep and broad algebraic results on the structure of hyperbolic groups and their subgroups. Gromov's paper and a recent work of the second author on the isomorphism problem [Se1] stress the need for understanding the structure of automorphisms of hyperbolic groups and more globally the structure of the automorphism group of a hyperbolic group.

The work of the first author on group actions on real trees [Ri] seems to have more and more applications since it was introduced. In this paper we adapt results from this work to the study of hyperbolic groups and their automorphisms. Our approach is an elaboration of the Bestvina-Paulin method ([Be], [Pa]) and we believe that besides the results we obtain our arguments should be applicable for future problems. The results we get serve as key points in our generalization of the solution to the isomorphism problem [Ri-Se].

We start by introducing a natural filtration of hyperbolic groups in terms of their possible actions on real trees. This filtration, although very simple, turns to be essential in understanding automorphisms and may serve as possible induction steps for future problems. In section 2 and 3 we bring an immediate application of the Bestvina-Paulin method for the Hopf and co-Hopf property for certain hyperbolic groups.

The automorphism group of a surface group is generated by Dehn twists and inner automorphisms. In general we call an automorphism generated by the above internal (the notion was suggested to us by Benjamin Weiss). In section 4 we start developing our machinary in order to show that for torsion-free hyperbolic groups, the group of internal automorphisms is of finite index. We do that by constructing a real tree equipped with an isometric group action in case the index of the internal subgroup is infinite and then show in sections 5 and 6 such a real tree cannot be obtained by our construction. Having a complete "proof scheme", we show how to get Gromov's theorem on freely indecomposable subgroups in section 7, and in the following section we prove the automorphism group of a hyperbolic group is finitely generated.

Further structural results on hyperbolic groups, their small splittings and automorphism group appear in a continuation paper by the second author [Se3]. Application of the techniques presented in this paper to (acylindrical) accessibility of finitely generated and finitely presented groups appear in [Se2]. In [Se4] we use a modification of our approach to study automorphisms of a free group.

1. Rigidity Tower

Actions of groups on real trees suggest a natural filtration for groups which turns to be essential in studying the structure of hyperbolic groups and their automorphism groups. The filtration seems to be a key point in the solution of the isomorphism problem ([Se1], [Ri-Se]), and some of the algebraic properties we discuss in this paper are proven only for certain levels in our filtration. We believe some of the techniques presented in this paper should serve as a tool for "climbing up" in our rigidity categories also for other algebraic properties of hyperbolic groups and their subgroups.

0. Kazhdan T-groups

Kazhdan's groups are known to have no non-trivial action on a real tree [Ha-Va]. Moreover, every measurable cocycle of such groups into the automorphism group of a simplicial tree is cohomologous to a cocycle with values in the isotropy group of a point of the tree [Ad-Sp].

Although we are not making use of the special properties of Kazhdan groups, they seem to be distinguished among our next category:

1. Strictly rigid groups

A group is called strictly rigid if it admits no non-trivial action on a real tree. In addition to Kazhdan's T-groups, fundamental groups of non-Haken 3-manifolds [Mo-Sh] and of Kähler hyperbolic manifolds [Gr-Sc] form examples for such groups. Clearly, like Kazhdan's T-property, strictly rigid is a property preserved under taking quotients. The algebraic structure of these (even hyperbolic) groups is unfortunately not yet completely clear, although they are described in [Ri] using R-trees of groups. Note that in particular strictly rigid groups do not admit a non-trivial Bass-Serre splitting.

2. Rigid groups

A small action of a group on a real tree is an action that satisfies the ACC condition [Ri] and edge stabilizers do not contain a free group (in the case of hyperbolic group they are, therefore, virtually cyclic). In [Ri] small actions of groups on real trees are studied in details and the existence of a small action for a group is shown to be equivalent to some algebraic properties of the group. Rigid groups are known to have no Bass-Serre splitting with virtually cyclic edge stabilizers [Ri], they have finite outer automorphism group [Pa] and solvable isomorphism problem [Se1]. Natural examples are fundamental groups of closed negatively curved manifolds.

Rigid hyperbolic groups have finite automorphism group. To study the structure of individual automorphisms and the automorphism group for general hyperbolic group, we need to introduce the following category.

3. Weakly rigid groups

A weakly rigid group is a group for which every small action on a real tree (in the above sense) is discrete.

4. Freely indecomposable groups

Groups which do not split as a non-trivial free product. From our discussion, using extensively the results of [Ri], we show that a freely indecomposable hyperbolic group is weakly rigid if and only if it does not contain "quadratically hanging" free subgroups (see 5.1 below).

To conclude, we would like to note that subclasses of weaker rigidity categories are sometime easier to handle than stronger ones. For example, the isomorphism problem for free products of rigid hyperbolic groups was shown to be solvable in [Se1], where weakly rigid hyperbolic ones require more [Ri-Se].

2. The Hopf Property

A group is called Hopf if every homomorphism of the group onto itself is an isomorphism. A simple application of the Paulin-Bestvina method ([Pa], [Be]) give us the follwing:

Theorem 2.1 *Strictly rigid hyperbolic groups are Hopf.*

Proof: Let $\Gamma = \langle G|R\rangle = \langle g_1, \cdots, g_t | r_1, \cdots, r_s\rangle$ be a strictly rigid δ-hyperbolic group, let $\Psi : \Gamma \to \Gamma$ be an onto homomorphism with kernel, and let X be the Cayley graph of Γ with respect to its set of generators G. The epimorphisms $\Psi^m : \Gamma \to \Gamma$ are non-conjugate. For each m we pick $\gamma_0 \in \Gamma$ for which:

$$\mu_m = \max_{1 \le j \le t} \left(id, \gamma_0 \Psi^m(g_j)\gamma_0^{-1}\right) = \min_{\gamma \in \Gamma} \max_{1 \le j \le t} \left(id, \gamma \Psi^m(g_j)\gamma^{-1}\right)$$

Since the $\{\Psi^m\}_{m=1}^{\infty}$ are non-conjugate there exists a subsequence (still denoted Ψ^m) for which $\mu_m \to \infty$. Let $\{(X_m, id)\}_{m=1}^{\infty}$ be the pointed metric spaces obtained from the Cayley graph X_m by dividing the metric on X by μ_m . (X_m, id) is endowed with a left isometric action of Γ via $\gamma_0 \Psi^m \gamma_0^{-1}$. At this stage we can apply the following.

Theorem 2.2 ([Pa], 2.3) *Let $\{X_m\}_{m=1}^{\infty}$ be a sequence of δ_m-hyperbolic spaces with $\delta_\infty = \underline{\lim}\delta_m < \infty$. Let G be a countable group isometrically acting on X_m . Suppose there exists a base point u_m in X_m such that for every finite subset P of G, the closed convex hull of the images of u_m under P is compact and these convex hulls are totally bounded metric spaces. Then there is a subsequence converging in the Gromov topology to a $50\delta_\infty$-hyperbolic space X_∞ endowed with an isometric action of G.*

Our spaces $\{(X_m, id.)\}_{m=1}^{\infty}$ satisfy the assumptions of the theorem (see [Pa]) and they are $\frac{\delta}{\mu_m}$ hyperbolic, so X_∞ is a real tree endowed with an isometric action of Γ , a contradiction to Γ being strictly rigid.

□

Corollary 2.3 *Let M be an irreducible 3-manifold with a Gromov hyperbolic fundamental group. Then $\pi_1(M)$ is Hopf.*

Proof: If M is Haken or $\pi_1(M)$ is elementary, $\pi_1(M)$ is residually finite and therefore Hopf. Otherwise the corollary follows from Morgan-Shalen [Mo-Sh1], and the previous theorem.

□

Question: Is every (torsion-free) hyperbolic group Hopf?

