ON K_1 OF A WALDHAUSEN CATEGORY

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Abstract. We give a simple representation of all elements in K_1 of a Waldhausen category and prove relations between these representatives which hold in K_1.

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Introduction

A general notion of K-theory, for a category $\mathbf{W}$ with cofibrations and weak equivalences, was defined by Waldhausen [Wal78] as the homotopy groups of the loop space of a certain simplicial category $wS.\mathbf{W}$,

$$K_n\mathbf{W} \;=\; \pi_n\Omega\,|wS.\mathbf{W}| \;\cong\; \pi_{n+1}\,|wS.\mathbf{W}|\,,\;\; n\geq 0.$$

Waldhausen K-theory generalizes the K-theory of an exact category $\mathbf{E}$, defined as the homotopy groups of $\Omega\,|Q\mathbf{E}|$, where $Q\mathbf{E}$ is a category defined by Quillen.

Gillet and Grayson defined in [GG87] a simplicial set $G.\mathbf{E}$ which is a model for $\Omega\,|Q\mathbf{E}|$. This allows one to compute $K_1\mathbf{E}$ as a fundamental group, $K_1\mathbf{E} = \pi_1\,|G.\mathbf{E}|$. Using the standard techniques for computing π_1 Gillet and Grayson produced algebraic representatives for arbitrary elements in $K_1\mathbf{E}$. These representatives were simplified by Sherman [She94, She98] and further simplified by Nenashev [Nen96]. Nenashev's representatives are pairs of short exact sequences on the same objects,

$$A \rightarrowtail B \twoheadrightarrow C.$$

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Such a pair gives a loop in $|G.\mathbf{E}|$ which corresponds to a 2-sphere in $|wS.\mathbf{E}|$, obtained by pasting the 2-simplices associated to each short exact sequence along their common boundary.

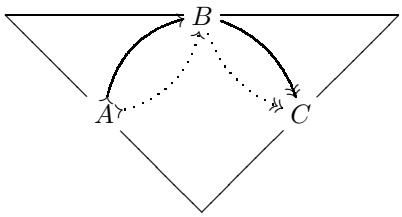
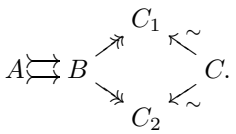


FIGURE 1. Nenashev's representative of an element in $K_1\mathbf{E}$.

Nenashev showed in [Nen98a] certain relations which are satisfied by these pairs of short exact sequences, and he later proved in [Nen98b] that pairs of short exact sequences together with these relations yield a presentation of $K_1\mathbf{E}$.

In this paper we produce representatives for all elements in K_1 of an arbitrary Waldhausen category $\mathbf{W}$ which are as close as possible to Nenashev's representatives for exact categories. They are given by pairs of cofiber sequences where the cofibrations have the same source and target. The cofibers, however, may be non-isomorphic, but they are weakly equivalent via a length 2 zig-zag.



This diagram is called a *pair of weak cofiber sequences*. It corresponds to a 2-sphere in $|wS.\mathbf{W}|$ obtained by pasting the 2-simplices associated to the cofiber sequences along the common part of the boundary. The two edges which remain free after this operation are filled with a disk made of two pieces corresponding to the weak equivalences.

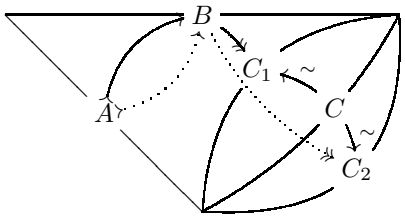


FIGURE 2. Our representative of an element in $K_1\mathbf{W}$.

In addition we show in Section 3 that pairs of weak cofiber sequences satisfy a certain generalization of Nenashev's relations.

Let $K_1^{\mathrm{wcs}}\ \mathbf{W}$ be the group generated by the pairs of weak cofiber sequences in $\mathbf{W}$ modulo the relations in Section 3. The results of this paper show in particular

that there is a natural surjection

$$K_1^{\mathrm{wcs}}\,\mathbf{W} \twoheadrightarrow K_1\mathbf{W}$$

which is an isomorphism in case $\mathbf{W} = \mathbf{E}$ is an exact category. We do not know whether it is an isomorphism for an arbitrary Waldhausen category $\mathbf{W}$, but we prove here further evidence which supports the conjecture: the homomorphism

$$\cdot\,\eta\colon K_0\mathbf{W} \otimes \mathbb{Z}/2 \to K_1\mathbf{W}$$

induced by the action of the Hopf map $\eta \in \pi_*^s$ in the stable homotopy groups of spheres factors as

$$K_0\mathbf{W} \otimes \mathbb{Z}/2 \longrightarrow K_1^{\mathrm{wcs}}\,\mathbf{W} \twoheadrightarrow K_1\mathbf{W}.$$

We finish this introduction with some comments about the method of proof. We do not rely on any previous similar result because we do not know of any generalization of the Gillet-Grayson construction for an arbitrary Waldhausen category $\mathbf{W}$. We use instead the algebraic model $\mathcal{D}_*\mathbf{W}$ defined in [MT06] for the 1-type of the Waldhausen K-theory spectrum $K\mathbf{W}$. The algebraic object $\mathcal{D}_*\mathbf{W}$ is a chain complex of non-abelian groups concentrated in dimensions $i = 0, 1$ whose homology is $K_i\mathbf{W}$.

$$\begin{array}{ccccccc} & & & & (\mathcal{D}_0\mathbf{W})^{ab} \otimes (\mathcal{D}_0\mathbf{W})^{ab} & & \\ & & & & \downarrow{\scriptstyle \langle\cdot,\cdot\rangle} & & \\ K_1\mathbf{W} & \hookrightarrow & \mathcal{D}_1\mathbf{W} & \xrightarrow{\partial} & \mathcal{D}_0\mathbf{W} & \twoheadrightarrow & K_0\mathbf{W}. \end{array}$$

This non-abelian chain complex is equipped with a bilinear map $\langle\cdot,\cdot\rangle$ which determines the commutators. This makes $\mathcal{D}_*\mathbf{W}$ a stable quadratic module in the sense of [Bau91]. This stable quadratic module is defined in [MT06] in terms of generators and relations. Generators correspond simply to objects, weak equivalences, and cofiber sequences in $\mathbf{W}$. The reader will notice that the simplicity of the presentation of $\mathcal{D}_*\mathbf{W}$ makes the proofs of the results in this paper considerably shorter than the proofs of the analogous results for exact categories.

Acknowledgement. The authors are very grateful to Grigory Garkusha for asking about the relation between the algebraic model for Waldhausen K-theory defined in [MT06] and Nenashev's presentation of K_1 of an exact category in [Nen98b].

1. The algebraic model for K_0 and K_1

Definition 1.1. A *stable quadratic module* C_* is a diagram of group homomorphisms

$$C_0^{ab} \otimes C_0^{ab} \xrightarrow{\langle\cdot,\cdot\rangle} C_1 \xrightarrow{\partial} C_0$$

such that given $c_i, d_i \in C_i$, $i = 0, 1$,

(1) $\partial\langle c_0, d_0\rangle = [d_0, c_0]$,
(2) $\langle\partial(c_1), \partial(d_1)\rangle = [d_1, c_1]$,
(3) $\langle c_0, d_0\rangle + \langle d_0, c_0\rangle = 0$.

Here $[x, y] = -x - y + x + y$ is the commutator of two elements $x, y \in K$ in any group K, and K^{ab} is the abelianization of K. It follows from the axioms that the image of $\langle\cdot,\cdot\rangle$ and $\operatorname{Ker}\partial$ are central in C_1, the groups C_0 and C_1 have nilpotency class 2, and $\partial(C_1)$ is a normal subgroup of C_0.

A *morphism* $f\colon C_* \to D_*$ of stable quadratic modules is given by group homomorphisms $f_i\colon C_i \to D_i$, $i = 0, 1$, compatible with the structure homomorphisms of C_* and D_*, i.e. $f_0\partial = \partial f_1$ and $f_1\langle\cdot,\cdot\rangle = \langle f_0, f_0\rangle$.

Stable quadratic modules were introduced in [Bau91, Definition IV.C.1]. Notice, however, that we adopt the opposite convention for the homomorphism $\langle\cdot,\cdot\rangle$.

