Chern Functors

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This is the second of four papers in which we try to come to terms with Deligne's problem of constructing a functorial Riemann-Roch isomorphism for the determinant line bundle of the cohomology of a proper smooth morphism p: $X \longrightarrow S$

$$\det \mathbb{R}p_*\mathfrak{F} \longrightarrow (I_{X/S}\mathfrak{ch}(\mathfrak{F})\mathfrak{Td}(T_{X/S}))^{(1)}. \qquad (1)$$

The first step in such a construction is to give live to the right hand side of (1). This was done by Deligne and Elkik ([D], [E]), who treated (1) as a global expression. It is our approach to give live to each ingrediant of the right hand side of (1), i.e., we can not only integrate the Chern functors along the fibres, we can also say what the Chern functors themselves are. Such an approach allows us to approach (1) by copying Grothendieck's proof of Riemann-Roch via embeddings into projective spaces, as we shall see in a forthcoming paper.

As the first step in this program, Chow categories as target categories for the Chern functors have been introduced in [F1]. Here we study the Chern functors themselves. Because of difficulties with the intersection product for non-smooth schemes over Spec($\mathbb{Z}$), we introduce $c_k(\mathfrak{F})$ not as a mere object of the Chow category $\mathfrak{CH}^k(X)$, but as a whole intersection product functor

$$c_k(\mathfrak{F})\cap A\colon \tilde{\mathfrak{CH}}^p(X) \longrightarrow \tilde{\mathfrak{CH}}^{p+k}(X). \qquad (2)$$

In the first five paragraphs of §1, we introduce $c_1(\mathcal{L})\cap A$ for a line bundle $\mathcal{L}$, using a functorial version of the product

$$H^1(X,K_1)\otimes E_2^{p,q}(X) \longrightarrow E_2^{p+1,q-1}(X),$$

where E_2 is the E_2-term of Quillen's spectral sequence. Starting from this point, in the remaining paragraphs of §1 we construct (2), copying Grothendieck's definition of the Chern classes. We also prove a Whitney isomorphism for the Chern functors. The second paragraph considers further properties of the Chern functors (like relation to the Gysin functor $\underline{f}^!$ constructed in [F1]), which are useful both for §3 and for the proof of functorial Riemann-Roch. In §3 we give an axiomatic characterization of the Chern functors, relating them to $c_1(\mathscr{L})\cap A$ for a line bundles by means of six natural isomorphisms (3.2.1.-4. and AX 0, AX 1) and four compatibilites AX 2-5 (of which the last one, AX 5, is very likely to be redundant) between these six isomorphisms. This is similar to the axioms for IC_2 in [D]. Finally we compare our functor $\underline{p}_*(c_2(\mathscr{E}))$ with Deligne's functor IC_2 and indicate how a similar comparison can be carried out for Elkik's line bundles.

The first paragraph almost coincides with §6 of [F2] (save for the correction of some sign errors) and has been announced in [F3]. I owe thanks to A.A. Beilinson, Ju.I. Manin, and A.N. Parchin for a number of helpful discussions. This paper has been finished during the author's stay at the Max-Planck-Institute in Spring 1989. I want to thank the MPI for its hospitality, and in particular G. Harder for his help in printing out the text.

Notations: We use all the notations of [F1] for the Chow categories $\mathfrak{CH}^k$ and $\tilde{\mathfrak{CH}}^k$, the functors $\underline{f}^*$, $\underline{f}^!$, $\underline{g}_*$, and $\mathfrak{sp}_\lambda$ between them, and for the E_2-term of Quillen's spectral sequence. In particular, $CH^k(X)=E_2^{k,-k}(X)$ and $G_k(X)=E_2^{k-1,-k}(X)$. The product in the higher algebraic K-theory is Waldhausen's. As we did in [F1], we suppose schemes to be noetherian, separated, and universally catenary.

1. Construction of the Chern functors

1.1. Some preparations: For a topological space X, a sheaf F on X, and a covering $\mathcal{U}=\{U_i\}_{i\in I}$ of X, let

$$\check{C}^*(\mathcal{U},F)= \coprod_{t=0} \prod_{i_0,\dots,i_t\in I} F(U_{i_0}\cap\dots\cap U_{i_t})$$

be the complex of Čech cochains. We denote by $\check{Z}^i(\mathcal{U},F)$ and $\check{B}^i(\mathcal{U},F)$ the groups of closed and exact cochains and by

$$(d_0\boldsymbol{\alpha})_{i_0,\dots,i_q} = \sum_{k=0}^{q}(-1)^k \alpha_{i_0,\dots,\hat{i}_k,\dots,i_q}$$

the Čech differential d_0. We will often denote q-cycles by bold and their evaluation on open subsets by usual letters.

For a complex of sheaves F^* with differential d_1 we put

$$\check{C}^i(\mathcal{U},F^*)= \bigoplus_{k+l=i} \check{C}^k(\mathcal{U},F^l)$$

$$d = (-1)^l d_0+d_1$$

and define $\check{B}^i$ and $\check{Z}^i$ by means of d.

Let $\mathcal{U}$ be a covering of X_{Zar}. To an element $\boldsymbol{\alpha}$ of $\check{Z}^q(\mathcal{U},E_1^{*,-q})$ we associate an object $\Phi(\boldsymbol{\alpha})$ of $\tilde{\mathfrak{CH}}^q(X)$ as follows. For an open subset W in $X_{(q)}$, we denote by $\mathcal{U}\cap W$ the covering of W by the $U_i\cap W$ and define

$$\Phi(\alpha)(W)=\{x\in\check{C}^{q-1}(\mathcal{U}\cap W,E_1^{*,-q} \mid d(x)=-\boldsymbol{\alpha}|_W\}/\check{B}^{q-1}(\mathcal{U}\cap W,E_1^{*,-q}). \qquad (1)$$

Since the sheaves $E_1^{p,q}$ are flabby, every $g\in G_q(W)$ defines $\tilde{g}\in H^{q-1}(\mathcal{U}\cap W,E_1^{*,-q})$ and acts on the set (1) by the rule $x\longmapsto x+\tilde{g}$. It is easy to see that $\Phi(\boldsymbol{\alpha})$ is an object of $\tilde{\mathfrak{CH}}^q(X)$.

Let $\boldsymbol{\alpha},\boldsymbol{\alpha}'\in\check{Z}^q(\mathcal{U},E_1^{*,-q})$ and $\boldsymbol{\gamma}\in(\check{C}^{q-1}/\check{B}^{q-1})(\mathcal{U},E_1^{*,-q})$ such that $d\boldsymbol{\gamma}=\boldsymbol{\alpha}'-\boldsymbol{\alpha}$. Then there is an isomorphism $\Phi(\boldsymbol{\gamma}):\Phi(\boldsymbol{\alpha})\longrightarrow\Phi(\boldsymbol{\alpha}')$ which sends x in (1) to $x-\boldsymbol{\gamma}|_W$.

Let $\mathcal{V}=\{V_j\}_{j\in J}$ be a refinement of $\mathcal{U}$, and let $\Phi:J\longrightarrow I$ be a function with $V_i\subseteq U_{\Phi(i)}$. It defines a homomorphism

$$\xi_{\Phi}:\check{C}^{*}(\mathcal{U},E_1^{*,q})\longrightarrow\check{C}^{*}(\mathcal{V},E_1^{*,q})$$

and a canonical isomorphism

$$\underline{\xi}_{\phi}:\ \mathbb{O}(\alpha)\longrightarrow\mathbb{O}(\xi_{\phi}(\alpha))\tag{2}$$

by the rule $x\longrightarrow\xi_{\phi}(x)$ in (1). If $\mathcal{W}$ is a refinement of $\mathcal{V}$ indexed by K and Ψ: $K\longrightarrow J$ an admissible function, then $\xi_{\Phi\Psi}=\xi_{\Psi}\xi_{\Phi}$ and $\underline{\xi}_{\Phi\Psi}=\underline{\xi}_{\Psi}\underline{\xi}_{\Phi}$.

Let E^{*} and F^{*} be complexes of sheaves on X, G a presheaf on X, and $\{.,.\}$: $E^{*}\otimes G\longrightarrow F^{*}$ a homomorphism of complexes. If $x\in\check{C}^{p}(\mathcal{U},E^{*})$ and $y\in\check{C}^{q}(\mathcal{U},G)$, we define $\{x,y\}$ by the usual formula

$$\{x,y\}_{i_0,\dots,i_r}=\{x_{i_0,\dots,i_{r-q}}|U_{i_0}\cap\dots U_{i_r},y_{i_{r-q},\dots,i_r}|U_{i_0}\cap\dots\cap U_{i_r}\}.\tag{3}$$

We have

$$d(\{x,y\})=\{d(x),y\}+(-1)^{p}\{x,d(y)\}.\tag{4}$$

<u>1.2. The functor $c_1(\mathcal{L})\cap A$:</u> Let $A\in Ob(\widetilde{\mathfrak{CH}}^{k}(X))$, $k\geq 0$, and $\mathcal{L}$ be a line bundle on X. We choose a covering $\mathcal{U}$ of X_{Zar} and non-vanishing sections ℓ_i of $\mathcal{L}$ on U_i. Let $\varphi=(\varphi_{ij})=(\ell_j/\ell_i)\in\mathcal{O}_X^{*}(U_i\cap U_j)\subseteq K_1(U_i\cap U_j)$ be the 1-cycle defined by the ℓ_i.

Let the product $\{.,.\}$: $E_1^{p,-q}(X)\otimes K_i(X)\longrightarrow E_1^{p,-q-i}$ be defined by

$$\{(a_x)_{x\in X_p},\varphi\}=(a_x\varphi)_{x\in X_p}.$$

Now (3) defines

$$\{.,.\}:\check{C}^{p}(\mathcal{U},E_1^{*,-k})\otimes\check{C}^{q}(\mathcal{U},K_1)\longrightarrow\check{C}^{p+q}(\mathcal{U},E_1^{*,-k-1}).$$

Let $a\in A_r(X)$ be a rational section. We put

$$(c_1(\mathcal{L})\cap A)_{\mathcal{U},\ell_i,a}=\mathbb{O}((-1)^{k}\{c(a),\varphi\}\in Ob(\widetilde{\mathfrak{CH}}^{k+1}(X))\tag{5}$$

where $c(a)$ is the cycle defined by a (cf. [F1, §3.3.]).

If a' is another rational section of A, then $a-a'\in E_1^{k-1,-k}(X)/im(d_1)$ and $d_1(a-a')=c(a)-c(a')$. Since $d(\varphi)=0$, we have by (4)

$$\mathbb{O}((-1)^{k}\{a'-a,\varphi\}):\ (c_1(\mathcal{L})\cap A)_{\mathcal{U},\ell_i,a}\longrightarrow(c_1(\mathcal{L})\cap A)_{\mathcal{U},\ell_i,a'}.\tag{6}$$

If $\tilde{\ell}_i$ are other trivializations of $\mathcal{L}$ on U_i, we put $\tilde{\varphi}_{ij}=\tilde{\ell}_j\tilde{\ell}_i^{-1}$ and $\psi_i=\tilde{\ell}_i\ell_i^{-1}$. Then $d(\psi)=\tilde{\varphi}\varphi^{-1}$, hence

$$\mathbb{O}(\{c(a),\psi\}):\ (c_1(\mathcal{L})\cap A)_{\mathcal{U},\ell_i,a}\longrightarrow(c_1(\mathcal{L})\cap A)_{\mathcal{U},\tilde{\ell}_i,a}.\tag{7}$$

If $\mathcal{V}$ is a refinement of $\mathcal{U}$ and Φ as in (2), we have

$$\xi_{\Phi}:\ (c_1(\mathcal{L})\cap A)_{\mathcal{U},\ell_i,a}\longrightarrow(c_1(\mathcal{L})\cap A)_{\mathcal{V},\ell_{\Phi(i)},a}\tag{8}$$

Of course, there are several compatibilities which must be checked.

For instance, if we replace a by a' and ℓ_i by $\tilde{\ell}_i$, we have

$$\begin{aligned}(-1)^k\{a'-a,\varphi\}+\{c(a'),\psi\}&=(-1)^k\{a'-a,\tilde{\varphi}\}+\{c(a),\psi\}-(-1)^k\{a'-a,d(\psi)\}+\\&\quad+\{d(a'-a),\psi\}\\&=(-1)^k\{a'-a,\tilde{\varphi}\}+\{c(a),\psi\}+d(\{a'-a,\psi\}\end{aligned}$$

hence

$$\Phi(\{c(a'),\psi\})\circ\Phi((-1)^k\{a'-a,\varphi\})=\Phi((-1)^k\{a'-a,\tilde{\varphi}\})\circ\Phi(\{c(a),\psi\});$$

which proves the compatibility of (6) and (7). The two other cases are verified in a similar manner.

By means of (6), (7), and (8), the objects $(c_1(\mathscr{L})\cap A)_{\mathcal{U},\ell_i,a}$ can be glued to one object $c_1(\mathscr{L})\cap A$. It defines a biadditive functor $\mathfrak{Pic}(X)\times\widetilde{\mathfrak{CH}}^k(X)\longrightarrow\mathfrak{CH}^{k+1}(X)$, where $\mathfrak{Pic}(X)$ is the gruppoid of line bundles on X. Biadditivity means that there are canonical isomorphisms

$$c_1(\mathscr{L}\otimes\mathscr{M})\cap A\longrightarrow c_1(\mathscr{L})\cap A\oplus c_1(\mathscr{M})\cap A$$
$$c_1(\mathscr{L})\cap(A\oplus B)\longrightarrow c_1(\mathscr{L})\cap A\oplus c_1(\mathscr{L})\cap B$$

which satisfy the additivity conditions of [DM, §1.8.] in each ofthe two variables and make the diagram

$$\begin{array}{ccc}c_1(\mathscr{L}\otimes\mathscr{M})\cap(A\oplus B) & \longrightarrow & c_1(\mathscr{L})\cap(A\oplus B)\oplus c_1(\mathscr{M})\cap(A\oplus B)\\ \downarrow & & \downarrow\\ c_1(\mathscr{L}\otimes\mathscr{M})\cap A\oplus c_1(\mathscr{L}\otimes\mathscr{M})\cap B & \longrightarrow & c_1(\mathscr{L})\cap A\oplus c_1(\mathscr{L})\cap B\oplus c_1(\mathscr{M})\cap A\oplus c_1(\mathscr{M})\cap B\end{array}\qquad(9)$$

commutative.

Let V and W be zariskoi-open subsets of X, $a\in A(V)$, ℓ a non-vanishing section of $\mathscr{L}$ on V. We want to define

$$\ell\cap a\in(c_1(\mathscr{L})\cap A)(V\cup W).\qquad(10)$$

Without loosing generality we may assume $X=V\cup W$. For a moment we also assume that V is open and dense in $X_{(k)}$, later we can get rid of this assumption. In the notations of (5),

$$(\{c(f),\ell/\ell_i\})_{i\in I}\ \operatorname{mod}(\check{B}^{k-1}(\mathcal{U},E_1^{*,-k}))\qquad(11)$$

defines an element of $\Phi(\{c(a),\varphi_{ij}\})(X)=(c_1(\mathscr{L})\cap A)_{\mathcal{U},\ell_i,a}(X)$. The product in (11) is well-defined because the supports of $c(a)$ and $\operatorname{div}(\ell)$ are disjoint. It is easy to see that (11) is compatible with (7), (8), and (9), hence it defines (10).

If g is a rational function on X which has no zeros or poles intersecting the support of $c(a)$, then there is a well-defined product $c(a)\cap g=(n_{c(a)}(x)g|_{\mathrm{Spec}\ k(x)})_{x\in X_k}\in G_{k+1}(X)$, and we have

$$(gl)\cap a-l\cap a=c(a)\cap g \tag{12}$$

Let $g\in G_k(U)$, where U is a Zariski-open subset of X containing $Z=\mathrm{supp}(\mathrm{div}(\ell))$. Since the sheaves $E_1^{p,q}$ are flabby, g defines a hypercohomology class in $H^{k-1}(U,E_1^{*,-k})$ which we denote by the same letter g. The section ℓ defines a cohomology class

$$(\mathcal{L},\ell)\in H_Z^1(X,\mathcal{O}_X^*) \longrightarrow H_Z^1(X,\mathcal{K}_1)$$

with support in Z. The product

$$\cap:\ H_Z^p(X,\mathcal{K}_1)\otimes H^q(U,E_1^{*,-k}) \longrightarrow H^{p+q}(X,E_1^{*,-k-1}) \tag{13}$$

defines

$$(\mathcal{L},\ell)\cap g\in G_{k+1}(X) \tag{14}$$

If U is open and dense in $X_{(k)}$, then

$$\ell\cap(g+a)-\ell\cap a=(\mathcal{L},\ell)\cap g. \tag{15}$$

Since in the definition of (14) we do not assume that U is open and dense in $X_{(k)}$, formula (15) may be used to define $\ell\cap a$ for $a\in A(U)$ without the assumption that the Zariski-open subset U is also open in $X_{(k)}$.

If $A\in\mathrm{Ob}(\mathfrak{CH}^0(X))$, then we define $c_1(\mathcal{L})\cap A$ by formula (5) with a=ß (cf. [F1, 3.5.]). Since there is only one rational section , we do not need (6). The transformations (7) and (8) are defined by the same formulas as above.

If k<0, the $c_1(\mathcal{L})\cap .:\ \tilde{\mathfrak{CH}}^k(X) \longrightarrow \tilde{\mathfrak{CH}}^{k+1}(X)$ is defined to be the only additive functor between these categories.

<u>1.3. Example:</u> Let X be a smooth curve over a normal base scheme S, and let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on X. We assume that ℓ and m are rational sections of $\mathcal{L}$ and $\mathcal{M}$ on X whose divisors do not intersect. We put

$$\langle\ell,m\rangle=p_*(\ell\cap m)\in p_*(c_1(\mathcal{L})\cap c_1(\mathcal{M}))(S),$$

where $p:X\longrightarrow S$ is the projection. In this case, (12) and (15) imply that $\langle\ell,m\rangle$ satisfies the transformation rules [D, (6.1.2.)], and consequently $p_*(c_1(\mathcal{L})\cap c_1(\mathcal{M}))$ can be identified with the line bundle $\langle\mathcal{L},\mathcal{M}\rangle$ defined in [D,§6].

<u>1.4. Compatibility with direkt and inverse images:</u> Let $p:X\Longrightarrow Y$ be a flat morphism, $\mathcal{U}$ a covering of X_{Zar}, and $\alpha\in\check{Z}(\mathcal{U},E_1^{*,-k})$. Let $p^{-1}(\mathcal{U})$ be the covering of Y by the sets $p^{-1}(U_i)$. There is a natural morphism $p^*: \check{C}^*(\mathcal{U},E_{1,X}^{*,-k})\longrightarrow\check{C}^*(\mathcal{U},E_{1,Y}^{*,-k})$ which on the cohomology groups defines the homomorphism p^* of [F1, §1]. There is an isomorphism in $\tilde{\mathfrak{CH}}^k(Y)$

$$\underline{p}^*(\Phi(\alpha))\longrightarrow\Phi(p^*(\alpha)) \tag{16}$$

sending x in (1) to $p^*(x)$.

Let $q: Y\longrightarrow X$ be proper of relative dimension d. Formula [F1, 1.(7)] defines a homomorphism of complexes

$$q_*: \check{C}^*(q^{-1}(\mathcal{U}),E_{1,Y}^{*,-k}) \longrightarrow \check{C}^*(\mathcal{U},E_{1,X}^{*,-k}).$$

If $\beta\in\check{Z}^k(p^{-1}(\mathcal{U}),E_{1,Y}^{*,-k})$, then there is an isomorphism

$$\underline{q}_*(\Phi(\beta))\longrightarrow\Phi(q_*(\beta)) \tag{17}$$

sending x in (1) to $q_*(x)$.

Let $\mathcal{L}$ be a line bundle on X, $\mathcal{U}$ a covering of X_{Zar} on which $\mathcal{L}$ is trivialized by sections ℓ_i, $\varphi_{ij}=\ell_j/\ell_i$, $p:Y\longrightarrow X$ a flat morphism, $q:Z\longrightarrow X$ a proper morphism of relative dimension d, $A\in Ob(\tilde{\mathfrak{CH}}^k(X))$, $B\in Ob(\tilde{\mathfrak{CH}}^k(Z))$, a and b rational sections of A and B.

Then $p^*(\{c(a),\varphi\})=\{c(a),p^*\varphi\}$, hence (16) defines

$$\underline{p}^*((c_1(\mathcal{L})\cap A)_{\mathcal{U},\ell_i,a}) \longrightarrow (c_1(p^*\mathcal{L})\cap A)_{p^{-1}(\mathcal{U}),p^*(\ell_i),p^*(a)} \tag{18}$$

It is easy to see that (18) is compatible with (6), (7), and (8), hence it defines

$$\underline{p}^*(c_1(\mathcal{L})\cap A)\longrightarrow c_1(p^*\mathcal{L})\cap A. \tag{19}$$

In a similar manner one constructs

$$\underline{q}_*((c_1(q^*\mathcal{L})\cap A)_{q^{-1}(\mathcal{U}),q^*(\ell_i),q^*(a)}) \longrightarrow (c_1(\mathcal{L})\cap\underline{q}_*A)_{\mathcal{U},\ell_i,a} \tag{20}$$

using (17) and the adjunction formula. We get

$$\underline{q}_*(c_1(q^*\mathcal{L})\cap A) \longrightarrow c_1(\mathcal{L})\cap\underline{q}_*A. \tag{21}$$

The isomorphisms (19) and (21) are compatible with composition of flat and proper morphisms and with the base change isomorphism of [F1,§3.12.]. More precisely, this means the following. If X-schemes are denoted $p: Y\longrightarrow X$, then $\tilde{\mathfrak{CH}}^{\cdot}(Y)$ is a bifibred Picard category over (X-schemes, proper morphisms of const. rel. dim., flat morphisms). Then it is easy to see that $c_1(p^*\mathcal{L})\cap.: \tilde{\mathfrak{CH}}^{\cdot}(Y)\longrightarrow\tilde{\mathfrak{CH}}^{\cdot}(Y)$, equiped with the transformations (19) and (21), is a biadmissible functor (in the sense of [F1, 3.11.]) between bifibred Picard categories.

Let $p: Y \longrightarrow X$ be flat, $\mathcal{L}$ a line bundle on X, $A \in Ob(\widetilde{\mathfrak{CH}}^k(X))$, ℓ and a rational sections of $\mathcal{L}$ and A. It is easy to see that the image of $p^*(\ell \cap a)$ by (19) is $p^*(\ell) \cap p^*(a)$. If p is proper, $B \in Ob(\widetilde{\mathfrak{CH}}^k(Y))$, $b \in B_r(Y)$, then $p_*(p^*(\ell) \cap b)$ is mapped to $\ell \cap p_*(b)$ by (21).

<u>1.5. Commutativity:</u> We want to define an isomorphism

$$\sigma_{\mathcal{L},\mathcal{M}}: c_1(\mathcal{L}) \cap c_1(\mathcal{M}) \cap A \longrightarrow c_1(\mathcal{M}) \cap c_1(\mathcal{L}) \cap A. \tag{22}$$

Let $\mathcal{U}$ be a covering of X_{Zar} on which $\mathcal{L}$ is trivialized by ℓ_i. Our first step is to define an isomorphism

$$c_1(\mathcal{L}) \cap \mathbb{O}(\alpha) \longrightarrow \mathbb{O}((-1)^k\{\alpha, \varphi\}) \tag{23}$$

for $\alpha \in \check{Z}^k(\mathcal{U}, E_1^{*,-k})$, where $\varphi_{ij} = \ell_i / \ell_j$. It will identify $c_1(\mathcal{L}) \cap \mathbb{O}(\gamma)$ with $\mathbb{O}((-1)^k\{\gamma, \varphi\})$ if $\gamma \in \check{C}^{k-1}(\mathcal{U}, E_1^{*,-k})$ and $d(\gamma) = \alpha' - \alpha$.

Let $g \in \mathbb{O}(\alpha)_r(X)$ be a rational section. By definition (1), one checks easily that g has a representative $g \in \check{C}^{k-1}(\mathcal{U}, E_1^{*,-k})$ with the property $d(g) + \alpha = c(g) \in E_1^{k,-k}(X) \subseteq \check{C}^k(\mathcal{U}, E_1^{*,-k})$. If x is a section on $W \subseteq X$ of $((c_1(\mathcal{L}) \cap \mathbb{O}(\alpha))_{\mathcal{U}, \ell_i, g}$, i.e., $x \in \check{C}^{k-1}(\mathcal{U}, E_1^{*,-k})$ and $d(x) = -(-1)^k\{c(g), \varphi\}$ on W, then $y = x + (-1)^k\{g, \varphi\}$ satisfies

$$d(y) = d(x) + (-1)^k\{d(g), \varphi\} = -(-1)^k\{c(g), \varphi\} + \{c(g), \varphi\} - (-1)^k\{\alpha, \varphi\},$$

hence $y \in \mathbb{O}((-1)^k\{\alpha, \varphi\})(W)$. It is easy to see that the function $x \longrightarrow y$ commutes with (6), hence it defines the isomorphism (23). The transformation (23) is compatible with the isomorphisms (7), (8), (19), and (21).