3. The co-Hopf property

A group is called co-Hopf if every monomorphism of the group into itself is an isomorphism. A simple application of the simplicial volume [Gr3] shows that the fundamental group of a closed negatively curved manifold is co-Hopf. Gnerealizing this observation we have:

3.1 *Let Γ be a rigid hyperbolic group. Then Γ is co-Hopf.*

Proof: Let φ be a monomorphism of Γ into itself which is not an isomorphism. Clearly we have:

$$\forall n \in N \quad \varphi^{n+1}(\Gamma) \subset \varphi^{n}(\Gamma) \quad \varphi^{n+1}(\Gamma) \neq \varphi(\Gamma) .$$

Since a rigid group admits only finitely many monomorphisms into a hyperbolic group up to conjugation ([Se1], 5.1), we have integers k and l and $g \in \Gamma$ such that for all $\gamma \in \Gamma$ we have:

$$g\varphi^{k}(\gamma)g^{-1} = \varphi^{k+l}(\gamma)$$

Therefore, conjugating $\varphi^{k}(\gamma)$ by g is equivalent to map it via φ^{l} , so we have:

$$g\varphi^{k+l}(\gamma)g^{-1} = \varphi^{l}\left(\varphi^{k+l}(\gamma)\right) =$$
$$\varphi^{l}\left(g\varphi^{k}(\gamma)g^{-1}\right) = \varphi^{l}(g)\varphi^{k+l}(\gamma)\varphi^{l}\left(g^{-1}\right)$$

which implies $\varphi^{l}(g) = g$ since Γ is not elementary. But this shows $g \in \varphi^{n}(\Gamma)$ for all integers n, so it acts on $\varphi^{k}(\Gamma)$ as inner automorphism, and we have:

$$\varphi^{k}(\Gamma) = \varphi^{k+l}(\Gamma)$$

a contradiction.

□

Remark: It seems very plausible that every (torsion-free) freely indecomposable hyperbolic group is co-Hopf. Our description of automorphisms of hyperbolic groups might help attacking this problem.

4. Internal Automorphisms

To certain extent the "structure" of a group is reflected in its automorphism group and vice versa. A rigid hyperbolic group has a finite outer automorphism group. The outer automorphism group of a closed surface on the other hand is finitely presented ([Ha-Th], [Mc], [Wa]) and generated by Dehn twists ([Li]). The notion of a Dehn twist can be made purely algebraic as an automorphism obtained naturally from an amalgamated product or an HNN extension over a cyclic group. The subgroup of the automorphism group generated by inner automorphisms and Dehn twists will be called (as suggested to us by Benjamin Weiss) the subgroup of internal automorphisms. Our goal in the next following sections is to show the group of internal automorphisms is of finite index in the automorphism group of a torsion-free hyperbolic group, and in parallel to get that the automorphism group of such groups is finitely generated. Our approach is a variation of the Bestvina-Paulin method ([Be], [Pa]) which is an elaborate application of the Gromov topology on metric spaces, joined with results from the work of the first author on (small) actions of groups on real trees [Ri]. This approach is adapted in [Ri-Se] in order to generalize the solution to the isomorphism problem given in [Se1], and to study acylindrical splittings of f.g. groups in [Se2].

Let $\Gamma = \langle G|R\rangle = \langle g_1, \cdots, g_t | r_1, \cdots, r_s\rangle$ be a torsion-free freely indecomposable δ-hyperbolic group, let X be the Cayley graph of Γ with respect to the generating set G and let I_Γ be the group of internal automorphisms of Γ. For each automorphism $\Psi \in \mathrm{Aut}(\Gamma)$ we pick a "shortest representative" in the left coset ΨI_Γ, $\widetilde{\Psi}$, that satisfy:

$$\max_{1\leq j\leq t} |\tilde{\Psi}(g_j)| = \min_{\varphi\in I_\Gamma} \max_{1\leq j\leq t} |\Psi \circ \varphi(g_j)|$$

Clearly, in a given left coset ΨI_Γ there are only finitely many shortest representatives. Now assume $[\mathrm{Aut}(\Gamma) : I_\Gamma] = \infty$ and let $\{\Psi_m\}_{m=1}^\infty$ be shortest representatives for distinct left cosets of I_Γ. Let:

$$\mu_m = \max_{1\leq j\leq t} |\Psi_m(g_j)|$$

Since Ψ_m is determined by the image of the generators $\{g_j\}_{j=1}^t$, and the Ψ_m do belong to distinct I_Γ left cosets, we have $\mu_m \underset{m\to\infty}{\longrightarrow} \infty$.

Let $(X_m, id.)$ denotes the pointed metric space obtained from the pointed metric space $(X, id.)$ by dividing the metric on X by μ_m. The space $(X_m, id.)$ is equipped with a Γ action via the automorphism Ψ_m.

Proposition 4.1 *There exists a subsequence (still denoted $(X_m, id.)$) that converges in the Gromov topology on metric spaces to a real tree (Y, y_0).*

Proof: $\{(X_m, id.)\}_{m=1}^\infty$ satisfy the assumptions of theorem 2.2 (see [Pa]) so there exists a converging subsequence. Any limit of a subsequence of the above spaces is 0-hyperbolic (since X_m is $\frac{\delta}{\mu_m}$ hyperbolic and $\mu_m \to \infty$), so it is a real tree equipped with a Γ-action. □

Proposition 4.2

(i) *Stabilizers of segments of Y are cyclic.*
(ii) *Stabilizers of tripods (convex hull of 3 points which are not on a segment) are trivial.*

Proof: (i) is identical with proposition 2.4 of [Pa]. To prove (ii) let $\{A, B, C\}$ be a tripod in Y and let N be the three valence vertex in the tripod $\{A, B, C\}$. Let $\gamma \in \Gamma$ fix our tripod and let $A_m, B_m, C_m \in X_m$ be triples of points converging to A, B, C in correspondance. Let:

$$l = \min\{d_Y(A,N),\ d_Y(B,N),\ d_Y(C,N)\}$$

From the convergence of the metric spaces $\{(X_m, id.)\}_{m=1}^\infty$ to (Y, y_0) we have for large enough m:

$$\max\{d_{X_m}(A_m, \Psi_m(\gamma)(A_m)),\ d_{X_m}(B_m, \Psi_m(\gamma)(B_m)),$$
$$d_{X_m}(C_m, \Psi_m(\gamma)(C_m))\} < \frac{l}{100\, t^{10\delta}}$$

Let N_m be three valence vertex of a (geodesic) approximating tree with vertices A_m, B_m, C_m . By the inequality above for m large we have:

$$d_{X_m}(\Psi_m(\gamma^s)(N_m), N_m) < \frac{8\delta}{\mu_m} \quad s = 1, \cdots, t^{10\delta}$$

Therefore, there exist $s_1 \neq s_2$ for which $\Psi_m(\gamma^{s_1})(N_m) = \Psi_m(\gamma^{s_2})(N_m)$ which clearly implies $\gamma^{s_2 - s_1} = 1$, but our group Γ was assumed torsion-free so $\gamma = 1$.

□

Proposition 4.3 *Let* $[y_1, y_2] \subset [y_3, y_4]$ *be segments of* Y *and assume* $\text{stab}([y_3, y_4]) \neq 1$. *Then:*

$$\text{stab}([y_1, y_2]) = \text{stab}([y_3, y_4])$$

Proof: By proposition 4.2 the stabilizer of $[y_3, y_4]$ is cyclic. Let $\gamma_1 \in \text{stab}([y_3, y_4])$ and $\gamma_2 \in \text{stab}([y_1, y_2]) \setminus \text{stab}([y_3, y_4])$. Clearly γ_1 commutes with γ_2 . On the other hand (assume w.l.o.g. $y_3 \notin \text{fix}(\gamma_2)$):

$$\gamma_2\gamma_1(y_3) = \gamma_2(y_3); \quad \gamma_2(y_3) \neq y_3$$

But if $\gamma_1\gamma_2(y_3) = \gamma_2(y_3)$ then γ_1 fixes the tripod $\{y_2, y_3, \gamma_2(y_3)\}$ which contradicts proposition 4.2.