Remark 1.2. There is a natural right action of C_0 on C_1 defined by

$$c_1^{c_0} \quad = \quad c_1 + \langle c_0, \partial(c_1)\rangle.$$

The axioms of a stable quadratic module imply that commutators in C_0 act trivially on C_1, and that C_0 acts trivially on the image of $\langle\cdot,\cdot\rangle$ and on $\operatorname{Ker}\partial$.

The action gives $\partial : C_1 \to C_0$ the structure of a crossed module. Indeed a stable quadratic module is the same as a commutative monoid in the category of crossed modules such that the monoid product of two elements in C_0 vanishes when one of them is a commutator, see [MT06, Lemma 4.15].

We assume the reader has certain familiarity with Waldhausen categories and related concepts. We refer to [Wei] for the basics, see also [Wal85]. The following definition was introduced in [MT06].

Definition 1.3. Let $\mathbf{C}$ be a Waldhausen category with distinguished zero object $*$, and cofibrations and weak equivalences denoted by $\rightarrowtail$ and $\xrightarrow{\sim}$, respectively. A generic cofiber sequence is denoted by

$$A \rightarrowtail B \twoheadrightarrow B/A.$$

We define $\mathcal{D}_*\mathbf{C}$ as the stable quadratic module generated in dimension zero by the symbols

- $[A]$ for any object in $\mathbf{C}$,

and in dimension one by

- $[A \xrightarrow{\sim} A']$ for any weak equivalence,
- $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence,

such that the following relations hold.

(R1) $\partial[A \xrightarrow{\sim} A'] = -[A'] + [A]$.
(R2) $\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$.
(R3) $[*] = 0$.
(R4) $[A \xrightarrow{1_A} A] = 0$.
(R5) $[A \xrightarrow{1_A} A \twoheadrightarrow *] = 0$, $[* \rightarrowtail A \xrightarrow{1_A} A] = 0$.
(R6) For any pair of composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

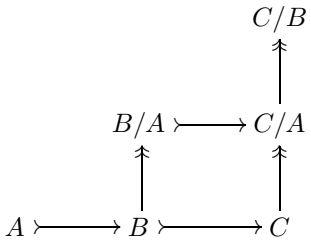
(R7) For any commutative diagram in $\mathbf{C}$ as follows

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \end{array}$$

we have

$$\begin{aligned}[A \xrightarrow{\sim} A'] + [B/A \xrightarrow{\sim} B'/A']^{[A]} &= -[A' \rightarrowtail B' \twoheadrightarrow B'/A'] \\ &\quad +[B \xrightarrow{\sim} B'] \\ &\quad +[A \rightarrowtail B \twoheadrightarrow B/A].\end{aligned}$$

(R8) For any commutative diagram consisting of four cofiber sequences in $\mathbf{C}$ as follows



we have

$$\begin{aligned}&[B \rightarrowtail C \twoheadrightarrow C/B] \\ +&[A \rightarrowtail B \twoheadrightarrow B/A] \quad = \quad [A \rightarrowtail C \twoheadrightarrow C/A] \\ &\qquad\qquad\qquad\quad +[B/A \rightarrowtail C/A \twoheadrightarrow C/B]^{[A]}.\end{aligned}$$

(R9) For any pair of objects A, B in $\mathbf{C}$

$$\langle [A], [B] \rangle = -[B \xrightarrow{i_2} A \vee B \xrightarrow{p_1} A] + [A \xrightarrow{i_1} A \vee B \xrightarrow{p_2} B].$$

Here

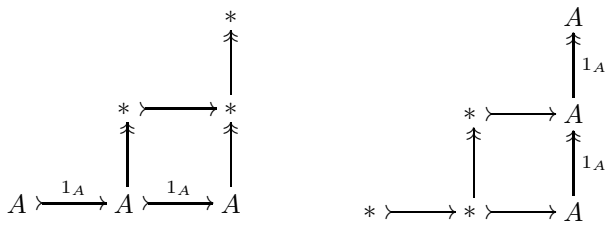
$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \vee B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

are the inclusions and projections of a coproduct in $\mathbf{C}$.

Remark 1.4. These relations are quite natural, and some illustration of their meaning is given in [MT06, Figure 2]. They are not however minimal: relation (R3) follows from (R2) and (R5), and (R4) follows from (R6). Also (R5) is equivalent to

(R5′) $[* \rightarrowtail * \twoheadrightarrow *] = 0$.

This equivalence follows from (R8) on considering the diagrams



We now introduce the basic object of study of this paper.

Definition 1.5. A *weak cofiber sequence* in a Waldhausen category $\mathbf{C}$ is a diagram

$$A \rightarrowtail B \twoheadrightarrow C_1 \xleftarrow{\sim} C$$

given by a cofiber sequence followed by a weak equivalence in the opposite direction.

We associate the element

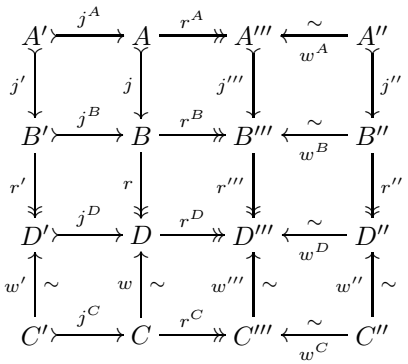
$$[A \rightarrowtail B \twoheadrightarrow C_1 \xleftarrow{\sim} C] \quad = \quad [A \rightarrowtail B \twoheadrightarrow C_1] + [C \xrightarrow{\sim} C_1]^{[A]} \in \mathcal{D}_*\mathbf{C} \tag{1.6}$$

to any weak cofiber sequence. By (R1) and (R2) we have

$$\partial[A \rightarrowtail B \twoheadrightarrow C_1 \xleftarrow{\sim} C] \quad = \quad -[B] + [C] + [A].$$

The following fundamental identity for weak cofiber sequences is a slightly less trivial consequence of the relations (R1)–(R9).

Proposition 1.7. *Consider a commutative diagram*



Denote the horizontal weak cofiber sequences by l^A, l^B, l^D and l^C, and the vertical ones by l', l, l''' and l''. Then the following equation holds in $\mathcal{D}_*\mathbf{C}$.

$$-[l^A] - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} + [l'] \quad = \quad \langle [C'], [A''] \rangle.$$

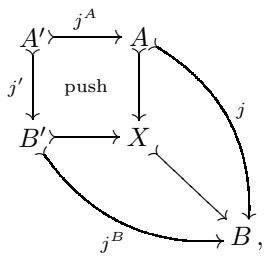
Proof. Applying (R7) to the two obvious equivalences of cofiber sequences gives

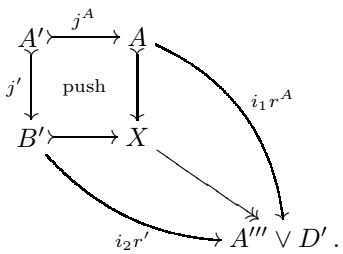
$$[D' \rightarrowtail D \twoheadrightarrow D'''] = [w] + [l^C] - [w^C]^{[C']} - [w''']^{[C']} - [w'], \tag{a}$$

$$[A''' \rightarrowtail B''' \twoheadrightarrow D'''] = [w^B] + [l''] - [w'']^{[A'']} - [w^D]^{[A'']} - [w^A], \tag{b}$$

using the notation of (1.6).

