Stability for the Axially Symmetric Pendent Drop

by

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I. Introduction

In this article we will discuss the axially symmetric pendent drop as it occurs in three different physical settings: Problem A (the siphon), Problem B (the medicine dropper), Problem C (drop pendent from a horizontal plate). Our goal is to exhibit a scheme for identifying those drops determining stable configurations and to show that they satisfy a strong minimizing property for the energy. First we describe the problems in more detail.

Problem A: The drop is suspended from a fixed circular opening of radius $\bar{r}$ located at the level $u = \bar{u}$ where u is the vertical coordinate with positive direction upward and $u = 0$ is the zero pressure level of the fluid. [See Figure 1] If X is the exposed body of the fluid and Ω is the liquid-air interface with $A(\Omega)$ its area, the potential energy of the configuration is

$$1.1) \qquad E_0(\Omega) = \sigma A(\Omega) + \rho g \int_X z dV .$$

σ is the surface tension of the liquid-air interface, ρ is the density of the fluid, and g is the gravitational constant. The condition for equilibrium is that the first variation of the potential energy $\partial E_0(\Omega, N) = 0$ for all normal perturbations N of Ω which vanish on the boundary. The Euler equations yield

$$1.2) \qquad 2H = -ku \quad \text{on} \quad \Omega \ , \ k = \rho g/\sigma .$$

H is the mean curvature of the surface measured so that it is positive at the drop tip. By a suitable scaling we may allow $k = 1$. The condition for stability is that the second variation is positive for all non-trivial normal perturbations.

1.3) $\partial^2 E_0(\Omega,N) > 0$ for all $N \not\equiv 0$, $N = 0$ on $\partial\Omega$.

Problem B: As in Problem A the fixed circular opening of radius $\bar{r}$ lies in a horizontal plane but the exposed volume is prescribed. [See Figure 1] The condition for equilibrium is that the first variation of the energy $\partial E_0(\Omega,N) = 0$ for all normal perturbations N of Ω vanishing on the boundary, and for which the first variation of the volume is zero. By the method of Lagrange multipliers we find

1.4) $\partial(E_0+\lambda V)(\Omega,N) = 0$ for some λ

and all normal perturbations N vanishing on $\delta\Omega$. This yields the condition

1.5) $2H = -ku + \lambda$, $k = \rho g/\sigma$.

By a vertical translation of coordinates we may take $\lambda = 0$ thus reducing (1.5) to the condition (1.2) while the vertical coordinate of the opening is at level $u = \bar{u}$. The condition for stability is that

1.6) $\partial^2(E_0 + \lambda V)(\Omega,N) > 0$

for all non-trivial normal perturbations N vanishing on the boundary and for which the first variation of the volume is zero.

Problem C: The drop is now pendent from a homogeneous horizontal plate. [See Figure 1] The potential energy is now

1.7) $E(\Omega) = E_0(\Omega) - \sigma\beta|\Sigma|$

where β is a physical constant and $|\Sigma|$ is the wetted area of the plate. Setting the first variation equal to zero for all volume preserving perturbations gives

1.8)
a) $2H = -ku + \lambda$ for some λ, $k = \rho g/\sigma$
b) $\beta = \cos\alpha$.

Here α is the angle of contact of the liquid-air interface with the horizontal plate measured interior to the fluid. Again we may choose $k = 1$ and by a vertical translation of coordinates may set $\lambda = 0$, with the horizontal plate at level $u = \bar{u}$. Clearly it is necessary for $|\beta| \le 1$ so that $0 \le \alpha \le \pi$. There are no pendent drops with $\alpha = \pi$ so we may consider $0 \le \alpha < \pi$ $(-1 < \beta \le 1)$. As in Problem B the condition for stability is that

1.9) $\partial^2(E + \lambda V)(\Omega, N) > 0$

for all non-trivial normal perturbations for which the first variation of the volume is zero.

There are two control variables appropriate to each of the problems we have described. For Problem A these variables are the radius of the circular opening $\bar{r}$ and the vertical coordinate $\bar{u}$ of this circle. The coordinate $-\bar{u}$ is just the pressure at the opening. We proceed to describe those value $(\bar{r},\bar{u})$ in the control space which correspond to stable configurations. We also show that any stable configuration is a minimizer for the energy functional $E_0(\Omega)$ in a strong sense. We do the same form of analysis for Problems B and C . For Problem B the control variables are $(\bar{r},\bar{V})$ where $\bar{r}$ in the radius of the circular opening and $\bar{V}$ is the exposed volume. For Problem C the appropriate variables are $(\bar{\psi},\bar{V})$ where ψ is the angle of contact and V in the volume.