□

The combination of propositions 4.1 and 4.2 shows the action of Γ on the real tree Y satisfies the ACC condition of [Ri] so it enables analyzing the action using the classification of small actions on real trees obtained in that paper. In [Ri] the real tree Y is divided into distinct components, where on each component a subgroup of Γ acts according to one of the following dynamics:

(i) Indiscrete action of the free group (e.g. Levitt type).
(ii) Interval exchange transformation.
(iii) Axial components.
(iv) Discrete action.

Our aim is to show that for each of these components either Γ splits as a non-trivial free product, or we obtain a contradiction to Ψ_m being shortest representatives in their cosets by constructing automorphisms $\varphi_m \in I_\Gamma$ (for large enough m) such that:

$$\max_{1 \leq j \leq t} |\Psi_m \varphi_m(g_j)| < \max_{1 \leq j \leq t} |\Psi_m(g_j)|$$

Remark A similar (although technically somewhat different) discussion appears in [Se2] for the study of acylindrical splittings of groups.

Indiscrete actions of the free group

Since stabilizers of tripods are trivial by proposition 4.3 and since the stabilizer of a segment is the stabilizer of the whole component in the above case by [Ri], the stabilizers of segments are trivial. Therefore, our group Γ splits as a free product $\Gamma = A * F_n$ where F_n is a free group on n generators. Γ was assumed freely indecomposable, so Y contains no components with an indiscrete action of the free group.

Axial components

If an axial component is not isometric to a real line, then by the above argument stabilizers of segments are trivial. Therefore, by [Ri] our group Γ has the form $\Gamma = A *_Z F_2$ in this case (since we assumed Γ is torsion-free and Γ contains no Z^2), so we treat it as a special case of an interval exchange transformation discussed in the following section.

The subgroup corresponds to an axial component which is isometric to a real line is solvable and it has Z^2 as a quotient. But the only solvable subgroups of torsion-free hyperbolic groups are cyclic, so real line axial components do not occur in Y.

In the next section we treat the IET components and show how to shorten all generators supported in part on these components, so we are left with the discrete case and the standard Bass-Serre theory. This last case is studied in section 6. The whole procedure described in this and the following two sections will serve us in getting other results about the structure of hyperbolic groups and their automorphism groups in the preceeding sections and in [Se3].

5. The IET components

Having our limit real tree (Y, y_0) , our aim is to find an automorphism $\varphi \in I_\Gamma$ such that all generators g_j supported in part on IET components will get shorter in Y, i.e.:

$$d_Y(\varphi(g_j)(y_0), y_0) < d_Y(g_j(y_0), y_0)$$

Achieving such a shortening automorphism we are left with the discrete case which is handled in the next section. Combining the two sections we get a sequence of automorphisms $\varphi_m \in I_\Gamma$ such that for large enough m :

$$\max_{1 \le j \le t} d_{X_m}(\Psi_m \circ \varphi_m(g_j), id.) < \max_{1 \le j \le t} d_{X_m}(\Psi_m(g_j), id.)$$

which clearly contradicts our choice of the automorphisms Ψ_m and, therefore, we obtain a contradiction to our initial assumption: $[\mathrm{Aut}(\Gamma) : I_\Gamma] = \infty$.

According to the work of the first author on group actions on real trees [Ri], the fundamental group S of an IET component T with trivial edge stabilizers is Fuchsian, and covered by a corresponding IET action of the free group:

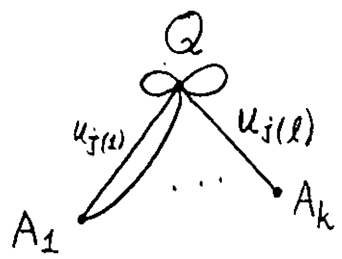
$$\begin{array}{ccc} F_n \times T_1 & \rightarrow & T_1 \\ (\nu, \mu) \downarrow & & \downarrow \mu \\ S \times T & \rightarrow & T \end{array}$$

where there exists a homomorphism $\nu : F_n \rightarrow S$ such that:

$$\forall f \in F_n \qquad \forall t_1 \in T_1 \qquad \mu(f(t_1)) = \nu(f)(\mu(t_1))$$

In this section we do not need to assume our hyperbolic group Γ and in particular the fundamental group of the IET component S are torsion-free, and in fact in [Se] we use the argument and results of this section in the context of f.g. groups with no 2-torsion. In our present situation of torsion-free groups we have the following notion which plays a central role in the study of dynamics of individual automorphisms and the algebraic structure of the automorphism group (see [Se3]).

Definition 5.1 *Let Γ be a freely indecomposable, torsion-free hyperbolic group. A quadratically hanging free group Q of Γ is a finitely generated free group $Q \simeq F_n$ such that Γ admits a graph of groups:*



where $j(i) \leq m$ and Q admits one of the following two presentations (the u_i 's are conjugate of the s_i 's):

(i)
$$Q = \left\langle s_1, \cdots, s_m, a_1, \cdots, a_g, b_1, \cdots, b_g \middle| \prod_{i=1}^{m} s_i \prod_{j=1}^{g} [a_j, b_j] = 1 \right\rangle$$

(ii)
$$Q = \left\langle s_1, \cdots, s_m, v_1, \cdots, v_g \middle| \prod_{i=1}^{m} s_i \prod_{j=1}^{g} v_j^2 = 1 \right\rangle$$

In section 9 we show that a freely indecomposable torsion-free hyperbolic group is weakly rigid if and only if it does not contain quadratically hanging free groups. For much stronger results on the role of quadratically hanging free groups in the structure of the automorphism group of a torsion-free hyperbolic group see [Se3].

Our aim is to shorten generators supported in part on IET components. To do that we find automorphisms of the fundamental group S of an IET component T such that the intersections between the segment $[y_0, g_j(y_0)]$ and the disjoint union of shifts of the corresponding IET components T are strictly shorter (if positive) for $[y_0, \varphi_T(g_j)(y_0)]$. The length of the intersection between $[y_0, g_j(y_0)]$ and the discrete part and other IET components remains unchanged. By setting $\varphi = \varphi_{T_1} \circ \cdots \circ \varphi_{T_q}$ where $T_1, \cdots, T_q$ denote all the (conjugacy classes of) IET components in Y we achieve our goal.

Let T be a fixed IET component with fundamental group S. Suppose $[y_0, g_j(y_0)]$ is supported in part on at least one of the conjugates of T and let:

$$\Gamma_T = \{\gamma \in \Gamma \mid \ |\gamma T \gamma^{-1} \cap [y_0, g_j(y_0)]| > 0\}$$
$$\varepsilon_T = \min_{\gamma \in \Gamma_T} |\gamma T \gamma^{-1} \cap [y_0, g_j(y_0)]|$$

and S be given by one of the standard presentations (for simplicity we assume no reflections):

$$\left\langle s_1, \cdots s_m, a_1, b_1, \cdots a_g, b_g \middle| s_1^{-h_1}, \cdots, s_m^{-h_m}, \prod_{i=1}^{m} s_i \prod_{j=1}^{g} [a_j, b_j] \right\rangle$$

or:

$$\left\langle s_1, \cdots s_m, v_1, \cdots, v_g | s_1^{-h_1}, \cdots, s_m^{-h_m}, \prod_{i=1}^{m} s_i \prod_{j=1}^{g} v_j^2 \right\rangle$$

