TERMINATION OF SUCCESSIVE BLOWING-UPS ALONG EXCEPTIONAL CURVES IN THREEFOLDS

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Introduction. Let X be a three-dimensional complex manifold, $C \subseteq X$ a closed compact smooth curve and let $\mu_1 : X_1 \longrightarrow X$ be the blowing-up of X along C. Then the exceptional divisor $E_1 = \mu_1^{-1}(C)$ is a ruled surface over C. There exist at most one section C_1 of the ruling $E_1 \longrightarrow C$ with $(C_1)^2_{E_1} < 0$. We call this section by a negative section. If E_1 has a negative section C_1, then let us consider the blowing-up $\mu_2 : X_2 \longrightarrow X_1$ along C_1. In this way, we have a sequence of blowing-ups

$$(B_k) : X_k \xrightarrow{\mu_k} X_{k-1} \xrightarrow{\mu_{k-1}} \cdots \longrightarrow X_1 \xrightarrow{\mu_1} X ,$$

the exceptional ruled surfaces E_i on $X_i (1 \leq i \leq k)$ and the negative sections C_i on E_i $(1 \leq i \leq k)$ such that the μ_j is just the blowing-up of X_{j-1} along C_{j-1} and $E_j = \mu_j^{-1}(C_{j-1})$ for $1 \leq j \leq k$. The purpose of this note is to prove that the normal bundle N_{C_k/X_k} is semi-stable for some k, if $C \subseteq X$ can be contracted to a point. In the case $C \cong \mathbb{P}^1$ and $N_{C/X} \cong \mathcal{O} \oplus \mathcal{O}(-2)$, M. Reid [5] has proved this and constructed the flip at C. Recently T. Ando [1] also treated this problem.

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§ 1. Preliminaries

Let E be a locally free sheaf of rank two on a smooth compact curve C .

Lemma (1.1). (1) If E is a semi-stable vector bundle, then there exist no curves Γ on the ruled surface $\mathbb{P}_C(E)$ with $\Gamma^2 < 0$.

(2) If E is unstable, then there exists a unique (up to isomorphisms) exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0 ,$$

which satisfies the following conditions:

(i) L and M are invertible sheaves on C ,

(ii) $\deg_C L > \deg_C M$.

Proof. (1). Let $\mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}_C(E)$ with respect to the E . Then E is semi-stable if and only if the line bundle $\mathcal{O}(2) \otimes \pi^*(\det E)^{-1}$ is nef on $\mathbb{P}_C(E)$, where π is the ruling $\mathbb{P}_C(E) \longrightarrow C$. (1) is an easy consequence of this fact.

(2). Since E is unstable, there exists an exact sequence satisfying (i) and (ii). Assume that there is another sequence

$$0 \longrightarrow L' \longrightarrow E \longrightarrow M' \longrightarrow 0$$

satisfying (i) and (ii). Since $\deg M' < \deg(\det E)$, the homomorphism $L \longrightarrow E \longrightarrow M'$ must be zero. Therefore $L' \simeq L$ and $M' \simeq M$. Q.E.D.

Definition (1.2). When E is unstable, we call the exact sequence

$$0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0$$

satisfying the above conditions (i) and (ii), the characteristic exact sequence of E. Here we also define $d^+(E) := \deg_C L$, $d^-(E) := \deg_C M$, and $\delta(E) := d^+(E) - d^-(E)$. When E is the conormal bundle $\check{N}_{C/X}$ of a curve $C \subseteq X$ as in the introduction, we simply denote $d^{\pm}(E)$ and $\delta(E)$ by $d^{\pm}(C)$ and $\delta(C)$, respectively.

Definition (1.3). A compact smooth curve C in a smooth three-fold X is called an <u>exceptional curve</u>, if there exists a proper bimeromorphic morphism $f : X \longrightarrow Z$ such that $f(C)$ is a point and that f is isomorphic outside C.

We have the following criterion.

Proposition (1.4). Let $C \subseteq X$ be a compact smooth curve in a smooth threefold. Then C is an exceptional curve if and only if there exists a coherent $\mathcal{O}_X$ - ideal J on a neighborhood of C satisfying the following condition

(E) : $\dim(\mathrm{Supp}(\mathcal{O}_X/J)) = 1$, $\mathrm{Supp}(\mathcal{O}_X/J) \supseteq C$,

and $(J \otimes_{\mathcal{O}_X} \mathcal{O}_C)/\mathrm{torsion}$ is an ample vector bundle on C .