Now we assume that $\mathcal{M}$ too is trivialized on $\mathcal{U}$, by sections m_i, with transition functions $\psi_{ij} = m_j / m_i$. Our next step is to construct an isomorphism

$$\chi_{\mathcal{U}, \mathcal{L}, \ell_i, \mathcal{M}, m_i, \alpha}: \mathbb{O}(-\{\alpha, \psi, \varphi\}) \longrightarrow \mathbb{O}(-\{\alpha, \varphi, \psi\}) \tag{24}$$

for $\alpha \in \check{Z}^k(\mathcal{U}, E_1^{*,-k})$. In (24), $\{.,.,.\}$ denotes the iteration of $\{.,.\}$. An easy calculation, using repeatedly the fact that $d(\varphi) = d(\psi) = 0$, shows

$$(\varphi\psi-\psi\varphi)_{ijk}=\varphi_{ij}\psi_{jk}-\psi_{ij}\varphi_{jk}=\varphi_{ij}\psi_{jk}+\varphi_{jk}\psi_{ij}$$
$$=\varphi_{ik}\psi_{jk}-\varphi_{jk}\psi_{jk}+\varphi_{jk}\psi_{ij}=\varphi_{ik}\psi_{ik}-\varphi_{jk}\psi_{jk}-\varphi_{ij}\psi_{ij}$$
$$=-d(\gamma)_{ijk}, \tag{25}$$

where $\gamma_{\alpha\beta}=\varphi_{\alpha\beta}\psi_{\alpha\beta}$. By (25), $\chi_{\mathcal{U},\mathcal{L},\ell_i,\mathcal{M},m_i,\alpha}$ may be defined by $\Phi((-1)^k\{\alpha,\gamma\})$. In the special case $\mathcal{L}=\mathcal{M}$ and $\ell_i=m_i$, the well-known identity between Steinberg symbols

$$\varphi_{\alpha\beta}\varphi_{\alpha\beta}=\varphi_{\alpha\beta}[-1] \quad \text{in } K_2(k(x))$$

can be used to compute $\chi_{\mathcal{U},\mathcal{L},\ell_i,\mathcal{L},\ell_i,\alpha}$ on $\Phi(\{\alpha,\varphi,\varphi\})$:

$$\chi_{\mathcal{U},\mathcal{L},\ell_i,\mathcal{L},\ell_i,\alpha}=\alpha\cap\varphi\cap[-1], \tag{26}$$

where the first product is (13) with X=S=U and the second product is

$$H^1(X,\mathcal{K}_1)\otimes K_1(X)\longrightarrow H^2(X,\mathcal{K}_2).$$

Now we are ready to define (22). For a rational section $a\in A_r(X)$, consider the isomorphism

$$c_1(\mathcal{L})\cap\Phi(\{c(a),\psi\})\longrightarrow\Phi(\{c(a),\psi,\varphi\})\longrightarrow\Phi(\{c(a),\varphi,\psi\})\longrightarrow$$
$$\longrightarrow c_1(\mathcal{M})\cap\Phi(\{c(a),\varphi\}), \tag{27}$$

where the first and the third arrow is of type (22) and the middle one is (23). We want to check that it commutes with the isomorphisms (6), (7), and (8). For (6), we have to prove the commutativity of

$$\begin{array}{ccc}
\Phi(-\{c(a),\psi,\varphi\}) & \xrightarrow{\Phi((-1)^k\{c(a),\gamma\})} & \Phi(-\{c(a),\varphi,\psi\}) \\
{\scriptstyle\Phi(-\{b-a,\psi,\varphi\})}\downarrow & & \downarrow{\scriptstyle\Phi(-\{b-a,\varphi,\psi\})} \\
\Phi(-\{c(b),\psi,\varphi\}) & \xrightarrow{\Phi((-1)^k\{c(b),\gamma\})} & \Phi(-\{c(b),\varphi,\psi\})
\end{array}.$$

But

$$(-1)^k\{c(a),\gamma\}-\{b-a,\varphi,\psi\}-(-1)^k\{c(b),\gamma\}+\{b-a,\psi,\varphi\}$$
$$=\{b-a,d(\gamma)\}-(-1)^k\{d(b-a),\gamma\} = -(-1)^k d(\{b-a,\gamma\}),$$

and the diagram commutes. For (7) the compatibility is verified in a similar manner, and for (8) it is trivial.

We have seen that the isomorphisms (27) fit together, defining (23). Since $\varphi\psi=-\psi\varphi$, we have $\sigma_{\mathcal{L},\mathcal{M}}\sigma_{\mathcal{M},\mathcal{L}}=\mathrm{Id}$. By (26), the action of

$\sigma_{\mathcal{L},\mathcal{L}}$ on $c_1(\mathcal{L})\cap c_1(\mathcal{L})\cap A$ is given by

$$\sigma_{\mathcal{L},\mathcal{L}}=[\mathcal{L}]\cap[-1]\cap[A], \tag{28}$$

where the products are the same as in (26). It is easy to see that $\sigma_{\mathcal{L},\mathcal{M}}$ is compatible with (19) and (21).

Let ℓ, m, and a be rational sections of $\mathcal{L}$, $\mathcal{M}$, A on X whose divisors and cycles meet properly, i.e., such that $\ell\cap a$, $m\cap a$, $\ell\cap m\cap a$, $m\cap\ell\cap a$ are rational sections of $c_1(\mathcal{L})\cap A,\dots,c_1(\mathcal{M})\cap c_1(\mathcal{L})\cap A$. We want to prove

$$\sigma_{\mathcal{L},\mathcal{M}}(\ell\cap m\cap a)=m\cap\ell\cap a. \tag{29}$$

The first thing we have do is to compute the image of $\ell\cap m\cap a$ under the isomorphism (23):

$$c_1(\mathcal{L})\cap(c_1(\mathcal{M})\cap A)_{\mathcal{U},m_i,a}\longrightarrow \mathbb{O}(\{c(f),\psi,\varphi\}).$$

$\mathbb{O}(\{c(f),\psi\})$ has a rational section $g=m\cap a$ given by $g=\{c(a),m/m_j\}$. Applying the definition of (23), we find that the image of $\ell\cap m\cap a$ in $\mathbb{O}(-\{c(f),\psi,\varphi\})$ is given by the class of

$$\{c(m\cap a),\ell/\ell_i\}-(-1)^k\{c(a),m/m_i,\varphi_{ij}\} \tag{30}$$

modulo $B^{k+1}(\mathcal{U},E_1^{*,-k-2})$. In a similar manner we find that the image of $m\cap\ell\cap a$ in $\mathbb{O}(\{c(f),\varphi,\psi\})$ is given by

$$\{c(\ell\cap a),m/m_i\}-(-1)^k\{c(a),\ell/\ell_i,\psi_{ij}\}. \tag{31}$$

By (30), (31), and the definition of $\sigma_{\mathcal{L},\mathcal{M}}$ the proof of (29) is reduced to the investigation of the difference

$$(-1)^k(\{c(a),\ell/\ell_i,\psi_{ij}\}-\{c(a),m/m_i,\varphi_{ij}\}-\{c(a),\varphi_{ij},\psi_{ij}\})+$$
$$+\{c(m\cap a),\ell/\ell_i\}-\{c(\ell\cap a),m/m_i\}$$

Since the supports of $\mathrm{div}(\ell)$, $\mathrm{div}(m)$, and $c(a)$ intersect properly, m/m_i and ℓ/ℓ_i have residue classes $(m/m_i)(x)\in k(x)^*$, $(\ell/\ell_i)(x)\in k(x)^*$ for $x\in(U_i)_k\cap\mathrm{supp}(c(a))$. Consequently, there is a well-defined element of $E_1^{k,-k-2}(U_i)$:

$$\lambda_i=\{c(a),\ell/\ell_i,m/m_i\} \tag{33}$$

where $c(a)=(n_x)_x$. Since the divisors of ℓ and m have no common component intersecting the support of $c(a)$, the tame symbol of (33) is given by

$$d_1(\lambda_i)=\{c(\ell\cap a),m/m_i\}-\{c(m\cap a),\ell/\ell_i\}.$$

For the Čech differntial of λ we find

$$d_0(\lambda)=\{c(a),m/m_i,\varphi_{ij}\}-\{c(a),\ell/\ell_i,\psi_{ij}\}+\{c(a),\varphi_{ij},\psi_{ij}\}$$

Consequently, the Čech hyperdifferential of λ is

$$(-1)^k(\{c(a),m/m_i,\varphi_{ij}\}-\{c(a),\ell/\ell_i,\psi_{ij}\}+\{c(a),\varphi_{ij},\psi_{ij}\}) \\ +\{c(\ell\cap a),m/m_i\}-\{c(m\cap a),\ell/\ell_i\}$$

and (32) is a complete differential. The proof of (29) is complete.

Let $\mathcal{L},\mathcal{M},\mathcal{N}$ be line bundles on X, and $A\in\widetilde{\mathfrak{C}\mathfrak{H}}^k(X)$. We want to prove the commutativity of

$$\begin{array}{ccc} c_1(\mathcal{L})\cap c_1(\mathcal{M})\cap c_1(\mathcal{N})\cap A & \longrightarrow & c_1(\mathcal{M})\cap c_1(\mathcal{L})\cap c_1(\mathcal{N})\cap A \\ \downarrow & & \downarrow \\ c_1(\mathcal{L})\cap c_1(\mathcal{N})\cap c_1(\mathcal{M})\cap A & & c_1(\mathcal{M})\cap c_1(\mathcal{N})\cap c_1(\mathcal{L})\cap A \\ \downarrow & & \downarrow \\ c_1(\mathcal{N})\cap c_1(\mathcal{L})\cap c_1(\mathcal{M})\cap A & \longrightarrow & c_1(\mathcal{N})\cap c_1(\mathcal{M})\cap c_1(\mathcal{L})\cap A. \end{array} \tag{34}$$

If A, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ have rational sections a, ℓ, m, n whose cycles and divisors meet properly (i.e., $\ell\cap m\cap n\cap a$ etc. are rational sections), then (34) follows from (29). In the general case, let p: $E\longrightarrow X$ be the fibre space of the bundle $\mathcal{L}\oplus\mathcal{M}\oplus\mathcal{N}$. Because $\underline{p}^*$ is an equivalence of categories, it suffices to prove (34) after base-change to E. On E, there are tautological sections ℓ, m, n of $p^*\mathcal{L}$, $p^*\mathcal{M}$, $p^*\mathcal{N}$. If A is any rational section of A on X, then the previuos remark can be applied to ℓ, m, n, and $p^*(a)$. The proof of (34) is complete.

Example: Let us return to the situation of 1.3.. By (29), $\underline{p}_*(\sigma_{\mathcal{L},\mathcal{M}})$ is the isomorphism $\langle\mathcal{L},\mathcal{M}\rangle\longrightarrow\langle\mathcal{M},\mathcal{L}\rangle$ which sends $\langle\ell,m\rangle$ to $\langle m,\ell\rangle$. Integrating (28) along the fibres, we get the well-known identity

$$\langle\ell,\tilde{\ell}\rangle=(-1)^{\deg(\mathcal{L})}\langle\tilde{\ell},\ell\rangle$$

in $\langle\mathcal{L}.\mathcal{L}\rangle$ (cf. /D, 6.2./).

<u>1.6. Lemma:</u> Let $\mathfrak{F}$ be a vector bundle of dimension e on X, $p:\mathbb{P}(\mathfrak{F})\longrightarrow X$ its projective fibration, and $\mathcal{O}(-1)\subset p^*\mathfrak{F}$ the tautological line bundle. It has a first Chern class $c_1(\mathcal{O}(1))\in H^1(\mathbb{P}(\mathfrak{F}),\mathcal{K}_1)$. Then the homomorphism

$$\prod_{j=0}^{e-1} E_2^{p-j,q+j}(X) \longrightarrow E_2^{p,q}(\mathbb{P}(\mathfrak{F})) \tag{35}$$

$$(\alpha_j) \longrightarrow \sum_{j=0}^{e-1} c_1(\mathcal{O}(1))^j p^*(\alpha_j)$$

is an isomorphism, where the product in (35) is the product

$$H^p(y,\mathcal{K}_q)\times E_2^{k,l}(Y) \longrightarrow E_2^{p+k,q-l}(Y)$$

defined in [G, p.281].

<u>Proof:</u> This is [G, Note (i) on p.287]. The proof is similar to the proof of [G, Theorerm 8.10.].

<u>1.7. Corollary:</u> The functor

$$\underset{j=0}{\overset{e-1}{\times}} \tilde{\mathfrak{CH}}^{p-j}(X) \longrightarrow \tilde{\mathfrak{CH}}^p(\mathbb{P}(\mathfrak{F})) \tag{36}$$

$$(A_j) \longrightarrow \bigoplus_{j=0}^{e-1} c_1(\mathcal{O}(1)^j \cap p^*(A_j)$$

is an equivalence of categories. Here the symbol $c_1(\mathcal{L})^j\cap B$ denotes the iteration $c_1(\mathcal{L})\cap c_1(\mathcal{L})\cap\ldots\cap B$.

<u>Proof:</u> It follows from 1.6. that (36) induces isomorphisms between π_1 and π_0 of the Picard categories on both sides of (36).

<u>1.8. Definition of theChern functors:</u> We amy compose the functor

$$\tilde{\mathfrak{CH}}^p(X) \longrightarrow \tilde{\mathfrak{CH}}^{p+e}(\mathbb{P}(\mathfrak{F}))$$

$$A \longrightarrow c_1(\mathcal{O}(1))^e \cap p^*(A)$$

with the inverse of (36) to obtain additive functors

$$c_i(\mathfrak{F})\cap A : \tilde{\mathfrak{CH}}^p(X) \longrightarrow \tilde{\mathfrak{CH}}^{p+i}(X), \quad 1\leq i\leq e$$

$$c_0(\mathfrak{F})\cap A=A$$

and an additive functor-isomorphism

$$\bigoplus_{j=0}^{e} c_1(\mathcal{O}(1))^{e-j} \cap p^*(c_j(\mathfrak{F})\cap A) \longrightarrow 0. \tag{37}$$

The Chern functors are unique up to unique functor-isomorphism: If $\tilde{c}_j(\mathcal{E})\cap.$ are other additive functors functors with $\tilde{c}_0(\mathcal{E})\cap A=A$ and an isomorphism (37), then there exists a unique functor-isomorphism $c_j(\mathcal{E})\cap A\longrightarrow\tilde{c}_j(\mathcal{E})\cap A$ compatible with (37) which is the identity if $j=0$.

If $\mathcal{E}$ is a line bundle, we have $\mathbb{P}(\mathcal{E})=X$ and $\mathcal{O}(1)=\mathcal{E}^{-1}$, and it follows easily that (37) is solved by the functor $c_1(\mathcal{E})\cap A$ defined in 1.2..

Let ϕ: $\mathcal{E}\longrightarrow\mathcal{E}'$ be an isomorphism of vector bundles. It induces an isomorphism $\mathbb{P}(\mathcal{E})\longrightarrow\mathbb{P}(\mathcal{E}')$, hence there is a unique isomorphism $c_j(\mathcal{E})\cap A\longrightarrow c_j(\mathcal{E}')\cap A$ which is compatible with (37) and is the identity if $j=0$.

Let $\mathcal{E}$ be a vector bundle on X, $f:Y\longrightarrow X$ a flat morphism and $g:Z\longrightarrow X$ a proper morphism of relative dimension d. Using the results of 1.4. it is not hard to construct natural isomorphisms

$$\underline{f}^*(c_j(\mathcal{E})\cap A)\longrightarrow c_j(f^*\mathcal{E})\cap\underline{f}^*A\ ,\ A\in\widetilde{\mathfrak{CH}}^{\cdot}(X) \qquad (38)$$

$$\underline{g}_*(c_j(g^*\mathcal{E})\cap B)\longrightarrow c_j(\mathcal{E})\cap\underline{g}_*B\ ,\ B\in\widetilde{\mathfrak{CH}}^{\cdot}(Z).$$

They satisfy the following compatibility with compoistion of flat and proper morphisms and the base-change isomorphism [F1,3.12.]: If for an X-scheme $f:Y\longrightarrow X$ we consider the functor

$$c_j(f^*\mathcal{E})\cap.:\ \widetilde{\mathfrak{CH}}^{\cdot}(X)\longrightarrow\widetilde{\mathfrak{CH}}^{\cdot}(Y),$$

then (38) defines on it the structure of a biadmissible functor between bifibred Picard categories over (X-schemes, proper morphisms, flat morphisms).

Our next steps aim at proving the functorial version of the Whitney sum formula. First we need the following isomorphisms:

<u>1.9.:</u> Let

$$\begin{array}{ccc} D & \xrightarrow{i} & X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & S & \end{array}$$

be a commutative diagram in which i is a regular immersion of codimension one and p and q are flat. We put $\mathcal{O}(D)=\mathcal{J}^{-1}$, where $\mathcal{J}$ is the sheaf of ideals defining D. There is a natural isomorphism

$$c_1(\mathcal{O}(D))\cap\underline{q}^*A\longrightarrow\underline{i}_*\underline{p}^*A \qquad (39)$$

which sends "1"$\cap q^*(a)$ to $i_*p^*(a)$, where a is a rational section of A on S and "1" is the canonical section of $\mathcal{O}(D)$. This isomorphisms is (in an obvious sense) compatible with flat base-changes $X'\longrightarrow X$, flat maps $S\longrightarrow S'$, and proper base-changes $S\longrightarrow S'$. If $\mathcal{L}$ is a line bundle on X, the diagram

$$\begin{array}{ccc} c_1(\mathcal{O}(D))\cap \underline{q}^*(c_1(\mathcal{L})\cap A) & \longrightarrow & \underline{i}_*\underline{p}^*(c_1(\mathcal{L})\cap A) \\ \downarrow & & \downarrow \\ c_1(q^*\mathcal{L})\cap c_1(\mathcal{O}(D))\cap \underline{q}^*A & \longrightarrow & c_1(q^*\mathcal{L})\cap \underline{i}_*\underline{p}^*A \end{array} \tag{40}$$

commutes.

Proof: We have only to check that the above definition of (39) is independent of the choice of a, i.e., that

$$c_1(\mathcal{O}(D))\cap \underline{q}^*(\gamma)=\underline{i}_*\underline{p}^*(\gamma)$$

for $\gamma\in \mathrm{Aut}_{\tilde{\mathfrak{G}\mathfrak{H}}^k(S)}(A)=G_k(S)$. If $\ell_i\in\mathcal{O}_X(U_i-D)$ are trivializations of $\mathcal{O}(D)$ on an open covering $\mathcal{U}$ of X, then the l.h.s. of the last equality is given by the cohomology class of $\{q^*\boldsymbol{\gamma},\boldsymbol{\varphi}\}\in \check{Z}^k(\mathcal{U},E_1^{*,-k})$, where $\boldsymbol{\gamma}\in E_1^{k-1,-k}(S)$ is a representative for γ, and $\varphi_{ij}=\ell_j/\ell_i$. If $q^*(\boldsymbol{\gamma})=(g_x)_{x\in X_k}$, then because D is flat over S and of codimension one in X, $g_x\neq 1$ implies that x does not belong to D, hence the image of ℓ_i in k(x) is well-defined. Consequently, the product

$$\boldsymbol{\alpha}=\left(q^*(\boldsymbol{\gamma})\ell_i\right)_{i\in I}=\left(g_x\ell_i\right)_{\substack{i\in I\\ x\in X_k}}$$

is well-defined. It satisfies

$$d(\boldsymbol{\alpha})=\{q^*(\boldsymbol{\gamma}),\boldsymbol{\varphi}\}+d_1(\ (q^*(\boldsymbol{\gamma})\ell_i)_{i\in I})=\{q^*(\boldsymbol{\gamma}),\boldsymbol{\varphi}\}-d_1(\ (\ell_i q^*(\boldsymbol{\gamma}))_{i\in I}).$$

By [F1, Lemma 1.6.], we have

$$d_1(\ell_i q^*(\boldsymbol{\gamma}))=i_*p^*(\boldsymbol{\gamma})\big|_{U_i} \quad \text{in } E_1^{k,-k-1}(U_i),$$

proving the desired identity.

1.10.: Now we are ready to construct the Whitney isomorphism for exacrt sequences of the form

$$0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow\mathcal{L}\longrightarrow 0$$

where $\mathcal{L}$ is a line bundle. We apply 1.9. to the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{F}) & \xrightarrow{i} & \mathbb{P}(\mathcal{E}) \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

There is a natural isomorphism $i^*\mathcal{O}(1)_{\mathcal{E}} \simeq \mathcal{O}(1)_{\mathcal{F}}$, so we may denote both line bundles by the same symbol $\mathcal{O}(1)$. Restricting $\mathcal{E}\longrightarrow\mathcal{L}$ to the subbundle $\mathcal{O}(-1)\subset p^*\mathcal{E}$ we obtain a section ξ of $\mathcal{H}om(\mathcal{O}(-1),p^*\mathcal{L})=p^*\mathcal{L}\otimes\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$. The subscheme defined by the vanishing of ξ is $\mathbb{P}(\mathcal{F})$. By 1.9., there is a canonical isomorphism

$$\underline{i}_*\underline{q}^*B \longrightarrow c_1(q^*\mathcal{L}\otimes\mathcal{O}(1))\cap\underline{q}^*B, \quad B\in\widetilde{\mathfrak{CH}}^{\cdot}(X)$$

which sends $i_*q^*(b)$ to $\xi\cap p^*(b)$. Applying this to (37) (for the bundle $\mathcal{F}$), we find an isomorphism

$$0 \longrightarrow \underline{i}_*\Big(\bigoplus_{j=0}^{f} c_1(\mathcal{O}(1))^{f-j}\cap\underline{q}^*(c_j(\mathcal{F})\cap A)\Big) \tag{41}$$

$$\longrightarrow \bigoplus_{j=0}^{f} c_1(\mathcal{O}(1))^{f-j}\cap\underline{i}_*\underline{q}^*(c_j(\mathcal{F})\cap A))$$

$$\longrightarrow \bigoplus_{j=0}^{f} c_1(\mathcal{O}(1))^{f-j}\cap\Big[c_1(p^*\mathcal{L})\cap\underline{p}^*(c_j(\mathcal{F})\cap A)\oplus c_1(\mathcal{O}(1)\cap\underline{p}^*(c_j(\mathcal{F})\cap A)\Big]$$

$$\longrightarrow \underline{p}^*(c_1(\mathcal{L})\cap c_f(\mathcal{F})\cap A)\oplus c_1(\mathcal{O}(1))^{e}\cap\underline{p}^*(c_0(\mathcal{F})\cap A)\oplus$$

$$\oplus\bigoplus_{j=1}^{f} c_1(\mathcal{O}(1))^{e-j}\cap\underline{p}^*(c_1(\mathcal{L})\cap c_{j-1}(\mathcal{F})\cap A\oplus c_j(\mathcal{F})\cap A)\,.$$

This is of the form (37) for the vector bundle $\mathcal{E}$.Sine (37) defines the Chern functors up to unique isomorphism, (41) defines the Whitney isomorphism we are looking for:

$$c_j(\mathcal{E})\cap A\longrightarrow\begin{cases} A=c_0(\mathcal{E})\cap A & \text{if } j=0 \\ c_1(\mathcal{L})\cap c_{j-1}(\mathcal{F})\cap A\oplus c_j(\mathcal{F})\cap A & \text{if } 1\le j\le f \\ c_1(\mathcal{L})\cap c_f(\mathcal{F})\cap A & \text{if } j=e\,. \end{cases} \tag{42}$$

We can write this in the shorter form

$$c.(\mathcal{E})\cap A \longrightarrow c.(\mathcal{L})\cap c.(\mathcal{F})\cap A,$$

where $c.(\mathcal{E})\cap A=\bigoplus_{j\ge 0} c_j(\mathcal{E})\cap A$ in $\widetilde{\mathfrak{CH}}^{\cdot}(X)$.

1.11. Symmetry: Before we can prove the analogue of (42) in the general case we have to define a symmetry isomorphism between the Chern functors and to explain its relation to (42).

Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles of dimension e and f on X. Let $r:\mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{F})\longrightarrow X$ be the projection. By applying 1.7. twice, we find that the following isomorphism in $\widetilde{\mathfrak{CH}}^{e+f}(\mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{F}))$ characterizes $c_k(\mathcal{E})\cap c_1(\mathcal{F})\cap A$ up to unique isomorphism:

$$\bigoplus_{j=0}^{e}\bigoplus_{k=0}^{f} c_1(q^*\mathcal{O}(1)_{\mathcal{F}})^{f-k}\cap c_1(p^*\mathcal{O}(1)_{\mathcal{E}})^{e-j}\cap \underline{r}^*(c_j(\mathcal{E})\cap c_k(\mathcal{F})\cap A) \longrightarrow 0$$

(p and q are the projections of $\mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{F})$ to $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{F})$). In a similar manner $c_k(\mathcal{F})\cap c_j(\mathcal{E})\cap A$ is characterized by the isomorphism

$$\bigoplus_{j=0}^{e}\bigoplus_{k=0}^{f} c_1(p^*\mathcal{O}(1)_{\mathcal{E}})^{e-j}\cap c_1(q^*\mathcal{O}(1)_{\mathcal{F}})^{f-k}\cap \underline{r}^*(c_j(\mathcal{E})\cap c_k(\mathcal{F})\cap A) \longrightarrow 0.$$

Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on a scheme and a and b integers. We define an isomorphism $c_1(\mathcal{L})^a\cap c_1(\mathcal{M})^b\longrightarrow c_1(\mathcal{M})^a\cap c_1(\mathcal{L})^b$ by the following prmutation of the factors:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & \ldots & a & a+1 & a+2 & \ldots & a+b \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \\ b+1 & b+2 & b+3 & \ldots & b+a & 1 & 2 & & b \end{array} \qquad (43)$$

In (43), 1;...;a are the factors of the product $c_1(\mathcal{L})^a$ and a+1;...;a+b are the factors of $c_1(\mathcal{M})^b$. In other words, we apply $\sigma_{\mathcal{L},\mathcal{M}}$ ab times but we never apply $\sigma_{\mathcal{L},L}\mathcal{L}$ or $\sigma_{\mathcal{M},\mathcal{M}}$. If an isomorphism $c_1(\mathcal{L})^a\cap c_1(\mathcal{M})^b\longrightarrow c_1(\mathcal{M})^a\cap c_1(\mathcal{L})^b$ is used without comment, it is supposed to be of the form (43).