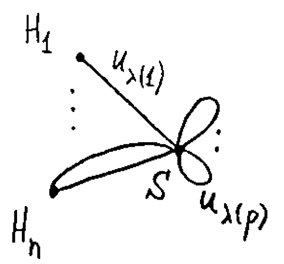
Now, let P_T be the set of all possible permutations σ on $2m+g$ or $m+g$ symbols (in correspondence) such that there exists an IET transformation (see [Ke] or [Ri] for definition) with permutation σ that gives a real tree T with S as fundamental group and $s_1, \cdots, s_m$ as non-conjugate stabilizers of vertices of T. With each such real tree T we get a natural presentation for the group S where the generators are the elements of S which correspond to the generators of the pseudogroup defined by the IET transformation. Clearly, this presentation depends only on the permutation $\sigma \in P_T$, so let χ_T be the maximal length of a generator in our standard presentations under all $\sigma \in P_T$:

$$\chi_T = \max_{\sigma \in P_T} \max_{i,j} (|s_i|, |a_j|, |b_j|)$$

or:

$$\chi_T = \max_{\sigma \in P_T} \max_{i,j} (|s_i|, |v_j|)$$

Let Λ be the graph of groups corresponds to the action of Γ on the real tree Y according to the first author's Bass-Serre theory for real trees [Ri]:



Let $x_r = \mathrm{Fix}(u_r) \cap T$, and suppose (w.l.o.g.) $y_0 \in T$ or x_{r_0} is the closest point on T to y_0. By taking appropriate conjugates we may assume:

$$\mathrm{diam}\{x_i\}_{i=1}^{p} < \frac{\varepsilon}{100}$$

and if $y_0 \in T$ then: $d_Y(y_0, x_1) < \frac{\varepsilon}{100}$ since the IET component T is minimal and orbits are dense ([Ri], [Ke]).

Each generator g_j can be represented as a word in the vertex stabilizers i.e.:

$$g_j = \prod_{i=1}^{k(j)} h_i^j w_i^j f_i^j$$

where $h_i^j \in \bigcup_{m=1}^{p} H_m$; $w_i^j \in S$ and f_i^j are generators of loops in the above grpah of groups (some of the h_i^j, w_i^j, f_i^j may be the identity). Let:

$$L = \max_{1 \le j \le t} \max_{1 \le i \le k(j)} |w_i^j|$$

where $|w_i^j|$ is the word length of w_i^j in the standard presentation for S.

Since the action of S on T is minimal (orbits are dense), in any subinterval we can represent the action of S as a pseudogroup, from which the action of S on T can be reconstructed [Ri]. By taking the interval to be of size $\frac{\varepsilon}{100L_{XT}}$ around y_0 if $y_0 \in T$ or around x_{r_0}, where x_{r_0} is the closest point to y_0 on T, we get an automorphism φ_T of S that satisfies:

$$d_Y\left(\varphi_T\left(w_i^j(x_{r_1}), x_{r_2}\right)\right) < \frac{\varepsilon}{20}$$

Therefore, we have strictly reduced the intersection between $[y_0, \varphi_T(g_j)(y_0)]$ and the conjugates of the IET component T in comparison with $[y_0, g_j(y_0)]$. Since all the automorphism group of S is internal (i.e. generated by Dehn twists and inner automorphisms [ZVC]), and the intersection of y_0 with the other IET components and with the discrete parts remains unchanged we get the following.

Theorem 5.2 *Let T be an IET component of the limit real tree Y. There exists an internal automorphism φ_T of Γ such that for generators $g_j \in G$ where $[y_0, g_j(y_0)]$ is supported in part on conjugates of T we have:*

$$d_Y(y_0, g_j(y_0)) > d_Y(y_0, \varphi_T(g_j)(y_0))$$

By composing automorphisms $\{\varphi_{T_i}\}_{i=1}^{q}$ for all distinct IET components we get:

Corollary 5.3 *There exists an internal automorphism φ of Γ such that for all generators $g_j \in G$ with $[y_0, g_j(y_0)]$ supported in part on IET components we have:*

$$d_Y(y_0, g_j(y_0)) > d_Y(y_0, \varphi(g_j)(y_0))$$

6. The discrete case

Showing how to make all generators supported in part on IET components shorter, we are left with a discrete action of Γ on (Y, y_0), the standard Bass-Serre theory. In this case we do not find an automorphism $\varphi \in I_\Gamma$ that makes the action on Y "shorter", but we do find automorphisms $\varphi_m \in I_\Gamma$ that makes $\Psi_m \circ \varphi_m$ "shorter" for large enough m. This again contradicts the way the automorphisms Ψ_m were chosen, and we obtain a contradiction to our basic assumption on the infinity of the index of the group of internal automorphisms I_Γ in $\mathrm{Aut}(\Gamma)$. The whole argument described in this section is very similar to the one given in [Se2] for the discrete case.

Since Γ is assumed freely indecomposable, stabilizers of edges can not be trivial, so by proposition 4.1 they are infinite cyclic. The treatment in this case is divided into several cases according to y_0 being in the interior of an edge of Y and in the first case we divide our argument to a splitting and non-splitting edge in the corresponding (Bass-Serre) graph of groups.

Case 1A $y_0 \in e$ and $\bar{e} \in Y/\Gamma$ is a splitting edge.

Let C be the cyclic subgroup of Γ that fixes the edge e. By the construction of the tree Y we get $\Gamma = A *_C B$ where C is strictly included in both A and B (in fact C is of infinite index in both). Given the above splitting, for each generator $g_j \in G$ we have the following presentation:

$$g_j = a_j^1 b_j^1 \cdots a_j^{n_j} b_j^{n_j} \qquad a_j^i \in A \qquad b_j^i \in B$$

(where a_j^1 and $b_j^{n_j}$ may be the identity element).

Let $z \in C$ be a generator of the cyclic subgroup C, and let ε be the minimum between the shortest edge of Y and the distances between y_0 and the vertices of e. By the convergence of the metric spaces $(X_m, id.)$ to (Y, y_0) in the Gromov topology on metric spaces, we have the following inequalities for large enough m :

$$
\begin{aligned}
& d_{X_m}(\Psi_m(z^s), id.) < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m(z^s a_j^i z^{-s}), id.\right) - d_Y\left(a_j^i(y_0), y_0\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m(z^s b_j^i z^{-s}), id.\right) - d_Y\left(b_j^i(y_0), y_0\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m\left(z^{s_1} a_{j_1}^{i_1} z^{-s_1}\right), \Psi_m(z^{s_2} a_{j_2}^{i_2} z^{-s_2})\right) \\
& \qquad - d_Y\left(z^{s_1} a_{j_1}^{i_1} z^{-s_1}(y_0), z^{s_2} a_{j_2}^{i_2} z^{-s_2}(y_0)\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m\left(z^{s_1} b_{j_1}^{i_1} z^{-s_1}\right), \Psi_m\left(z^{s_2} b_{j_2}^{i_2} z^{-s_2}\right)\right) \\
& \qquad - d_Y\left(z^{s_1} b_{j_1}^{i_1} z^{-s_1}(y_0), z^{s_2} b_{j_2}^{i_2} z^{-s_2}(y_0)\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m\left(z^{s_1} a_{j_1}^{i_1} z^{-s_1}\right), \Psi_m\left(z^{s_2} b_{j_2}^{i_2} z^{-s_2}\right)\right) \\
& \qquad - d_Y\left(z^{s_1} a_{j_1}^{i_1} z^{-s_1}(y_0), z^{s_2} b_{j_2}^{i_2} z^{-s_2}(y_0)\right)| < \varepsilon_1
\end{aligned}
\tag{6.1}
$$

where $\varepsilon_1 = \frac{\varepsilon}{20\delta \cdot t^{2\delta} 400}$; $\quad 0 \leq |s| < 20\delta \cdot t^{2\delta}$

Lemma 6.1 *Let* $w_m \in \left[id., \Psi_m\left(a_j^1\right)\right]$; $\quad w_m^{'} \in \left[id., \Psi_m\left(b_j^1\right)\right]$ *satisfy:*