Now consider the following commutative diagrams





We now have four commutative diagrams of cofiber sequences as in (R8).

c)

$$\begin{array}{ccccc} & & D' & & \\ & & \twoheaduparrow & & \\ A''' & \rightarrowtail & A''' \vee D' & & \\ \twoheaduparrow & & \twoheaduparrow & & \\ A' & \rightarrowtail & A & \rightarrowtail & X \end{array}$$

d)

$$\begin{array}{ccccc} & & A''' & & \\ & & \twoheaduparrow & & \\ D' & \rightarrowtail & A''' \vee D' & & \\ \twoheaduparrow & & \twoheaduparrow & & \\ A' & \rightarrowtail & B' & \rightarrowtail & X \end{array}$$

e)

$$\begin{array}{ccccc} & & D''' & & \\ & & \twoheaduparrow & & \\ D' & \rightarrowtail & D & & \\ \twoheaduparrow & & \twoheaduparrow & & \\ A & \rightarrowtail & X & \rightarrowtail & B \end{array}$$

f)

$$\begin{array}{ccccc} & & D''' & & \\ & & \twoheaduparrow & & \\ A''' & \rightarrowtail & B''' & & \\ \twoheaduparrow & & \twoheaduparrow & & \\ B' & \rightarrowtail & X & \rightarrowtail & B \end{array}$$

Therefore we obtain the corresponding relations

$$\begin{aligned} \text{(c)} \qquad & [A \rightarrowtail X \twoheadrightarrow D'] + [l^A] - [w^A]^{[A']} \\ & \qquad = [A' \rightarrowtail X \twoheadrightarrow A''' \vee D'] + [A''' \rightarrowtail A''' \vee D' \twoheadrightarrow D']^{[A']}, \\ \text{(d)} \qquad & [B' \rightarrowtail X \twoheadrightarrow A'''] + [l'] - [w']^{[A']} \\ & \qquad = [A' \rightarrowtail X \twoheadrightarrow A''' \vee D'] + [D' \rightarrowtail A''' \vee D' \twoheadrightarrow A''']^{[A']}, \\ \text{(e)} \qquad & [X \rightarrowtail B \twoheadrightarrow D'''] + [A \rightarrowtail X \twoheadrightarrow D'] \\ & \qquad = [l] - [w]^{[A]} + [D' \rightarrowtail D \twoheadrightarrow D''']^{[A]} \\ & \qquad = [l] + ([l^C] - [w^C]^{[C']} - [w''']^{[C']} - [w'])^{[A]}, \\ \text{(f)} \qquad & [X \rightarrowtail B \twoheadrightarrow D'''] + [B' \rightarrowtail X \twoheadrightarrow A'''] \\ & \qquad = [l^B] - [w^B]^{[B']} + [A''' \rightarrowtail B''' \twoheadrightarrow D''']^{[B']} \\ & \qquad = [l^B] + ([l''] - [w'']^{[A'']} - [w^D]^{[A'']} - [w^A])^{[B']}. \end{aligned}$$

Here we have used equations (a) and (b). Now $-$(c) + (d) and $-$(e) + (f) yield

$$\begin{aligned} \text{(g)} \quad & [w^A]^{[A']} - [l^A] - [A \rightarrowtail X \twoheadrightarrow D'] + [B' \rightarrowtail X \twoheadrightarrow A'''] + [l'] - [w']^{[A']} = \\ & \quad - [A''' \rightarrowtail A''' \vee D' \twoheadrightarrow D']^{[A']} + [D' \rightarrowtail A''' \vee D' \twoheadrightarrow A''']^{[A']} = \langle [D'], [A'''] \rangle, \end{aligned}$$

$$\begin{aligned} \text{(h)} \quad & - [A \rightarrowtail X \twoheadrightarrow D'] + [B' \rightarrowtail X \twoheadrightarrow A'''] = \\ & [w']^{[A]} + a^{[C']+[A]} - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} - a^{[A'']+[B']} - [w^A]^{[B']}, \end{aligned}$$

where we use (R9) and Remark 1.2, and we write $a = [w'''] + [w^C] = [w^D] + [w'']$, by (R6). In fact it is easy to check by using Remark 1.2 and the laws of stable quadratic modules that the terms $a^{[C']+[A]}$ and $a^{[A'']+[B']}$ cancel in this expression, since

$$\partial \Big(- [l^C]^{[A]} - [l] \ + \ [l^B] + [l'']^{[B']} \Big) \ = \ -([C'] + [A]) + ([A''] + [B']).$$

Substituting (h) into the left hand side of (g) we obtain

$$\begin{aligned}[w^A]^{[A']} - [l^A] + [w']^{[A]} - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} - [w^A]^{[B']} + [l'] - [w']^{[A']} \\ = \langle [D'], [A'''] \rangle,\end{aligned}$$

which, using again Definition 1.1 and Remark 1.2, may be rewritten as

$$\begin{aligned}- [l^A] - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} + [l'] \\ = \ - [w']^{[A'']+[A']} - [w^A]^{[A']} - \langle [A'''], [D'] \rangle + [w']^{[A']} + [w^A]^{[C']+[A']}.\end{aligned}$$

The result then follows from the identity

$$\langle \partial[w^A], \partial[w'] \rangle - \langle [A''], \partial[w'] \rangle + \langle [C'], \partial[w^A] \rangle = \langle [C'], [A''] \rangle + \langle [A'''], [D'] \rangle.$$

□

2. Pairs of weak cofiber sequences and K_1

Let $\mathbf{C}$ be a Waldhausen category. A *pair of weak cofiber sequences* is a diagram given by two weak cofiber sequences with the first, second, and fourth objects in common

$$\begin{array}{ccccc} & & C_1 & & \\ & \nearrow & & \nwarrow^{\sim} & \\ A \rightrightarrows B & & & & C. \\ & \searrow & & \swarrow_{\sim} & \\ & & C_2 & & \end{array}$$

We associate to any such pair the following element

(2.1)

$$\left\{ \begin{array}{ccccc} & & C_1 & & \\ & \overset{r_1}{\nearrow} & & \overset{\sim}{\nwarrow} & \\ A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B & & \underset{w_2}{\overset{w_1}{}} & & C \\ & \underset{r_2}{\searrow} & & \underset{\sim}{\swarrow} & \\ & & C_2 & & \end{array} \right\} \begin{aligned} &= -[A \overset{j_1}{\rightarrowtail} B \overset{r_1}{\twoheadrightarrow} C_1 \underset{w_1}{\overset{\sim}{\leftarrow}} C] + [A \overset{j_2}{\rightarrowtail} B \overset{r_2}{\twoheadrightarrow} C_2 \underset{w_2}{\overset{\sim}{\leftarrow}} C] \\ &= [A \overset{j_2}{\rightarrowtail} B \overset{r_2}{\twoheadrightarrow} C_2 \underset{w_2}{\overset{\sim}{\leftarrow}} C] - [A \overset{j_1}{\rightarrowtail} B \overset{r_1}{\twoheadrightarrow} C_1 \underset{w_1}{\overset{\sim}{\leftarrow}} C], \end{aligned}$$

which lies in $K_1\mathbf{C} = \operatorname{Ker} \partial$. For the second equality in (2.1) we use that $\operatorname{Ker} \partial$ is central, see Remark 1.2, so we can permute the terms cyclically.

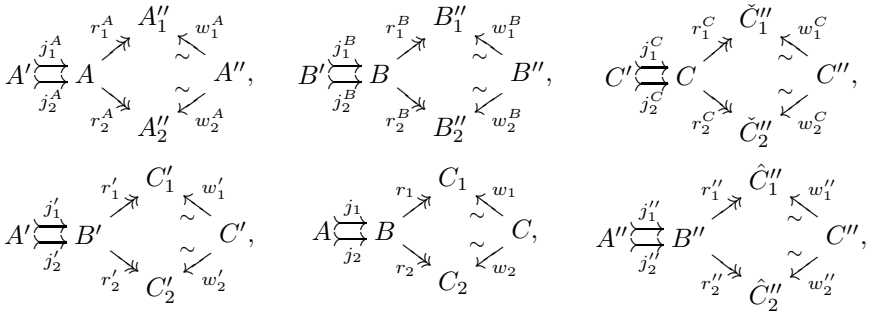
The following theorem is one of the main results of this paper.

Theorem 2.2. *Any element in $K_1\mathbf{C}$ is represented by a pair of weak cofiber sequences.*

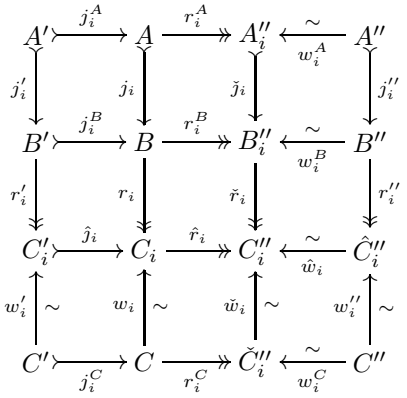
This theorem will be proved later in this paper. We first give a set of useful relations between pairs of weak cofiber sequences and develop a sum-normalized version of the model $\mathcal{D}_*\mathbf{C}$.

3. Relations between pairs of weak cofiber sequences

Suppose that we have six pairs of weak cofiber sequences in a Waldhausen category $\mathbf{C}$



that we denote for the sake of simplicity as λ^A, λ^B, λ^C, λ', λ, and λ'', respectively. Moreover, assume that for $i = 1, 2$ there exists a commutative diagram



Notice that six of the eight weak cofiber sequences in this diagram are already in the six diagrams above. The other two weak cofiber sequences are just assumed to exist with the property of making the diagram commutative.