An early work was that of E. Pitts [9] who was interested in Problem B. Suppose that the radius $\bar{r}$ of the circular opening is sufficiently small so that the solution $u \equiv 0$ is stable and let there be given a one parameter family of symmetric pendent drops spanning the circle and parameterized by drop height. Let $V(h)$ be the exposed volume. Pitts showed that if $V'(h)$ is positive for $0 \leq h < \bar{h}$ and is negative for $h > \bar{h}$ then the corresponding drops are (symmetrically) stable for $0 \leq h < \bar{h}$ and unstable for $h > \bar{h}$. An informative but not mathematically rigorous discussion is to be found in the paper of E.A. Boucher, M.J.B. Evans, and H.J. Kent [3]. Their paper includes graphs depicting the regions of stability in the control domain for each of our problems.

Another approach is found in the paper of E. Gonzales, U. Massari, and I. Tamanini [7]. In each of the problems one readily observes that the energy functional has no lower bound and thus a stable pendent drop can be at best a local minimizer.

Their approach is to put a floor underneath the apparatus and to restrict the fluid to remain above this floor. This puts a lower bound on the energy functional and it follows that the variational problem always has a solution. The difficulty is that the minimizer might be a connected drop which contacts the floor or might consist of two components, one pendent and one sessile. For problem C they show that for small enough volumes the solution is the pendent liquid drop. It seems unlikely that this procedure will identify all stable drops. For example, if one takes a stable pendent drop and puts a floor at the level of the drop tip it does not follow that we have the minimizer. It seems to me that for drops of larger volume the minimizer would contact the floor.

Much of the work in the present article may be found in [11] where proofs of many of the technical results to be quoted are proven. A later discussion may also be found in [12].

II. Description of the Profile Curves

Suitably normalized, the differential equation for the profile curve whose surface of revolution represents the liquid-air interface satisfying (1.2) with $k = 1$ is

$$\text{(2.1)}\quad \begin{aligned} &\text{a)}\quad r'(s) = \cos\psi && r(0) = 0 \\ &\text{b)}\quad u'(s) = \sin\psi && u(0) = u_0 = -2\kappa \\ &\text{c)}\quad \psi'(s) = -(\sin\psi/r) - u && \psi(0) = 0\ . \end{aligned}$$

The set of solutions to this system has been carefully discussed by P. Concus and R. Finn [5]. There is a unique solution $\{r(s,\kappa),\ u(s,\kappa),\ \psi(s,\kappa)\}$ to the system satisfying the initial conditions $r(0,\kappa) = 0$, $u(0,\kappa) = -2\kappa = u_0$, $\psi(0,\kappa) = 0$, where κ is the curvature at the drop tip. The solutions exist for all s and κ being analytic in both variables. We note that $u \equiv 0$ is a solution and that a reflection of any solution about the r-axis yields another solution. Drops with $u_0 < 0$ represent pendent drops while solutions with $u_0 > 0$ represent "emerging" bubbles. We now list other important properties of the family. [Figure 2]

1. For "small" $u_0 < 0$ the solutions can be expressed non-parametrically with u as a function of r over the entire r-axis and $u(r) \sim u_0 J_0(r)$ where $J_0(r)$ is the zero order Bessel function.

2. There is a value $u_0^*(\sim -2.5678)$ such that the profile curve with drop tip at u_0^* attains a simultaneous vertical tangent and inflection point at (r_1^*, u_1^*) where $r_1^* \cong .91$ and $u_1^* \cong 1.1$. For $0 < r < r_1^*$ the curve is convex while for r greater than r_1^* the curve may again be expressed non-parametrically in terms of r.