There is a unique isomorphism $\sigma_{\mathcal{E},\mathcal{F}}$: $c_j(\mathcal{E})\cap c_k(\mathcal{F})\cap A\longrightarrow c_k(\mathcal{F})\cap c_j(\mathcal{E})\cap A$ which is the identity if jk=0 and makes the diagram

$$\begin{array}{ccc}
\bigoplus_{j=0}^{e}\bigoplus_{k=0}^{f} c_1(q^*\mathcal{O}(1)_{\mathcal{F}})^{f-k}\cap c_1(p^*\mathcal{O}(1)_{\mathcal{E}})^{e-j}\cap \underline{r}^*(c_j(\mathcal{E})\cap c_k(\mathcal{F})\cap A) & \longrightarrow & \\
\downarrow & & \downarrow \\
\bigoplus_{j=0}^{e}\bigoplus_{k=0}^{f} c_1(p^*\mathcal{O}(1)_{\mathcal{E}})^{e-j}\cap c_1(q^*\mathcal{O}(1)_{\mathcal{F}})^{f-k}\cap \underline{r}^*(c_j(\mathcal{E})\cap c_k(\mathcal{F})\cap A) & & 0 \\
\downarrow & & \uparrow \\
\bigoplus_{j=0}^{e}\bigoplus_{k=0}^{f} c_1(p^*\mathcal{O}(1)_{\mathcal{E}})^{e-j}\cap c_1(q^*\mathcal{O}(1)_{\mathcal{F}})^{f-k}\cap \underline{r}^*(c_j(\mathcal{E})\cap c_k(\mathcal{F})\cap A) & \longrightarrow &
\end{array} \tag{44}$$

commutative.

We have an analogue of (34) and the identity $\sigma_{\mathcal{E},\mathcal{F}}\sigma_{\mathcal{F},\mathcal{E}}=1$ because these properties are satisfied for line bundles. It is also clear that $\sigma_{\mathcal{E},\mathcal{F}}$ is compatible with flat pull-back and proper push-forward. Let $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow\mathcal{L}\longrightarrow 0$ be an exact sequence with $\dim(\mathcal{L})=1$, and let $\mathcal{G}$ be a vector bundle on X. It follows from (40) that the diagram

$$\begin{array}{ccc}
c.(\mathcal{L})\cap c.(\mathcal{F})\cap c_j(\mathcal{G})\cap A & \longrightarrow & c.(\mathcal{E})\cap c_j(\mathcal{G})\cap A \\
\downarrow{\scriptstyle \sigma_{\mathcal{F},\mathcal{G}}} & & \\
c.(\mathcal{L})\cap c_j(\mathcal{G})\cap c.(\mathcal{F})\cap A & & \downarrow{\scriptstyle \sigma_{\mathcal{E},\mathcal{G}}} \\
\downarrow{\scriptstyle \sigma_{\mathcal{L},\mathcal{G}}} & & \\
c_j(\mathcal{G})\cap c.(\mathcal{L})\cap c.(\mathcal{F})\cap A & \longrightarrow & c_j(\mathcal{G})\cap c.(\mathcal{E})\cap A
\end{array} \tag{45}$$

commutes.

<u>1.12.</u> Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X of dimensions e and f, and let $A=(A_l)_{0\le l<\infty}\in \mathrm{Ob}(\widetilde{\mathfrak{CH}}^{\cdot}(X))$. We define an element

$$\tau.(\mathcal{E},\mathcal{F},A)=(\tau_k(\mathcal{E},\mathcal{F},A))_{1\le k<\infty}\in\coprod_{k\ge 1} G_k(X) \tag{46}$$

by

$$\tau_k(\mathcal{E},\mathcal{F},A)=\sum_{l=0}^{k}\ \sum_{i+j=k-l}(e-i)(f-j)c_i(\mathcal{E})c_j(\mathcal{F})[A][-1], \tag{47}$$

where $c_i(\mathcal{E})[A]$ is defined as the isomorphism class of $c_i(\mathcal{E})\cap A$, and $[-1]$ is $-1\in K_1$.

The aim of 1.12. is to prove the following formula: Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \xrightarrow{\pi} \mathcal{L}\oplus\mathcal{M} \longrightarrow 0$$

be an exact sequence of vector bundles, with $\dim(\mathcal{L})=\dim(\mathcal{M})=1$. We put $\mathcal{G}=\pi^{-1}(\mathcal{M})$, $\mathcal{H}=\pi^{-1}(\mathcal{L})$. Let $A\in Ob(\widetilde{\mathfrak{CH}}^{k}(X))$. Then the following diagram commutes:

$$\begin{array}{ccccc} c.(\mathcal{E})\cap A & \xrightarrow{\alpha} & c.(\mathcal{L})\cap c.(\mathcal{G})\cap A & \xrightarrow{\beta} & c.(\mathcal{L})\cap c.(\mathcal{M})\cap c.(\mathcal{F})\cap A \\ \Big\downarrow{\scriptstyle \tau.(\mathcal{L},\mathcal{M},c.(\mathcal{E})\cap A)} & & & & \Big\downarrow{\scriptstyle \sigma_{\mathcal{L},\mathcal{M}}} \\ c.(\mathcal{E})\cap A & \xrightarrow{\gamma} & c.(\mathcal{M})\cap c.(\mathcal{H})\cap A & \xrightarrow{\delta} & c.(\mathcal{M})\cap c.(\mathcal{L})\cap c.(\mathcal{F})\cap A, \end{array} \qquad (48)$$

where the horizontal arrows are in an obvious manner constructed from the Whitney sum isomorphism.

Let $t:Y\longrightarrow X$ be the fibre space of the bundle $\mathcal{L}\oplus\mathcal{M}\oplus\mathcal{E}^{\vee\oplus(e-1)}$, where $\mathcal{E}^{\vee}$ is the dual of $\mathcal{E}$. The e-2 $\mathcal{E}^{\vee}$-coordinates define sections $\lambda_3,\ldots,\lambda_e$ of $t^*\mathcal{E}^{\vee}$. The L- and M-coordinates define sections ℓ and m of $t^*\mathcal{L}$ and $t^*\mathcal{M}$ on Y. We define rational sections λ_1, λ_2 of $t^*\mathcal{E}^{\vee}$ by $\lambda_1(\mathcal{G})=0$, $\lambda_1(\ell)=1$ and $\lambda_2(\mathcal{H})=0$, $\lambda_2(m)=1$.

We have a cartesian diagram of projective fibrations over Y

$$\begin{array}{ccccc} & & \mathbb{P}(\mathcal{F}) & & \\ & \swarrow{\scriptstyle i'} & & \searrow{\scriptstyle j'} & \\ \mathbb{P}(\mathcal{G}) & & & & \mathbb{P}(\mathcal{H}) \\ & \searrow{\scriptstyle i} & & \swarrow{\scriptstyle j} & \\ & & \mathbb{P}(\mathcal{E}) & & \end{array}$$

and denote the projections from $\mathbb{P}(\mathcal{E})$, $\mathbb{P}(\mathcal{H})$, $\mathbb{P}(\mathcal{G})$, and $\mathbb{P}(\mathcal{F})$ to Y by p, q, r, and s. By the isomorphism between $\mathcal{E}^{\vee}$ and $p_*\mathcal{O}(1)_{\mathcal{E}}$, λ_i defines sections Λ_i of $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ (rational sections if i=1 or i=2). To avoid awkward expressions, we denote the restrictions of Λ_i and of $\mathcal{O}(1)_{\mathcal{E}}$ to one of the projective subspaces of $\mathbb{P}(\mathcal{E})$ by the same letters Λ_i and $\mathcal{O}(1)$.

Let a_i be a rational section of $\mathcal{F})\cap A$ on X. Then

$$\varepsilon=\bigoplus_{n=0}^{e-2}\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap s^*t^*(a_n)$$

is a rational section of

$$\bigoplus_{n=0}^{e-2}c_1(\mathcal{O}(1))^{e-2-n}\cap \underline{s}^*\underline{t}^*(c_n(\mathcal{F})\cap A)$$

on $\mathbb{P}(\mathcal{F})$. Let $b\in(G_{k+e-2})_r(\mathbb{P}(t^*\mathcal{F})$ be the image of ε by (37). We denote the isomorphism $c.(\mathcal{M})\cap c.(\mathcal{F})\longrightarrow c.(\mathcal{G})$ by ζ. It is easy to see that $\Lambda_2\otimes m$ is the canonical section ξ of $p^*t^*\mathcal{M}\otimes\mathcal{O}(1)$ which has a simple zero along $\mathbb{P}(\mathscr{K})$. Using this and (41), we see that the image of

$$\bigoplus_{n=0}^{e-2}\Big[\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_2\cap\zeta(a_n)\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\zeta(a_n)\Big]\in \tag{49}$$

$$\in\Big(\bigoplus_{n=0}^{e-1}c_1(\mathcal{O}(1))^{e-1-n}\cap\underline{r}^*\underline{t}^*(c_n(\mathscr{G})\cap A)\Big)_r(\mathbb{P}(\mathscr{G}))$$

in $(G_{1+e-1})_r(\mathbb{P}(t^*\mathscr{G}))$ is $i'_*(b)$. The lax notation (49) means more precisely

$$\zeta'\Big(\bigoplus_{n=0}^{e-2}\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_2\cap r^*t^*(a_n)\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap r^*t^*(a_n)\Big),$$

where

$$\zeta':\ \bigoplus_{n=0}^{e-2}\Big[c_1(\mathcal{O}(1))^{e-1-n}\cap\underline{r}^*\underline{t}^*(c_n(\mathscr{F})\cap A)\oplus$$

$$\oplus c_1(\mathcal{O}(1))^{e-2-n}\cap\underline{r}^*\underline{t}^*(c_1(\mathscr{M})\cap c_n(\mathscr{F})\cap A)\Big]$$

$$\downarrow$$

$$\bigoplus_{m=0}^{e-1}c_1(\mathcal{O}(1))^{e-1-n}\cap\underline{r}^*\underline{t}^*(c_m(\mathscr{G})\cap A)$$

is the isomorphism derived from ζ. A similar computation can be applied to the image of (53) by $\underline{i}_*$. Its result is that the image of

$$\bigoplus_{n=0}^{e-2}\Big[\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_2\cap\Lambda_1\cap\alpha^{-1}\beta^{-1}(a_n)\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_2\cap\alpha^{-1}\beta^{-1}(\ell\cap a_n)\oplus \tag{50}$$

$$\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_1\cap\alpha^{-1}\beta^{-1}(m\cap a_n)\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\alpha^{-1}\beta^{-1}(\ell\cap m\cap a_n)\in$$

$$\in\Big(\bigoplus_{m=0}^{e}c_1(\mathcal{O}(1))^{e-n}\cap\underline{r}^*\underline{t}^*(c_m(\mathscr{E})\cap A)\Big)_r(\mathbb{P}(t^*\mathscr{E}))$$

in $(G_{k+e})_r(\mathbb{P}(t^*\mathscr{E}))$ is $i_*i'_*(b)$. The meaning of (50) is similar to that of (49), α and β are isomorphisms in (48). Applying the same method to the embeddings j and j', we find that the image of

$$\bigoplus_{n=0}^{e-2}\Big(\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_1\cap\Lambda_2\cap\gamma^{-1}\delta^{-1}(a_n)\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_1\cap\gamma^{-1}\delta^{-1}(\mathcal{m}\cap a_n)\oplus$$
$$\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_2\cap\gamma^{-1}\delta^{-1}(\ell\cap a_n)\oplus\Lambda_e\cap\ldots\cap\Lambda_{n+3}\gamma^{-1}\delta^{-1}(\mathcal{m}\cap\ell\cap a_n)\tag{51}$$

in $(G_{k+e})_r(\mathbb{P}(t^*\mathcal{E}))$ is $j_*j'_*(b)=i_*i'_*(b)$. It follows that (50) and (51) are eaqual. By (29) and 1.7., this reduces the proof of (48) to the proof of

$$\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_1\cap\Lambda_2\cap p^*p^*(a_n)-\Lambda_e\cap\ldots\cap\Lambda_{n+3}\cap\Lambda_2\cap\Lambda_1\cap p^*p^*(a_n)=\tag{52}$$
$$=c_1(\mathcal{O}(1))^{e-n-1}\cap\underline{p}^*\underline{t}^*(\tau_{k+n+1}(\mathcal{L},\mathcal{M},c.(\mathcal{F})\cap A)).$$

By (28) and (29), the difference in (52) is

$$c_1(\mathcal{O}(1))^{e-2-j}\cap\Big[[-1]\cap[\mathcal{O}(1)]\cap p^*t^*([c_j(\mathcal{F})\cap A]\Big]$$
$$=c_1(\mathcal{O}(1))^{e-1-j}\cap\underline{p}^*\underline{p}^*(\tau_{k+n+1}(\mathcal{L},\mathcal{M},c.(\mathcal{F})\cap A)),$$

and the proof of (48) is complete.

1.13. Let

$$\Sigma:\ 0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\xrightarrow{\ \pi\ }\mathcal{G}\longrightarrow 0$$

be an exact sequence of vector bundles, and let $0=\mathcal{G}_0\subset\mathcal{G}_1\subset\ldots\subset\mathcal{G}_g=\mathcal{G}$ be a filtration of $\mathcal{G}$ with $\dim(\mathcal{G}_i/\mathcal{G}_{i-1})=1$. Let $\mathcal{E}_i=\pi^{-1}\mathcal{G}_i$ and $\mathcal{L}=\mathcal{G}_i/\mathcal{G}_{i-1}$. A successive application of 1.10. gives us isomorphisms

$$c.(\mathcal{G})\cap A\longrightarrow c.(\mathcal{L}_g)\cap c.(\mathcal{G}_{g-1})\cap A\longrightarrow\ldots\longrightarrow c.(\mathcal{L}_g)\cap\ldots\cap c.(\mathcal{L}_1)\cap A$$
$$c.(\mathcal{F})\cap A\longrightarrow c.(\mathcal{L}_g)\cap c.(\mathcal{E}_{g-1})\cap A\longrightarrow\ldots\longrightarrow c.(\mathcal{L}_g)\cap\ldots\cap c.(\mathcal{L}_1)\cap c.(\mathcal{F})\cap A$$

We want to prove that the isomorphism

$$\Phi_{\mathcal{G}.}:\ c.(\mathcal{G})\cap c.(\mathcal{F})\cap A\longrightarrow c.(\mathcal{E})\cap A\tag{53}$$

is independent of the filtration $\mathcal{G}.$ of $\mathcal{G}$. We proceed by induction on the dimension of $\mathcal{F}$.

1.13.1.: Let $\mathcal{F}$ be a line bundle. The sheaf $\mathcal{M}$ of splittings of the exact sequence Σ is a principal homogeneous sheaf for $\mathcal{H}om(\mathcal{G},\mathcal{F})$, hence it is representable by a smooth X-scheme M and [G, Theorem 8.3.] asserts that pull-back to M is an isomorphism on $E_2^{p,q}$ of the Quillen spectral sequence, such that it is sufficient to prove our assertion after pull-back to M.

We may thus achieve that Σ has a splitting $s\colon \mathcal{G} \longrightarrow \mathcal{E}$. Let

$$\Sigma'\colon\ 0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

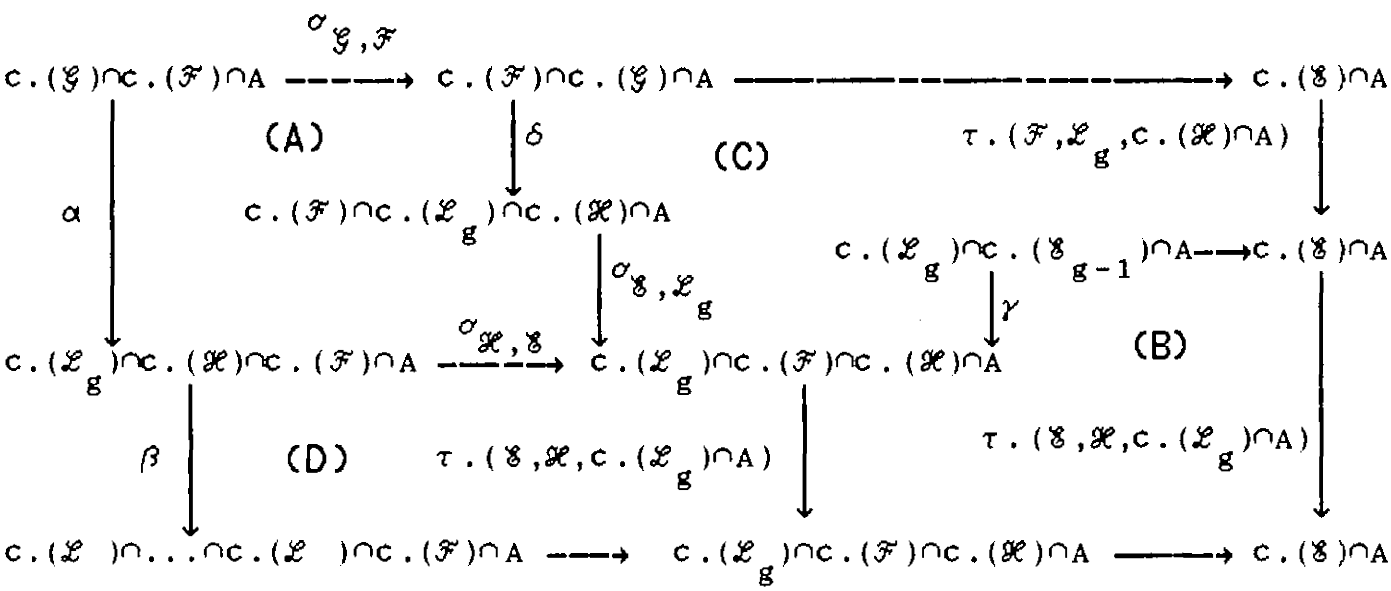
be the exact sequence defined by this splitting. We want to prove that

$$\begin{array}{ccccc} c.(\mathcal{G})\cap c.(\mathcal{F})\cap A & \xrightarrow{\sigma_{\mathcal{F},\mathcal{G}}} & c.(\mathcal{F})\cap c.(\mathcal{G})\cap A & \longrightarrow & c.(\mathcal{E})\cap A \\ \downarrow & & & & \downarrow{\scriptstyle \tau.(\mathcal{F},\mathcal{G},A)} \\ c.(\mathcal{L}_g)\cap\ldots\cap c.(\mathcal{L}_1)\cap c.(\mathcal{F})\cap A & & \longrightarrow & & c.(\mathcal{E})\cap A \end{array} \qquad (54)$$

commutes. Since the arrows in the upper row are independent of $\mathcal{G}.$, we conclude that (53) is independent of $\mathcal{G}.$.

We prove (54) by induction on g. If $\mathcal{G}$ is a line bundle, (54) is (48) in the special case where the line bundle occuring in (48) is zero. If $g>1$, we put $\mathcal{H}=\mathcal{G}_{g-1}$ and consider the following diagram:

(55)



The arrows α and δ in (55) are defined by $0\longrightarrow\mathcal{H}\longrightarrow\mathcal{G}\longrightarrow\mathcal{L}_g\longrightarrow 0$, β is defined by by the ascending filtration $(\mathcal{G}_k)_{0\le k\le g-1}$ of $\mathcal{H}$, γ is defined by the sequence $0\longrightarrow\mathcal{H}\xrightarrow{s}\mathcal{F}_{g-1}\longrightarrow\mathcal{E}\longrightarrow 0$. The commutativity of (A) is consequence of (45), (B) is the induction assumption, (C) is (48), and (D) is trivial. An easy calculation shows

$$\tau.(\mathcal{F},\mathcal{L}_g,c.(\mathcal{H})\cap A)+\tau.(\mathcal{E},\mathcal{H},c.(\mathcal{L}_g)\cap A)=\tau.(\mathcal{F},\mathcal{G},A).$$

It follows that the outer contour of (55) is (54), and the proof of (54) is complete.

<u>1.13.2.: The splitting principle:</u> Let $\mathcal{E}$ be an e-dimensional vector bundle on X, and let $p\colon Y\longrightarrow X$ be its flag fibration parametrizing maximal flags.

(a) $\underline{p}^*: \tilde{\mathfrak{CH}}^P(X) \longrightarrow \tilde{\mathfrak{CH}}^P(Y)$ is a faithful functor.

(b) Let $p_{1,2}$ be the projections of $Y \times_X Y$ to its factors and $r=pp_1=pp_2$. If A and B are objects of $\tilde{\mathfrak{CH}}^P(X)$ and if $\varphi:\underline{p}^*A \longrightarrow \underline{p}^*B$ is as isomorphism, then f is of the form $\underline{p}^*(\psi)$ for a (unique) $\psi:A \longrightarrow B$ if and only if

$$\underline{p}^*(\varphi)=\underline{p}^*(\varphi) \tag{56}$$

in $\mathrm{Hom}(\underline{r}^*A,\underline{r}^*B)$.

<u>Proof</u>: The projection p admits a factorization

$$Y=Y_0 \xrightarrow{p^{(1)}} Y_1 \longrightarrow \ldots \xrightarrow{p^{(e-1)}} Y_{e-1}=X$$

into j-dimensional projective fibratiobns $p^{(j)}$. By 1.7., $\underline{p}^{(j)*}$ is faithful, and (a) follows. Condition (54) in (b) is certainly necessary. If f exists we have $p^*([A])=p^*([B])$ in $CH^k(Y)$. By 1.6., p^* is injective on the Chow groups, and we conclude that there is a homomorphism $h:A \longrightarrow B$. Then $f'=f-\underline{p}^*(h) \in G_k(Y)$ satisfies

$$p_1^*(f')=p_2^*(f') \text{ in } G_k(Y \times Y). \tag{57}$$

Let $0=\mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_e=p^*\mathcal{E}$ be the universal flag of $p^*\mathcal{E}$, and let $\mathcal{L}_i=\mathcal{E}_i/\mathcal{E}_{i-1}$. Iterating 1.6., we have

$$G_k(Y)=\bigoplus_{j_1=0}^{e-1} \ldots \bigoplus_{j_{e-1}=0}^{1} c_1(\mathcal{L}_1)^{j_1} \cap \ldots \cap c_1(\mathcal{L}_{e-1})^{j_{e-1}} \cap p^*\left(G_{k-\sum j_i}(X)\right) \tag{58}$$

$$G_k(Y \times Y)=\bigoplus_{j_1;j_1'=0}^{e-1} \quad \bigoplus_{j_{e-1},j_{e-1}'=0}^{1} \quad c_1(p_1^*\mathcal{L}_1)^{j_1} \cap c_1(p_2^*\mathcal{L}_1)^{j_1'} \cap \ldots \tag{59}$$

$$\ldots \cap c_1(p_1^*\mathcal{L}_{e-1})^{j_{e-1}} \cap c_1(p_2^*\mathcal{L}_{e-1})^{j_{e-1}'} \cap r^*\left(G_{k-\sum j_i-\sum j_i'}(X)\right) .$$

If we represent f' in the form (58), then (59) implies that (57) is valid if and only if all compoonents of f' are zero save for the component belonging to $(j_1,\ldots,j_{e-1})=(0,\ldots,0)$ in (58), i.e., if and only if $f'=p^*(g')$, and (b) follows.

<u>1.13.3.:</u> Now we are ready to perform the induction argument announced at the beginning of 1.13.. Let $\dim(\mathcal{F})=f>1$, and assume that our claim (i.e., that (53) is independent of the filtration) has already been verified for bundles of dimension less than e. By part (a) of the splitting principle we may assume that $\mathcal{F}$ has a subbundle $\mathcal{H}$ of dimension f-1. We consider the following commutative diagram, in which each arrow is in the obvious manner constructed from (42):

$$\begin{array}{ccccc}
c.(\mathcal{G})\cap c.(\mathcal{F})\cap A & \xrightarrow{\alpha} & c.(\mathcal{L}_g)\cap\ldots\cap c.(\mathcal{L}_1)\cap c.(\mathcal{F})\cap A & \xrightarrow{\beta} & c.(\mathcal{E})\cap A \\
\gamma\downarrow & & \downarrow & & \uparrow\zeta \\
c.(\mathcal{G})\cap c.(\mathcal{F}/\mathcal{H})\cap A & \xrightarrow{\delta} & c.(\mathcal{L}_g)\cap\ldots\cap c.(\mathcal{L}_1)\cap c.(\mathcal{F}/\mathcal{H})\cap c.(\mathcal{H})\cap A & & \\
 & & \uparrow\varepsilon & & \\
 & & c.(\mathcal{E}/\mathcal{H})\cap c.(\mathcal{H})\cap A & &
\end{array}$$

By the induction assumption, $\zeta\varepsilon$ is independent of the filtration $\mathcal{F}_i$. By the result of 1.13.1., the same is true about $\varepsilon^{-1}\delta$. It follows that $\beta\alpha$ is independent of the $\mathcal{F}_i$. Since $\beta\alpha$ is (53), we are through.