Proof. First we assume that C is an exceptional curve. Then there exist two effective Cartier divisors S_1 and S_2 on a neighborhood of C such that $(S_1 \cdot C) < 0$, $(S_2 \cdot C) < 0$, and $\dim(S_1 \cap S_2) = 1$. Let J be the ideal $\mathcal{O}_X(-S_1) + \mathcal{O}_X(-S_2)$. Then we have

$$J \otimes \mathcal{O}_C \cong (\mathcal{O}_C \otimes \mathcal{O}_X(-S_1)) \oplus (\mathcal{O}_C \otimes \mathcal{O}_X(-S_2)) \ .$$

Thus J satisfies the condition (E) .

Next we assume that there is an $\mathcal{O}_X$ - ideal J satisfying the condition (E) . By considering the primary decomposition of J , we have an $\mathcal{O}_X$ - ideal $J_0 \supseteq J$ such that $\mathrm{Supp}(\mathcal{O}_X/J_0) = C$ and $\mathrm{Supp}(J_0/J) \not\supseteq C$. Hence there is an injection $(J \otimes \mathcal{O}_C/\mathrm{torsion}) \longrightarrow (J_0 \otimes \mathcal{O}_C/\mathrm{torsion})$, where $\mathrm{rank}(J \otimes \mathcal{O}_C/\mathrm{torsion}) = \mathrm{rank}(J_0 \otimes \mathcal{O}_C/\mathrm{torsion})$. Therefore J_0 also satisfies the condition (E) . Let $\mu : V \to X$ be the blowing-up by the ideal J_0 , i.e., $V := \underline{\mathrm{Projan}}_X(\bigoplus_{d\geq 0} J_0^d)$. we have an exceptional Cartier divisor $E := \underline{\mathrm{Projan}}_X(\bigoplus_{d\geq 0} J_0^d/J_0^{d+1})$. Let W be a component of E . If $\mu(W)$ is a point, then $\mathcal{O}_W \otimes \mathcal{O}_V(-E)$ is ample, since $\mathcal{O}_V(-E)$ is μ - ample. If $\mu(W)$ is not a point, then $\mu(W) = C$ and W is also a component of $\underline{\mathrm{Proj}}_C\,(\bigoplus_{d\geq 0} J_0^d \otimes \mathcal{O}_C/\text{torsion})$. Since $(J_0 \otimes \mathcal{O}_C/\text{torsion})$ is an ample vector bundle, $\mathcal{O}_W \otimes \mathcal{O}_V(-E)$ is also ample. Therefore $\mathcal{O}_E(-E)$ is an ample invertible sheaf. Then by the contraction criterion (cf. [2], [3]), we have a morphism $\nu : V \longrightarrow Z$ such that $\nu(E)$ is a point and ν is an isomorphism outside E . Therefore we have the contraction $f : X \longrightarrow Z$ of C .

Q.E.D.

Lemma (1.5). Let $C \subseteq X$ be an exceptional curve.

(1) If the conormal bundle $N^{\vee}_{C/X} \cong I_C/I_C^2$ is semi-stable, then I_C/I_C^2 is an ample vector bundle.

(2) If I_C/I_C^2 is unstable, then $d^+(C) > 0$.

Proof. Take an ideal J satisfying (E) and the maximal integer k such that $J \subseteq I_C^k$. Then we have an injection

$$J/J \cap I_C^{k+1} \hookrightarrow I_C^k/I_C^{k+1} \simeq \mathrm{Sym}^k\,(I_C/I_C^2) .$$

By the condition (E), $J/J \cap I_C^{k+1}$ is an ample vector bundle. Therefore we have proved (1) and (2). Q.E.D.

Let $C \subseteq X$ be an exceptional curve such that I_C/I_C^2 is unstable. Let us consider the blowing-up

$\mu_1 : X_1 \longrightarrow X$, $E_1 = \mu_1^{-1}(C)$, and the negative section C_1 corresponding to the characteristic exact sequence of I_C/I_C^2 .

Lemma (1.6). $C_1 \subseteq X_1$ is also an exceptional curve.