3. For $u_0^* < u_0 < 0$ the solutions may be expressed in non-parametric form $u = u(r)$ for all r.

4. For $u_0 < u_0^*$ the profile curves attain a vertical tangent at a point (r_1, u_1) where $0 < r_1 < r_1^*$ and $u_1 < u_1^*$. The curves

form a bulge at this point and r decreases (with increasing s) to a value r_2 forming a neck at (r_2,u_2) where $u_2 < 0$. r_1 and u_1 are increasing functions of u_0 for $u_0 < u_0^*$ with limit $r_1 = 0$ and limit $u_1 = -\infty$ as u_0 approaches $-\infty$.

5. For $u_0 << u_0^*$ the solution curves form a sequence of bulges and necks until it crosses the r-axis with $r'(s)$ and $u'(s)$ both positive. From this point the curves continue non-parametrically $u = u(r)$ out to $r = +\infty$. The entire curve has no self intersections.

6. The first inflection point on a profile curve with tip at $u_0 < 0$ occurs at a point $(\hat{r},\hat{u})$ where $\hat{u} < 0$. $\hat{r}$ and $\hat{u}$ are monotonically increasing functions of u_0 for $u_0 < 0$. If $u_0 < u_0^*$ so that the profile curve has both a neck and a bulge, then the first inflection point lies between the first bulge and neck.

III. Analysis of Stability

Our method for determining the stable configurations for each of the problems proceeds as follows. Take a given profile curve $\{r(s,\kappa), u(s,\kappa), \psi(s,\kappa)\}$ satisfying (2.1) and let $(\bar{r},\bar{u})$ be a point on the curve $\bar{r} = r(\bar{s},\bar{\kappa})$ and $\bar{u} = u(\bar{s},\bar{\kappa})$. The curve from its tip to this point generates a possible pendent drop whose exposed volume V can be calculated.

3.1) V = volume of drop = $\pi\bar{r}(\bar{r}\bar{u} + 2\sin\bar{\psi})$.

The volume gives us a forth function of the parameters (s,κ) , $V = V(s,\kappa)$. For each of the three problems two of the four functions are prescribed. This generates a mapping from the (s,κ)-plane (the parameter space) into the appropriate two-dimensional "control" space. The analysis of this map determines stability in each case.

IV. Problem A (The Siphon)

The appropriate map here is $A(s,\kappa)$ defined by

4.1) $A(s,\kappa) = (r(s,\kappa),u(s,\kappa))$.

The control space is the (r,u)-plane. One can easily check that the derivative of A , $DA(s,\kappa)$ is invertible when $s = 0$. Let O be the set of all points in the (s,κ)-plane where $DA(s,\kappa)$ is invertible.

Definition 4.1. $O_S \subset O$ is that component of O in the parameter space containing the line $s = 0$.

Theorem 4.1.: Every point $(\bar{s},\bar{\kappa})$ in O_S determines a stable pendent drop for Problem A. (i.e. the drop generated by the profile curve $(r(s,\bar{\kappa}),u(s,\bar{\kappa}))$ $0 \leq s \leq \bar{s}$. Any point outside O_S determines an unstable pendent drop for Problem A.

The control set $A(O_S)$ is an open set in the (r,u)-plane. A point $(\bar{r},\bar{u})$ determines a stable configuration for Problem A only if it is in this set which we now wish to describe. Note that $A(O_S)$ is symmetric about both axes so we may restrict ourselves to the case $r \geq 0$, $u \geq 0$.

Theorem 4.2.: [11, p. 434]: Let $\bar{\kappa} > 0$ and consider the curve $(r(s,\bar{\kappa}),\ u(s,\bar{\kappa}))$ for $s \geq 0$. There is a smallest value s_Λ such that $(s,\bar{\kappa})$ is in O_S for $0 < s < s_\Lambda$ while $(s_\Lambda,\bar{\kappa})$ is on the boundary of O_S . On the interval $0 < s \leq s_\Lambda$ we have $r'(s) = \cos\psi$ positive so that $0 < \psi(s) < \pi/2$.

The corresponding profile curve can thus be expressed in non-parametric form $u = f(r,\bar{\kappa})$ for $0 < r \leq r_\Lambda$ where $r_\Lambda = r(s_\Lambda,\bar{\kappa})$ and $u_\Lambda = f(r_\Lambda,\bar{\kappa})$. The point (r_Λ,u_Λ) lies on the boundary of $A(O_S)$ and in the point conjugate to the drop tip along this curve.