<u>1.14. The Whitney isomorphism:</u> Let

$$\Sigma:\ 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \xrightarrow{\pi} \mathcal{G} \longrightarrow 0$$

be an exact sequence of vector bundles on X. We denote by $p:Y\longrightarrow X$ the flag manifold of $\mathcal{G}$, by $p_{1;2}:Y\times_X Y\longrightarrow Y$ the projections to the factors and put $r=pp_1=pp_2$. Let $\mathcal{G}.=(0=\mathcal{G}_0\subset\mathcal{G}_1\subset\ldots\subset\mathcal{G}_g=p^*\mathcal{G})$ be the universal flag of $\mathcal{G}$. It defines an isomorphism (53)

$$\Phi_{\mathcal{G}.}:\ c.(p^*\mathcal{G})\cap c.(p^*\mathcal{F})\cap \underline{p}^*A \longrightarrow c.(p^*\mathcal{E})\cap\underline{p}^*A.$$

Since (as one proves easily) (53) is compatible with flat base change, we have

$$\underline{p}_i^*(\Phi_{\mathcal{G}.})=\Phi_{p_i^*\mathcal{G}.}:\ c.(r^*\mathcal{G})\cap c.(r^*\mathcal{F})\cap\underline{r}^*A \longrightarrow c.(r^*\mathcal{E})\cap\underline{r}^*A.$$

By the main result of 1.13., this is independent of $i\in\{1;2\}$. By part (b) of the splitting principle 1.13.2., we conclude that there exists a unique

$$\Phi_{\Sigma}: c.(p^*\mathcal{G})\cap c.(p^*\mathcal{F})\cap \underline{p}^*A \longrightarrow c.(p^*\mathcal{E})\cap \underline{p}^*A \qquad (60)$$

with $\Phi_{\mathcal{G}}=\underline{p}^*(\Phi_{\Sigma})$. If $\mathcal{G}$ is a line bundle, we have Y=X and (60) coincides with (42).

If $\mathcal{G}$ has a flag $0=\Gamma_0\subset\Gamma_1\subset\ldots\subset\Gamma_g=\mathcal{G}$ with one-dimensional quotients $\Lambda_i=\Gamma_i/\Gamma_{i-1}$, then the diagram

$$\begin{array}{ccc} c.(\mathcal{E})\cap c.(\mathcal{F})\cap A & \xrightarrow{\alpha} & c.(\Lambda_g)\cap\ldots\cap c.(\Lambda_1)\cap c.(\mathcal{E})\cap A \\ & \searrow^{\Phi_{\Sigma}} \quad c.(\mathcal{E})\cap A \quad {}^{\beta}\swarrow & \end{array} \qquad (61)$$

commutes. Indeed, $\beta\alpha=\Phi_{\Gamma.}$, hence $\underline{p}^*(\beta\alpha)=\Phi_{p^*\Gamma.}=\Phi_{\mathcal{G}.}=\underline{p}^*(\Phi_{\Sigma})$ by the main result of 1.13., and (61) follows from the splitting principle.

Let $0\subset\mathcal{G}\subset\mathcal{F}\subset\mathcal{E}$ be a filtration of $\mathcal{E}$. Then the diagram

$$\begin{array}{ccc} c.(\mathcal{E})\cap A & \longrightarrow & c.(\mathcal{E}/\mathcal{F})\cap c.(\mathcal{G})\cap A \\ \downarrow & & \downarrow \\ c.(\mathcal{E}/\mathcal{G})\cap c.(\mathcal{G})\cap A & \longrightarrow & c.(\mathcal{E}/\mathcal{F})\cap c.(\mathcal{F}/\mathcal{G})\cap A \end{array} \qquad (62)$$

commutes. By the splitting principle, it suffices to prove (62) in the case that $\mathcal{E}/\mathcal{F}$ and $\mathcal{F}/\mathcal{G}$ have flags with one-dimensional quotients, in which (62) follows from (61).

It is easy to see that (64) is compatible with isomorphism $\Sigma\longrightarrow\Sigma'$ of short exact sequences and with flat and proper base-changes $Y\longrightarrow X$. By (61) and the splitting principle it is possible to extend (45) to the case $\dim(\mathcal{L})\geq 1$.

Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X. We have sequences

$$\Sigma_1:\ 0\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}\oplus\mathcal{F}\longrightarrow\mathcal{F}\longrightarrow 0$$

$$\Sigma_2:\ 0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\oplus\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow 0.$$

The diagram

$$\begin{array}{ccc} c.(\mathcal{F})\cap c.(\mathcal{E})\cap A & \xrightarrow{\varphi_{\Sigma_1}} & c.(\mathcal{E}\oplus\mathcal{F})\cap A \\ \downarrow{\scriptstyle \sigma_{\mathcal{E},\mathcal{F}}} & & \downarrow{\scriptstyle \tau.(\mathcal{E},\mathcal{F},A)} \\ c.(\mathcal{E})\cap c.(\mathcal{F})\cap A & \xrightarrow{\varphi_{\Sigma_2}} & c.(\mathcal{E}\oplus\mathcal{F})\cap A \end{array} \qquad (63)$$

commutes. This is a consequence of 1.12., (61) and the splitting principle.

<u>1.15.:</u> If $i>0$ and $A\in Ob(\widetilde{\mathfrak{CH}}^k(X))$, then $c_i(\mathcal{E})\cap A\in Ob(\mathfrak{CH}^{k+i}(X))$. For X has a Zariski covering on which $\mathcal{E}$ and hence $c_i(\mathcal{E})\cap A$ are trivial, and we apply [F1, 3.8.].

<u>1.16.</u>: Let

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ & S & \end{array}$$

be a commutative diagram with p and q flat and i a regular closed immersion of codimension one. If $\mathcal{E}$ is a vector bundle on S, then the diagram

$$\begin{array}{ccccc}
\underline{i}_*\underline{q}^*(c_k(\mathcal{E})\cap A) & \longrightarrow & \underline{i}_*(c_k(q^*\mathcal{E})\cap\underline{q}^*A) & \longrightarrow & c_k(p^*\mathcal{E})\cap\underline{i}_*\underline{q}^*A \\
\big\downarrow & & & & \big\downarrow \\
c_1(\mathcal{O}_X(D))\cap\underline{p}^*(c_k(\mathcal{E})\cap A) & & & & c_k(p^*\mathcal{E})\cap c_1(\mathcal{O}_X(D))\cap\underline{p}^*A \\
& \searrow & & \nearrow & \\
& & c_1((\mathcal{O}_X(D))\cap c_k(p^*\mathcal{E})\cap\underline{p}^*A & &
\end{array} \qquad (64)$$

commutes.

<u>Proof</u>: If $\mathcal{E}$ is a line bundle, (64) coincides with (40). Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

be an exact sequence such that (64) is true for $\mathcal{F}$ and $\mathcal{G}$. Then we have the diagram

$$\begin{array}{ccc}
\underline{i}_*\underline{q}^*(c.(\mathcal{E})\cap A) & \longrightarrow & c_1(\mathcal{O}_X(D))\cap\underline{p}^*(c.(\mathcal{E})\cap A) \\
\big\downarrow & \text{NT} & \big\downarrow \\
\underline{i}_*\underline{q}^*(c.(\mathcal{G})\cap c.(\mathcal{F})\cap A) & \longrightarrow & c_1(\mathcal{O}_X(D))\cap\underline{p}^*(c.(\mathcal{G})\cap c.(\mathcal{F})\cap A) \\
\big\downarrow & & \big\downarrow \\
\underline{i}_*\left[c.(q^*\mathcal{G})\cap\underline{q}^*(c.(\mathcal{F})\cap A)\right] & \text{(A)} & c_1(\mathcal{O}_X(D))\cap c.(p^*\mathcal{G})\cap\underline{p}^*(c.(\mathcal{F})\cap A) \\
\big\downarrow & & \big\downarrow \\
c.(p^*\mathcal{G})\cap\underline{i}_*\underline{q}^*(c.(\mathcal{F})\cap A) & \longrightarrow & c.(p^*\mathcal{G})\cap c_1(\mathcal{O}_X(D))\cap\underline{p}^*(c.(\mathcal{F})\cap A) \\
\big\downarrow & & \big\downarrow \\
c.(p^*\mathcal{G})\cap\underline{i}_*\left[c.(q^*\mathcal{F})\cap\underline{q}^*A\right] & \text{(B)} & c.(p^*\mathcal{G})\cap c_1(\mathcal{O}_X(D))\cap c.(p^*\mathcal{F})\cap\underline{p}^*A \\
\big\downarrow & & \big\downarrow \\
c.(p^*\mathcal{G})\cap c.(p^*\mathcal{F})\cap\underline{i}_*\underline{q}^*A & \longrightarrow & c.(p^*\mathcal{G})\cap c.(p^*\mathcal{F})\cap c_1(\mathcal{O}_X(D))\cap\underline{p}^*A \\
\big\downarrow & \text{NT} & \big\downarrow \\
c.(\mathcal{E})\cap\underline{i}_*\underline{q}^*A & \longrightarrow & c.(p^*\mathcal{E})\cap c_1(\mathcal{O}_X(D))\cap\underline{p}^*A
\end{array}$$

.

(A) and (B) are (64) for $\mathcal{G}$ and $\mathcal{F}$. As we did in [F1], we used the label NT to denote squares which commute just because the arrows involved in them are natural transformations. By the biadmissibility of the Whitney isomorphism $c.(\mathcal{E})\cap A \longrightarrow c.(\mathcal{G})\cap c.(\mathcal{F})\cap A$, the composition of the left column is the top row of (64). By the generalization of (45), the composition of the right column is the bottom row of (64). It follows that (64) is true for $\mathcal{E}$. Thus it is possible to prove (64) by induction on $\dim(\mathcal{E})$, using the splitting principle.

2. Further Properties of the Chern functors

2.1. Relation between c_1 and specialization: Let $D\subset X$ be a closed subscheme of X whose sheaf of ideals is in some neighbourhood of D generated by f. For a line bundle $\mathscr{L}$ on X and $A\in Ob(\tilde{\mathfrak{CH}}^k(X-D))$, we want to construct an isomorphism

$$\alpha_{\mathscr{L},f}: c_1(\mathscr{L}|_D)\cap \mathfrak{sp}_f(A)\longrightarrow \mathfrak{sp}_f(c_1(\mathscr{L}|_{X-D})\cap A). \qquad (1)$$

To this end we fix a covering $\mathcal{U}=\bigcup_i U_i$ on which $\mathscr{L}$ is trivialized by non-vanishing sections ℓ_i. We denote by $\mathcal{U}|_{X-D}$, $\mathcal{U}|_D$ the coverings $D=\bigcup (D\cap U_i)$ and $X-D=\bigcup ((X-)\cap U_i)$. Let $C^*(\mathcal{U},E_1^{\cdot,-p})$ be the absolute Čech complex with differential d. The closed and exact Čech chains are denoted $Z^*(\mathcal{U},E_1^{\cdot,-p})$ and $B^*(\mathcal{U},E_1^{\cdot,-p})$. For $c\in Z^p(\mathcal{U},E_1^{\cdot,-p})$ $\mathbb{O}(c)\in\tilde{\mathfrak{CH}}^p(X)$ has been defined in 1.(1). In our situation, we have a homomorphism

$$sp_f:\ E_1^{p,q}(X-D)\longrightarrow E_1^{p,q}$$

(cf. [F2,§1.?]). The induced homomorphism

$$sp_f: C^*(\mathcal{U}|_{X-D})\longrightarrow C^*(\mathcal{U}|_D)$$

turns easily out to be a homomorphism of complexes. Consequently we have an homomorphism

$$\mathfrak{sp}_f(\mathbb{O}(c))\longrightarrow \mathbb{O}(sp_f(c)) \qquad (2)$$

for $c\in C^p(\mathcal{U},E_1^{*,-p})$. If $V\subseteq (X-D)_p$ is open and $a\in\mathbb{O}(C)(V)$ is given by x as in 1.(1), then (2) maps $sp_f(a)\in\mathfrak{sp}_f\mathbb{O}(c)(D-\overline{(X-D-V)})$ to the section of $\mathbb{O}(sp_f(c))$ defined by the Čech cycle $sp_f(x)\in Z^p(\mathcal{U}|_{D-\overline{(X-D-V)}})$.

Now we are ready to define the isomorphisms (1). If $a\in A_r(X-D)$ and ϕ_{ij} denotes the Čech cycle ℓ_i/ℓ_j, then

$$(c_1(\mathscr{L})\cap A)_{\mathcal{U},\ell,a}=\mathbb{O}(\{\phi_{ij},c(a)\}).$$

Since $sp_f(\{\phi_{ij},c(a)\})=\{\phi_{ij}|_D,sp_f(c(a))\}=\{\phi_{ij}|_D,c(sp_f(a))\}$, (2) defines an isomorphism

$$sp_f(c_1(\mathscr{L}\cap A)_{\mathcal{U},l,a})=sp_f(\mathbb{O}(\{\phi_{ij},c(a)\})\to\mathbb{O}(\{\phi_{ij}|_D,c(sp_f(a))\})=$$
$$=(c_1(\mathscr{L}|_D)\cap sp_f(A))_{\mathcal{U}|_D,l|_D,sp_f(a)}.$$

It is easy to see that these isomorphisms are compatible with the isomorphisms for changing l or a and refining $\mathcal{U}$ (cf. §1.2.). Consequently they define (1).

2.2. Relation between c_k and specialization: Let X, D, and f be the same as before, and let $\mathscr{E}$ be a vector bundle of dimension e on X. We denote by $\mathbb{P}(\mathscr{E})$ the corresponding projective fibration and by $p:\mathbb{P}(\mathscr{E})\to X$ the projection. If $A\in\tilde{\mathfrak{CH}}^P(X-D)$, then from the isomorphism

$$\bigoplus_{k=0}^{e} c_1(\mathcal{O}(1))^{e-k}\cap \underline{p}^*(c_k(\mathscr{E})\cap A)\longrightarrow 0$$

in $\tilde{\mathfrak{CH}}^{k+e}(\mathbb{P}(\mathscr{E}))$ we derive by (1) an isomorphism

$$\bigoplus_{k=0}^{e} c_1(\mathcal{O}(1))^{e-k}\cap \underline{p}^*(sp_f(c_k(\mathscr{E})\cap A))\longrightarrow$$

$$\to\bigoplus_{k=0}^{e} c_1(\mathcal{O}(1)|_{X-D})^{e-k}\cap sp_{p^*(f)}(\underline{p}^*(c_k(\mathscr{E})\cap A))\longrightarrow$$

$$\to sp_{p^*(f)}\left(\bigoplus_{k=0}^{e} c_1(\mathcal{O}(1))^{e-k}\cap \underline{p}^*(c_k(\mathscr{E})\cap A)\right)\longrightarrow 0 \quad .$$

Since $sp_f(c_0(\mathscr{E})\cap A)=sp_f(A)=c_0(\mathscr{E}|_D)\cap sp_f(A)$, this isomorphism and the definition of the Chern functors in §1.8. give an isomorphism

$$\alpha_{\mathscr{E},f}:sp_f(c_k(\mathscr{E})\cap A)\longrightarrow c_k(\mathscr{E}|_D)\cap sp_f(A). \qquad (3)$$

2.3. Properties of the isomorphism (3): The following properties are easily verified:

2.3.1. Compatibility with pull-back and push-forward: Let K_{sp} be the category defined in [F2,§3.13?]. Let objects of $K_{sp}\backslash(X,D,f)$ be denoted by $(q:Y\rightarrow X, q^{-1}(D), q^{*}(f))$, and let $\mathbb{K}_{sp}\backslash(X,D)$ be $(K_{sp}\backslash(X,D)$, flat morphisms, proper morphisms of c.r.d.). Then $\widetilde{\mathfrak{CH}}^{\cdot}(Y)$ and $\widetilde{\mathfrak{CH}}^{\cdot}(q^{-1}(D))$ are bifibred over $\mathbb{K}_{sp}\backslash(X,D,f)$, and the functors

$$sp_{q^{*}(f)} : \widetilde{\mathfrak{CH}}^{\cdot}(Y) \longrightarrow \widetilde{\mathfrak{CH}}^{\cdot}(q^{-1}(D))$$

$$c_k(q^{*}\mathcal{E})\cap . : \widetilde{\mathfrak{CH}}^{\cdot}(Y) \longrightarrow \widetilde{\mathfrak{CH}}^{\cdot}(Y)$$

$$c_k(q^{*}\mathcal{E}|_D)\cap . : \widetilde{\mathfrak{CH}}^{\cdot}\left(q^{-1}(Y))\right) \longrightarrow \widetilde{\mathfrak{CH}}^{\cdot}\left(q^{-1}(Y)\right)$$

are biadmissible. The property is that the isomorphism

$$sp_{q^{*}(f)}\left(c_k(q^{*}\mathcal{E})\cap .\right) \longrightarrow c_k(q^{*}\mathcal{E}|_D)\cap sp_{q^{*}(f)}$$

is biadmissible.

2.3.2. Compatibility with the Whitney sum isomorphism: If

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0$$

is an exact sequence of vector bundles on X, then the diagram

$$\begin{array}{ccc} c.(\mathcal{F}|_D)\cap c.(\mathcal{G}|_D)\cap sp_f(A) & \longrightarrow & c.(\mathcal{E}|_D)\cap sp_f(A) \\ \downarrow & & \\ c.(\mathcal{F}|_D)\cap sp_f(c.(\mathcal{G})\cap A) & & \downarrow \\ \downarrow & & \\ sp_f(c.(\mathcal{F})\cap c.(\mathcal{G})\cap A) & \longrightarrow & sp_f(c\ (\mathcal{E})\cap A) \end{array} \qquad (4)$$

commutes.

2.3.3. Let $D_i \subset X$ $(i \in \{1;2\})$ be regular closed immersions of coidmension one, with sheaf of ideals trivialized by f_i. We assume that the sequence $\{f_1;f_2\}$ is regular in a neighbourhood of $D_1 \cap D_2$. If $\mathcal{E}$ is a vector bundle on X and $A \in Ob(\widetilde{\mathfrak{CH}}^1(X-D_1-D_2))$, then the diagram

$$\begin{array}{ccc} sp_{f_1}(sp_{f_2}(c_k(\mathcal{E})\cap A)) & \longrightarrow & sp_{f_2}(sp_{f_1}(c_k(\mathcal{E})\cap A)) \\ \downarrow & & \downarrow \\ sp_{f_1}(c_k(\mathcal{E}|_{D_2})\cap sp_{f_2}(A)) & & sp_{f_2}(c_k(\mathcal{E}|_{D_1})\cap sp_{f_1}(A)) \\ \downarrow & & \downarrow \\ c_k(\mathcal{E}|_{D_1\cap D_2})\cap sp_{f_1}(sp_{f_2}(A)) & \longrightarrow & c_k(\mathcal{E}|_{D_1\cap D_2})\cap sp_{f_2}(sp_{f_1}(A)) \end{array} \qquad (5)$$

commutes. The horizontal arrows have been defined in [F,§3.15].

2.3.4. If in the commutative triangle

$$\begin{array}{ccc} D & \longrightarrow & X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & Z & \end{array}$$

p and q are flat and $\mathcal{E}$ is a vector bundle on Z, then the diagram

$$\begin{array}{ccccc} sp_f\left[\underline{q}^*(c_k(\mathcal{E})\cap A)\right] & \longrightarrow & sp_f(c_k(q^*\mathcal{E})\cap \underline{q}^*A)) & \xrightarrow{\alpha_{f,q^*\mathcal{E}}} & c_k(p^*\mathcal{E})\cap sp_f\underline{p}^*A) \\ \downarrow & & & & \downarrow \\ \underline{p}^*c_k(\mathcal{E})\cap A & & \longrightarrow & & c_k(p^*\mathcal{E})\cap\underline{p}^*A \end{array} \qquad (6)$$

commutes.

2.3.5.: Let $\mathcal{L}$ be a line bundle on X, $A \in Ob(\widetilde{\mathfrak{CH}}^k(X-D))$, $a \in A_r(X-D)$. We assume that ℓ is a rational section of $\mathcal{L}$ on X whose divisor meets $c(a)$, D, and $D \cap \overline{supp(c(a))}$ properly. Then

$$\alpha_{\mathcal{L},\lambda}(sp_\lambda(\ell \cap a)) = \ell|_D \cap sp_\lambda(a) \in (c_1(\mathcal{L}|_D)\cap sp_\lambda A)_r(D).$$

2.4. Relation between c_k and $f^!$:

Proposition: There exists a unique collection of isomorphisms

$$\beta_{f,\mathcal{E}}: \underline{f}^!(c_k(\mathcal{E})\cap A) \longrightarrow c_k(f^*\mathcal{E})\cap \underline{f}^!A \qquad (7)$$

for a local complete intersection morphism $f:X\to Y$ *which admits an immersion into a smooth* Y*-scheme (abbreviated: an slci-morphism* $f:X\to Y$*), a vector bundle* $\mathcal{E}$ *on Y, and* $A\in\widetilde{\mathfrak{CH}}^k(Y)$ *such that the following properties are satisfied:*

2.4.1. Compatibility with pull-back and push-forward: Let S be a scheme and $\mathcal{E}$ be a vector bundle on S. Let $\mathbb{K}_{lci,S}$ be defined by replacing "scheme" by "S-scheme" in the definition of $\mathbb{K}_{lci}$ (cf. [F2,§4.7.]). If objects of this bicategory are denoted $f:X\to Y$, then $\mathcal{E}_X$ and $\mathcal{E}_Y$ refer to the pull-backs of $\mathcal{E}$ to X and Y. Then

$$c_k(\mathcal{E}_X)\cap.:\widetilde{\mathfrak{CH}}^{\cdot}(X)\longrightarrow\widetilde{\mathfrak{CH}}^{\cdot}(X),$$

$$c_k(\mathcal{E}_Y)\cap.:\widetilde{\mathfrak{CH}}^{\cdot}(Y)\longrightarrow\widetilde{\mathfrak{CH}}^{\cdot}(Y),$$

and

$$\underline{f}^!:\widetilde{\mathfrak{CH}}^{\cdot}(Y)\longrightarrow\widetilde{\mathfrak{CH}}^{\cdot}(X)$$

are biadmissible functors between bifibred Picard categories over $\mathbb{K}_{lci,S}$. The condition is that

$$\beta_{f,\mathcal{E}_Y}:\underline{f}^!c_k(\mathcal{E}_Y)\longrightarrow c.(\mathcal{E}_X)\cap\underline{f}^!$$

is a biamissible functor-isomorphism.

2.4.2. Compatibility with composition: If $X\xrightarrow{f}Y\xrightarrow{g}Z$ are lci-morphisms such that g and gf (and hence f too) are slci, then the diagram

$$\begin{array}{ccccc}
\underline{f}^!\underline{g}^!(c_k(\mathcal{E})\cap A) & \longrightarrow & \underline{f}^!(c_k(g^*\mathcal{E})\cap\underline{g}^!A) & \longrightarrow & c_k(f^*g^*\mathcal{E})\cap\underline{f}^!\underline{g}^!A \\
\downarrow & & & & \downarrow \\
\underline{(gf)}^!(c_k(\mathcal{E})\cap A) & & \longrightarrow & & c_k\left((gf)^*\mathcal{E}\right)\cap\underline{(gf)}^!A
\end{array} \qquad (8)$$

commutes for every vector bundle $\mathcal{E}$ on Z.

2.4.3. Compatibility with specialization: Let (f,X,Y,D,λ) be an object of $K_{lci,sp}$ (cf. [F2,§4.7.]). It is given by a Cartesian diagram

$$\begin{array}{ccc} f^{-1}(D) & \subset & X \\ {\scriptstyle f_D}\downarrow & & \downarrow{\scriptstyle f} \\ D & \subset & Y \end{array}$$

and a function λ *in a neighbourhood of* D *defining* D. *If* $\mathfrak{E}$ *is a vector bundle on* Y *and* $A \in Ob(\mathfrak{C}\tilde{\mathfrak{H}}^{1}(Y))$, *then the diagram*

$$\begin{array}{ccc} sp_{f^*(\lambda)}\, \underline{f}^!(c_k(\mathfrak{E})\cap A) & \longrightarrow & \underline{f}_D^!(sp_\lambda(c_k(\mathfrak{E})\cap A) \\ \Big\downarrow {\scriptstyle sp_{f^*(\lambda}(\beta_{f,\mathfrak{E}})} & & \Big\downarrow {\scriptstyle \underline{f}^!(\alpha_{\mathfrak{E},\lambda})} \\ sp_{f^*(\lambda)}\left[c_k(f^*\mathfrak{E})\cap \underline{f}^! A\right] & & \underline{f}_D^!\left[c_k(\mathfrak{E}|_D)\cap sp_\lambda\, A\right] \\ \Big\downarrow {\scriptstyle \alpha_{f^*\mathfrak{E},f\ (\lambda)}} & & \Big\downarrow {\scriptstyle \beta_{f_D,\mathfrak{E}|_D}} \\ c_k(f_D^*\mathfrak{E}|_D)\cap sp_{f^*(\lambda)}\, \underline{f}^! A & \longrightarrow & c_k(f_D^*\mathfrak{E}|_D)\cap \underline{f}_D^!\, sp_\lambda A \end{array} \qquad (9)$$

commutes.