Theorem 3.1. *In the situation above the following equation holds*

$$\{\lambda^A\} - \{\lambda^B\} + \{\lambda^C\} = \{\lambda'\} - \{\lambda\} + \{\lambda''\}\,. \tag{S1}$$

Proof. For $i = 1, 2$ let λ_i^A, λ_i^B, λ_i^C, λ_i', λ_i, and λ_i'' be the upper and the lower weak cofiber sequences of the pairs above, respectively. By Proposition 1.7

$$\begin{aligned} -[\lambda_1^A] - [\lambda_1^C]^{[A]} - [\lambda_1] + [\lambda_1^B] + [\lambda_1'']^{[B']} + [\lambda_1'] &= \langle [C'], [A''] \rangle \\ &= -[\lambda_2^A] - [\lambda_2^C]^{[A]} - [\lambda_2] + [\lambda_2^B] + [\lambda_2'']^{[B']} + [\lambda_2']. \end{aligned}$$

Since $\partial(\lambda_1^C) = \partial(\lambda_2^C)$ and $\partial(\lambda_1'') = \partial(\lambda_2'')$ the actions cancel,

$$-[\lambda_1^A] - [\lambda_1^C] - [\lambda_1] + [\lambda_1^B] + [\lambda_1''] + [\lambda_1'] = -[\lambda_2^A] - [\lambda_2^C] - [\lambda_2] + [\lambda_2^B] + [\lambda_2''] + [\lambda_2'].$$

Now one can rearrange the terms in this equation, by using that pairs of weak cofiber sequences are central in $\mathcal{D}_1\mathbf{C}$, obtaining the equation in the statement. □

Theorem 3.1 encodes the most relevant relation satisfied by pairs of weak cofiber sequences in $K_1\mathbf{C}$. They satisfy a further relation which follows from the very

definition in (2.1), but which we would like to record as a proposition by analogy with [Nen98a].

Proposition 3.2. *A pair of weak cofiber sequences given by two times the same weak cofiber sequence is trivial.*

(S2)
$$\left\{ A \underset{j}{\overset{j}{\rightrightarrows}} B \overset{r}{\twoheadrightarrow} C_1 \overset{w}{\xleftarrow{\sim}} C \overset{w}{\xleftarrow{\sim}} C_1 \overset{r}{\twoheadleftarrow} B \right\} = 0.$$

We establish a further useful relation in the following proposition.

Proposition 3.3. *Relations* (S1) *and* (S2) *imply that the sum of two pairs of weak cofiber sequences coincides with their coproduct.*

$$\left\{ A \vee \bar{A} \rightrightarrows B \vee \bar{B} \twoheadrightarrow C_1 \vee \bar{C}_1 \xleftarrow{\sim} C \vee \bar{C} \xleftarrow{\sim} C_2 \vee \bar{C}_2 \twoheadleftarrow B \vee \bar{B} \right\} = \left\{ A \rightrightarrows B \twoheadrightarrow C_1 \xleftarrow{\sim} C \xleftarrow{\sim} C_2 \twoheadleftarrow B \right\} + \left\{ \bar{A} \rightrightarrows \bar{B} \twoheadrightarrow \bar{C}_1 \xleftarrow{\sim} \bar{C} \xleftarrow{\sim} \bar{C}_2 \twoheadleftarrow \bar{B} \right\}.$$

Proof. Apply relations (S1) and (S2) to the following pairs of weak cofiber sequences

$$A \rightrightarrows B \twoheadrightarrow C_1 \xleftarrow{\sim} C \xleftarrow{\sim} C_2 \twoheadleftarrow B, \quad A \vee \bar{A} \rightrightarrows B \vee \bar{B} \twoheadrightarrow C_1 \vee \bar{C}_1 \xleftarrow{\sim} C \vee \bar{C} \xleftarrow{\sim} C_2 \vee \bar{C}_2 \twoheadleftarrow B \vee \bar{B}, \quad \bar{A} \rightrightarrows \bar{B} \twoheadrightarrow \bar{C}_1 \xleftarrow{\sim} \bar{C} \xleftarrow{\sim} \bar{C}_2 \twoheadleftarrow \bar{B},$$

$$A \underset{i_1}{\overset{i_1}{\rightrightarrows}} A \vee \bar{A} \overset{p_2}{\twoheadrightarrow} \bar{A} \overset{1}{\xleftarrow{\sim}} \bar{A}, \quad B \underset{i_1}{\overset{i_1}{\rightrightarrows}} B \vee \bar{B} \overset{p_2}{\twoheadrightarrow} \bar{B} \overset{1}{\xleftarrow{\sim}} \bar{B}, \quad C \underset{i_1}{\overset{i_1}{\rightrightarrows}} C \vee \bar{C} \overset{p_2}{\twoheadrightarrow} \bar{C} \overset{1}{\xleftarrow{\sim}} \bar{C}.$$

□

4. Waldhausen categories with functorial coproducts

Let $\mathbf{C}$ be a Waldhausen category endowed with a symmetric monoidal structure $+$ which is strictly associative

$$(A + B) + C \quad = \quad A + (B + C),$$

strictly unital

$$* + A \quad = \quad A \quad = \quad A + *,$$

but not necessarily strictly commutative, such that

$$A \;=\; A + * \longrightarrow A + B \longleftarrow * + B \;=\; B$$

is always a coproduct diagram. Such a category will be called a *Waldhausen category with a functorial coproduct*. Then we define the sum-normalized stable quadratic module $\mathcal{D}^+_*\mathbf{C}$ as the quotient of $\mathcal{D}_*\mathbf{C}$ by the extra relation

(R10) $[B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A] = 0.$

Remark 4.1. In $\mathcal{D}^+_*\mathbf{C}$, the relations (R2) and (R10) imply that

$$[A + B] = [A] + [B]\,.$$

Furthermore, relation (R9) becomes equivalent to

(R9′) $\langle [A], [B] \rangle = [\tau_{B,A} \colon B + A \xrightarrow{\cong} A + B].$

Here $\tau_{B,A}$ is the symmetry isomorphism of the symmetric monoidal structure. This equivalence follows from the commutative diagram

$$\begin{array}{ccccc} A & \overset{i_2}{\rightarrowtail} & B + A & \overset{p_1}{\twoheadrightarrow} & B \\ \| & & \downarrow{\scriptstyle \tau_{B,A}}\ {\scriptstyle \cong} & & \| \\ A & \underset{i_1}{\rightarrowtail} & A + B & \underset{p_2}{\twoheadrightarrow} & B \end{array}$$

together with (R4), (R7) and (R10).

Theorem 4.2. *Let* $\mathbf{C}$ *be a Waldhausen category with a functorial coproduct such that there exists a set* $\mathbb{S}$ *which freely generates the monoid of objects of* $\mathbf{C}$ *under the operation* $+$*. Then the projection*

$$p \colon \mathcal{D}_*\mathbf{C} \twoheadrightarrow \mathcal{D}^+_*\mathbf{C}$$

is a weak equivalence. Indeed, it is part of a strong deformation retraction.

Waldhausen categories satisfying the hypothesis of Theorem 4.2 are general enough as the following proposition shows.

Proposition 4.3. *For any Waldhausen category* $\mathbf{C}$ *there is another Waldhausen category* $\mathrm{Sum}(\mathbf{C})$ *with a functorial coproduct whose monoid of objects is freely generated by the objects of* $\mathbf{C}$ *except from* $*$*. Moreover, there are natural mutually inverse exact equivalences of categories*

$$\mathrm{Sum}(\mathbf{C}) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathbf{C}\ .$$

Proof. The statement already says which are the objects of $\mathrm{Sum}(\mathbf{C})$. The functor φ sends an object Y in $\mathrm{Sum}(\mathbf{C})$, which can be uniquely written as a formal sum of non-zero objects in $\mathbf{C}$, $Y = X_1 + \cdots + X_n$, to an arbitrarily chosen coproduct of these objects in $\mathbf{C}$

$$\varphi(Y) \quad = \quad X_1 \vee \cdots \vee X_n.$$

For $n = 0, 1$ we make special choices, namely for $n = 0$, $\varphi(Y) = *$, and for $n = 1$ we set $\varphi(Y) = X_1$. Morphisms in $\mathrm{Sum}(\mathbf{C})$ are defined in the unique possible way so that φ is fully faithful. Then the formal sum defines a functorial coproduct on $\mathrm{Sum}(\mathbf{C})$. The functor ψ sends $*$ to the zero object of the free monoid and any other object in $\mathbf{C}$ to the corresponding object with a single summand in $\mathrm{Sum}(\mathbf{C})$, so that $\varphi\psi = 1$. The inverse natural isomorphism $1 \cong \psi\varphi$,

$$X_1 + \cdots + X_n \quad \cong \quad (X_1 \vee \cdots \vee X_n),$$

is the unique isomorphism in $\mathrm{Sum}(\mathbf{C})$ which φ maps to the identity. Cofibrations and weak equivalences in $\mathrm{Sum}(\mathbf{C})$ are the morphisms which φ maps to cofibrations and weak equivalences, respectively. This Waldhausen category structure in $\mathrm{Sum}(\mathbf{C})$ makes the functors φ and ψ exact. □

Let us recall the notion of homotopy in the category of stable quadratic modules.