Since $r'(s)$ is positive we can use r as an independent variable rather than s . Points (r_Λ,u_Λ) on the boundary of $A(O_S)$ are determined by the condition that $D\tilde{A}(r,\kappa)$ be singular where $\tilde{A}(r,\kappa) = (r,f(r,\kappa))$ with $u = f(r,\kappa)$ the non-parametric representation of the profile curves. This occurs when $f_\kappa(r,\kappa) = 0$ which means that (r,κ) is on the envelope of the family of extremals.

Theorem 4.3.[11, p. 434]: The first envelope Γ_Λ of the family of profile curves $u = f(r,\kappa)$ for $\kappa \geq 0$ $(u_0 \leq 0)$ and r-positive, is the graph of a smooth analytic function $u = e(r)$ for $0 < r \leq \alpha_0$ where α_0 is the first positive zero of the Bessel function $J_0(r)$. This function has the following properties.

$$\text{limit } e(r) = -\infty \quad \text{as} \quad r \longrightarrow 0$$

$$\text{limit } e(r) = 0 \quad \text{as} \quad r \longrightarrow \alpha_0 \ .$$

The derivative $e'(r)$ is positive on the interval $0 < r < \alpha_0$ with $e'(\alpha_0) = 0$ and limit $e'(r) = +\infty$ as r approaches 0 .

The entire envelope is a smooth curve without self intersections with a cusp at $(\alpha_0, 0)$.[Figure 3]

Consequences:

1. The map $A(s,\kappa)$ is a diffeomorphism of O_S onto its image $A(O_S)$.

2. For $(\bar{r},\bar{u})$ in $A(O_S)$ where $\bar{r} < r_1^*$ the profile curve of the corresponding stable pendent drop is convex. On the other hand if $\bar{r}$ is near α_0 $(\bar{r} < \alpha_0)$ and $(\bar{r},\bar{u})$ is in $A(O_S)$ then the profile curve will contain an inflection point so that the corresponding pendent drop loses convexity. [Figure 3]

3. There are no "inaccessible" stable pendent drops for Problem A. The vertical line $r = \bar{r}$ intersects $A(O_S)$ in a connected interval. Thus the stable pendent drop corresponding to the point $(\bar{r},\bar{u})$ can be reached from the zero pressure solution $u \equiv 0$ corresponding to the point $(\bar{r},0)$ in $A(O_S)$ merely by increasing the pressure $p = -u$ from 0 to $-\bar{u}$ [Figure 3].

The set of stable extremals for Problem A furnish a strong minimum for the functional $E_0(\Omega)$ as the following theorem will show. Take the region of stability $A(O_S)$ for Problem A symmetric with respect to the r and u axes, and rotate it about the u-axis generating a simply connected region $U \subset R^3$. The boundary of U resembles a vertical cylinder becomming narrow at the ends and possessing a cuspoidal curve in the plane $u = 0$. This region is foliated by our family of axially symmetric pendent drops which are extremals for the functional $E_0(\Omega)$.

These surfaces are oriented by choosing the unit normal vector ξ to point downward at the drop tip (i.e. the outward normal to the fluid). We rewrite the functional $E_0(\Omega)$ as follows. Let $P(\bar{x})$ satisfy $\operatorname{div} P(\bar{x}) = z$ and set

$$4.1) \quad \tilde{E}_0(\Omega) = A(\Omega) + k\int_\Omega P(\bar{x}) \cdot \xi \, dA \qquad k = \rho g/\sigma .$$

From the divergence theorem it follows that if Ω_1, Ω_2 are two oriented surfaces with the same boundary then
$E_0(\Omega_2) - E_0(\Omega_1) = \tilde{E}_0(\Omega_2) - \tilde{E}_0(\Omega_1)$.

<u>Theorem 4.4.</u>: Let $U \subset R^3$ be a simply connected region smoothly foliated by extremals of the functional $\tilde{E}_0(\Omega)$.

Let L be any leaf of the foliation and consider a domain $\Sigma_0 \subset L$ with boundary $\partial\Sigma_0$. Let Σ be any other oriented surface in U with the same boundary. We claim that $\tilde{E}_0(\Sigma) \geq \tilde{E}_0(\Sigma_0)$ with equality only if $\Sigma = \Sigma_0$.