Proof. Let $0 \longrightarrow L \longrightarrow I_C/I_C^2 \longrightarrow M \longrightarrow 0$ be the characteristic exact sequence. Assume that I_C/I_C^2 is ample. Then from the natural exact sequence

$$0 \longrightarrow \mathcal{O}_{C_1} \otimes \mathcal{O}_{X_1}(-E_1) \longrightarrow I_{C_1}/I_{C_1}^2 \longrightarrow \mathcal{O}_{C_1} \otimes \mathcal{O}_{E_1}(-C_1) \longrightarrow 0 ,$$
$$\mathcal{O}_{C_1} \otimes \mathcal{O}_{X_1}(-E_1) \cong M, \qquad \mathcal{O}_{C_1} \otimes \mathcal{O}_{E_1}(-C_1) \cong L \otimes M^{-1}$$

and the condition $\deg L > \deg M > 0$, we see that $I_{C_1}/I_{C_1}^2$ is also ample. Next assume that I_C/I_C^2 is not ample. Then $\deg M \leq 0$. Take an $\mathcal{O}_X$ - ideal J satisfying the condition (E) for $C \hookrightarrow X$. Let us consider the $\mathcal{O}_{X_1}$ - ideal $J' := \text{Image}\ (\mu_1^* J \longrightarrow \mathcal{O}_{X_1})$. Since $J \subseteq I_C$, we have $J' \subseteq \mathcal{O}_{X_1}(-E_1)$. Take the maximal integer ℓ such that $J' \subseteq \mathcal{O}_{X_1}(-\ell E_1)$ and let $J_1 := J' \otimes \mathcal{O}_{X_1}(\ell E_1) \hookrightarrow \mathcal{O}_{X_1}$. We shall prove that the J_1 satisfies the condition (E) for $C_1 \hookrightarrow X_1$. Since $(J \otimes \mathcal{O}_C/\text{torsion})$ is ample on C , $(J' \otimes \mathcal{O}_{C_1}/\text{torsion})$ is also ample on C_1 . Now we have a natural homomorphism

$$J' \otimes \mathcal{O}_{C_1} \longrightarrow \mathcal{O}_{X_1}(-\ell E_1) \otimes \mathcal{O}_{C_1} \otimes \mathcal{O}_{C_1} \cong M^{\otimes \ell} .$$

Since $\deg M \leq 0$, this homomorphism must be zero. Therefore $J_1 \subseteq I_{C_1}$. On the other hand, $(J_1 \otimes \mathcal{O}_{C_1}/\text{torsion})$ is ample, because

$$J_1 \otimes \mathcal{O}_{C_1} \cong (J' \otimes \mathcal{O}_{C_1}) \otimes (\mathcal{O}_{X_1}(\ell E_1) \otimes \mathcal{O}_{C_1})$$

$$\cong (J' \otimes \mathcal{O}_{C_1}) \otimes M^{\otimes(-\ell)} .$$

Therefore J_1 satisfies the condition (E). Q.E.D.

Lemma (1.7). Let $C \subseteq X$ be an exceptional curve and let J be an $\mathcal{O}_X$ - ideal satisfying the condition (E) for $C \subseteq X$. Then it is impossible to construct an infinite descending filtration $I^{(k)}$ $(k \geq 0)$ of the defining ideal I_C which satisfies the following two conditions (α) and (β) :

(α) $I^{(k)}$ is a coherent $\mathcal{O}_X$ - ideal for all $k \geq 0$ and $J \nsubseteq \bigcap_{k\geq 0} I^{(k)}$,

(β) $I^{(k)}/I^{(k+1)}$ is an $\mathcal{O}_C$ - invertible sheaf and not ample for all $k \geq 0$.

Proof. By (α) , we can take the maximal integer k such that $J \subseteq I^{(k)}$. Then we have an injection $J/J \cap I^{(k+1)} \rightarrow I^{(k)}/I^{(k+1)}$. Since $(J \otimes \mathcal{O}_C/\text{torsion})$ is ample, $J/J \cap I^{(k+1)}$ is also ample. This contradicts to (β) . Q.E.D.

§ 2. Termination

Let $C \subseteq X$ be an exceptional curve such that I_C/I_C^2 is unstable. Then we have the characteristic exact sequence:

$$0 \longrightarrow L \longrightarrow I_C/I_C^2 \longrightarrow M \longrightarrow 0 \qquad \text{(e.1)}.$$

Let us consider $\mu_1 : X_1 \longrightarrow X$ of (B_k), E_1, and C_1 (see the introduction). Then we have an exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{O}_{C_1} \otimes \mathcal{O}_{X_1}(-E_1) & \longrightarrow & I_{C_1}/I_{C_1}^2 & \longrightarrow & \mathcal{O}_{C_1} \otimes \mathcal{O}_{E_1}(-C_1) & \longrightarrow 0 & \qquad \text{(e.2)}. \\ & \wr\!\wr & & & & \wr\!\wr & & \\ & M & & & & L \otimes M^{-1} & & \end{array}$$

Assume that $I_{C_1}/I_{C_1}^2$ is also unstable. Then we have the characteristic exact sequence

$$0 \longrightarrow L_1 \longrightarrow I_{C_1}/I_{C_1}^2 \longrightarrow M_1 \longrightarrow 0 \qquad \text{(e.3)}.$$

The following lemma is easily proved.