Definition 4.4. Given morphisms $f, g : C_* \to C'_*$ of stable quadratic modules, a *homotopy* $f \overset{\alpha}{\simeq} g$ from f to g is a function $\alpha\colon C_0 \to C'_1$ satisfying

(1) $\alpha(c_0 + d_0) = \alpha(c_0)^{f_0(d_0)} + \alpha(d_0)$,
(2) $g_0(c_0) = f_0(c_0) + \partial\alpha(c_0)$,
(3) $g_1(c_1) = f_1(c_1) + \alpha\partial(c_1)$.

The following lemma then follows from the laws of stable quadratic modules.

Lemma 4.5. *A homotopy α as in Definition 4.4 satisfies*

(a) $$\alpha([c_0, d_0]) = -\langle f_0(d_0), f_0(c_0)\rangle + \langle g_0(d_0), g_0(c_0)\rangle,$$

(b) $$\alpha(c_0) + g_1(c_1) = f_1(c_1) + \alpha(c_0 + \partial(c_1)).$$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. In order to define a strong deformation retraction,

$$\alpha \circlearrowright \mathcal{D}_*\mathbf{C} \underset{j}{\overset{p}{\rightleftarrows}} \mathcal{D}^+_*\mathbf{C} \ , \qquad 1 \overset{\alpha}{\simeq} jp, \quad pj = 1,$$

the crucial step will be to define the homotopy $\alpha\colon \mathcal{D}_0\mathbf{C} \to \mathcal{D}_1\mathbf{C}$ on the generators $[A]$ of $\mathcal{D}_0\mathbf{C}$. Then one can use the equations

(1) $\alpha(c_0 + d_0) = \alpha(c_0)^{d_0} + \alpha(d_0)$,
(2) $jp(c_0) = c_0 + \partial\alpha(c_0)$,
(3) $jp(c_1) = c_1 + \alpha\partial(c_1)$,

for $c_i, d_i \in \mathcal{D}_i\mathbf{C}$ to define the composite jp and the homotopy α on all of $\mathcal{D}_*\mathbf{C}$. It is a straightforward calculation to check that the map jp so defined is a homomorphism and that α is well defined. In particular, Lemma 4.5 (a) implies that α vanishes on commutators of length 3. In order to define j we must show that jp factors through the projection p, that is,

$$jp([B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A]) \ = \ 0.$$

If we also show that

$$p\alpha \ = \ 0$$

then equations (2) and (3) above say $pjp(c_i) = p(c_i) + 0$, and since p is surjective it follows that the composite pj is the identity.

The set of objects of $\mathbf{C}$ is the free monoid generated by objects $S \in \mathbb{S}$. Therefore we can define inductively the homotopy α by

$$\alpha([S + B]) = [B \overset{i_2}{\rightarrowtail} S + B \overset{p_1}{\twoheadrightarrow} S] + \alpha([B])$$

for $B \in \mathbf{C}$ and $S \in \mathbb{S}$, with $\alpha([S]) = 0$. We claim that a similar relation then holds for all objects A, B of $\mathbf{C}$,

$$\begin{aligned} \alpha([A + B]) &= [B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A] + \alpha([A] + [B]) \\ &= [B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A] + \alpha([A])^{[B]} + \alpha([B]). \end{aligned} \tag{4.6}$$

If $A = *$ or $A \in \mathbb{S}$ then (4.6) holds by definition, so we assume inductively it holds for given A, B and show it holds for $S + A$, B also:

$$\begin{aligned}\alpha([S+A+B]) &= [A+B \rightarrowtail S+A+B \twoheadrightarrow S] + \alpha([A+B]) \\ &= [A+B \rightarrowtail S+A+B \twoheadrightarrow S] + [B \rightarrowtail A+B \twoheadrightarrow A] \ + \ \alpha([A])^{[B]} + \alpha([B]) \\ &= [B \rightarrowtail S+A+B \twoheadrightarrow S+A] + [A \rightarrowtail S+A \twoheadrightarrow S]^{[B]} + \alpha([A])^{[B]} + \alpha([B]) \\ &= [B \rightarrowtail S+A+B \twoheadrightarrow S+A] + \alpha([S+A])^{[B]} + \alpha([B])\,.\end{aligned}$$

Here we have used (R8) for the composable cofibrations

$$B \ \rightarrowtail \ A+B \ \rightarrowtail \ S+A+B\,.$$

It is clear from the definition of α that the relation $p\alpha = 0$ holds. It remains to see that jp factors through $\mathcal{D}^+_*\mathbf{C}$, that is,

$$jp(c_1) = 0 \quad \text{if} \quad c_1 = [B \rightarrowtail A+B \twoheadrightarrow A].$$

Let $c_0 = [A+B]$ so that $c_0 + \partial(c_1) = [A] + [B]$. Then by Lemma 4.5 (b) we have

$$\begin{aligned}jp(c_1) &= -\alpha(c_0) + c_1 + \alpha(c_0 + \partial(c_1)) \\ &= \ -\alpha([A+B]) \ + \ [B \rightarrowtail A+B \twoheadrightarrow A] \ + \ \alpha([A]+[B])\,.\end{aligned}$$

This is zero by (4.6). □

We finish this section with four lemmas which show useful relations in $\mathcal{D}^+_*\mathbf{C}$.

Lemma 4.7. *The following equality holds in* $\mathcal{D}^+_*\mathbf{C}$.

$$\begin{aligned}&[A+A' \ \rightarrowtail \ B+B' \ \twoheadrightarrow \ B/A+B'/A' \ \xleftarrow{\sim} \ C+C'] \\ &\quad = \ [A \rightarrowtail B \twoheadrightarrow B/A \xleftarrow{\sim} C]^{[B']} + [A' \rightarrowtail B' \twoheadrightarrow B'/A' \xleftarrow{\sim} C'] + \langle [A], [C'] \rangle.\end{aligned}$$

Proof. Use (R4), (R10) and Proposition 1.7 applied to the commutative diagram

$$\begin{array}{ccccccc}
A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' & \xleftarrow{\sim} & C' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A+A' & \rightarrowtail & B+B' & \twoheadrightarrow & B/A+B'/A' & \xleftarrow{\sim} & C+C' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \rightarrowtail & B & \twoheadrightarrow & B/A & \xleftarrow{\sim} & C \\
\| & & \| & & \| & & \| \\
A & \rightarrowtail & B & \twoheadrightarrow & B/A & \xleftarrow{\sim} & C
\end{array}$$

□

As special cases we have

Lemma 4.8. *The following equality holds in* $\mathcal{D}^+_*\mathbf{C}$.

$$\begin{aligned}&[A+A' \rightarrowtail B+B' \twoheadrightarrow B/A+B'/A'] \\ &\qquad = \ [A \rightarrowtail B \twoheadrightarrow B/A]^{[B']} + [A' \rightarrowtail B' \twoheadrightarrow B'/A'] + \langle [A], [B'/A'] \rangle.\end{aligned}$$

Lemma 4.9. *The following equality holds in* $\mathcal{D}^+_*\mathbf{C}$.

$$[A+B \xrightarrow{\sim} A'+B'] \ = \ [A \xrightarrow{\sim} A']^{[B']} + [B \xrightarrow{\sim} B'].$$

We generalize (R9′) in the following lemma.