Proof: We have a vector field defined on U by choosing for each $p \in U$ the oriented unit normal vector $\xi(p)$ to the leaf of the foliation containing p. One has the formula $2H = \mathrm{div}_\Sigma \xi$, the surface divergence of the unit normal vector field ξ. Making use of the fact that $\mathrm{div}\, P(\bar{x}) = z$ and $\xi \cdot \nabla_\xi \xi = 0$ so that $\mathrm{div}_\Sigma \xi = \mathrm{div}\, \xi$, we may rewrite Euler's equation.

4.2) $\mathrm{div}(\xi + k\, P(\bar{x})) = 0$.

The vector field $\xi + k\, P(\bar{x})$ is a calibration. Let $\Sigma, \Sigma_0 \subset U$ with $\Sigma_0 \subset L$ be as in the statement of the theorem. We calculate

$$\tilde{E}_0(\Sigma) = \int_\Sigma (\nu + kP(\bar{x})) \cdot \nu \, dA$$

where ν is the oriented unit normal to Σ .

$$\tilde{E}_0(\Sigma_0) = \int_{\Sigma_0} (\xi_0 + kP(x)) \cdot \xi_0 \, dA = \int_\Sigma (\xi + kP(x)] \cdot \nu \, dA$$

by (4.2) and the divergence theorem. Thus we find

4.3) $\tilde{E}_0(\Sigma) - \tilde{E}_0(\Sigma_0) = \int_\Sigma [1 - (\nu \cdot \xi)] dA \geq 0$

with equality iff $\nu \cdot \xi \equiv 1$ or $\Sigma = \Sigma_0$.

Q.E.D.

V. Stability for Problem B (The Medicine Dropper)

The constraints are now the radius of the tube r, and the exposed volume V. The control space is the (r,V)-plane and we are led to study the map $B(s,\kappa)$ from the parameter space to the control space defined by

5.1) $$B(s,\kappa) = (r(s,\kappa),\ V(s,\kappa))$$

where V is given by (3.1). As in Problem A we let O' be the open set in the (s,κ)-plane where the derivative $DB(s,\kappa)$ is invertible. Since $B(0,\kappa) = (0,0)$ we see that the line $s = 0$ lies outside of O'.

Fix $\bar{\kappa}$ and consider the map $B(s,\bar{\kappa})$ for positive s. There exists a smallest positive value s_B such that the derivative $DB(s,\bar{\kappa})$ is invertible for $0 < s < s_B$ but singular at $s_B = s_B(\bar{\kappa})$. Let (r_B,u_B) be the corresponding point on the profile curve where $r_B = r(s_B,\bar{\kappa})$ and $u_B = u(s_B,\bar{\kappa})$. It is a classical result that if $(\bar{r},\bar{u})$ is a point on the profile curve prior to (r_B,u_B) then the pendent drop generated by the profile curve up to $(\bar{r},\bar{u})$ is "symmetricly" stable for Problem B. If $(\bar{r},\bar{u})$ is chosen to be beyond the point (r_B,u_B) then the generated drop is unstable for Problem B.

Definition 5.1.: The point (r_B,u_B) is called the "Volume-constrained" conjugate point on the profile curve relative to the drop tip.

Note: The axisymmetric pendent drop is said to be symmetricly stable if the second variation (1.6) is positive for all non-trivial symmetric normal perturbations N of Ω vanishing on the boundary of Ω and for which the first variation of volume is zero. If the profile curve can be expressed in the form $r = r(u)$ then symmetric stability implies stability [11, p. 464]. In this case we observe that the angle of inclination must be nonnegative. However, if the angle of inclination becomes negative on some portion of the profile curve (the corresponding drop is of reentrant type) then the drop is unstable for Problem B due to a non-symmetric perturbation. This fact was noted by D.H. Michael and P.G. Williams [8]. For an alternative discussion see [11].

Let O'_S be the subset of O' consisting of all points (s,κ) where $0 < s < s_B(\kappa)$. It follows that (s,κ) determines a stable pendent frop for Problem B if it lies in O'_S and an unstable drop if outside O'_S. We now describe O'_S and its image $B(O'_S)$ in the control space. It is clear that O'_S is an open set in the parameter space and $B(O'_S)$ is an open set in the control space.