<u>*2.4.4.:*</u> *If in*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & Z & \end{array}$$

p *and* q *are flat and* f *is slci, then the diagram*

$$\begin{array}{ccccc} \underline{f}^!\underline{p}^*\left[c_k(\mathfrak{E})\cap A\right] & \longrightarrow & \underline{f}^!\left[c_k(p^*\mathfrak{E})\cap \underline{p}^* A\right] & \xrightarrow{\beta_{f,p^*\mathfrak{E}}} & c_k(q^*\mathfrak{E})\cap \underline{f}^*\underline{p}^* A) \\ \Big\downarrow & & & & \Big\downarrow \\ \underline{q}^*\left[c_k(\mathfrak{E})\cap A\right. & & \longrightarrow & & c_k(q^*\mathfrak{E})\cap \underline{q}^* A \end{array} \qquad (10)$$

commutes for every vector bundle $\mathfrak{E}$ *on* Z.

<u>2.5. Proof of 2.4.:</u> We proceed in four steps. In 2.5.1.-3. we prove that $\beta_{i,\mathfrak{E}}$ exists and is unique for regular immersions i. In 2.5.4., we extend this to the general case.

<u>2.5.1.:</u> Let $i:X_0 \to X_1$ be a regular closed immersion. We denote by $m:M_0 \to M_1$ its deformation to the normal bundle (cf. [Fu,§5] or [F2,§4.2.]. This is a regular closed immersion

$$\begin{array}{ccc} M_0 & \xrightarrow{m} & M_1 \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ X_1 \times \mathbb{P}^1 & \xrightarrow{i\times\mathbb{P}^1} & X_1 \times \mathbb{P}^1 . \end{array}$$

The following properties are satisfied:

(i) π_0 is an isomorphism. Let the superscript $^{(a)}$ denote the restriction of morphisms with source M_i to $M_i^{(a)} = \pi_i^{-1}(X \times \mathbb{P}^1)$. Then $\pi_1^{(a)}: M_1^{(a)} \longrightarrow X_1 \times \mathbb{A}^1$ is an isomorphism.

(ii) Let p_i denote the composition $M_i \xrightarrow{\pi_i} X_i \times \mathbb{P}^1 \to X_i$.

(iii) Let the superscript $^{(\infty)}$ denote the restriction of morphisms with source M_i to $M_i^{(\infty)} = \pi_i^{-1}(\infty) \subset M_i$. Then $p_1^{(\infty)}$ factors over a map $p_\infty: M_1^{(\infty)} \longrightarrow X_0$, and p_∞ is the projection of a vector bundle with zero section $m^{(\infty)}: M_0^{(\infty)} \longrightarrow M_1^{(\infty)}$. Hence $\underline{p}_\infty^*: \tilde{\mathfrak{C}\mathfrak{H}}^\cdot(X_0) \longrightarrow \tilde{\mathfrak{C}\mathfrak{H}}^\cdot(M_1^{(\infty)})$ is an equivalence of categories.

(iv) The formation of M is compatible with any base change $Y_1 \to X_1$ after which i remains regular of the same codimension.

Let $\lambda \in \Gamma(\mathbb{P}^1 - \{0\}, \mathcal{O}_{\mathbb{P}^1})$ be the inverse of the coordinate function. For the sake of simplicity it is denoted by the same letter λ for all projective lines over an arbitrary scheme. There is a canonical isomorphism

$$\underline{i}^! A \to sp_\lambda \underline{p}_0^{(a)*} \underline{i}^! A \to sp_\lambda \underline{m}^{(a)!} \underline{p}_1^{(a)*} A \to \underline{m}^{(\infty)} sp_\lambda \underline{p}_1^{(a)*} A \to$$

$$\longrightarrow (\underline{p}_\infty^*)^{-1} sp_\lambda \underline{p}_1^{(a)*} A. \qquad (11)$$

(cf. [F2,§4.4]) for $A \in Ob(\tilde{\mathfrak{C}\mathfrak{H}}^\cdot(X_1))$. For a vector bundle $\mathfrak{F}$ on X_1, we define $\beta_{i,\mathfrak{F}}$ by the composition

$$\underline{i}^!\left[c_k(\mathcal{E})\cap A\right] \to (\underline{p}_\infty^*)^{-1}\left[sp_\lambda\left[\underline{p}_1^{(a)*}(c_k(\mathcal{E})\cap A)\right]\right] \to$$

$$\to (\underline{p}_\infty^*)^{-1}\left[sp_\lambda(c_k(p_1^*\mathcal{E})\cap \underline{p}_1^{(a)*}A\right] \xrightarrow{\alpha_{\lambda,p_1^*\mathcal{E}}}$$

$$\to (\underline{p}_\infty^*)^{-1}\left[c_k(p_\infty^*\mathcal{E}|_{X_o})\cap sp_\lambda(\underline{p}_1^{(a)*}A)\right] \to$$

$$\to c_k(\mathcal{E}|_{X_o})\cap(\underline{p}_\infty^*)^{-1}\left[sp_\lambda \underline{p}_1^{(a)*}A\right] \to c_k(\mathcal{E}|_{X_o})\cap\underline{i}^!A. \qquad (12)$$

By applying 2.4.1., (9), and (10) to the isomorphisms in (11), we see that a system of isomorphisms $\beta_{i,\mathcal{E}}$ satisfying 2.4.1., 2.4.3, and 2.4.4. for regular closed immersions must be given by (12). Conversely, since (12) contains only transformations compatible with flat and proper base change and with specialization, 2.4.1. and 2.4.3. are consequences of (iv). 2.4.4. follows from 2.3.5. by an easy computation.

2.5.2.: It remains to prove that $\beta_{i,\mathcal{E}}$ satisfies2.4.2. in the case of regular closed immersions. First we prove (8) in the following case: A and B are the bundle spaces of vector bundles $\mathcal{A}$ and $\mathcal{B}$ on X, $f:X \to A$ is the zero section, $g: A \to B$ is an injective homomorphism of vector bundles, and $\mathcal{E}=r^*\mathcal{F}$, where $r:B \to X$ is the bundle projection and $\mathcal{F}$ is a vector bundle on X.

Without loosing generality we may assume that there is a projection $p: B \to A$ of vector bundles. Otherwise we consider the X-scheme

$$\pi: Z=\{\text{projections from } \mathcal{A} \text{ to } \mathcal{B}\} \longrightarrow X,$$

which is a principal homogeneous space for the vector bundle $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{B}/\mathcal{A},\mathcal{A})$. Since $\underline{\pi}^*: \tilde{\mathfrak{C}\mathfrak{H}}^\cdot(X) \to \tilde{\mathfrak{C}\mathfrak{H}}^\cdot(Z)$ is an equivalence of categories, it suffices to verify (8) after base-change to Z, where the desired projection p exists.

Now we consider the projections $B \xrightarrow{p} A \xrightarrow{q} X$. Then $\underline{f}^!\cong(\underline{q}^*)^{-1}$, $\underline{g}^!\cong(\underline{p}^*)^{-1}$, $\underline{(gf)}^!\cong(\underline{(qp)}^*)^{-1}$, and the diagrams

$$\begin{array}{ccc} \underline{f}^!\underline{g}^! & \longrightarrow & (\underline{q}^*)^{-1}(\underline{p}^*)^{-1} \\ \downarrow & & \downarrow \\ \underline{(gf)}^! & \longrightarrow & (\underline{(pq)}^*)^{-1} \end{array} \qquad \begin{array}{ccc} \underline{f}^!(c_k(\mathcal{E})\cap .) & \longrightarrow & c_k(f^*\mathcal{E})\cap \underline{f}^!(.) \\ \downarrow & & \downarrow \\ (\underline{q}^*)^{-1}(c_k(q^*{}_{\circ})\cap .) & \longrightarrow & c_k(\mathcal{F})\cap(\underline{q}^*)^{-1} \end{array}$$

commute (for the right one, this is 2.4.4.). Since the analogues of the right diagram for g and gf are also commutative, our claim follows from the properties of the isomorphisms 1.(38).

2.5.3.: To prove (8) in the case of arbitrary regular immersions $X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2$, we consider the deformation to the normal bundle

$$\begin{array}{ccccc} M_0 & \xrightarrow{m_0} & M_1 & \xrightarrow{m_1} & M_2 \\ \pi_0\downarrow & & \pi_1\downarrow & & \pi_2\downarrow \\ X_0\times\mathbb{P}^1 & \rightarrow & X_1\times\mathbb{P}^1 & \rightarrow & X_2\times\mathbb{P}^1 \end{array}$$

with the following properties (cf [F2,§4.2.]):

(i) π_0 is an isomorphism, and if the superscript $^{(a)}$ denotes restriction of morphisms to $M_i^{(a)}=\pi_i^{-1}(X_i\times\mathbb{A}^1)$, then $\pi_i^{(a)}:M_i^{(a)}\rightarrow X_i\times\mathbb{A}^1$ is an isomorphism.

(ii) We denote by p_i the projections $M_i \xrightarrow{\pi_i} X_i\times\mathbb{P}^1 \rightarrow X_i$.

(iii) 2.5.2. is applicable to the composition

$$M_0^{(\infty)} \xrightarrow{m_0^{(\infty)}} M_1^{(\infty)} \xrightarrow{m_1^{(\infty)}} M_2^{(\infty)}.$$

Let λ be the same as in 2.5.1. By the construction of the isomorphism $\underline{f}^!\underline{g}^! \longrightarrow \underline{(gf)}^!$ in [F2,§4.8.], the diagram

$$\begin{array}{ccccc} \underline{f}^!\underline{g}^!A & \longrightarrow & sp_\lambda \underline{p}_0^{(a)*}\underline{f}^!\underline{g}^!A & \longrightarrow & m_0^{(\infty)!}sp_\lambda \underline{p}_1^{(a)*}\underline{g}^!A \\ \downarrow & & & & \downarrow \\ \underline{(gf)}^!A & & & & \underline{m}_0^{(\infty)!}m_1^{(\infty)!}sp_\lambda \underline{p}_2^{(a)*}A \\ \downarrow & & & & \downarrow \\ sp_\lambda \underline{p}_0^{(a)*}\underline{gf)}^!A & & \longrightarrow & & \underline{(m_0^{(\infty)}m_1^{(\infty)})}^! sp_\lambda \underline{p}_2^{(a)*}A \end{array}$$

commutes. Using this, we can deduce (8) from 2.5.2., 2.4.1., and 2.4.3..

2.5.4.: We have proven that $\beta_{i,\mathcal{E}}$ exists and is unique for regular immersions i. Let f: X $\longrightarrow$ Y be an slci-morphism, and let $\mathcal{E}$ be a vector bundle on Y. We choose a factorization of f

$$\sigma: X \xrightarrow{i} S \xrightarrow{p} Y,$$

where p is smooth. Then $\underline{f}^! \cong \underline{f}_\sigma^! = \underline{i}^! \underline{p}^*$ (cf. [F2, §4.10]). We define $\beta_{f,\mathcal{E}}$ by

$$\underline{i}^!\underline{p}^* c_k(\mathcal{E})\cap A \longrightarrow \underline{i}^!(c_k(p^*\mathcal{E})\cap \underline{p}^*A) \xrightarrow{\beta_{i,p^*\mathcal{E}}} c_k(f^*\mathcal{E})\cap \underline{i}^!\underline{p}^*A. \qquad (13)$$

By 2.4.4., 2.4.2., and our result about the uniqueness of $\beta_{i,p^*\mathcal{E}}$, a system of isomorphisms $\beta_{f,\mathcal{E}}$ satisfying 2.4.1.-4. must be given by (13) if it exists.

Our first task is to prove that (13) is indepenent of σ. This follows from 2.4.4. (applied in the case of regular immersions) and the construction of the change of factoriazation isomorphism $\underline{f}_{\sigma_1}^! \rightarrow \underline{f}_{\sigma_2}^!$ in [F2,§4.9-10]. Now 2.4.1., 2.4.3, and 2.4.4. can immediately be reduced to the case of regular immersions.

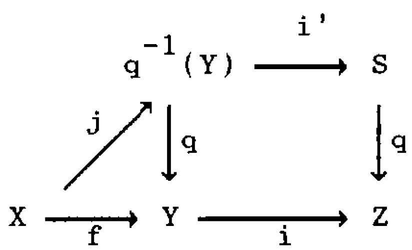
The proof of (8) can be split up into the following four cases:

(α) f and g are regular closed immersions. This case has alreay been dealt with.

(β) f is a regular closed immersion, and g is smooth. This case follows from definition (13) and [F2,§4.12.,Sublemma 1]. We note that this is the only case of 2.4.2. which does not follow from the other points of 2.4..

(γ) f and g are smooth. This case follows immediately from (13).

(δ) f is smooth, and g=i is a regular immersion. By our assumption, if factors over a smooth z-scheme S. Consider the diagram



The square is Cartesian; i, i', and j are regular immersions, p, q, and f are smooth. By the definition of the isomorphism (?): $\underline{f}^!\underline{g}^! \longrightarrow \underline{(gf)}^!$ in [F2, § 4.11.], the following diagram commutes:

$$\begin{array}{ccc}
\underline{f}^!\underline{i}^!A & \xrightarrow{(?)} & \underline{(if)}^!A \\
\Big\downarrow (a) & & (d)\Big\uparrow \\
\underline{j}^!\underline{p}^*\underline{i}^!A & & \underline{(ji')}^!\underline{q}^*A \\
& \overset{(b)}{\searrow} \quad \overset{(c)}{\nearrow} & \\
& \underline{j}^!\underline{i}'^!\underline{q}^*A &
\end{array}$$

The compatibility of the isomorphisms $\beta_{\cdot,\cdot^*\mathscr{L}}$ with the arrows (a), (b), (c), (d) follows from case (β), 2.4.1., case (α), and case (β). It follows that these isomorphisms are compatible with (?), which is (11). The proof of 2.4. is complete.

2.6. For our axiomatic characterization of Chern functors we need some further properties of the isomorphisms $\beta_{f,\mathscr{L}}$.

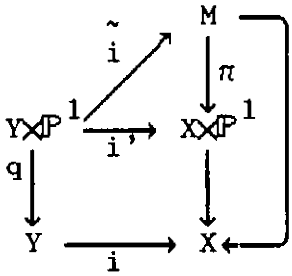
2.6.1. Let $i: X \longrightarrow Y$ be a regular closed immersion, $\mathscr{L}$ a line bundle on X, and $A\in\widetilde{\mathfrak{CH}}^k(X)$. We assume that a and ℓ are rational sections of A and $\mathscr{L}$ on X such that c(a), div(ℓ) and Y meet properly. Then $\ell\cap a\in(c_1(\mathscr{L})\cap A)_r(X)$, $i^!(\ell\cap a)\in(\underline{i}^!(c_1(\mathscr{L})\cap A)_r(Y)$, and $i^*(\ell)\cap i^!(a)\in(c_1(i^*\mathscr{L})\cap\underline{i}^!A)_r(Y)$. We claim that the isomorphism $\beta_{i,\mathscr{L}}$: $\underline{i}^!(c_1(\mathscr{L})\cap A) \longrightarrow c_1(i^*\mathscr{L})\cap\underline{i}^!A$ maps $i^!(\ell\cap a)$ to $i^*(\ell)\cap i^!(a)$.

Proof: *Step 1:* First we assume that we are in the following situation:

- X is a vector bundle over Y, with bundle projection p.
- $\mathscr{L}=p^*\mathscr{L}_1$, $\ell=p^*(\ell_1)$, $A=\underline{p}^*A_1$, $a=p^*(a_1)$ for some $\mathscr{L}_1$ and A_1 on Y.

Then the assumption follows from 2.4.4.

Step 2: In the general case we consider the deformation to the normal bundle



, $M^{(\infty)}=\pi^{-1}(\infty)$, $M^{(a)}=\pi^{-1}(X\times\mathbb{A}^1)$, superscripts $^{(a)}$ or $^{(\infty)}$=restriction to $M^{(a)}$ or $M^{(\infty)}$

and denote by λ a coordinate function on $\mathbb{P}^1$ as in 2.5.1.. We consider the rational section $sp_\lambda(p^{(a)*}(a))$ of $sp_\lambda p^{(a)*}A$. In the following computatuion we will use the canonical isomorphism $sp_\lambda q^{(a)*}A\simeq A$ without warning. By the axioms of 2.4., we have

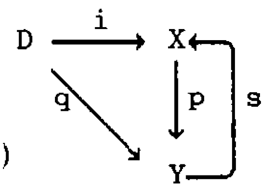
$$\beta_{i,\mathcal{L}}(i^!(\ell\cap a)) = sp_\lambda p^{(a)*}(\beta_{i,\mathcal{L}}(i^!(\ell\cap a))) \tag{14}$$
$$= sp_\lambda \beta_{\tilde{i},p^*\mathcal{L}}(\tilde{i}^!(p^{(a)*}\ell)\cap p^{(a)*}(a)))$$
$$= \beta_{\tilde{i}^{(\infty)},p^{(\infty)*}\mathcal{L}}(\tilde{i}^{(\infty)!}\ p^{(\infty)*}(\ell)\cap sp_\lambda p^{(\infty)*}(a))).$$

We have used 2.4.1. in line 2 and 2.4.3. and 2.3.5. in line 3. Now we note that $sp_\lambda p^{(a)*}(a)=p^{(\infty)*}i^!(a)$, where $p^{(\infty)}:M \longrightarrow Y$ is the restriction of p. This allows us to apply step 1, hence (14) is eaqual to

$$i^*\ell\cap\tilde{i}^{(\infty)!}(sp_\lambda p^{(a)*}(a))=i^*(\ell)\cap i^!(a),$$

where the last equality holds for similar reasons as in (14). This proves 2.6.1..

<u>2.6.2.</u> Let

(15) 

be a commutative diagram with p and q smooth, i a regular immersion of codimension one, and s a section of p. We assume that $s^{-1}(D)$ is regular of codimension one in Y and denote by i_Y: $s^{-1}(D) \longrightarrow Y$, s_D: $s^{-1}(D) \longrightarrow D$ the restrictions of i and s. Furthermore we assume that there is a flat map r: $Y\longrightarrow Z$ whose restriction r_D to D remains flat:

$$\begin{array}{ccc} s^{-1}(D) & \xrightarrow{i_Y} & Y \\ & \searrow^{r_D} & \downarrow r \\ & & Z \end{array} .$$

Note that there are isomorphisms

$$\begin{array}{lcl} c_1(\mathcal{O}(D))\cap\underline{p}^*\underline{r}^*A & \longrightarrow & \underline{i}_*\underline{q}^*\underline{r}^*A \\ c_1(\mathcal{O}(s^{-1}(D)\cap\underline{r}^*A & \longrightarrow & \underline{i}_{Y*}\underline{r}_D^*A \end{array} . \qquad 1.(39)$$

We assert that the diagram

$$\begin{array}{ccc} \underline{s}^!(c_1(\mathcal{O}(D))\cap\underline{p}^*\underline{r}^*A) & \longrightarrow & \underline{s}^!\underline{i}_*\underline{q}^*\underline{r}^*A \\ \downarrow & & \downarrow \\ c_1(\mathcal{O}(s^{-1}(D))\cap\underline{s}^!\underline{p}^*\underline{r}^*A & & \underline{i}_{Y*}\underline{s}_D^!\underline{q}^*\underline{r}^*A \\ \downarrow & & \downarrow \\ c_1(\mathcal{O}(s^{-1}\ D))\cap\underline{r}^*A & \longrightarrow & \underline{i}_{Y*}\underline{r}_D^*\ A \end{array} \qquad (16)$$

commutes. The lower vertical arrows are of type [F1, 4.7.1.], and the right upper vertical isomorphism is the base-change isomorphism for $\underline{s}^!$ provided by [F1, 4.7.].

Proof: Let a be a rational section of A, and let "1" be the cononical section of $\mathcal{O}(D)$ (resp. of $\mathcal{O}(s^{-1}(D)))$ which has a zero along D (resp. $s^{-1}(D)$). Then by 2.6.1., the construction of 1.(39), and the construction of the remaining arrows in [F1], the diagram (16) acts on $s^!(\text{"1"}\cap p^*r^*(a))$ as follows:

$$\begin{array}{ccc} s^!(\text{"1"}\cap p^*r^*(a)) & \longrightarrow & s^!(i_*q^*r^*(a)) \\ \downarrow & & \downarrow \\ \text{"1"}\cap s^!\ {}^*r^*(a) & & i_{Y*}s_D^!q^*r^*(a) \\ \downarrow & & \downarrow \\ \text{"1"}\cap\ {}^*(a) & \longrightarrow & i_{Y*}r_D^*(a) \end{array} ,$$

which proves our claim.

2.6.3.: We consider again (15) under the assumption that p and q are smooth, i is a regular immersion of codimension one, and s is a section of p. Now we assume that the images of i and s are disjoint. Then $\underline{s}^{!}\underline{i}_{*}$ has a canonical trivialization. On the other side, $s^{*}\mathcal{O}(D)$ is trivialized by $s^{*}("1")$, and we obtain another trivialization

$$\underline{s}^{!}\underline{i}_{*}\underline{q}^{*}A \longrightarrow \underline{s}^{!}(c_1(\mathcal{O}(D)\cap\underline{p}^{*}A) \longrightarrow c_1(s^{*}\mathcal{O}(D))\cap\underline{s}^{!}\underline{p}^{*}A \xrightarrow{s^{*}("1")} 0 .$$

We claim that these trivializations coincide.

Proof: This is similar to 2.6.2.. The first trivialization maps $s^{!}i_{*}q^{*}(a)$ to zero, while the second one maps it to

$$s^{!}i_{*}q^{*}(a) \longrightarrow s^{!}("1"\cap p^{*}(a)) \longrightarrow s^{*}("1")\cap s^{!}p^{*}(a) \longrightarrow 0.$$

2.6.4. Let $f: X \longrightarrow Y$ be slci, $\mathcal{E}$ and $\mathcal{F}$ be vecrtor bundles on Y, and $A\in Ob(\tilde{\mathfrak{CH}}^{k}(Y))$. Then the diagram

$$\begin{array}{ccccc}
\underline{f}^{!}(c_k(\mathcal{E})\cap c_l(\mathcal{F})\cap A) & \longrightarrow & c_k(f^{*}\mathcal{E})\cap\underline{f}^{!}(c_l(\mathcal{F})\cap A) & \longrightarrow & c_k(f^{*}\mathcal{E})\cap c_l(f^{*}\mathcal{F})\cap\underline{f}^{!}A \\
\downarrow{\scriptstyle \sigma_{\mathcal{E},\mathcal{F}}} & & & & \downarrow{\scriptstyle \sigma_{\mathcal{E},\mathcal{F}}} \\
\underline{f}^{!}(c_l(\mathcal{F})\cap c_k(\mathcal{E})\cap A) & \longrightarrow & c_l(f^{*}\mathcal{F})\ \underline{f}^{!}(c_k(\mathcal{E})\cap A) & \longrightarrow & c_l(f^{*}\mathcal{F})\cap c_k(f^{*}\mathcal{E})\cap\underline{f}^{!}A
\end{array}$$

commutes.

Proof: By deformation to the normal bundle, we can reduce this to the case of the zero section of a vector bundle, which is clear from 2.4.4. because $\sigma_{\mathcal{E},\mathcal{F}}$ is compatible with flat pull-back.

2.6.5. Compatibility with the Whitney isomorphism: Let $f:X\longrightarrow Y$ be sclci, and let $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow\mathcal{G}\longrightarrow 0$ be an exact sequence of vector bundles on Y. Then for $A\in Ob(\tilde{\mathfrak{CH}}^{k}(Y))$ the diagram

$$\begin{array}{ccccc}
\underline{f}^{!}(c.(\mathcal{E})\cap A) & & \longrightarrow & & c.(f^{*}\mathcal{E})\cap\underline{f}^{!}A \\
\downarrow & & & & \downarrow \\
\underline{f}^{!}(c.(\mathcal{G})\cap c.(\mathcal{F})\cap A) & \longrightarrow & c.(f^{*}\mathcal{G})\cap\underline{f}^{!}(c.(\mathcal{F})\cap A) & \longrightarrow & c.(f^{*}\mathcal{G})\cap c.(f^{*}\mathcal{F})\cap\underline{f}^{!}A
\end{array}$$

commutes.

Proof: If f is smooth, this is clear. This reduces us to the case of a regular closed immersion f. In this case the diagram commutes because (12) contains only transformations which are compatible with the Whitney isomorphism (cf. for instance 2.3.3.).