Lemma 4.10. *Let* $\mathbf{C}$ *be a Waldhausen category with a functorial coproduct and let* $A_1, \dots, A_n$ *be objects in* $\mathbf{C}$*. Given a permutation* $\sigma \in \mathrm{Sym}(n)$ *of* n *elements we denote by*

$$\sigma_{A_1,\dots,A_n} \colon A_{\sigma_1} + \cdots + A_{\sigma_n} \xrightarrow{\cong} A_1 + \cdots + A_n$$

the isomorphism permuting the factors of the coproduct. Then the following formula holds in $\mathcal{D}^+_*\mathbf{C}$*.*

$$[\sigma_{A_1,\dots,A_n}] \quad = \quad \sum_{\substack{i>j \\ \sigma_i<\sigma_j}} \langle [A_{\sigma_i}], [A_{\sigma_j}] \rangle.$$

Proof. The result holds for $n = 1$ by (R4). Suppose $\sigma \in \mathrm{Sym}(n)$ for $n \geq 2$, and note that the isomorphism $\sigma_{A_1,\dots,A_n}$ factors naturally as

$$\begin{aligned}(\sigma'_{A_1,\dots,A_{n-1}} + 1)(1 + \tau_{A_n,B}) \colon A_{\sigma_1} + \cdots + A_{\sigma_n} &\longrightarrow A_{\sigma'_1} + \cdots + A_{\sigma'_{n-1}} + A_n \\ &\longrightarrow A_1 + \cdots + A_{n-1} + A_n\end{aligned}$$

where $(\sigma'_1, \dots, \sigma'_{n-1}, n) = (\sigma_1, \dots, \widehat{\sigma_k}, \dots, \sigma_n, \sigma_k)$ and $B = A_{\sigma_{k+1}} + \cdots + A_{\sigma_n}$. Therefore by (R4), (R6) and Lemma 4.9,

$$\begin{aligned}[\sigma_{A_1,\dots,A_n}] &= [\sigma'_{A_1,\dots,A_{n-1}} + 1] + [1 + \tau_{A_n,B}] \\ &= ([\sigma'_{A_1,\dots,A_{n-1}}]^{[A_n]} + 0) + (0 + [\tau_{A_n,B}]).\end{aligned}$$

By induction, (R9′) and Remark 4.1 this is equal to

$$\sum_{\substack{p>q \\ \sigma'_p<\sigma'_q}} \langle [A_{\sigma'_p}], [A_{\sigma'_q}] \rangle + \langle [B], [A_n] \rangle = \sum_{\substack{i>j \\ \sigma_i<\sigma_j<n}} \langle [A_{\sigma_i}], [A_{\sigma_j}] \rangle + \sum_{\substack{i>j \\ \sigma_i<\sigma_j=n}} \langle [A_{\sigma_i}], [A_{\sigma_j}] \rangle$$

as required. □

5. Proof of Theorem 2.2

The morphism of stable quadratic modules $\mathcal{D}_*F \colon \mathcal{D}_*\mathbf{C} \to \mathcal{D}_*\mathbf{D}$ induced by an exact functor $F \colon \mathbf{C} \to \mathbf{D}$, see [MT06], takes pairs of weak cofiber sequences to pairs of weak cofiber sequences,

$$(\mathcal{D}_*F)\left\{ \begin{array}{ccccc} & & C_1 & & \\ A & \rightrightarrows & B & & C \\ & & C_2 & & \end{array} \right\} = \left\{ \begin{array}{ccccc} & & F(C_1) & & \\ F(A) & \rightrightarrows & F(B) & & F(C) \\ & & F(C_2) & & \end{array} \right\}.$$

By [MT06, Theorem 3.2] exact equivalences of Waldhausen categories induce homotopy equivalences of stable quadratic modules, and hence isomorphisms in K_1. Therefore if the theorem holds for a certain Waldhausen category then it holds for all equivalent ones. In particular by Proposition 4.3 it is enough to prove the theorem for any Waldhausen category $\mathbf{C}$ with a functorial coproduct such that the monoid of objects is freely generated by a set $\mathbb{S}$. By Theorem 4.2 we can work in the sum-normalized construction $\mathcal{D}^+_*\mathbf{C}$ in this case.

Any element $x \in \mathcal{D}^+_1\mathbf{C}$ is a sum of weak equivalences and cofiber sequences with coefficients $+1$ or -1. By Lemma 4.9 modulo the image of $\langle \cdot, \cdot \rangle$ we can collect on the one hand all weak equivalences with coefficient $+1$ and on the other all weak

equivalences with coefficient -1. Moreover, by Lemma 4.8 we can do the same for cofiber sequences. Therefore the following equation holds modulo the image of $\langle\cdot,\cdot\rangle$.

$$\begin{aligned}
x &= -[V_1 \xrightarrow{\sim} V_2] - [X_1 \rightarrowtail X_2 \twoheadrightarrow X_3] \\
&\quad +[Y_1 \rightarrowtail Y_2 \twoheadrightarrow Y_3] + [W_1 \xrightarrow{\sim} W_2] \\
\text{(R4), (R10), 4.9, 4.8} &= -[V_1 + W_1 \xrightarrow{\sim} V_2 + W_1] \\
&\quad -[X_1 + Y_1 \rightarrowtail X_2 + Y_3 + Y_1 \twoheadrightarrow X_3 + Y_3] \\
&\quad +[X_1 + Y_1 \rightarrowtail X_3 + X_1 + Y_2 \twoheadrightarrow X_3 + Y_3] \\
&\quad +[V_1 + W_1 \xrightarrow{\sim} V_1 + W_2] \\
\text{renaming} &= -[L \xrightarrow{\sim} L_1] - [A \rightarrowtail E_1 \twoheadrightarrow D] \\
&\quad +[A \rightarrowtail E_2 \twoheadrightarrow D] + [L \xrightarrow{\sim} L_2] \mod \langle\cdot,\cdot\rangle.
\end{aligned}$$

Suppose that $\partial(x) = 0$ modulo commutators. Then modulo $[\cdot,\cdot]$

$$\begin{aligned}
0 &= -[L] + [L_1] - [A] - [D] + [E_1] \\
&\quad -[E_2] + [D] + [A] - [L_2] + [L] \\
&= [L_1] + [E_1] - [E_2] - [L_2] \mod [\cdot,\cdot], \\
\text{(5.1)} \qquad \text{i.e. } [L_1 + E_1] &= [L_2 + E_2] \mod [\cdot,\cdot].
\end{aligned}$$

The quotient of $\mathcal{D}_0^+\mathbf{C}$ by the commutator subgroup is the free abelian group on $\mathbb{S}$, hence by (5.1) there are objects $S_1, \ldots, S_n \in \mathbb{S}$ and a permutation $\sigma \in \mathrm{Sym}(n)$ such that

$$\begin{aligned}
L_1 + E_1 &= S_1 + \cdots + S_n, \\
L_2 + E_2 &= S_{\sigma_1} + \cdots + S_{\sigma_n},
\end{aligned}$$

so there is an isomorphism

$$\sigma_{S_1,\ldots,S_n} \colon L_2 + E_2 \xrightarrow{\sim} L_1 + E_1.$$

Again by Lemmas 4.9 and 4.8 modulo the image of $\langle\cdot,\cdot\rangle$

$$\begin{aligned}
x &= -[L + D \xrightarrow{\sim} L_1 + D] - [A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_1 + D] \\
&\quad +[A \rightarrowtail L_2 + E_2 \twoheadrightarrow L_2 + D] + [L + D \xrightarrow{\sim} L_2 + D] \\
4.10 &= -[L + D \xrightarrow{\sim} L_1 + D] - [A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_1 + D] \\
&\quad +[\sigma_{S_1,\ldots,S_n} \colon L_2 + E_2 \xrightarrow{\sim} L_1 + E_1] \\
&\quad +[A \rightarrowtail L_2 + E_2 \twoheadrightarrow L_2 + D] + [L + D \xrightarrow{\sim} L_2 + D] \\
\text{(R7)} &= -[L + D \xrightarrow{\sim} L_1 + D] - [A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_1 + D] \\
&\quad +[A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_2 + D] + [L + D \xrightarrow{\sim} L_2 + D] \\
\text{renaming} &= -[C \xrightarrow{\sim} C_1] - [A \rightarrowtail B \twoheadrightarrow C_1] \\
&\quad +[A \rightarrowtail B \twoheadrightarrow C_2] + [C \xrightarrow{\sim} C_2] \\
&= -[C \xrightarrow{\sim} C_1]^{[A]} - [A \rightarrowtail B \twoheadrightarrow C_1] \\
&\quad +[A \rightarrowtail B \twoheadrightarrow C_2] + [C \xrightarrow{\sim} C_2]^{[A]} \\
&= -[A \rightarrowtail B \twoheadrightarrow C_1 \xleftarrow{\sim} C] + [A \rightarrowtail B \twoheadrightarrow C_2 \xleftarrow{\sim} C] \mod \langle\cdot,\cdot\rangle,
\end{aligned}$$

i.e. there is $y \in \mathcal{D}_1^+\mathbf{C}$ in the image of $\langle\cdot,\cdot\rangle$ such that

$$x \quad = \quad \left\{ \quad A \rightrightarrows B \begin{array}{c} \nearrow C_1 \nwarrow \sim \\ \searrow C_2 \swarrow \sim \end{array} C \quad \right\} + y.$$

Now assume that $\partial(x) = 0$. Since the pair of weak cofiber sequences is also in the kernel of ∂ we have $\partial(y) = 0$. In order to give the next step we need a technical lemma.