Lemma (2.1). (1) If $\deg L < 2 \deg M$, then (e.2) is isomorphic to (e.3).

(2) If $\deg L \geq 2 \deg M$, then $\deg M \leq \deg M_1$ and

$\deg L_1 \leq \deg L - \deg M$. Here $\deg M = \deg M_1$ (or equivalently $\deg L_1 = \deg L - \deg M$) , if and only if (e.2) is split.

Definition (2.2). Let $C \subseteq X$ be an exceptional curve. C is called of <u>type S</u> , if I_C/I_C^2 is a semi-stable vector bundle. C is called of <u>type P</u> (resp. <u>type N</u>) , if I_C/I_C^2 is unstable and ample (resp. not ample). C is called of <u>type I</u> , if there exist two prime divisors S_1 and S_2 on a neighborhood of C such that C is just the scheme-theoretic intersection $S_1 \cap S_2$.

Lemma (2.3). If C is of type P , then one of the following conditions are satisfied:

(i) C_1 is of type S ,

(ii) C_1 is of type P and C_2 is of type I ,

(iii) C_1 is of type P and $0 < \delta(C_1) < \delta(C)$.

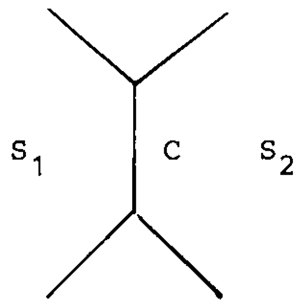
Proof. Assume that C_1 is not of type S . Then by (e.2), C_1 is of type P . If $d^+(C) < 2d^-(C)$, then by Lemma (2.1) - (1), C_2 is just the intersection of E_2 and the proper transform E_1' of E_1 on X_2 . Therefore the condition (ii) is satisfied. If $d^+(C) \geq 2d^-(C)$, then by Lemma (2,1) - (2), we have

$$d^-(C) \leq d^-(C_1) < d^+(C_1) \leq \delta(C) < d^+(C) .$$

Therefore the condition (iii) is satisfied. Q.E.D.

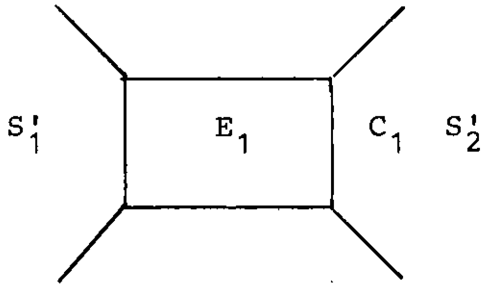
Proposition (2.4). If C is of type P and of type I, then there is a positive integer k such that C_k is of type S.

Proof. Let S_1 and S_2 be prime divisors with $S_1 \cap S_2 = C$. Then S_1 and S_2 are smooth surfaces near C, and $I_C/I_C^2 \cong \mathcal{O}_C(-S_1) \oplus \mathcal{O}_C(-S_2)$.



(Fig 1)

Assume that $(S_1 \cdot C) > (S_2 \cdot C)$. Then we have $d^+(C) = -(S_2 \cdot C) = -(C)^2_{S_1} > d^-(C) = -(S_1 \cdot C) = -(C)^2_{S_2}$. Let us consider the $\mu_1 : X_1 \longrightarrow X$ and let S_i' be the proper transform of S_i on X_1 for $i = 1,2$. Then C_1 is just the complete intersection $S_2' \cap E_1$, and $I_{C_1}/I_{C_1}^2 \cong \mathcal{O}_{C_1}(-E_1) \oplus \mathcal{O}_{C_1}(-S_2')$.



(Fig. 2)

Here we have

$$d^{+}(C_1) = \max\ (\delta(C)\ ,\ d^{-}(C))\ ,$$

$$d^{-}(C_1) = \min\ (\delta(C)\ ,\ d^{-}(C))\ .$$

Therefore C_k is of type S for some k . Q.E.D.

Lemma (2.5). If C is of type N , then one of the following conditions are satisfied:

(i) C_1 is of type S ,

(ii) C_1 is of type P ,

(iii) C_1 is of type N and $0 \geq d^{-}(C_1) > d^{-}(C)$,

(iv) (e.2) is split.