<u>2.7: Relation between c_1 and the determinant:</u> For a vector bundle $\mathfrak{E}$ of dimension e, we denote by $\det(\mathfrak{E})=\Lambda^{e}\mathfrak{E}$ its determinant line bundle. If Σ: $0\to\mathcal{F}\to\mathfrak{E}\to\mathfrak{G}\to 0$ is an exact sequence of vector bundles, then there is a canonical isomorphism

$$i_{\Sigma}:\ \det(\mathcal{F})\otimes\det(\mathfrak{G}) \longrightarrow \det(\mathfrak{E}).$$

Proposition: There is a unique system of isomorphisms

$$\iota_{\mathfrak{E}}:\ c_1(\det(\mathfrak{E}))\cap A \longrightarrow c_1(\mathfrak{E})\cap A \qquad (17)$$

for $A\in Ob(\tilde{\mathfrak{CH}}^{k}(X))$ *and a vector bundle* $\mathfrak{E}$ *on* X *such that the following properties are satisfied:*

<u>2.7.1: Compatibility with pull-back and push-forward:</u> If X-schemes are denoted $p:Y\to X$, *then* $\tilde{\mathfrak{CH}}^{\cdot}(Y)$ *is a bifibred Picard category over the bicategory (X-schemes, proper morphisms of c.r.d, flat morphisms), and* $c_1(\det(p^*\mathfrak{E}))\cap.$ *and* $c_1(p^*\mathfrak{E})\cap.:\tilde{\mathfrak{CH}}^{\cdot}(Y)\to\tilde{\mathfrak{CH}}^{\cdot}(Y)$ *are biadmissible functors. The condition is that*

$$\iota_{p^*\mathfrak{E}}:\ c_1(\det(p^*\mathfrak{E}))\cap. \longrightarrow c_1(p^*\mathfrak{E})\cap.$$

is a biadmissible functor-isomorphism.

<u>2.7.2. Compatibility with the Whitney isomorphism:</u> If

$$\Sigma:\ 0 \longrightarrow \mathcal{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{G} \longrightarrow 0$$

is an exact sequence of vector bundles on X, *then the diagram*

$$\begin{array}{ccccc}
c_1(\mathfrak{G})\cap A\oplus c_1(\mathcal{F})\cap A & & \xrightarrow{\ \varphi_{\Sigma}\ } & & c_1(\mathfrak{E})\cap A \\
\uparrow{\scriptstyle \iota_{\mathfrak{G}}\cup\iota_{\mathcal{F}}} & & & & \uparrow{\scriptstyle \iota_{\mathfrak{E}}} \\
c_1(\det(\mathfrak{G}))\cap A\oplus c_1(\det(\mathcal{F}))\cap A & \longrightarrow & c_1(\det(\mathfrak{G})\otimes\det(\mathcal{F}))\cap A & \xrightarrow[i_{\Sigma}]{} & c_1(\det(\mathfrak{E}))\cap A
\end{array}$$

commutes.

<u>2.7.3. Normalization:</u> If $\mathcal{L}$ *is a line bundle,* $\iota_{\mathcal{L}}$ *is the identity.*

These conditions characterize $\iota_{\mathfrak{E}}$ *uniquely. In addition, the following properties are satisfied:*

<u>2.7.4.:</u> If $\mathfrak{E}$ *and* $\mathcal{F}$ *are vector bundles on* X, *then the diagram*

$$\begin{array}{ccc} c_1(\det(\mathscr{E}))\cap c_1(\det(\mathscr{F}))\cap A & \xrightarrow{\sigma_{\det(\mathscr{E}),\det(\mathscr{F})}} & c_1(\det(\mathscr{F}))\cap c_1(\det(\mathscr{E}))\cap A \\ \downarrow{\scriptstyle \iota_{\mathscr{E}}\cap\iota_{\mathscr{F}}} & & \downarrow{\scriptstyle \iota_{\mathscr{F}}\cap\iota_{\mathscr{E}}} \\ c_1(\mathscr{E})\cap c_1(\mathscr{F})\cap A & \xrightarrow{\sigma_{\mathscr{E},\mathscr{F}}} & c_1(\mathscr{F})\cup c_1(\mathscr{E})\cap A \end{array}$$

commutes.

__2.7.5.:__ The isomorphisms $\iota_{\mathscr{E}}$ and $\beta_{f,E}$ (for a lci-morphism f) are compatible.

Proof: By the splitting principle it is clear that 2.7.1-3. characterize $\iota_{\mathscr{E}}$ uniquely and that 2.7.4. and 2.7.5. can be reduced to the case of line bundles in which they are clear.

It remains to construct an isomorphism $\iota_{\mathscr{E}}$ with 2.7.1-3.. Let $\mathscr{E}.:0=\mathscr{E}_0\subset\mathscr{E}_1\subset\ldots\subset\mathscr{E}_e=\mathscr{E}$ be a full flag of $\mathscr{E}$ with quotients $\mathscr{L}_i=\mathscr{E}_i/\mathscr{E}_{i-1}$. We have an isomorphism

$$c_1(\det(\mathscr{E}))\cap A \longrightarrow \bigoplus_{i=1}^{e} c_1(\mathscr{L})\cap A \longrightarrow c_1(\mathscr{E})\cap A, \tag{18}$$

where the first isomorphism is derived from ι_{Σ} and the second isomorphism is derived from the isomorphisms φ_{Σ}. It suffices to prove that (18) is independent of the filtration $\mathscr{E}.$, for then we can use 1.13.2.(b) to descent (18) from the flag manifold of $\mathscr{E}$ to X (cf. the construction of 1.(60)). Because A is isomorphic to an object $\underline{i}_*B$ for $B\in Ob(\tilde{\mathfrak{CH}}^0(Z))$ and i: Z⟶X a closed subscheme of codimension k and since (18) conatains only biadmissible transformations, we may assume $A\in Ob(\tilde{\mathfrak{CH}}^0(X))$. Then the restriction functor

$$\tilde{\mathfrak{CH}}^1(X) \longrightarrow \prod_{\eta\in X_0} \tilde{\mathfrak{CH}}^1(\mathrm{Spec}\ k(\eta))$$

is faithful, so we may assume X is the spectrum of a field. Let p:F⟶X be the full flag manifold of $\mathscr{E}$. Because (18) contains only transformations which are compatible with the functor $\underline{s}^!$ for s:X⟶F a section of p, it suffices to prove that the isomorphism between line bundles

$$c_1(p^*(\det(\mathscr{E})))\cap 1 \rightarrow c_1(p^*\mathscr{E})\cap 1$$

is constant on F. This is clear because F is a proper variety.

<u>2.8. Transition to the virtual category:</u> For an exact category $\mathfrak{P}$, we denote by $\mathfrak{K}(\mathfrak{P})$ its virtual category in the sense of [D, §4]. For a scheme X, we denote by $\mathfrak{K}(X)=K(\mathfrak{P}(X))$ the category of virtual vector bundles on X. By the universal properties of the virtual category ([D,§4.3.]), there exist unique (up to unique functor-isomorphism) additive functors

$$c_i(\mathcal{E})\cap A:\ \mathfrak{K}(X)\times\tilde{\mathfrak{CH}}^{k}(X) \longrightarrow \tilde{\mathfrak{CH}}^{k+i}(X),\ i\geq 0$$

$$c_0(\mathcal{E})\cap A=A$$

together with additive (in A) functor-isomorphisms

$$c_i(0)\cap A \longrightarrow 0 \text{ if } i>0$$

$$c.(\mathcal{E}\oplus\mathcal{F})\cap A \longrightarrow c.(\mathcal{E})\cap c.(\mathcal{F})\cap A$$

such that

(i) $c_i(\mathcal{E})\cap A=c_i([\mathcal{E}])\cap A$ if $\mathcal{E}$ is a vector bundle and $[\mathcal{E}]$ the corresponding virtual bundle.

(ii) If Σ: $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow\mathcal{G}\longrightarrow 0$ is a short exact sequence of vector bundles, then the following diagram involving the Whithney sum isomorphism and the isomorphism $[\mathcal{E}]\longrightarrow[\mathcal{G}]\oplus[\mathcal{F}]$ induced by Σ commutes:

$$\begin{array}{ccc} c.(\mathcal{E})\cap A & \longrightarrow & c.(\mathcal{G})\cap c.(\mathcal{F})\cap A \\ \downarrow & & \downarrow \\ c.([\mathcal{G}]\oplus[\mathcal{F}])\cap A & \longrightarrow & c.([\mathcal{G}])\cap c.([\mathcal{F}])\cap A. \end{array}$$

In the rest of this paper, we will for the sake of simplicity not distinguish between vector bundles themselfes and the virtual vector bundles defined by them. Using the universal property of the virtual category, we get isomorphisms

$$c_i(\mathcal{E})\cap c_j(\mathcal{F})\cap A \longrightarrow c_j(\mathcal{F})\cap c_i(\mathcal{E})\cap A$$

$$c_i(f^*\mathcal{E})\cap \underline{f}^*A \longrightarrow \underline{f}^*(c_i(\mathcal{E})\cap A)$$

$$c_i(f^*\mathcal{E})\cap \underline{f}^!A \longrightarrow \underline{f}^!(c_i(\mathcal{E})\cap A)$$

$$c_i(\mathcal{E})\cap g_*A \longrightarrow g_*(c_i(g^*\mathcal{E})\cap A)$$

$$sp_\lambda(c_i(\mathcal{E})\cap A) \longrightarrow c_i(\mathcal{E}|_D)\cap sp_\lambda A$$

$$c_1(\mathcal{E})\cap A \longrightarrow c_1(\det(\mathcal{E}))\cap A$$

because the corresponding isomorphisms for "real" bundles are compatible with the Whitney sum isomorphism. These isomorphisms for virtual vector bundles satisfy the same compatibilities as the corresponding isomorphisms between "real" vector bundles.

2.9. Polynomials in the Chern functors: Let $P(c_i(\mathcal{E}_j)$ be a polynomial with integral coeffitients in the Chern classes of vector bundles $\mathcal{E}_j$, $j \in J$ on X. For a total ordering $j_1 < j_2 < \ldots < j_N$ of J and virtual vector bundles $\mathcal{E}_j$ we put

$$[P(c_i(\mathcal{E}_j))]_< \cap A = \tag{19}$$

$$= \bigoplus_\alpha n_\alpha c_1(\mathcal{E}_{i_1})^{\alpha_{1,i_1}} \cap c_2(\mathcal{E}_{i_1})^{\alpha_{2,i_1}} \cap \ldots \cap c_1(\mathcal{E}_{i_2})^{\alpha_{1,i_2}} \cap c_2(\mathcal{E}_{i_2})^{\alpha_{2,i_2}} \cap \ldots \cap A,$$

where

$$P = \sum_\alpha n_a \prod_{i,j} c_i(\mathcal{E}_j)^{\alpha_{ij}}.$$

This means, all monomials of the polynomial P are ordered lexicographically according to the indices j (coming first) and i. If α and β are multi-indices, then there exists a unique isomorphism

$$\begin{aligned} & c_1(\mathcal{E}_{i_1})^{\alpha_{1,i_1}} \cap c_2(\mathcal{E}_{i_1})^{\alpha_{2,i_1}} \cap \ldots \cap c_1(\mathcal{E}_{i_2})^{\alpha_{1,i_2}} \cap c_2(\mathcal{E}_{i_2})^{\alpha_{2,i_2}} \cap \ldots \cap \\ & \cap c_1(\mathcal{E}_{i_1})^{\beta_{1,i_1}} \cap c_2(\mathcal{E}_{i_1})^{\beta_{2,i_1}} \cap \ldots \cap c_1(\mathcal{E}_{i_2})^{\beta_{1,i_2}} \cap c_2(\mathcal{E}_{i_2})^{\beta_{2,i_2}} \cap \ldots \cap A \\ & \downarrow \\ & c_1(\mathcal{E}_{i_1})^{\alpha_{1,i_1}+\beta_{1,i_1}} \cap c_2(\mathcal{E}_{i_1})^{\alpha_{2,i_1}+\beta_{2,i_1}} \cap \ldots \ldots \\ & \ldots \cap c_1(\mathcal{E}_{i_2})^{\alpha_{1,i_2}} \cap c_2(\mathcal{E}_{i_2})^{\alpha_{2,i_2}} \cap \ldots \cap A \end{aligned} \tag{20}$$

defined by applying the transformations $\sigma_{\cdot,\cdot}$ to the permutation which brings all factors into the right order with the minimal number of transpositions, i.e., without interchanging identical factors $c_i(\mathfrak{X}_j)\cap c_i(\mathfrak{X}_j)$. From the isomorphism (20) we derive a canonical isomorphism

$$[P(c_i(\mathfrak{X}_j)]_<\cap[Q(c_i(\mathfrak{X}_j)]_<\cap A \longrightarrow [(PQ)(c_i(\mathfrak{X}_j)]_<\cap A. \tag{21}$$

The diagram

$$\begin{array}{ccc} [(PQR)(c_i(\mathfrak{X}_j)]_<\cap A & \longrightarrow & [P(c_i(\mathfrak{X}_j)]_<\cap[(QR)(c_i(\mathfrak{X}_j)]_<\cap A \\ \downarrow & & \downarrow \\ [(PQ)(c_i(\mathfrak{X}_j)]_<\cap[R(c_i(\mathfrak{X}_j)]_<\cap A & \longrightarrow & [R(c_i(\mathfrak{X}_j)]_<\cap[R(c_i(\mathfrak{X}_j)]_<\cap[R(c_i(\mathfrak{X}_j)]_<\cap A \end{array} \tag{22}$$

commutes.

Let $\ll$ be another ordering of J and $\pi:J\longrightarrow J$ be the prmutation with $\pi(i)\ll\pi(j)$ iff $i<j$. For each monomial there exists a unique permutation

$$\begin{array}{c} c_1(\mathfrak{X}_{i_1})^{\alpha_{1,i_1}}\cap c_2(\mathfrak{X}_{i_1})^{\alpha_{2,i_1}}\cap\ldots\cap c_1(\mathfrak{X}_{i_2})^{\alpha_{1,i_2}}\cap c_2(\mathfrak{X}_{i_2})^{\alpha_{2,i_2}}\cap\ldots\cap A \\ \downarrow \\ c_1(\mathfrak{X}_{\pi(i_1)})^{\alpha_{1,\pi(i_1)}}\cap c_2(\mathfrak{X}_{\pi(i_1)})^{\alpha_{2,\pi(i_1)}}\cap\ldots \\ \ldots\cap c_1(\mathfrak{X}_{\pi(i_2)})^{\alpha_{1,\pi(i_2)}}\cap c_2(\mathfrak{X}_{\pi(i_2)})^{\alpha_{2,\pi(i_2)}}\cap\ldots\cap A \end{array}$$

defined by the permutation which brings all factors to the right order with the minimal number of transpositions. We get a canonical isomorphism

$$[P(c_i(\mathfrak{X}_j)]_<\cap A \longrightarrow [P(c_i(\mathfrak{X}_j)]_\ll\cap A. \tag{23}$$

These isomorphisms satisfy the necessary compatibility to glue the objects $[P(c_i(\mathfrak{X}_j)]_<\cap A$ to one object $P(c_i(\mathfrak{X}_j)\cap A$. If confusions are impossible, we will also write $\mathfrak{P}(\mathfrak{X}_j)\cap A$ for $P(c_i(\mathfrak{X}_j)\cap A$. The isomorphisms (21) and (23) commute, giving a canonical isomorphism

$$\mathcal{N}:P(c_i(\mathfrak{X}_j)\cap Q(c_i(\mathfrak{X}_j)\cap A \longrightarrow (PQ)(c_i(\mathfrak{X}_j)\cap A \tag{24}$$

satisfying the analogue of (22).

Let us stress that $\mathfrak{P}(\mathcal{E}_j)\cap A$ behaves bad if we identify some of the vector bundles $\mathcal{E}_i$. For instance, if $P(c_i\mathcal{E}),c_j(\mathcal{F}))$ is a polynomial in two vector bundles and if $Q(c_i(\mathcal{E})):=P(c_i\mathcal{E}),c_j(\mathcal{E}))$, then there is now canonical isomorphism

$$\mathfrak{P}(\mathcal{E},\mathcal{E})\cap A \longrightarrow \mathfrak{Q}(\mathcal{E})\cap A$$

unless we fix an order of the two variables in P.

There is, however, the following substitution principle: Let $\mathcal{F}(\mathcal{G}_1)$ be a functor in virtual bundles $\mathcal{G}_1$, and let a functor-isomorphism

$$\alpha:\ c_k(\mathcal{F}(\mathcal{G}_1))\cap A \longrightarrow Q_k(c_m(\mathcal{G}_1))\cap A$$

be given. If $P(c_i(\mathcal{F}),c_i(\mathcal{E}_j))$ is a polynomial in Chern classes, then α induces a canonical isomorphism

$$P(c_i(\mathcal{F}(\mathcal{G}_1)),c_i(\mathcal{E}_j))\cap A \longrightarrow R(c_i(\mathcal{E}_j),c_m(\mathcal{G}_1))\cap A, \qquad (25)$$

where

$$R(c_i(\mathcal{E}_j),c_m(\mathcal{G}_1))=P(Q_k(c_m(\mathcal{G}_1)),c_i(\mathcal{E}_j)).$$

The isomorphism (25) is independent of the choice of order of the varibles $\mathcal{G}_k$, $\mathcal{E}_1$.

If our polynomials have the more general size $P(\dim(\mathcal{E}_j),c_i(\mathcal{E}_j))$, then these methods apply also. We get a functor

$$P(\dim(\mathcal{E}_j),c_i(\mathcal{E}_j))\cap A = \mathfrak{P}(\mathcal{E}_j)\cap A$$

in virtual vector bundles $\mathcal{E}_j$ and $A\in Ob(\tilde{\mathfrak{CH}}^k(X))$ satisfying similar properties as above.

<u>2.10. Twist by a line bundle:</u> Let

$$P_j(\dim(\mathcal{E}),c_k(\mathcal{E}),c_1(\mathcal{L}))=\sum_{l=0}^{j}\binom{\dim(\mathcal{E})+l-j}{l}\ c_1(\mathcal{L})^{j-l}\cap c_l(\mathcal{E}) \qquad (26)$$

be the polynomial with the property

$$c_j(\mathcal{E}\otimes\mathcal{L})=P_j(\dim(\mathcal{E}),c_k(\mathcal{E}),c_1(\mathcal{L})).$$

We have the obvious identities

$$P_j(\dim(\mathcal{E}),c_k(\mathcal{E}),c_1(\mathcal{L})+c_1(\mathcal{M}))= \qquad (27)$$
$$=P_j(\dim(\mathcal{E}),P_j(\dim(\mathcal{E}),c_k(\mathcal{E}),c_1(\mathcal{M})),c_1(\mathcal{L})).$$

$$P_i(\dim(\mathcal{E})+\dim(\mathcal{E}'),c_1(\mathcal{L}),\sum_{k+l=i}c_k(\mathcal{E}'')c_l(\mathcal{E}'))= \qquad (28)$$

$$=\sum_{k+l=j} P_k(\dim(\mathcal{E}'),c_m(\mathcal{E}'),c_1(\mathcal{L}))P_l(\dim(\mathcal{E}''),c_n(\mathcal{E}''),c_1(\mathcal{L})).$$

Theorem: There exists a unique functor isomorphism

$$c_j(\mathfrak{E}\otimes\mathcal{L})\cap A \longrightarrow \mathfrak{P}_j(\mathcal{L},\mathfrak{E})\cap A \qquad (29)$$

with the following properties:

2.10.1. Compatibility with direct and inverse images: If $\mathcal{L}$ *and* $\mathfrak{E}$ *are a line bundle and a virtual bundle on* S *and if* S*-schemes are denoted* $p: X\longrightarrow S$*, then*

$$c_j(p^*\mathfrak{E}\otimes p^*\mathcal{L})\cap . \ : \widetilde{\mathfrak{CH}}^{\cdot}(X) \longrightarrow \widetilde{\mathfrak{CH}}^{\cdot}(X)$$

and

$$\mathfrak{P}_j(\mathcal{L},\mathfrak{E})\cap . \ : \widetilde{\mathfrak{CH}}^{\cdot}(X) \longrightarrow \widetilde{\mathfrak{CH}}^{\cdot}(X)$$

are biadmissible functors between bifibred Picard categories over S*-schemes. Then (29) is supposed to be biadmissible.*

2.10.2. Normalization: If $\mathfrak{E}$ *is a line bundle, then (29) in dimension zero is the identity of* A*, (29) in dimension one is the canonical isomorphism* $c_1(\mathcal{L}\otimes\mathfrak{E})\cap A\longrightarrow c_1(\mathcal{L})\cap A\oplus c_1(\mathfrak{E})\cap A$*, and (29) in dimension larger than one is the identity of the zero object.*

2.10.3. Compatibility with the Whitney sum isomorphism: If $\mathfrak{E}$ *and* $\mathcal{F}$ *are virtual vector bundles, then the diagram*

$$\begin{array}{ccc} c_k((\mathfrak{E}\oplus\mathcal{F})\otimes\mathcal{L})\cap A & \longrightarrow & \bigoplus_{i+j=k} c_i(\mathfrak{E}\otimes\mathcal{L})\cap c_k(\mathcal{F}\otimes\mathcal{L})\cap A \\ \downarrow & & \downarrow \\ \mathfrak{P}_k(\mathfrak{E}\oplus\mathcal{F},\mathcal{L})\cap A & \longrightarrow & \bigoplus_{i+j=k} \mathfrak{P}_i(\mathfrak{E},\mathcal{L})\cap\mathfrak{P}_k(\mathcal{F},\mathcal{L})\cap A \end{array} \qquad (30)$$

commutes up to a correcting sign

$$\Delta_k(\mathfrak{E},\mathcal{F},\mathcal{L},A)=c_1(\mathcal{L})\cap\tau_{k-1}(\mathfrak{E}\otimes\mathcal{L},\mathcal{F}\otimes\mathcal{L},A) \qquad (31)$$

$$=c_1(\mathcal{L})\cap\sum_{n+m=k-2}(\dim(\mathfrak{E})-n)(\dim(\mathcal{F})-m)c_n(\mathfrak{E})\cap c_m(\mathcal{F})\cap[A]\cap[-1].$$

The lower horizontal arrow is defined by (28), (25), (24) and the Whitney sum isomorphism.

These properties suffice to characterize (29). The following properties are also satisfied:

2.10.4. If $\mathcal{M}$ is another line bundle, then the diagram

$$\begin{array}{ccc} & c_i(\mathcal{E}\otimes\mathcal{L}\otimes\mathcal{M})\cap A & \\ \swarrow & & \searrow \\ P_i(\dim(e),c_1(\mathcal{L})\oplus c_1(\mathcal{M}),c_j(\mathcal{E}))\cap A & & P_i(\dim(\mathcal{E}\otimes\mathcal{L}),c_1(\mathcal{M}),c_k(\mathcal{E}\otimes\mathcal{L}))\cap A \\ \searrow & & \swarrow \\ & P_i(\dim(\mathcal{E}),c_1(\mathcal{M}),P_k(\dim(\mathcal{E}),c_1(\mathcal{L}),c_1(\mathcal{E})))\cap A & \end{array} \tag{32}$$

commutes.

2.10.5. Compatibility with $f^!$: The diagram

$$\begin{array}{ccc} c_k(f^*(\mathcal{E}\otimes\mathcal{L}))\cap \underline{f}^! A & \longrightarrow & \underline{f}^!(c_k(\mathcal{E}\otimes\mathcal{L})\cap A) \\ \downarrow & & \downarrow \\ \mathfrak{P}_k(f^*\mathcal{E},f^*\mathcal{L})\cap\underline{f}^! A & \longrightarrow & \underline{f}^!(\mathfrak{P}_k(\mathcal{E},\mathcal{L})\cap A) \end{array} \tag{33}$$

commutes.

Proof: *Step 1:* It follows from the splitting principle that 2.10.1.-2.10.3. characterize (29) uniquely. 2.10.5. for a line bundle $\mathcal{E}$ follows from 2.10.2., and the general case of 2.10.5. follows from this case and 2.10.3. by the splitting principle. It remains to construct an isomorphism with the properties 2.10.1-4.. It suffices to consider "real" vector bundles $\mathcal{E}$ and to consider short exact sequences $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{H}\longrightarrow\mathcal{E}\longrightarrow 0$ in 2.10.3..

Step 2: To construct (29) for a vector bundle $\mathcal{E}$ we use the identifications $\mathbb{P}(\mathcal{E}\otimes\mathcal{L})=\mathbb{P}(\mathcal{E})$ and $\mathcal{O}(1)_{\mathcal{E}\otimes\mathcal{L}}=\mathcal{O}(1)_{\mathcal{E}}\otimes\mathcal{L}^{-1}$. Let $p\colon \mathbb{P}(\mathcal{E})\longrightarrow X$ be the projection. We have canonical isomorphisms

$$\begin{array}{l} \bigoplus_{j=0}^{e} c_1(\mathcal{O}(1)_{\mathcal{E}\otimes\mathcal{L}})^{e-j}\cap \underline{p}^*\left[\bigoplus_{k=0}^{j}\binom{e+k-j}{k}c_1(\mathcal{L})^k\cap c_{j-k}(\mathcal{E})\cap A\right] \\ \qquad\downarrow \\ \bigoplus_{j=0}^{e}\bigoplus_{k=0}^{j}\binom{e+k-j}{k}c_1(\mathcal{O}(1)_{\mathcal{E}\otimes\mathcal{L}})^{e-j}\cap c_1(\underline{p}^*\mathcal{L})^k\cap\underline{p}^*(c_{j-k}(\mathcal{E})\cap A) \\ \qquad\downarrow \\ \bigoplus_{l=0}^{e} c_1(\mathcal{O}(1)_{\mathcal{E}})^{e-l}\cap\underline{p}^*(c_l(\mathcal{E})\cap A) \xrightarrow{1.(37)} 0 \quad , \end{array}$$

defining (29). The proofs of 2.10.1., 2.10.2., and 2.10.4. are straightforward. It remains to prove 2.10.3..