Lemma 5.2. *Let C_* be a stable quadratic module such that C_0 is a free group of nilpotency class 2. Then any element $y \in \operatorname{Ker}\partial \cap \operatorname{Image}\langle\cdot,\cdot\rangle$ is of the form $y = \langle a, a\rangle$ for some $a \in C_0$.*

Proof. For any abelian group A let $\hat{\otimes}^2 A$ be the quotient of the tensor square $A \otimes A$ by the relations $a \otimes b + b \otimes a = 0$, $a, b \in A$, and let $\wedge^2 A$ be the quotient of $A \otimes A$ by the relations $a \otimes a = 0$, $a \in A$, which is also a quotient of $\hat{\otimes}^2 A$. The projection of $a \otimes b \in A \otimes A$ to $\hat{\otimes}^2 A$ and $\wedge^2 A$ is denoted by $a \hat{\otimes} b$ and $a \wedge b$, respectively.

There is a commutative diagram of group homomorphisms

$$\begin{array}{ccccc} & & C_0^{ab} \otimes C_0^{ab} & & \\ & \swarrow & \downarrow & \searrow & \\ C_0^{ab} \otimes \mathbb{Z}/2 & \xrightarrow{\bar{\tau}} & \hat{\otimes}^2 C_0^{ab} & \xrightarrow{q} & \wedge^2 C_0^{ab} \\ & \searrow^{\langle\cdot,\cdot\rangle} & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_0} \\ & & C_1 & \xrightarrow{\partial} & C_0 \end{array}$$

where $\bar{\tau}(a \otimes 1) = a \hat{\otimes} a$, the factorization c_1 of $\langle\cdot,\cdot\rangle$ is given by $c_1(a \hat{\otimes} b) = \langle a, b\rangle$, which is well defined by Definition 1.1 (3), and $c_0(a \wedge b) = [b, a]$ is well known to be injective in the case C_0 is free of nilpotency class 2.

Moreover, C_0^{ab} is a free abelian group, hence the middle row is a short exact sequence, see [Bau91, I.4]. Therefore any element $y \in C_1$ which is both in the image of c_1 and the kernel of ∂ is in the image of $c_1\bar{\tau}$ as required. □

The group $\mathcal{D}_0^+\mathbf{C}$ is free of nilpotency class 2, therefore by the previous lemma $y = \langle a, a\rangle$ for some $a \in \mathcal{D}_0^+\mathbf{C}$. By the laws of a stable quadratic module the element $\langle a, a\rangle$ only depends on a mod 2, compare [MT06, Definition 1.6], and so we can suppose that a is a sum of basis elements with coefficient $+1$. Hence by Remark 4.1 we can take $a = [M]$ for some object M in $\mathbf{C}$,

$$y \quad = \quad \langle [M], [M] \rangle.$$

The element $\langle [M], [M]\rangle$ is itself a pair of weak cofiber sequences

$$\langle [M], [M]\rangle \quad = \quad \left\{ \quad M \underset{i_1}{\overset{i_2}{\rightrightarrows}} M + M \begin{array}{c} \nearrow^{p_1} M \nwarrow^{\sim} \\ \searrow_{p_2} M \swarrow_{\sim} \end{array} M \quad \right\},$$

therefore x is also a pair of weak cofiber sequences, by Proposition 3.3, and the proof of Theorem 2.2 is complete.

6. Comparison with Nenashev's approach for exact categories

For any Waldhausen category $\mathbf{C}$ we denote by $K_1^{\mathrm{wcs}}\mathbf{C}$ the abelian group generated by pairs of weak cofiber sequences

$$\left\{ \quad A \rightarrowtail B \twoheadrightarrow \begin{array}{ccc} & C_1 & \\ \nearrow & & \nwarrow^{\sim} \\ C & & \\ \searrow & & \swarrow_{\sim} \\ & C_2 & \end{array} \quad \right\}$$

modulo the relations (S1) and (S2) in Section 3. Theorem 3.1 and Proposition 3.2 show the existence of a natural homomorphism

$$K_1^{\mathrm{wcs}}\mathbf{C} \twoheadrightarrow K_1\mathbf{C} \tag{6.1}$$

which is surjective by Theorem 2.2.

Given an exact category $\mathbf{E}$ Nenashev defines in [Nen98a] an abelian group $D(\mathbf{E})$ by generators and relations which surjects naturally to $K_1\mathbf{E}$. Moreover, he shows in [Nen98b] that this natural surjection is indeed a natural isomorphism

$$D(\mathbf{E}) \quad \cong \quad K_1\mathbf{E}. \tag{6.2}$$

Generators of $D(\mathbf{E})$ are pairs of short exact sequences

$$\{\ A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B \underset{r_2}{\overset{r_1}{\rightrightarrows}} C\ \}.$$

They satisfy two kind of relations. The first, analogous to (S2), says that a generator vanishes provided $j_1 = j_2$ and $r_1 = r_2$. The second is a simplification of (S1): given six pairs of short exact sequences

$$A' \underset{j_2^A}{\overset{j_1^A}{\rightrightarrows}} A \underset{r_2^A}{\overset{r_1^A}{\rightrightarrows}} A'', \qquad B' \underset{j_2^B}{\overset{j_1^B}{\rightrightarrows}} B \underset{r_2^B}{\overset{r_1^B}{\rightrightarrows}} B'', \qquad C' \underset{j_2^C}{\overset{j_1^C}{\rightrightarrows}} C \underset{r_2^C}{\overset{r_1^C}{\rightrightarrows}} C'',$$

$$A' \underset{j_2'}{\overset{j_1'}{\rightrightarrows}} B' \underset{r_2'}{\overset{r_1'}{\rightrightarrows}} C', \qquad A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B \underset{r_2}{\overset{r_1}{\rightrightarrows}} C, \qquad A'' \underset{j_2''}{\overset{j_1''}{\rightrightarrows}} B'' \underset{r_2''}{\overset{r_1''}{\rightrightarrows}} C'',$$

denoted for simplicity as λ^A, λ^B, λ^C, λ', λ, and λ'', such that the diagram

$$\begin{array}{ccccc} A' & \overset{j_i^A}{\rightarrowtail} & A & \overset{r_i^A}{\twoheadrightarrow} & A'' \\ \downarrow{\scriptstyle j_i'} & & \downarrow{\scriptstyle j_i} & & \downarrow{\scriptstyle j_i''} \\ B' & \overset{j_i^B}{\rightarrowtail} & B & \overset{r_i^B}{\twoheadrightarrow} & B'i \\ \downarrow{\scriptstyle r_i'} & & \downarrow{\scriptstyle r_i} & & \downarrow{\scriptstyle r_i''} \\ C' & \underset{j_i^C}{\rightarrowtail} & C & \underset{r_i^C}{\twoheadrightarrow} & C'' \end{array}$$

commutes for $i = 1, 2$ then

$$\{\lambda^A\} - \{\lambda^B\} + \{\lambda^C\} \quad = \quad \{\lambda'\} - \{\lambda\} + \{\lambda''\}.$$

Proposition 6.3. *For any exact category* $\mathbf{E}$ *there is a natural isomorphism*

$$D(\mathbf{E}) \quad \cong \quad K_1^{\mathrm{wcs}}\mathbf{E}.$$

Proof. There is a clear homomorphism $D(\mathbf{E}) \to K_1^{\mathrm{wcs}}\,\mathbf{E}$ defined by

$$\left\{ A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B \underset{r_2}{\overset{r_1}{\twoheadrightarrow}} C \right\} \quad \mapsto \quad \left\{ \begin{tikzcd} & C & \\ A \arrow[r, shift left, "j_1"] \arrow[r, shift right, "j_2"'] & B \arrow[u, two heads, "r_1"] \arrow[d, two heads, "r_2"'] & C \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & C & \end{tikzcd} \right\}$$