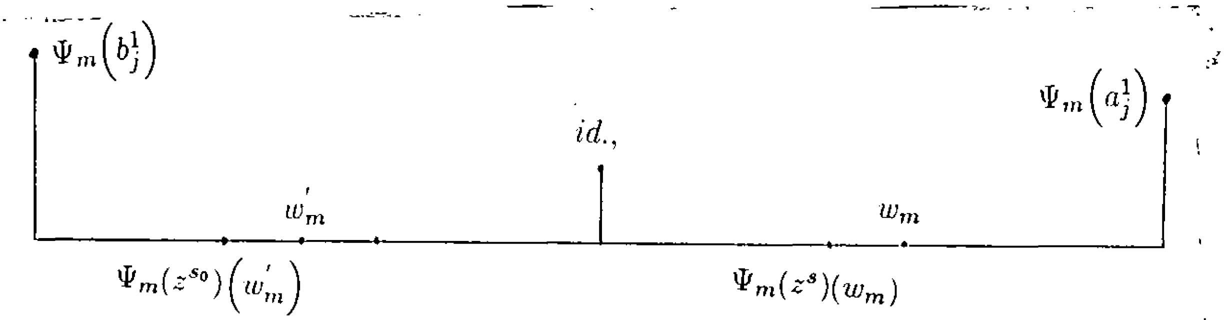
$$d_{X_m}(w_m, id.) = d_{X_m}\left(w_m^{'}, id.\right) = \frac{\varepsilon}{2} .$$

Then for m large enough (so that inequalities (6.1) hold) and for some s_0, $\; 1 \leq s_0 \leq 20\delta t^{2\delta}$:

$$
\begin{aligned}
& d_X(w_m, \Psi_m(z^{s_0})(w_m)) > 10\delta \\
& d_X\left(w_m^{'}, \Psi_m(z^{s_0})\left(w_m^{'}\right)\right) > 10\delta .
\end{aligned}
$$

Proof: $\Psi_m(z^s)(w_m)$ is 2δ-close to a geodesic segment $\left[id., \Psi_m\left(a_j^1\right)\right]$. Therefore, a simple pigeon-hole argument proves the lemma.

□



Proposition 6.2 *Assume (w.l.o.g.) that for the* s_0 *of the previous lemma:*

$$d_X(id., \Psi_m(z^{s_0})(w_m)) < d_X(id., w_m) - 8\delta$$
$$d_X\left(id., \Psi_m(z^{s_0})\left(w_m'\right)\right) > d_X\left(id., w_m'\right) + 8\delta$$

Then for all $1 \le j \le t;\quad 1 \le i \le n_j$ *we have:*

$$d_X\left(id., \Psi_m\left(z^{s_0} a_j^i z^{-s_0}\right)\right) < d_X\left(id., \Psi_m\left(a_j^i\right)\right) - 8\delta$$
$$d_X\left(id., \Psi_m\left(z^{-s_0} b_j^i z^{s_0}\right)\right) < d_X\left(id., \Psi_m\left(b_j^i\right)\right) - 8\delta$$

Proof: By inequalities (6.1):

$$\begin{aligned} d_X\left(id., \Psi_m\left(z^{s_0} a_j^i z^{-s_0}\right)\right) &\le d_X(id., \Psi_m(z^{s_0})(w_m)) \\ &+ d_X\left(\Psi_m(z^{s_0})(w_m), \Psi_m\left(z^{s_0} a_j^i\right)(w_m)\right) + \\ &+ d_X\left(\Psi_m\left(z^{s_0} a_j^i\right)(w_m), \Psi_m\left(z^{s_0} a_j^i z^{-s_0}\right)\right) \\ &< d_X(id., w_m) + d_X\left(w_m, \Psi_m\left(a_j^i\right) w_m\right) + \\ &+ d_X\left(\Psi_m\left(a_j^i\right)(w_m), \Psi_m\left(a_j^i\right)\right) - 16\delta \\ &< d_X\left(id., \Psi_m\left(a_j^i\right)\right) - 8\delta \end{aligned}$$

A similar argument proves the inequality for the b_j^i 's.

□

Theorem 6.3 *Let* φ *be the* Γ*-automorphism defined by:*

$$\begin{array}{ll} \forall a \in A & \varphi(a) = z^{s_0} a z^{-s_0} \\ \forall b \in B & \varphi(b) = z^{-s_0} b z^{s_0} \end{array}$$

Then:

$$\max_{1 \le j \le t} d_X(id., \Psi_m \circ \varphi(g_j)) < \max_{1 \le j \le t} d_X(id., \Psi_m(g_j))$$

Proof: Clearly both maxima are obtained for $g_j \notin C$. From the inequalities (6.1) and proposition 6.2 we have:

$$d_X\left(id., \Psi_m \circ \varphi\left(a_j^1 \cdots b_j^i\right)\right) < d_X\left(id., \Psi_m\left(a_j^1 \cdots b_j^i\right)\right) - 2i \cdot (4\delta)$$

and the theorem follows (cf. [Se2] ch. 2).

□

Theorem 6.3 contradicts our choice of the automorphism Ψ_m . Therefore, our pointed limit tree (Y, y_0) does not fall into case 1A.

Case 1B $y_0 \in e$ and $\overline{e} \in Y/\Gamma$ is a non-splitting edge.

Let C be the cyclic subgroup that fixes the edge e, and let $\Gamma = A*_C$. Let z be a generator of C, and let $f \in \Gamma$ be a (Bass-Serre) element corresponding to a simple loop containing the edge $\overline{e}$. For each generator $\{g_j\}_{j=1}^t$ we have:

$$g_j = a_j^1 f^{k_j^1} \cdots a_j^{n_j} f^{k_j^{n_j}}$$

where a_j^1 or $f^{k_j^{n_j}}$ may be the identity element.

Let $n = \max_{i,j}\left(k_j^i\right)$ and let ε be the minimum between the shortest edge of Y and the distances between y_0 and the vertices of e. By the convergence of the metric spaces $(X_m, id.)$ to (Y, y_0) in the Gromov topology on metric spaces, we have the following inequalities for large enough m :

$$
\begin{aligned}
& d_{X_m}(\Psi_m(z^s), id.) < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m\left(a_j^i\right), id.\right) - d_Y\left(a_j^i(y_0), y_0\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m\left((fz^s)^k\right), id.\right) - d_Y\left(f^k(y_0), y_0\right)| < \varepsilon_1 \\
((6.2)\qquad & |d_{X_m}\left(\Psi_m\left(a_j^1(fz^s)^{k_j^i} \cdots a_j^i(fz^s)^{k_j^i}\right), id.\right) - \\
& \qquad d_Y\left(a_j^1 f^{k_j^i} \cdots a_j^i f^{k_j^i}(y_0), y_0\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m\left(a_{j_1}^{i_1}\right), \Psi_m\left(a_{j_2}^{i_2}\right)\right) - d_Y\left(a_{j_1}^{i_1}(y_0), a_{j_2}^{i_2}(y_0)\right)| < \varepsilon_1 \\
& |d_{X_m}\left(\Psi_m\left(a_j^i\right), \Psi_m\left((fz^s)^k\right)\right) - d_Y\left(a_j^i(y_0), f^k(y_0)\right)| < \varepsilon_1
\end{aligned}
$$

where $\varepsilon_1 = \frac{\varepsilon}{60\delta \cdot t^{2\delta} \cdot 100}$; $\quad -n \leq k \leq n$; $\quad |s| < 20\delta \cdot t^{2\delta}$.