Step 3: First we prove (30) in the case of an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \longrightarrow 0 \qquad (33)$$

of vector bundles with $\dim(\mathcal{G})=1$. We have the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{F}) & \xrightarrow{\ i\ } & \mathbb{P}(\mathcal{H}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ & X & \end{array}$$

of projective fibrations. We consider the diagram

$$\begin{array}{ccc} \underline{i}_*\left[\bigoplus_{j=0}^{f} c_1(\mathcal{O}(1)_{\mathcal{F}\otimes\mathcal{L}})^{f-j}\cap \underline{q}^*(\mathfrak{P}_j(\mathcal{F},\mathcal{L})\cap A)\right] & \longrightarrow & \\ \downarrow \alpha & (A) & \underline{i}_*\left[\bigoplus_{j=0}^{f} c_1(\mathcal{O}(1)_{\mathcal{F}})^{f-j}\cap \underline{q}^*(c_j(\mathcal{F})\cap A)\right] \\ \bigoplus_{j=0}^{f} c_1(\mathcal{O}(1)_{\mathcal{H}\otimes\mathcal{L}})^{f-j}\cap c_1(\mathcal{G}\otimes\mathcal{L}(1))\cap \underline{p}^*(\mathfrak{P}_j(\mathcal{F},\mathcal{L})\cap A) & \xrightarrow{\ \gamma\ } & \downarrow \beta \\ \downarrow \delta & (B) & \bigoplus_{j=0}^{f} c_1(\mathcal{O}(1)_{\mathcal{H}})^{f-j}\cap c_1(\mathcal{G}\otimes\mathcal{L}(1))\cap \underline{p}^*(c_j(\mathcal{F})\cap A) \\ \bigoplus_{j=0}^{e} c_1(\mathcal{O}(1)_{\mathcal{H}\otimes\mathcal{L}})^{e-j}\cap \underline{p}^*(\mathfrak{P}_j(\mathcal{H},\mathcal{L})\cap A) & & \downarrow \varepsilon \\ & \longrightarrow & \bigoplus_{j=0}^{e} c_1(\mathcal{O}(1)_{\mathcal{H}})^{e-j}\cap \underline{p}^*(c_j(\mathcal{G})\cap A) \end{array} \quad . \qquad (34)$$

The isomorphisms α and β interchange $\underline{i}_*$ and $c_1(\mathcal{O}(1))$ and apply 1.(39). The two lower vertical arrows are built of (24), (25), and the Whitney sum isomorphism. The isomorphism γ interchanges $c_1(\mathcal{L}^{-1})^k$ with $c_1(\mathcal{G}\otimes\mathcal{L}(1))$ and applies (24). The two other arrows are of type (24).

The commutativity of (A) follows from 1.(40). If $c_1(\mathcal{L}^{-1})^l$ occurs (with multiplicity $\binom{f-j}{l}$) in the binomial resolution of the power $(c_1(\mathcal{G}(1)\otimes\mathcal{L}))^{f-j}$, then γ involves interchanging $c_1(\mathcal{L})$ l times with $c_1(\mathcal{L}^{-1})$. Since the other arrows in (B) use only minimal permutations, (B) commutes up to the sign

$$\sum_{j=0}^{f}\sum_{l=0}^{f-j} l\binom{f-j}{l} c_1(\mathcal{O}(1)_{\mathcal{H}})^{f-j-l}\cap c_1(\mathcal{L}^{-1})^{l}\cap p^*(P_j(\dim(\mathcal{F}),c_i(\mathcal{F}),c_1(\mathcal{L})\cap[-1]\cap[A])=$$

$$=\sum_{j=0}^{f-1} (f-j)(c_1(\mathcal{O}(1)_{\mathcal{H}\otimes\mathcal{L}})^{f-j-1}\cap c_1\mathcal{L})\cap p^*(c_j(\mathcal{F}\otimes\mathcal{L})\cap[-1]\cap[A]).$$

By 1.6., the definition of (29) in step 2, and the definition of the Whitney isomorphism in 1.10. we conclude that (30) commutes up to the sign

$$(f+2-k)c_1(\mathcal{L})\cap c_{k-2}(\mathcal{F}\otimes\mathcal{L})\cap[-1]\cap[A]=\Delta_k(\mathcal{E},\mathcal{F},\mathcal{L},A),$$

proving (30) in the special case of an exact sequence (33) with a line bundle $\mathcal{E}$. For an arbitrary exact sequence (33), (30) follows by induction on the dimension of $\mathcal{E}$, using the splitting principle. For arbitrary virtual bundles $\mathcal{E},\mathcal{F}$ (30) follows by the universal property of Deligne's virtual category. The proof of 2.10. is complete.

3. Axiomatic Characterization of the Chern Functors

<u>3.1. The trivialization $T_{\mathcal{E},s}$:</u> Let $\mathcal{E}$ be a vector bundle of dimension e on X. A non-vanishing global section s of $\mathcal{E}$ defines a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0 \qquad (1)$$

and hence a trivialization

$$T_{\mathcal{E},s}: c_e(\mathcal{E})\cap A \longrightarrow c_{e-1}(\mathcal{E})\cap c_1(\mathcal{O}_X)\cap A \longrightarrow 0.$$

<u>*Proposition: (i)*</u> *Let* σ: $X \longrightarrow \mathbb{P}(\mathcal{E})$ *be the section of* $\mathbb{P}(\mathcal{E})$ *defined by* s. *Because* $\underline{\sigma}^*\mathcal{O}(1)\simeq\mathcal{O}_X$ *canonically,* $\underline{\sigma}^!(c_1(\mathcal{O}(1)\cap \underline{p}^*A)\simeq c_1(\mathcal{O}_X)\cap A$ *has a canonical trivialization if* k>0. *Hence by applying* $\underline{\sigma}^!$ *to the morphism*

$$\bigoplus_{j=0}^{e} c_1(\mathcal{O}(1)^{e-j}\cap \underline{p}^*(c_j(\mathcal{E})\cap A) \longrightarrow 0 \qquad 1.(37)$$

defining $c.(\mathcal{E})\cap A$, *we obtain a trivialization*

$$0 \longrightarrow \underline{\sigma}^!\left[\bigoplus_{j=0}^{e} c_1(\mathcal{O}(1)^{e-j}\cap \underline{p}^*(c_j(\mathcal{E})\cap A)\right] \longrightarrow \sigma^!\underline{p}^*(c_e(\mathcal{E})\cap A) \longrightarrow c_e(\mathcal{E})\cap A \qquad (2)$$

of $c_e(\mathcal{E})\cap A$. *We claim that this trivialization coincides with* $T_{\mathcal{E},s}$.

(ii) Let $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\xrightarrow{\pi}\mathcal{L}\longrightarrow 0$ *be an exact sequence of vector bundles on* X, *with* dim($\mathcal{E}$)=1. *We assume that* $\mathcal{E}$ *has a non-zero section* s *such that* i:D$\longrightarrow$X *is a regular immersion of codimension one, where* D *is the subscheme defined by the vanishing of* $\pi(s)$. *Then* $\pi(s)$ *defines an isomorphism* $\mathcal{L}\simeq\mathcal{O}_X(D)$. *We assume also that* $r: X\longrightarrow Z$ *is a flat morphism whose restriction* r_D *to* D *is also flat.*

Then

$$
\begin{aligned}
c_e(\mathcal{E})\cap \underline{r}^*A &\longrightarrow c_1(\mathcal{L})\cap c_f(\mathcal{F})\cap \underline{r}^*A & (3)\\
&\xrightarrow{\sigma_{\mathcal{L},\mathcal{F}}} c_f(\mathcal{F})\cap c_1(\mathcal{L})\cap \underline{r}^*A \\
&\xrightarrow{\pi(s)} c_f(\mathcal{F})\cap c_1(\mathcal{O}_X(D))\cap \underline{r}^*A (f=g-1)\\
&\xrightarrow{1.(39)} c_f(\mathcal{F})\cap \underline{i}_*\underline{r}_D^*A \\
&\longrightarrow \underline{i}_*\left[c_f(\mathcal{F}|_D)\cap \underline{r}_D^*A\right]\\
&\xrightarrow{T_{\mathcal{F}|_D,s|_D}} \underline{i}_*(0)=0
\end{aligned}
$$

defines a trivialization of $c_e(\mathcal{E})\cap \underline{r}^*A$*. We claim that this trivialization coincides with* $T_{\mathcal{E},s}$.

<u>Proof of (i)</u>:Without loosing genrality we may assume that (1) splits, because this can be achieved by passing to a certain p.h.s. for the dual of $\mathcal{G}$. By 1.(63), the diagram

$$
\begin{array}{ccc}
c_e(\mathcal{E})\cap A & \longrightarrow & c_1(\mathcal{O}_X)\cap c_{e-1}(\mathcal{G})\cap A\\
& \searrow \quad\quad \nearrow {\scriptstyle \sigma_{\mathcal{G},\mathcal{O}_X}} & \\
& c_{e-1}(\mathcal{G})\cap c_1(\mathcal{O}_X)\cap A &
\end{array}
$$

commutes up to the sign

$$\left[\dim(\mathcal{G})-(e-1)\right]\cap c_{e-1}(\mathcal{G})\cap[-1]\cap[A]=0$$

Hence, $T_{\mathcal{E},s}$ coincides with the trivialization defined by the complementary sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}=\mathcal{O}_X \longrightarrow 0 \qquad (4)$$

and $c_1(\mathcal{O}_X)\cap c_{e-1}(\mathcal{G})\cap A \longrightarrow 0$.

Now we consider the diagram

$$\begin{array}{ccc} D=\mathbb{P}(\mathcal{G}) & \xrightarrow{\ \underline{i}\ } & \mathbb{P}(\mathcal{E}) \\ & \searrow^{q} & \uparrow p \quad \Big\uparrow s \\ & & X \end{array} .$$

By our construction of the Whitney isomorphism associated to (4) (cf. 1.10.), the diagram

$$\begin{array}{ccc}
\underline{\sigma}^!\Big[\bigoplus_{j=0}^{e} c_1(\mathcal{O}(1))^{e-j}\cap \underline{p}^*(c_j(\mathcal{E})\cap A)\Big] & \xrightarrow{\ \underline{\sigma}^!(1.(37))\ } & 0 \\
\Big\downarrow 1.(42) & & \\
\underline{\sigma}^!\Big[\bigoplus_{j=0}^{e-1} c_1(\mathcal{O}(1))^{e-1-j}\cap c_1(\mathcal{O}(1))\cap \underline{p}^*(c_j(\mathcal{F})\cap A)\oplus & & \\
\oplus \bigoplus_{j=0}^{e-1} c_1(\mathcal{O}(1))^{e-1-j}\cap \underline{p}^*(c_1(\mathcal{L})\cap c_j(\mathcal{G})\cap A)\Big] & & \uparrow \\
\Big\downarrow 1.(39) & & \\
\sigma^!\Big[\bigoplus_{j=0}^{e-1} c_1(\mathcal{O}(1))^{e-1-j}\cap \underline{i}_*\underline{q}^*(c_j(\mathcal{F})\cap A)\Big] & & \\
\Big\downarrow 1.(38) & & \\
\underline{\sigma}^!\underline{i}_*\Big[\bigoplus_{j=0}^{e-1} c_1(\mathcal{O}(1))^{e-1-j}\cap \underline{q}^*(c_j(\mathcal{F})\cap A)\Big] & \xrightarrow{\ \underline{\sigma}^!\underline{i}_*(1.(37))\ } &
\end{array}$$

commutes. By the additivity of the canonical functor-isomorphism $\underline{\sigma}^!\underline{i}_*\to 0$, the right vertical isomorphism $\underline{\sigma}^!\underline{i}_*(1.(37))$ coincides with the canonical trivialization of $\underline{\sigma}^!\underline{i}_*$. Using this and 2.6.3.,we conclude that the composition

$$\bigoplus_{j=0}^{e-1} c_1(\underline{\sigma}^*\mathcal{O}(1))^{e-1-j} \cap c_1(\sigma^*\mathcal{L}(1)) \cap \underline{\sigma}^! \underline{p}^*(c_j(\mathcal{F}) \cap A)$$

$$\downarrow$$

$$\underline{\sigma}^!\Big[\bigoplus_{j=0}^{e-1} c_1(\mathcal{O}(1))^{e-1-j} \cap c_1(\mathcal{O}(1)) \cap \underline{p}^*(c_j(\mathcal{F}) \cap A) \oplus$$

$$\oplus \bigoplus_{j=0}^{e-1} c_1(\mathcal{O}(1))^{e-1-j} \cap \underline{p}^*(c_1(\mathcal{L}) \cap c_j(\mathcal{G}) \cap A)\Big]$$

$$\Big\downarrow \ \underline{\sigma}^! \underline{i}_*(1.(37)) \circ 1.(38) \circ 1.(39)$$

$$0$$

coincides with the trivialization defined by the canonical isomorphism $\sigma^*\mathcal{L}(1) \simeq \mathcal{O}_X$. We get the commutative diagram

$$\begin{array}{ccc} \underline{\sigma}^!\Big[\bigoplus_{j=0}^{e} c_1(\mathcal{O}(1))^{e-j} \cap \underline{p}^*(c_j(\mathcal{G}) \cap A)\Big] & \xrightarrow{\underline{\sigma}^!(1.(37))} & 0 \\ \Big\downarrow 1.(42) & \nearrow & \sigma^*(\mathcal{L}(1)) \xrightarrow{\text{canonical}} \mathcal{O}_X \\ \bigoplus_{j=0}^{e-1} c_1(\underline{\sigma}^*\mathcal{O}(1))^{e-1-j} \cap c_1(\sigma^*\mathcal{L}(1)) \cap \underline{\sigma}^! \underline{p}^*(c_j(\mathcal{F}) \cap A) & & \end{array} \tag{5}$$

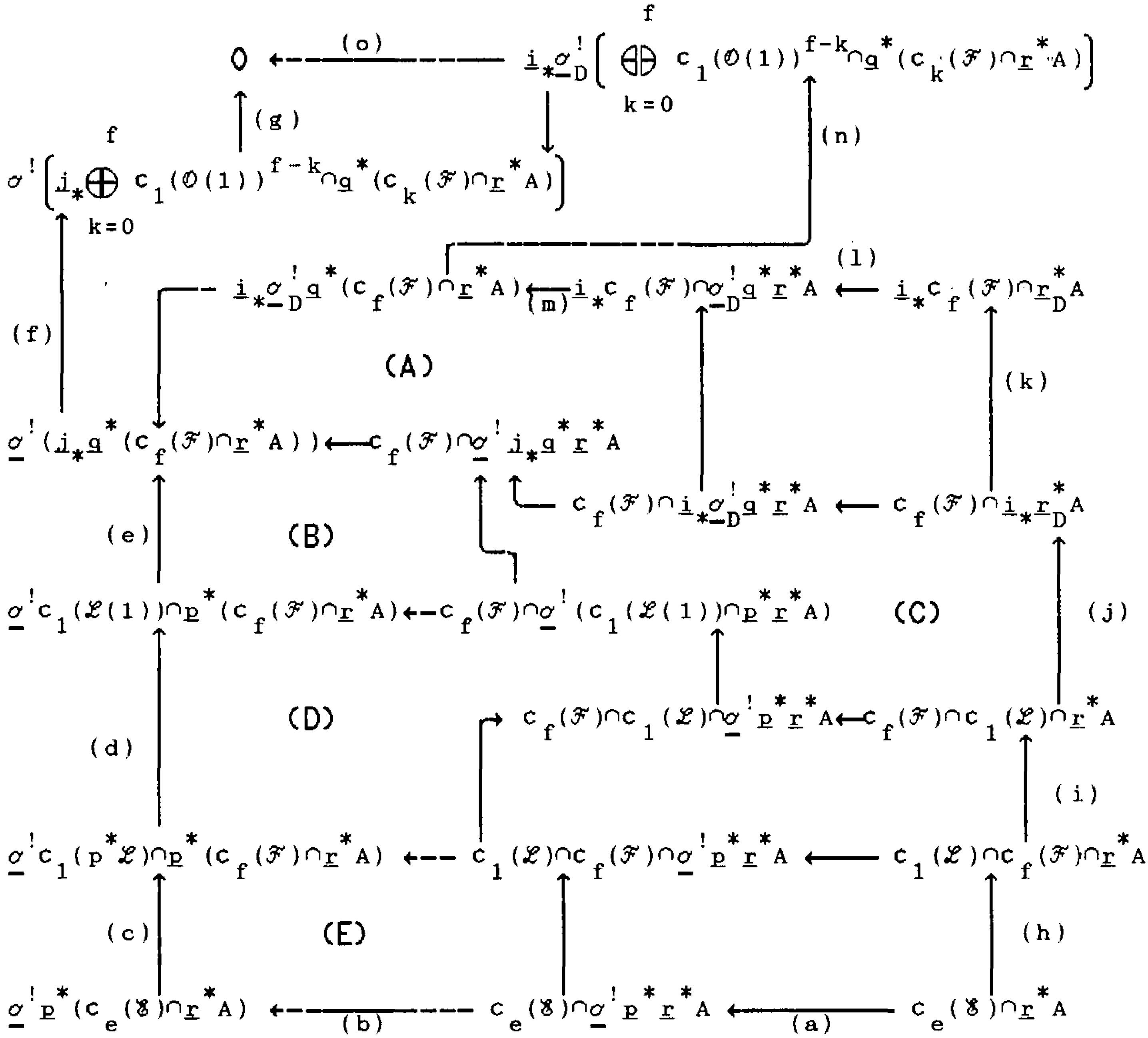
(we have not yet used the fact that $\mathcal{L} \simeq \mathcal{O}_X$ in (4)). Now, by the canonical isomorphisms $\mathcal{L} \simeq \mathcal{O}_X$ and $\sigma^*\mathcal{O}(1) \simeq \mathcal{O}_X$, the vertical arrow in the last diagram is (2), and the composition of the two other ones is $T_{\mathcal{G},s}$.

Proof of (ii): We consider the fibre square

$$\begin{array}{ccc} D & \xrightarrow{i} & X \\ \underline{\sigma}_D \downarrow & & \downarrow \underline{\sigma} \\ \mathbb{P}(\mathcal{F}) & \xrightarrow{j} & \mathbb{P}(\mathcal{G}). \end{array}$$

The projections from $\mathbb{P}(\mathcal{G})$ and $\mathbb{P}(\mathcal{F})$ to X are denoted by p and q. The assertion is proved by applying 2.6.2. to this situation. In fact, let us consider diagram (6) on page 3-5. In the definition of the arrows (d), (f), and (n) we have used the canonical isomorphism $\sigma^*\mathcal{O}(1) \simeq \mathcal{O}_X$ defined by the section s. The commutativity of (A) is essentially 2.4.1., (B) follows from 1.16., (C) is 2.6.2., (D) is essentially 2.6.4. and (E) is a consequence of 2.6.5.. The commutativity of the other squares is more or less obvious.

Diagram(6):(cf.p.3-4)



By part (i) of the proposition, the composition (o)∘(n)∘(m)∘(l) coincides with $T_{\mathcal{F}|_{D},s|_{D}}$. Consequently, the composition (o)∘...∘(h) is (3). On the other side, it follows from the explicit description of the Whitney isomorphism in 1.10. and part (i) of the proposition that the composition (g)∘...∘(a) is $T_{\mathcal{E},s}$. It follows that (3) and $T_{\mathcal{E},s}$ coincide. The proof of the proposition is complete.

3.2.The axioms: We assume that for a vector bundle $\mathcal{E}$ on X

$$c_j(\mathcal{E})\cap. \; : \; \mathfrak{CH}^{\tilde{k}}(X) \longrightarrow \mathfrak{CH}^{\tilde{k+j}}(X) \qquad (7)$$

is an additive functor, and that the following natural transformations are given:

3.2.1.: A symmety isomorphism

$$\sigma_{\mathcal{E},\mathcal{F}}: c_j(\mathcal{E})\cap c_k(\mathcal{F})\cap A \longrightarrow c_k(\mathcal{F})\cap c_j(\mathcal{E})\cap A \qquad (8)$$

3.2.2.: For a flat morphism f, an isomorphism

$$c_j(f^*\mathcal{E})\cap \underline{f}^*A \longrightarrow \underline{f}^*(c_j(\mathcal{E})\cap A). \qquad (9)$$

3.2.3.: For a proper morphism g of constant relative dimension, an isomorphism

$$c_j(\mathcal{E})\cap g_*A \longrightarrow g_*(c_j(g^*\mathcal{E})\cap A). \qquad (10)$$

3.2.4.: A Whitney sum isomorphism

$$c.(\mathcal{E})\cap A \longrightarrow c.(\mathcal{G})\cap c.(\mathcal{F})\cap A \qquad (11)$$

for every short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0 \qquad (12)$$

of vector bundles (the case $\mathcal{F}$=0 or $\mathcal{G}$=0 is not excluded).

The following axioms must be satisfied:

AX 0 (Vanishing): $c_j(\mathcal{E})\cap.$ is the zero functor if j<0 or j>dim($\mathcal{E}$), and $c_0(\mathcal{E})\cap A=A$. For $\mathcal{G}$=0, this implies $c.(\mathcal{G})\cap A=A$.

AX 1 (Normalization): $c_1(\mathcal{L})\cap A = \mathrm{c}_1(\mathcal{L})\cap A$ for a line bundle $\mathcal{L}$, where $\mathrm{c}_1(\mathcal{L})\cap.$ is the additive functor introduced in 1.2.. If $\mathcal{F}$ and $\mathcal{G}$ are line bundles, then (8) coincides with the symmetry introduced in 1.5.. Also, int the case of line bundles (9) and (10) coincide with the isomorphisms introduced in 1.4..

AX 2 (Compatibilities for (9) and (10)): If $\mathscr{E}$ is a vector bundle on X snd if X-schemes are denoted by $p:Y\longrightarrow X$, then (9) and (10) define for the additive functors

$$c_j(p^*\mathscr{E})\cap.: \tilde{\mathfrak{C}\mathfrak{H}}^{\cdot}(Y) \longrightarrow \tilde{\mathfrak{C}\mathfrak{H}}^{\cdot}(Y)$$

the structur of a biadmissible functor between bifibred Picard categories over (X-schemes). If $\mathscr{E}$ and $\mathscr{F}$ are vector bundles on X, then the isomorphism (8)

$$c_j(p^*\mathscr{E})\cap c_k(p^*\mathscr{F})\cap. \longrightarrow c_k(p^*\mathscr{F})\cap c_j(p^*\mathscr{E})\cap$$

is a biadmissible functormorphism. Similar, if (12) is an exact sequence of vector bundles on X, then the isomorphism (11)

$$c.(p^*\mathscr{E})\cap. \longrightarrow c.(p^*\mathscr{G})\cap c.(p^*\mathscr{F})\cap.$$

is biadmissible.

AX 3: The analogues of 1.(45) (for vector bundles of arbitrary dimension) and of 1.(62) for the isomorphisms (8) and (11) commute. Note that this would allow us to apply 2.8. to the functors $c_j(\mathscr{E})\cap.$, but it will not be necessary to do so.

<u>Corollary:</u> $\sigma_{\mathscr{E},\mathscr{F}}\sigma_{\mathscr{F},\mathscr{E}}=\mathrm{Id}$, and the analogue of 1.(34) for the isomorphisms (8) commutes. Note that this will enable us to apply 2.9. to the functors c_j. Hence for a polynomial P in Chern classes the polynomial $P(\dim(\mathscr{E}_j),c_i(\mathscr{E}_j))\cap A=\mathscr{P}(\mathscr{E}_j)\cap A$ in Chern functors is well-defined. This will be important for the formulation of AX 4.

<u>Proof</u>: This is clear from AX 1 if all the vector bundles involved are line bundles. The general case follows by induction on the dimension of the vector bundles, using AX 3 and the splitting principle.

AX 4 (Twist by a line bundle): The analogue of 2.10.1.-4. for the functors c_i and the isomorphisms (8)-(11) is true, i.e., there exists an isomorphism

$$c_j(\mathscr{E}\otimes\mathscr{L})\cap A \longrightarrow \mathscr{P}_j(\mathscr{E},\mathscr{L})\cap A \qquad (13)$$

satisfying 2.10.1.-2.10.4. (The uniqueness of such an isomorphism is clear).

AX 5: By the previous axioms, the definition of the trivialization $T_{\mathcal{E},s}$ of $c_e(\mathcal{E})\cap.$ defined by a global non-vanishing section s of $\mathcal{E}$ works for the functors c_j. We assume that Proposition 3.1.(ii) remains true for the functors c_j.

<u>Remark:</u> It seems very likely that AX 5 is a consequence of the other axioms. I hope I will be able to return to that subject in the forthcoming paper on functorial Riemann-Roch.

<u>*3.3. Theorem:*</u> *If $c_j(\mathcal{E})\cap.$ satisfies the properties listed in 3.2., then there is a unique additive functor-isomorphism*

$$c_j(\mathcal{E})\cap A \longrightarrow \mathrm{c}_j(\mathcal{E})\cap A \tag{14}$$

which commutes with the transformations 3.2.1.-4. and is the identity if $j\le 0$, $j>\dim(\mathcal{E})$, or if $\mathcal{E}$ is a line bundle.