Exact categories regarded as Waldhausen categories have the particular feature that all weak equivalences are isomorphisms, hence one can also define a homomorphism $K_1^{\mathrm{wcs}}\,\mathbf{E} \to D(\mathbf{E})$ as

$$\left\{ \begin{tikzcd} & C_1 & \\ A \arrow[r, shift left, "j_1"] \arrow[r, shift right, "j_2"'] & B \arrow[u, two heads, "r_1"] \arrow[d, two heads, "r_2"'] & C \arrow[ul, "\cong"'] \arrow[dl, "\cong"] \\ & C_2 & \end{tikzcd} \right\} \quad \mapsto \quad \left\{ A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B \underset{w_2^{-1}r_2}{\overset{w_1^{-1}r_1}{\rightrightarrows}} C \right\}.$$

We leave the reader to check the compatibiliy with the relations and the fact that these two homomorphisms are inverse of each other. □

Remark 6.4. The composite of the isomorphism in Proposition 6.3 with the natural epimorphism (6.1) for $\mathbf{C} = \mathbf{E}$ coincides with Nenashev's isomorphism (6.2), therefore the natural epimorphism (6.1) is an isomorphism when $\mathbf{C} = \mathbf{E}$ is an exact category.

7. Weak cofiber sequences and the stable Hopf map

After the previous section one could conjecture that the natural epimorphism

$$K_1^{\mathrm{wcs}}\,\mathbf{C} \twoheadrightarrow K_1\mathbf{C}$$

in (6.1) is an isomorphism not only for exact categories but for any Waldhausen category $\mathbf{C}$. In order to support this conjecture we show the following result.

Theorem 7.1. *For any Waldhausen category* $\mathbf{C}$ *there is a natural homomorphism*

$$\phi\colon K_0\mathbf{C} \otimes \mathbb{Z}/2 \longrightarrow K_1^{\mathrm{wcs}}\,\mathbf{C}$$

which composed with (6.1) yields the homomorphism $\cdot\,\eta\colon K_0\mathbf{C} \otimes \mathbb{Z}/2 \to K_1\mathbf{C}$ *determined by the action of the stable Hopf map* $\eta \in \pi_*^s$ *in the stable homotopy groups of spheres.*

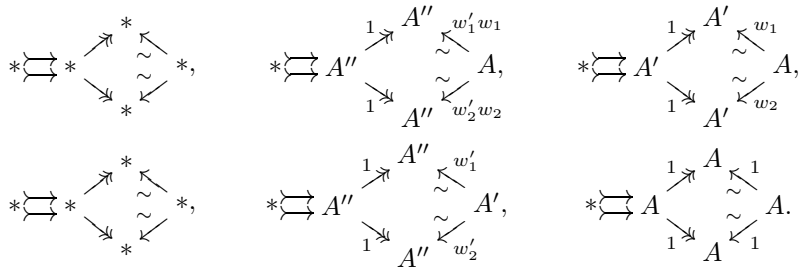
For the proof we use the following lemma.

Lemma 7.2. *The following relation holds in* $K_1^{\mathrm{wcs}}\,\mathbf{C}$

$$\left\{ \begin{tikzcd} & A'' & \\ * \arrow[r, shift left] \arrow[r, shift right] & A'' \arrow[u, two heads, "1"] \arrow[d, two heads, "1"'] & A \arrow[ul, "w_1'w_1"'] \arrow[dl, "w_2'w_2"] \\ & A'' & \end{tikzcd} \right\} = \left\{ \begin{tikzcd} & A' & \\ * \arrow[r, shift left] \arrow[r, shift right] & A' \arrow[u, two heads, "1"] \arrow[d, two heads, "1"'] & A \arrow[ul, "w_1"'] \arrow[dl, "w_2"] \\ & A' & \end{tikzcd} \right\}$$

$$+ \left\{ \begin{tikzcd} & A'' & \\ * \arrow[r, shift left] \arrow[r, shift right] & A'' \arrow[u, two heads, "1"] \arrow[d, two heads, "1"'] & A' \arrow[ul, "w_1'"'] \arrow[dl, "w_2'"] \\ & A'' & \end{tikzcd} \right\}.$$

Proof. Use relations (S1) and (S2) applied to the following pairs of weak cofiber sequences



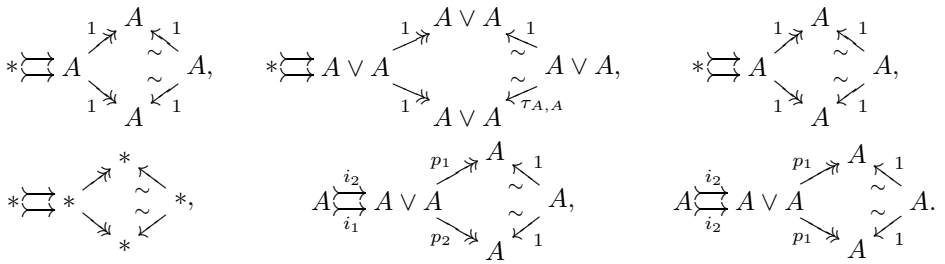
□

Now we are ready to prove Theorem 7.1.

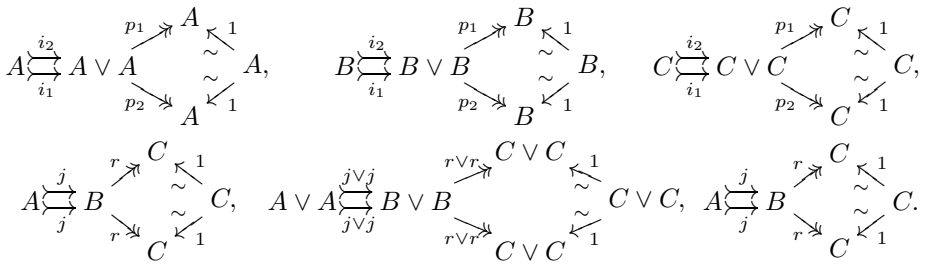
Proof of Theorem 7.1. We define the homomorphism by

$$\phi[A] = \left\{ A \underset{i_1}{\overset{i_2}{\rightrightarrows}} A \vee A \overset{p_1}{\underset{p_2}{\twoheadrightarrow}} A \xleftarrow{\sim} A \right\} = \left\{ * \rightrightarrows A \vee A \overset{1}{\underset{1}{\twoheadrightarrow}} A \vee A \xleftarrow{\sim} A \vee A \right\}.$$

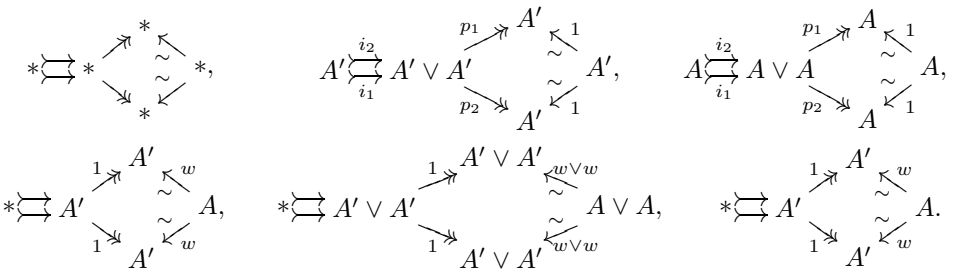
Here the second equality follows from (S1) and (S2) applied to the following pairs of weak cofiber sequences



Given a cofiber sequence $A \overset{j}{\rightarrowtail} B \overset{r}{\twoheadrightarrow} C$ the equation $\phi[C] + \phi[A] = \phi[B]$ follows from (S1) and (S2) applied to the following pairs of weak cofiber sequences



Given a weak equivalence $w\colon A \to A'$ the equation $\phi[A] = \phi[A']$ follows from (S1) and (S2) applied to the following pairs of weak cofiber sequences



The equation $2\phi[A] = 0$ follows from (S2) and Lemma 7.2 by using the second formula for $\phi[A]$ above and the fact that $\tau^2_{A,A} = 1$.

We have already shown that ϕ is a well-defined homomorphism. The composite of ϕ and (6.1) coincides with the action of the stable Hopf map by the main result of [MT06]. □

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