<u>**Lemma 6.4**</u> *Let* $w_m \in [id., \Psi_m(f)]$; $\quad w_m' \in \left[id., \Psi_m\left(a_j^1\right)\right]$ *satisfy:*

$$d_{X_m}(w_m, id.) = d_{X_m}\left(w_m', id.\right) = \frac{\varepsilon}{2}$$

Then for m *large enough (so that inequalities (6.2) hold) and for some* s_0; $\;1 \leq s_0 \leq 20\delta t^{2\delta}$:

$$
\begin{aligned}
& d_X(w_m, \Psi_m(z^{s_0})(w_m)) > 20\delta \\
& d_X\left(w_m', \Psi_m(z^{s_0})\left(w_m'\right)\right) > 20\delta
\end{aligned}
$$

Proof: identical to the proof of lemma 6.1.

□

<u>**Proposition 6.5**</u> *Assume (w.l.o.g.) that for the* s_0 *of the previous lemma:*

$$
\begin{aligned}
& d_X(id., \Psi_m(z^{s_0})(w_m)) < d_X(id., w_m) - 18\delta \\
& d_X\left(id., \Psi_m(z^{s_0})\left(w_m'\right)\right) > d_X\left(id., w_m'\right) + 18\delta\ .
\end{aligned}
$$

Then for all $1 \leq j \leq t$; $\;1 \leq i \leq n_j$ *we have:*

(i) $$d_X\left(\Psi_m(z^{s_0}), \Psi_m\left(fz^{3s_0}\right)(\Psi_m(z^{s_0}))\right) < |\Psi_m(f)| - 20\delta$$

(ii) $$d_X\left(\Psi_m(z^{s_0}), \Psi_m\left(a_j^i\right)(\Psi_m(z^{s_0}))\right) < |\Psi_m\left(a_j^i\right)| - 20\delta$$

Proof: By inequalities (6.2):

$$
\begin{aligned}
& d_X\left(\Psi_m(z^{s_0}), \Psi_m\left(fz^{3s_0}\right)\right) \leq d_X(\Psi_m(z^{s_0}), w_m) + \\
& \quad + d_X\left(w_m, \Psi_m(f)\left(w_m'\right)\right) + d_X\left(\Psi_m(f)\left(w_m'\right), \Psi_m\left(fz^{3s_0}\right)\right) \\
& \leq |w_m| + d_X(\Psi_m(z^{s_0})(w_m), w_m) + \\
& \quad + d_X\left(w_m, \Psi_m(f)\left(w_m'\right)\right) + |\Psi_m\left(z^{-3s_0}\right)\left(w_m'\right)| \\
& \leq |w_m| + d_X\left(w_m, \Psi_m(f)\left(w_m'\right)\right) + \\
& \quad + |w_m'| - 30\delta \leq |\Psi_m(f)| - 20\delta
\end{aligned}
$$

To prove (ii) we have:

$$\begin{aligned}
&d_X\left(\Psi_m(z^{s_0}), \Psi_m\left(a_j^i z^{s_0}\right)\right) \le d_X\left(w_m^{'}, \Psi_m(z^{s_0})\right)\\
&+ d_X\left(w_m^{'}, \Psi_m\left(a_j^i\right)\left(w_m^{'}\right)\right) + d_X\left(\Psi_m\left(a_j^i\right)\left(w_m^{'}\right), \Psi_m\left(a_j^i z^{s_0}\right)\right)\\
&\le 2|w_m^{'}| + d_X\left(w_m^{'}, \Psi_m\left(a_j^i\right)\left(w_m^{'}\right)\right) - 28\delta \le\\
&\le \left|\Psi_m\left(a_j^i\right)\right| - 20\delta
\end{aligned}$$

□

Theorem 6.6 *Let φ be the Γ-automorphism defined by:*

$$\forall a \in A \qquad \varphi(a) = a$$
$$\varphi(f) = f z^{3s_0}$$

Then:

$$\max_{1\le j\le t} d_X(\Psi_m(z^{s_0}), \Psi_m \circ \varphi(g_j)(\Psi_m(z^{s_0}))) < \max_{1\le j\le t} d_X(id., \Psi_m(g_j))$$

Proof: Assume $\tau = \tau_0 a_j^i$ is a subword of the reduced form for g_j and:

$$d_X(\Psi_m(z^{s_0}), \Psi_m \circ \varphi(\tau_0)(\Psi_m(z^{s_0}))) \le d_X(id., \Psi_m(\tau_0))$$
$$d_X(\Psi_m(z^{s_0}), \Psi_m \circ \varphi(\tau)(\Psi_m(z^{s_0}))) < d_X(id., \Psi_m(\tau))\ .$$

First suppose $n_j^i > 0$. Then:

$$\begin{aligned}
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right) \le\\
&d_X(\Psi_m(z^{s_0}), \Psi_m \circ \varphi(\tau)(w_m)) +\\
&\quad + d_X\left(w_m, \Psi_m\left(f z^{3s_0}\right)^{n_j^i}(\Psi_m(z^{s_0}))\right)\\
&\quad < d_X(id., \Psi_m(\tau)(w_m)) + d_X\left(w_m, \Psi_m(f)^{n_j^i}\right)\\
&\quad - 12\delta < d_X\left(id., \Psi_m\left(\tau f^{n_j^i}\right)\right) - 8\delta\ .
\end{aligned}$$

Now, suppose $n_j^i < 0$. Then:

$$\begin{aligned}
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau_0 a_j^i f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right)\\
&\quad \le d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau_0 a_j^i\right)(\Psi_m(z^{s_0}))\right) +\\
&\quad + d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(a_j^i f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right) -\\
&\quad - d_X\left(\Psi_m(z^{s_0}), \Psi_m\left(a_j^i\right)(\Psi_m(z^{s_0}))\right) + 4\delta\\
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(a_j^i f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right) -\\
&\quad - d_X\left(\Psi_m(z^{s_0}), \Psi_m\left(a_j^i\right)(\Psi_m(z^{s_0}))\right)\\
&\quad \le d_X\left(id., \Psi_m\left(a_j^i f^{n_j^i}\right)\right) - d_X\left(id., \Psi_m\left(a_j^i\right)\right) - 15\delta
\end{aligned}$$

Therefore:

$$\begin{aligned}
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau_0 a_j^i f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right)\\
&\quad < d_X\left(id., \Psi_m\left(\tau f^{n_j^i}\right)\right) + d_X\left(id., \Psi_m\left(a_j^i f^{n_j^i}\right)\right) -\\
&\quad - d_X\left(id., \Psi_m\left(a_j^i\right)\right) - 10\delta < d_X\left(id., \Psi_m\left(\tau f^{n_j^i}\right)\right) - 5\delta
\end{aligned}$$

We are left with $\tau = \tau_0 f^{n_j^i}$ as a subword of the reduced form for g_j and we assume:

$$d_X(\Psi_m(z^{s_0}), \Psi_m \circ \varphi(\tau_0)(\Psi_m(z^{s_0}))) \leq d_X(id., \Psi_m(\tau_0))$$
$$d_X(\Psi_m(z^{s_0}), \Psi_m \circ \varphi(\tau)(\Psi_m(z^{s_0}))) < d_X(id., \Psi_m(\tau))$$

First suppose $n_j^i < 0$. Then:

$$\begin{aligned}
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau a_j^{i+1}\right)(\Psi_m(z^{s_0}))\right) \leq \\
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi(\tau)\left(w_m'\right)\right) + \\
&d_X\left(w_m', \Psi_m\left(a_j^{i+1}\right)(\Psi_m(z^{s_0}))\right) \\
&\quad \leq d_X\left(id., \Psi_m(\tau)\left(w_m'\right)\right) + d_X\left(w_m', \Psi_m\left(a_j^{i+1}\right)\right) - 30\delta \\
&\quad \leq d_X\left(id., \Psi_m\left(\tau a_j^{i+1}\right)\right) - 20\delta
\end{aligned}$$