<u>Proof</u>: The uniqueness of (14) is clear. To prove the existence, we first consider some consequences of the axioms:

Step 1: Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles and $\mathcal{E}$ an arbitrary vector bundle. We want to check that the diagram

$$\begin{array}{ccc} c_j(\mathcal{E}\otimes\mathcal{L})\cap c_1(\mathcal{M})\cap A & \xrightarrow{\sigma_{\mathcal{E},\mathcal{F}}} & c_1(\mathcal{M})\cap c_j(\mathcal{E}\otimes\mathcal{L})\cap A \\ \downarrow & & \downarrow \\ \mathcal{P}_j(\mathcal{E},\mathcal{L})\cap c_1(\mathcal{M})\cap A & \longrightarrow & c_1(\mathcal{M})\cap\mathcal{P}_j(\mathcal{E},\mathcal{L})\cap A \end{array}$$

commutes (cf. AX 4. The lower horizontal arrow is 2.(24)). By 2.10.2., this is clear if $\mathcal{E}$ is a line bundle. The general case follows by induction on the dimension of $\mathcal{E}$, using 2.10.3. and the splitting principle. Here 1.(45) is used again, because the induction argument involves using the Whitney isomorphism.

Step 2: Let

$$\begin{array}{ccc} D & \xrightarrow{i} & X \\ {\scriptstyle q}\searrow & & \swarrow{\scriptstyle p} \\ & S & \end{array}$$

be a commutative diagramwith p and q flat and ia regular closed immersionof codimension one. Fora vector bundle $\mathcal{E}$ on S, we want to check that the diagram

$$\begin{array}{ccccc}
\underline{i}_*\underline{q}^*(c_j(\mathcal{E})\cap A) & \longrightarrow & \underline{i}_*(c_j(q^*\mathcal{E})\cap \ \underline{q}^*A) & \longrightarrow & c_j(\underline{p}^*\mathcal{E})\cap\underline{p}^*A \\
\downarrow {\scriptstyle 1.(39)} & & & & \downarrow {\scriptstyle 1.(39)} \\
c_1(\mathcal{O}_X(D))\cap\underline{p}^*(c_j(\mathcal{E})\cap A) & & & & c_j(p^*\mathcal{E})\cap c_1(\mathcal{O}_X(D))\cap\underline{p}^*A \\
& \searrow & c_1(\mathcal{O}_X(D))\cap c_j(p^*\mathcal{E})\cap\underline{p}^*A & \nearrow &
\end{array}$$

commutes.

If $\mathcal{E}$ is a line bundle, this is 1.(40). The general case follows by the splitting principle from 1.(45) (cf. AX 3) and AX 2. The details have been presented in 1.16..

Step 3: Now we are ready to construct (14). For a vector bundle $\mathcal{E}$ of dimension e on X, we denote by p: $\mathbb{P}(\mathcal{E})\longrightarrow X$ the projective fibration. Then

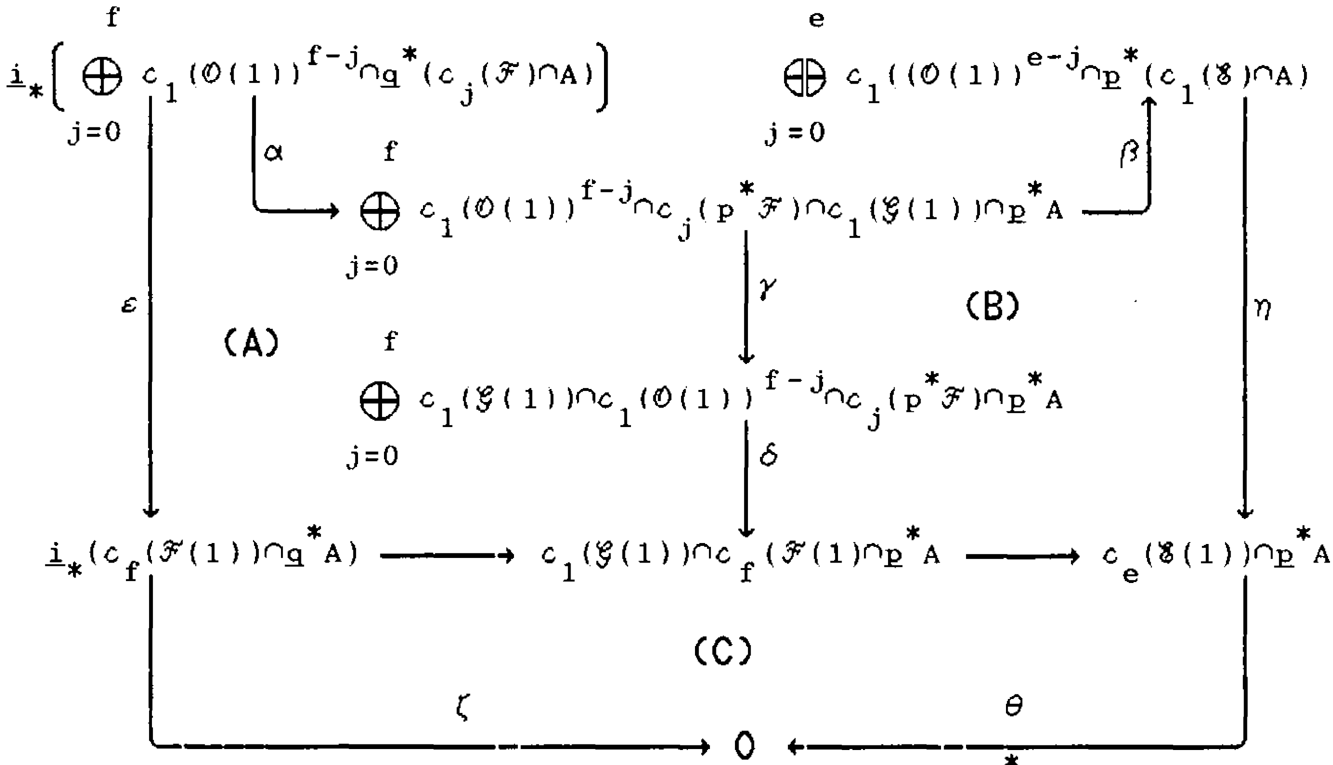
$$\mathcal{E}(1):=p^*\mathcal{E}\otimes\mathcal{O}(1)$$

has a canonical non-vanishing global section s, defining a trivialization

$$\bigoplus_{j=0}^{e} c_1(\mathcal{O}(1))^{e-j}\cap\underline{p}^*(c_j(\mathcal{E})\cap A) \xrightarrow{\text{AX 1, AX 4}} c_e(\mathcal{E}(1))\cap A) \xrightarrow{T_{\mathcal{E}(1),s}} 0 \qquad (15)$$

Because the isomorphism 1.(37) characterizes $c_j(\mathcal{E})\cap A$ up to unique isomorphism, (15) defines an isomorphism (14). The compatibility of (14) with the isomorphisms 3.2.2. and 3.2.3. is clear because (14) contains only biadmissible transformations. It remains to prove that (14) commutes with 3.2.1. and 3.2.4..

Step 4: The hard part is the compatibility with (11). By the splitting principle and because 1.(62) was supposed to be commutativ, it suffices to consider short exact sequences (12) with $\dim(\mathcal{G})=1$. We consider the following diagram.



Here α is given by first interchanging $c_j(\mathcal{F})\cap.$ and $\underline{q}^*$, then interchanging both $c_j(\mathcal{F})\cap.$ and $c_1(\mathcal{O}(1))\cap.$ with $\underline{i}_*$, and finally applying 1.(37). By step 3, the result of first interchanging $c_1(\mathcal{O}(1))$ with $\underline{i}_*$, applying $\underline{i}_*\underline{q}^*(c_j(\mathcal{F})\cap A) \longrightarrow c_1(\mathcal{G}(1))\cap\underline{p}^*(c_j(\mathcal{F})\cap A)$, and then bringing $c_j(\mathcal{F})$ to the left side would be the same. Consequently, $\beta\alpha$ is 1.(41). The pentagon (A) commutes by a combination of 2.10.2. (cf. AX 4) and step 2. The pentagon (B) would by 2.10.3. (i.e., AX 4) commute up to the sign

$$c_1(\mathcal{O}(1))\cap c_{f-1}(\mathcal{F}(1))\cap[-1]\cap[\underline{p}^*A] = \sum_{l=0}^{f-1}(l+1)c_1(\mathcal{O}(1))^{l+1}\cap c_{f-l-1}(p^*\mathcal{F})\cap[-1]\cap[\underline{p}^*A] \tag{16}$$

if $\delta\gamma$ was the bottom horizontal arrow in 2.(30). However, δ involves (f-j) times interchanging $c_1(\mathcal{O}(1))$ with itsself, whereas the arrow in 2.(30) uses only minimal permutations. Hence δ produces the additional sign

$$\sum_{j=0}^{f}(f-j)c_1(\mathcal{O}(1))^{f-j}\cap c_j(p^*\mathcal{F})\cap[-1]\cap[\underline{p}^*A]$$

cancelling (16). Hence (B) commutes.

By AX 5, (C) also commutes. Now $\zeta\varepsilon$ is (15) for $\mathcal{F}$ and $\theta\eta$ is (15) for $\mathcal{E}$, whereas $\beta\alpha$ is 1.(41). This proves compatibility between (14) and the Whitney isomorphism.

Step 5: The compatibility between (14) and $\sigma_{\mathcal{E},\mathcal{F}}$ follows now by induction on $\dim(\mathcal{E})$ and $\dim(\mathcal{F})$, using AX 1 for the start and the result of step 4, 1.(45) (cf. AX 3), and the splitting principle for the induction argument. The proof of 3.3. is complete.

<u>3.4. Comparison with Deligne's IC_2:</u> Let p: X⟶S be a proper smooth morphism of relative dimension one, where S is normal and locally factorial. For line bundles $\mathcal{L}$, $\mathcal{M}$ on X, put

$$\langle\mathcal{L},\mathcal{M}\rangle = p_*\left[c_1(\mathcal{L})\cap c_1(\mathcal{M})\right]. \tag{17}$$

Note that by 1.3. this is the line bundle on S constructed in [D]. If ℓ and m are rational sections of $\mathcal{L}$ and $\mathcal{M}$ whose divisors do not intersect, then

$$\langle\ell,m\rangle = p_*(l\cap m) \tag{18}$$

is a section of $\langle\mathcal{L},\mathcal{M}\rangle$ on S satisfying the transformation rules of [D]. For a virtual vector bundle $\mathcal{E}$ on X, put

$$IC_2(\mathcal{E}) = p_*(c_2(\mathcal{E})). \tag{19}$$

This functor has the following two structures:

<u>3.4.1.:</u> For a line bundle $\mathcal{L}$ on X, a canonical isomorphism $IC_2(\mathcal{L})\simeq\mathcal{O}_X$ defined by the canonical trivialization $c_2(\mathcal{L})\longrightarrow 0$ in $\mathfrak{C}\tilde{\mathfrak{H}}^2(X)$.

<u>3.4.2.:</u> A Whitney isomorphism

$$IC_2(\mathcal{E}\oplus\mathcal{F}) \longrightarrow IC_2(\mathcal{E})\otimes IC_2(\mathcal{F})\otimes\langle\det(\mathcal{E}),\det(\mathcal{F})\rangle \tag{20}$$

defined by the following arrows:

$$\begin{aligned} p_*(c_2(\mathcal{E}\oplus\mathcal{F})) &\longrightarrow p_*(c_2(\mathcal{E}))\otimes p_*(c_2(\mathcal{F}))\otimes p_*(c_1(\mathcal{E})\cap c_1(\mathcal{F})) \\ &\longrightarrow IC_2(\mathcal{E})\otimes IC_2(\mathcal{F})\otimes p_*\left[c_1(\det(\mathcal{E}))\cap c_1(\det(\mathcal{F}))\right] \\ &\longrightarrow IC_2(\mathcal{E})\otimes IC_2(\mathcal{F})\otimes\langle\det(\mathcal{E}),\det(\mathcal{F})\rangle, \end{aligned} \tag{21}$$

where the first arrow is the Whitney isomorphism for the functors c_k, the second arrow is 2.7. plus definition (19), and the third one <u>differs by the sign</u>

$$(-1)^{\deg(\det(\mathcal{E}))\dim(\mathcal{F})}$$

from the tautological arrow given by (17).

Proposition: IC_2, *together with the isomorphisms 3.4.1. and 3.4.2., satisfies the axioms of [D, Proposition 9.4.]. Concequently, it is canonically isomorphic to the functor which Deligne named* IC_2.

Proof: *Step 1:* It is immediately verified that if $T(\mathscr{E})$ is defined as in [D, 9.5.], then T becomes an additive functor between Picard categories. In particular the compatibility of $T(\mathscr{E})$ with the symmetries of its source and target categories follows from 1.(63) in view of the sign convention we made in (21). It remains to verify assumption (iii) of [D,9.5.]. We start with a preparation for this.

Step 2: By 2.10., we have an isomorphism

$$IC_2(\mathscr{E}\otimes\mathscr{L}) \simeq IC_2(\mathscr{E})\otimes\langle\det(\mathscr{E}),\mathscr{L}\rangle^{e-1}\otimes\langle\mathscr{L},\mathscr{L}\rangle^{e(e-1)/2},\ e=\dim(\mathscr{E}). \quad (22)$$

Let us consider the square

$$\begin{array}{ccc} IC_2\left[(\mathscr{E}\oplus\mathscr{F})\otimes\mathscr{L}\right] & \longrightarrow & IC_2(\mathscr{E}\otimes\mathscr{L})\otimes IC_2(\mathscr{F}\otimes\mathscr{L})\otimes\langle\det(\mathscr{E}\otimes\mathscr{L}),\det(\mathscr{F}\otimes\mathscr{L})\rangle \\ \downarrow & & \downarrow \\ \begin{array}{l} IC_2(\mathscr{E}\oplus\mathscr{F})\otimes\langle\det(\mathscr{E})\otimes\det(\mathscr{F}),\mathscr{L}\rangle^{e+f-1}\otimes \\ \otimes\langle\mathscr{L},\mathscr{L}\rangle^{(e+f)(e+f-1)/2} \end{array} & \longrightarrow & \begin{array}{l} IC_2(\mathscr{E})\otimes IC_2(\mathscr{F})\otimes\langle\det(\mathscr{E}),\det(\mathscr{F})\rangle\otimes \\ \otimes\langle\det(\mathscr{E}),\mathscr{L}\rangle^{e-1}\otimes\langle\det(\mathscr{F}),\mathscr{L}\rangle^{f-1}\otimes \\ \otimes\langle\det(\mathscr{E}),\mathscr{L}\rangle^{f}\otimes\langle\det(\mathscr{F}),\mathscr{L}\rangle^{e}\otimes \\ \otimes\langle\mathscr{L},\mathscr{L}\rangle^{ef+e(f-1)/2+f(e-1)/2} \end{array} \end{array} \quad (23)$$

By 2.10.3., the correct sign for (23) would be

$$(-1)^{\deg(\mathscr{L})ef} \quad (24)$$

if there was no sign convention in (24). However, the sign convention modifies the top arrow by

$$(-1)^{f(\deg(\det(\mathscr{E}))+\deg(\mathscr{L})e)}$$

and the bottom arrow by

$$(-1)^{f\deg(\det(\mathscr{E}))},$$

cancelling (24). Consequently, (23) commutes on the nose.

If $\mathscr{E}$ is a line bundle, (22) reduces to a canonical isomorphism $IC_2(\mathscr{E}\otimes\mathscr{L})\simeq IC_2(\mathscr{E})$, and this isomorphism respects the canonical trivializations of both sides.

Step 3: Now we are ready to verify assumption (iii) in [D,9.5.]. We recall that this signifies the following:

Let Q' be a line bundle on S, s a section of p, $Q=s_*Q'$ in $\mathfrak{K}(X)$, $\mathscr{L}$ a line bundle on X together with an isomorphism $Q'\simeq s^*\mathscr{L}$. Then the isomorphisms

$$\mathcal{O}_S \longrightarrow IC_2(\mathscr{L}) \longrightarrow IC_2\Big(\mathscr{L}(-s(S))\Big)\otimes IC_2(Q)\otimes\langle \det(Q),\mathscr{L}(-s(S))\rangle \longrightarrow$$
$$\longrightarrow IC_2(Q)\otimes s^*\mathscr{L}(-s(S)),$$

where the second arrow is given by (20) and the exact sequence $0\longrightarrow\mathscr{L}(-s(S))\longrightarrow\mathscr{L}\longrightarrow Q\longrightarrow 0$, define

$$I_{\mathscr{L}}:\ IC_2(Q) \longrightarrow Q'^{-1}\otimes s^*\mathcal{O}_X(-s(S)).$$

The condition is that $I_{\mathscr{L}}$ is independant of $\mathscr{L}$.

Because S is integral, we may assume S=Spec(k) for a field k. Then $Q=\mathcal{O}_S$, and $\mathscr{L}$ has a trivialization at s. If $\mathscr{M}$ is another line bundle on S, we have to check $I_{\mathscr{L}}=I_{\mathscr{M}}$. We claim that this condition depends only on $\mathscr{N}=\mathscr{M}\otimes\mathscr{L}^{-1}$. Indeed, by step 2 we have the commutative square

$$\begin{array}{ccc} IC_2(\mathscr{L}\otimes\mathscr{N}) & \longrightarrow & IC_2(\mathscr{L}\otimes\mathscr{N}(-s))\otimes\langle\mathscr{L}\otimes\mathscr{N}(-s),\mathcal{O}_X(s)\rangle\otimes IC_2(Q\otimes\mathscr{N}) \\ \downarrow & \circlearrowleft & \downarrow \\ IC_2(\mathscr{L}) & \longrightarrow & IC_2(\mathscr{L}(-s))\otimes\langle\mathscr{L}(-s),\mathcal{O}_X(s)\rangle\otimes IC_2(Q)\otimes \\ & & \otimes\langle\mathscr{N},\mathcal{O}_X(s)\rangle\otimes\langle\mathscr{N},\mathcal{O}_X(s)\rangle^{-1} \end{array},$$

reducing the proof of $I_{\mathscr{L}}=I_{\mathscr{M}}$ to the commutativity of

$$\begin{array}{ccc} IC_2(Q) & \longrightarrow & IC_2(Q\otimes\mathscr{N}) \\ & \searrow & \downarrow \\ & & IC_2(Q)\otimes\langle\mathscr{N},\mathcal{O}_X(s)\rangle^{-1} \end{array},$$

where the horizontal arrow is given by the trivialization of $\mathscr{N}$ at the point s, the vertical arrow is (22), and the slanted arrow is given by the trivialization of $\langle\mathscr{N},\mathcal{O}_X(s)\rangle$ which sends $\langle n,\text{"1"}\rangle$ to $n(s)$, where "1" is the canonical section of $\mathcal{O}_X(s)$ with a zero at s, and n is any rational section of $\mathscr{N}$ whose order at s is zero. It is clear that the last diagram depends only on $\mathscr{N}$, which proves our claim.

Because the condition we have to verify depends only on $\mathcal{L}\otimes\mathcal{M}^{-1}$, we may assume that $\mathcal{L}$ and $\mathcal{M}$ have global section ℓ and m with $\ell(s)=m(s)=1$ and whose divisors do not intersect. Then $\sigma=(\ell,-m)$ is a global non-vanishing section of $\mathcal{E}=\mathcal{L}\oplus\mathcal{M}$, and $\oplus$ is contained in

$$\mathcal{F}=\ker(\mathcal{E}=\mathcal{L}\oplus\mathcal{M} \longrightarrow Q) \tag{25}$$
$$\langle\lambda,\mu\rangle \longrightarrow \lambda(s)+\mu(s).$$

We have a commutative diagram in $\mathfrak{R}(X)$

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F}\oplus Q \\ \downarrow & & \downarrow \\ \mathcal{L}\oplus\mathcal{M} & & \mathcal{L}(-s)\oplus\mathcal{M}\oplus Q \\ \llcorner\!\!\to & \mathcal{L}(-s)\oplus Q\oplus\mathcal{M} & \to\!\!\lrcorner\uparrow \end{array} \tag{26}$$

The right vertical arrow is given by $0\longrightarrow\mathcal{L}(-s)\longrightarrow\mathcal{F}\longrightarrow\mathcal{M}\longrightarrow 0$, and the horizontal arrow by $0\longrightarrow\mathcal{F}\longrightarrow\mathcal{E}\longrightarrow Q\longrightarrow 0$, with $\mathcal{E}\longrightarrow Q$ defined in (25). The non-vanishing global section σ defines trivializations T_σ of $IC_2(\mathcal{E})$ and $IC_2(\mathcal{F})$. Now the top row of (26) defines isomorphisms

$$\begin{array}{ccccc} 0 & \xrightarrow{T_\sigma} & IC_2(\mathcal{E}) & \longrightarrow & IC_2(\mathcal{F})\otimes IC_2(Q)\otimes\langle\det(\mathcal{F}),\mathcal{O}_X(s)\rangle \\ & & & & \downarrow T_\sigma \\ & & & & IC_2(Q)\otimes\langle\det(\mathcal{F}),\mathcal{O}_X(s)\rangle, \end{array}$$

where the second arrow uses the canonical isomorphism $\det(Q)\simeq\mathcal{O}_X(s)$. In view of the canonical isomorphism $\langle\mathcal{L}\otimes\mathcal{M},\mathcal{O}_X(s)\rangle\longrightarrow\mathcal{O}_S$, this defines

$$I_{\mathcal{E}}:\ IC_2(Q) \longrightarrow s^*\mathcal{O}_X(s).$$

We want to compare $I_{\mathcal{E}}$ and $I_{\mathcal{L}}$. (26) gives us a commutative diagram

$$\begin{array}{ccc} IC_2(\mathcal{E}) & \longrightarrow & IC_2(\mathcal{F})\otimes IC_2(Q)\otimes\langle\det(\mathcal{F}),\mathcal{O}_X(s)\rangle \\ \downarrow & & \downarrow \\ IC_2(\mathcal{L})\otimes IC_2(\mathcal{M})\otimes & \longrightarrow & IC_2(\mathcal{L}(-s))\otimes IC_2(Q)\otimes\langle\mathcal{L}(-s),\mathcal{O}_X(s)\rangle\otimes \\ \otimes\langle\mathcal{L},\mathcal{M}\rangle & & \otimes\langle\mathcal{L}(-s),\mathcal{M}\rangle\otimes\langle\mathcal{M},\mathcal{O}_X(s)\rangle\otimes IC_2(\mathcal{M}) \end{array} .$$

By applying 3.1.(ii) to the exact sequence $0\longrightarrow\mathcal{L}\longrightarrow\mathcal{E}\longrightarrow\mathcal{M}\longrightarrow 0$ and the global section σ of $\mathcal{E}$, we find that by the left vertical arrow the trivialization T_σ of $IC_2(\mathcal{E})$ corresponds to the trivialization of $\langle\mathcal{L},\mathcal{M}\rangle$ which maps

$$\langle\ell,-m\rangle \text{ to } (-1)^{\deg(\mathcal{M})} \qquad (27)$$

(of course, multiply by the canonical trivializations of $IC_2(\mathcal{L})$ and $IC_2(\mathcal{M})$. The sign in (27) comes in because of the sign convention in (21)).

In a similar manner, applying 3.1. to the short exact sequence $0\longrightarrow\mathcal{L}(-s)\longrightarrow\mathcal{F}\longrightarrow\mathcal{M}\longrightarrow 0$ and the global section σ of $\mathcal{F}$, we conclude that the trivialization T_σ of $IC_2(\mathcal{F})$ corresponds by the right horizontal arrow to the trivialization of $\langle\mathcal{L}(-s),\mathcal{M}\rangle$ which maps

$$\langle\ell,-m\rangle \text{ to } (-1)^{\deg(\mathcal{M})}, \qquad (28)$$

where ℓ is viewed as a rational section of $\mathcal{L}(-s)$ which is singular at s but regualar at the divisor of m.

If we identify both $\langle\det(\mathcal{F}),\mathcal{O}_X(s)\rangle$ and $\langle\mathcal{L}(-s),\mathcal{O}_X(s)\rangle$ to $s^*\mathcal{O}_X(s)^{-1}$, then under the right vertical arrow these identifications differ by the trivialization of $\langle\mathcal{M},\mathcal{O}_X(s)\rangle$ which maps

$$\langle -m,\text{"}1\text{"}\rangle \text{ to } -1. \qquad (29)$$

By (27), (28), and (29), we conclude that $I_{\mathcal{E}}=-I_{\mathcal{L}}$. In a similar manner, using the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}(-s) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Q & = & Q & & & &
\end{array}$$

of exact sequences, one proves $I_{\mathcal{M}}=-I_{\mathcal{E}}$. Consequently, $I_{\mathcal{L}}=I_{\mathcal{M}}$, completing the verification of [D,9.5.(iii)] and the proof of the proposition.

In a similar manner one can compare the integrals of our Chern functors over the fibres of a higher-dimensional morphism with Elkiks line bundles (in case both functors are defined). The starting point is the comparition for an integral of a product of first Chern classes of line bundles, where one has to verify that the integral of the product of c_1's satisfies the descent condition used by Elkik. The extension of this isomorphism for c_1 to the general case is esier than the comparision with Deligne's IC_2 carried out in the above proposition, because the construction we used in 1.10. is also used by Elkik.

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