Proof. Assume that C_1 is not of type S . Since C is of type N , we have $d^-(C) \leq 0$. Therefore $d^+(C) > 2d^-(C)$ by Lemma (1.5) - (2). Hence by Lemma (2,1) - (2), we have $d^-(C) \leq d^-(C_1)$. Here the equality holds if and only if (e.2) is split. Q.E.D.

Proposition (2.6). There exist no pseudo-exceptional curves $C \subseteq X$ of type N such that C_k satisfies the condition (2.5) - (iv) for all k .

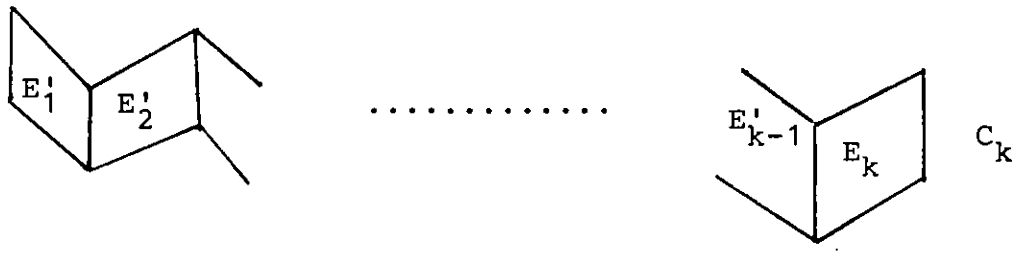
Proof. Assume the contrary. Let D_k be the effective divisor:

$$E_k + \mu_k^* E_{k-1} + \mu_k^* \mu_{k-1}^* E_{k-2} + \dots + \mu_k^* \dots \mu_2^* E_1$$

on X_k . Then we have

$$K_{X_k} = \mu_k^* \dots \mu_1^* K_X + D_k \qquad (*)_k .$$

Let E_i' be the proper transform of E_i on X_k for $i \leq k$. Then by the condition (2.5) - (iv), we can prove that $E_i' \cap E_j' = \phi$ for $|i - j| \geq 2$ and that all the double curves $E_i' \cap E_{i+1}'$ are disjoint from each other for $i \leq k-1$. Further the negative section C_k on E_k has no intersections with E_i' $(i \leq k-1)$.



(Fig. 3)

Therefore $-D_k$ is relatively nef over X and

$$(-D_k)\cdot C_k = -(E_k\cdot C_k) \leqq 0 \qquad (**)_k .$$

Let $I^{(k)}$ be the ideal $(\mu_1\circ\dots\circ\mu_{k+1})_*\mathcal{O}_{X_{k+1}}(-D_{k+1})$. Then we have an infinite sequence of descending filtration $I^{(k)}$ of $I_C = I^{(0)}$. By the formula $(*)_{k+1}$, we have $I^{(k)}/I^{(k+1)} \cong (\mu_1\circ\dots\circ\mu_{k+1})_*(\mathcal{O}_{C_{k+1}}\otimes\mathcal{O}_{X_{k+1}}(-D_{k+1}))$. Hence by $(**)_{k+1}$, the filtration $I^{(k)}$ satisfies the condition (1.7) - (β) . Thus by Lemma (1.7) , we have $\bigcap_{k\geq 0} I^{(k)} \supseteqq J$ for any $\mathcal{O}_X$ - ideal J satisfying the condition (E) for $C\subseteq X$. Let $x\in C$ be a general point and let H be a general smooth divisor on a neighborhood of x in X such that $H\cap C = \{x\}$ and this intersection is transversal. Then $H_k := \mu_k^*\dots\mu_1^* H$ is also a smooth divisor on X_k . Let a be an element of $(\bigcap_{k\geq 0} I^{(k)}\mathcal{O}_H)_x$ and let $\Delta := \mathrm{div}(a)$ on H . Then the proper transform Δ_k of Δ in H_k always contains the point $C_k\cap H_k$. Therefore $(\bigcap_{k\geq 0} I^{(k)}\mathcal{O}_H)_x$ is a prime ideal generated by one element.

Since dim Supp $(\mathcal{O}_X/J) = 1$, we have dim Supp $(\mathcal{O}_H/J \cdot \mathcal{O}_H) = 0$ for general H . This is a contradiction. Q.E.D.

By (1.6), (2.3), (2.4), (2.5), (2.6), we finally proved the following:

Theorem. If $C \subseteq X$ is an exceptional curve, then C_k is of type S for some k .

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