Now suppose $n_j^i > 0$. Then:

$$\begin{aligned}
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau_0 f^{n_j^i} a_j^{i+1}\right)(\Psi_m(z^{s_0}))\right) \\
&\quad \leq d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau_0 f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right) \\
&\quad + d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(f^{n_j^i} a_j^{i+1}\right)(\Psi_m(z^{s_0}))\right) \\
&\quad - d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right) + 4\delta \\
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(f^{n_j^i} a_j^{i+1}\right)(\Psi_m(z^{s_0}))\right) \\
&\quad - d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(f^{n_j^i}\right)(\Psi_m(z^{s_0}))\right) \\
&\quad \leq d_X\left(id., \Psi_m\left(f^{n_j^i} a_j^{i+1}\right)\right) - d_X\left(id., \Psi_m\left(f^{n_j^i}\right)\right) - 15\delta
\end{aligned}$$

Therefore:

$$\begin{aligned}
&d_X\left(\Psi_m(z^{s_0}), \Psi_m \circ \varphi\left(\tau_0 f^{n_j^i} a_j^{i+1}\right)(\Psi_m(z^{s_0}))\right) \\
&\quad < d_X\left(id., \Psi_m\left(\tau a_j^{i+1}\right)\right) + d_X\left(id., \Psi_m\left(f^{n_j^i} a_j^{i+1}\right)\right) \\
&\quad - d_X\left(id., \Psi_m\left(f^{n_j^i}\right)\right) - 10\delta < d_X\left(id., \Psi_m\left(\tau a_j^{i+1}\right)\right)
\end{aligned}$$

Clearly, a finite induction argument finishes the proof of the theorem.

□

Theorem 6.6 contradicts our choice of the automorphisms Ψ_m for large enough m, so joint with theorem 6.3 we conclude that y_0 does not belong to the interior of an edge of the limit (discrete) tree Y, and is, therefore, a vertex of Y. Since y_0 is a vertex of Y we have no distinguished edge which we should try to make "shorter", but rather make all the edges adjacent to the vertex y_0 shorter, and by that complete a contradiction to the whole construction of the tree Y.

Let $\overline{y}_0$ be the vertex corresponding to the orbit of y_0 in Y/Γ , let $\overline{e}_1, \cdots, \overline{e}_p$ be the edges adjacent to $\overline{y}_0$ in Y/Γ , let $c_1, \cdots, c_p$ be their stabilizers and $z_q \in C_q$ be their generators. As we did in the case y_0 lies in the interior of an edge, we split our treatment into two cases.

Case 2A $\overline{e}_q$ is a splitting edge in Y/Γ .

This case is naturally parallel to case 1A and our approach is, therefore, very similar. The group Γ splits as $\Gamma = A *_{C_q} B$ where $\text{stab}(y_0) < A$ and C_q is of infinite index in both A and B. For each j let:

$$g_j = a_j^1 b_j^1 \cdots a_j^{n_j} b_j^{n_j}$$

(where a_j^1 or $b_j^{n_j}$ may be the identity element). By the convergence of the pointed metric spaces $(X_m, id.)$ to (Y, y_0) in the Gromov topology we may assume that the inequalities (6.1) hold for m large enough, where ε in these inequalities is the length of the shortest edge of Y.

Lemma 6.7 *Let* $w_m \in \left[id., \Psi_m\left(b_j^1\right)\right]$; $|w_m| = \frac{\varepsilon}{2}\mu_m$. *Then for* m *large (such that inequalities (6.1) hold):*

$$d_X\left(w_m, \Psi_m\left(z_q^{s_0}\right)(w_m)\right) > 20\delta$$

for some s_0 $1 \leq |s_0| \leq 20\delta t^{2\delta}$.

Proof: identical to the proof of lemma 6.1.

□

Proposition 6.8 *Let* s_0; $1 \leq |s_0| \leq 20\delta t^{2\delta}$ *satisfy the conclusion of lemma 6.7, and suppose (w.l.o.g.):*

$$d_X(id., w_m) > d_X\left(id., \Psi_m\left(z_q^{s_0}\right)(w_m)\right)$$

Let φ_q be a Γ-automorphism defined by:

$$\begin{aligned} &\forall a \in A \qquad \varphi_q(a) = a \\ &\forall b \in B \qquad \varphi_q(b) = z_q^{s_0} b\, z_q^{-s_0} \end{aligned}$$

Then:

$$d_X\left(id., \Psi_m \circ \varphi_q\left(b_j^i\right)\right) < d_X\left(id., \Psi_m\left(b_j^i\right)\right) - 20\delta$$

Proof:

$$\begin{aligned} d_X\left(id., \Psi_m\left(z_q^{s_0} b_j^i z_q^{-s_0}\right)\right) &\leq d_X\left(id., \Psi_m\left(z_q^{s_0}\right)(w_m)\right) \\ &+ d_X\left(w_m, \Psi_m\left(b_j^i\right)(w_m)\right) + d_X\left(w_m, \Psi_m\left(z_q^{-s_0}\right)\right) \\ &< d_X(id., w_m) + d_X\left(w_m, \Psi_m\left(b_j^i\right)(w_m)\right) + d_X(id., w_m) - 30\delta \\ &< d_X\left(id., \Psi_m\left(b_j^i\right)\right) - 20\delta \end{aligned}$$

□

Theorem 6.9 *Let* $\{g_j\}_{j=1}^t$ *be the generators of* Γ , *and let* Γ *be given by the splitting above* $\Gamma = A *_{C_q} B$. *Then:*

(i) $$g_j \in A \Rightarrow |\Psi_m(g_j)| = |\Psi_m \circ \varphi_q(g_j)|$$

(ii) $$g_j \notin A \Rightarrow |\Psi_m \circ \varphi_q(g_j)| < |\Psi_m(g_j)| - 20\delta$$

Proof: (i) is immediate since $\varphi_q(g_j) = g_j$ if $g_j \in A$. The proof of (ii) is identical with the proof of theorem 6.3.

□

Case 2B $\bar{e}_q$ is a non-splitting edge in Y/Γ .

This last case is naturally similar to case 1B. In particular Γ splits as $\Gamma = A*_{C_q}$, and each g_j admits a presentation:

$$g_j = a_j^1 f^{k_j^1} \cdots a_j^{n_j} f^{k_j^{n_j}}$$

where $a_j^i \in A$ and a_j^1 or $f^{k_j^{n_j}}$ may be the identity element. By the convergence of the pointed metric spaces $(X_m, id.)$ to (Y, y_0) in the Gromov topology we may assume inequalities (6.2) hold for m large enough, where ε in these inequalities is the length of the shortest edge of Y.

Lemma 6.10 *Let* $w_m \in [id., \Psi_m(f)]$; $|w_m| = \frac{\varepsilon}{2}\mu_m$. *Then for* m *large (such that inequalities (6.1) hold):*

$$d_X\big(w_m, \Psi_m\big(z_q^{s_0}\big)(w_m)\big) > 20\delta$$

for some s_0; $1 \leq |s_0| \leq 20\delta t^{2\delta}$

Proof: identical to the proof of lemma 6.4.

Proposition 6.11 *Let* s_0; $1 \leq |s_0| \leq 20\delta t^{2\delta}$ *satisfy the conclusion of lemma 6.10 and suppose (w.l.o.g.):*

$$d_X(id., w_m) > d_X\big(id., \Psi_m\big(z_q^{s_0}\big)(w_m)\big)$$

Let φ_q be a Γ-automorphism given by:

$$\forall a \in A \qquad \varphi_q(a) = a$$
$$\varphi_q(f) = f z_q^{2s_0}$$

Then:

$$d_X(id., \Psi_m \circ \varphi_q(f)) < d_X(id., \Psi_m(f)) - 20\delta$$

Proof:

$$\begin{aligned} d_X\big(id., \Psi_m\big(f z_q^{2s_0}\big)\big) &\leq d_X\big(id., \Psi_m\big(z_q^{2s_0}\big)(w_m)\big) \\ &+ d_X\big(\Psi_m\big(f z_q^{2s_0}\big), \Psi_m\big(z_q^{2s_0}\big)(w_m)\big) \\ &< d_X(id., w_m) + d_X(\Psi_m(f), w_m) - 30\delta \\ &< d_X(id., \Psi_m(f)) - 20\delta \end{aligned}$$

□

Theorem 6.12 *Let* $\{g_j\}_{j=1}^t$ *be the generators of* Γ , *and let* Γ *be given by the splitting above* $\Gamma = A *_{C_q}$. *Then:*

(i) $$g_j \in A \Rightarrow |\Psi_m(g_j)| = |\Psi_m \circ \varphi_q(g_j)|$$

(ii) $$g_j \notin A \Rightarrow |\Psi_m \circ \varphi_q(g_j)| < |\Psi_m(g_j)| - 10\delta$$

Proof: similar to the proof of theorem 6.6.

□

<u>**Theorem 6.13**</u> *Let* $\varphi = \varphi_1 \circ \cdots \circ \varphi_p$. *Then for* m *large enough:*

$$\max_{1 \le j \le t} |\Psi_m \circ \varphi(g_j)| < |\Psi_m(g_j)|$$

Proof: Clearly for m large the maximum is obtained for some $g_j \notin \mathrm{stab}(y_0)$. Such g_j has to become "shorter" by at least one of the φ_q by theorems 6.9 and 6.12. Since all the φ_q do not increase the length of the g_j the theorem follows.

□

Contradicting all the possibilities appear in [Ri] for the dynamics of components of the real tree Y, we get a contradiction to its existence. Since the construction was based on I_Γ being infinite index in $\mathrm{Aut}(\Gamma)$, for Γ torsion-free freely indecomposable hyperbolic group, we get the main result of the last three sections:

<u>**Theorem 6.14**</u> *Let* Γ *be a torsion-free hyperbolic group. Then* I_Γ , *the group of internal automorphisms of* Γ *is of finite index in* $\mathrm{Aut}(\Gamma)$.

Proof: Since the automorphism group of a free group is generated by Dehn twists and inner automorphisms the theorem follows.

□

The argument presented in the last three sections is going to be used in the preceeding sections to obtain additional structure results for hyperbolic groups, their subgroups and their automorphism groups.

<u>7. Freely indecomposable subgroups</u>

In this section we prove the following theorem, which is originally due to Gromov ([Gr], 5.3 C') :

<u>**Theorem 7.1**</u> *Let* Γ *be a hyperbolic group and let* Γ_1 *be a f.g., torsion-free, freely indecomposable (non-cyclic) subgroup. Then* Γ *contains at most finitely many conjugacy classes of subgroups isomorphic to* Γ_1 .

Proof: Let $\Gamma_1 = \langle G_1 \rangle$ and suppose there are infinitely many conjugacy classes of subgroups isomorphic to Γ_1 in Γ . For each such different embedding we choose a monomorphism $\varphi : \Gamma_1 \to \Gamma$ such that:

$$\max_{1 \le j \le t} |\varphi(g_j)| = \min_{\substack{\gamma \in \Gamma \\ \Psi \in \mathrm{Aut}(\Gamma_1)}} \max_{1 \le j \le t} |\varphi \Psi(g_j)|$$

Let $\{\varphi_m\}_{m=1}^{\infty}$ be a sequence of these shortest imbeddings and let:

$$\mu_m = \max_{1 \le j \le t} |\varphi_m(g_j)|$$

Then by the infinity assumption $\mu_m \to \infty$ and by the limiting argument described in section 4 we obtain a real tree Y equipped with a (small) left Γ_1 action, which is the Gromov limit of a subsequence of the metric spaces $\varphi_m(\Gamma_1)$ (with metric inherited from Γ). Now, Γ_1 being freely indecomposable and the shortening arguments described in sections 5 and 6 give a contradiction to the construction of our real tree Y. Therefore, we have a contradiction to the infinity of the sequence of shortest embeddings φ_m .

□

Remark: Note that in our statement of theorem 7.1 only the subgroup Γ_1 is assumed torsion-free, and we assume Γ_1 is f.g. and not f.p. .

8. Finite generation of the automorphism group

The general scheme presented in sections 4, 5, 6 enable us to get the following:

Theorem 8.1 *Let Γ be a torsion-free hyperbolic group. Then $\mathrm{Aut}(\Gamma)$ is finitely generated.*

Proof: First note that $\mathrm{Inn}(\Gamma)$ clearly is f.g. and it is enough to prove the theorem for Γ freely indecomposable. Suppose $\mathrm{Aut}(\Gamma)$ is not f.g. for Γ torsion-free, freely indecomposable hyperbolic group.

On the set of conjugacy classes of small splittings of Γ , we may define a natural height function. A splitting $\Gamma = A *_C B$ (and in correspondence an HNN extension) is of height at most ν if $A = \langle a_1, \cdots, a_r \rangle$; $B = \langle b_1, \cdots, b_s \rangle$ and $|a_i| \leq \nu$; $|b_i| \leq \nu$ and the generator of C is also of length not exceeding ν . The height of a conjugacy clas of a small splitting, therefore, is the minimum possible height under the action of $\mathrm{Inn}(\Gamma)$. Clearly, given a height ν_0 , there are only finitely many conjugacy classes of small splittings with such height. Let DT_m^Γ denote the subgroup of $\mathrm{Aut}(\Gamma)$ generated by $\mathrm{Inn}(\Gamma)$ and Dehn twists obtained from splittings of height at most m.

Now let φ_m be one of the "shortest" automorphisms of Γ which are not in DT_m^Γ , i.e.:

$$\max_{1 \leq j \leq t} |\varphi_m(g_j)| \leq \min_{\varphi \in \mathrm{Aut}(\Gamma) \setminus DT_m^\Gamma} \max_{1 \leq j \leq t} |\varphi(g_j)|$$

At this stage we are able to use the argument presented in section 4 once again and get an action of Γ on a real tree Y. But by the arguments presented in sections 5 and 6 there exist finite (fixed) Dehn twists so that the group generated by them makes φ_m (after taking a subsequence) shorter for all $m > m_1$. On the other hand for some m_2 this finite set belongs to $DT_{m_2}^\Gamma$ a contradiction.

□

Remark: This rather simple argument is in fact a key point in our generalized approach to the isomorphism problem [Ri-Se]. On finite presentability see [Se3].

9. Weakly rigid hyperbolic groups

Weakly rigid hyperbolic groups are of main importance for the understanding of the automorphism group of a hyperbolic group and in connection with the isomorphism problem [Ri-Se]. The main observation for their significance is the following:

Theorem 9.1 *A torsion-free freely indecomposable hyperbolic group Γ is weakly rigid if and only if Γ does not contain quadratically hanging free groups (see definition 5.1).*

Proof: If Γ as above is not weakly rigid then by the discussion appears in section 4 Γ admits a small action on a real tree Y, where Y contains IET components. The stabilizer of a segment includes in such an IET component is the stabilizer of the whole component by [Ri]. If a stabilizer of such a segment is cyclic, then Γ contains a subgroup H having the short exact sequence:

$$1 \to Z \to H \to N \to 1$$

where N is free or a surface group. The normalizer of a cyclic group in Γ is cyclic so we get a contradiction. On the other hand if Γ contains a quadratically hanging subgroup then acting on Γ with a pseudo-Anosov automorphism of the quadratically hanging subgroup and take a Gromov limit we get a real tree with a small action of Γ and an IET component (the whole limiting tree Y has IET dynamics in this case).

□

Much stronger results on the structure of weakly rigid hyperbolic groups and their automorphism group appear in [Se3].

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