Theorem 5.1.: [12] O'_S is a connected open set in the (s,κ)-plane bounded on the left by the line $s = 0$ and on the right by an analytic curve γ_B which is the graph of a positive analytic function $s_B = \sigma(\kappa)$.

Theorem 5.2.: [11] Let (r_B,u_B) be the volume-constrained conjugate point on the profile curve $(r(s,\bar{\kappa}),u(s,\bar{\kappa}))$. At the point (r_B,u_B) the derivative $r_s(s,\bar{\kappa})$ is positive. The point (r_B,u_B) is located between the first and second inflection points of the curve. If the profile curve possesses a bulge (and hence a neck) then (r_B,u_B) is located above the neck. As $\kappa \longrightarrow 0$ the point (r_B,u_B) approaches the point $(\alpha_1,0)$ where α_1 is a root of the equation $r\,J_0(r) + 2\,J_0'(r) = 0$.[See Figure 4]

By Theorem 5.1 the curve γ_B is an analytic arc parameterized by κ. Its image $B(\gamma_B)$ is a parameterized curve in the (r,V)-plane and is the envelope Γ_B of the family of curves $(r(s,\kappa),V(s,\kappa))$. Thus Γ_B can be expressed in the form $(r(\kappa),V(\kappa))$ where $r(\kappa) = r(\sigma(\kappa),\kappa)$ and $V(\kappa) = V(\sigma(\kappa),\kappa)$ are analytic functions of κ. Furthermore

$$\text{limit } (r(\kappa),V(\kappa)) = (0,0) \quad \text{as} \quad \kappa \longrightarrow +\infty$$

$$\text{limit } (r(\kappa),V(\kappa)) = (\alpha_1,0) \text{ as } \kappa \longrightarrow 0 \quad .$$

From Theorem 5.2 a given curve $B(s,\bar{\kappa})$ touches the envelope Γ_B at a point where $r'(s)$ is positive. Thus in a neighborhood of this point we may express the curve $B(s,\bar{\kappa})$ in the form $V = g(r,\bar{\kappa})$. If the envelope is smooth it would be tangent to this family of curves and itself would have a non-parametric representation $V = G(r)$. A point on the envelope of the family $V = g(r,\kappa)$ is determined by the condition $g_\kappa(r,\kappa) = 0$ while the condition for smoothness is $g_{\kappa\kappa}(r,\kappa) \neq 0$. Since the envelope Γ_B is an analytic parameterized curve it will be smooth except perhaps at isolated points where the derivatives

$r'(\kappa)$ and $V'(\kappa)$ both vanish. At such points the possibility of a cusp arises. One such cusp occurs at $(\alpha_1, 0)$.

<u>Conjecture</u>. That part of the envelope Γ_B which lies in the half space $V > 0$ is a smooth curve which may be expressed as a graph of a function $V = G(r)$, $0 < r < \alpha_1$, with $G(0) = G(\alpha_1) = 0$ and $G'(\alpha_1) = 0$. There is a single value r^* $0 < r^* < \alpha_1$ where $G'(r^*) = 0$.

Computer calculations strongly indicate that the conjecture is true but a complete proof is lacking. [Figure 5]

If the envelope is a smooth curve then it follows that the map $B(s,\kappa)$ is a diffeomorphism of O_S' onto its image $B(O_S')$. In this case any vertical line $r = \bar{r}$ in the control space would intersect $B(O_S')$ in a connected interval. The stable pendent drop corresponding to $(\bar{r}, \bar{V})$ is accessible from the flat drop $u \equiv 0$ corresponding to the point $(\bar{r}, 0)$ through a family of stable pendent drops of increasing volume and fixed radius for the aperature until a maximum volume is reached.

If the envelope were not smooth then the possibility arises that the map B is not a diffeomrphism of O_S' onto its image or that for some $\bar{r}$ the intersection of the line $r = \bar{r}$ with the set $B(O_S')$ is not connected. In either case there would exist stable pendent drops corresponding to some $(\bar{r}, \bar{V})$ in $B(O_S')$ which could not be connected to $(\bar{r}, 0)$ in the manner described above.

If we follow the procedure of describing those drops accessible from $(\bar{r},0)$ corresponding to $u \equiv 0$ through stable drops of increasing volume, then we have the following result.

Theorem 5.3 [11]: (a) Suppose that $\bar{r} \leq r_1^*$ where (r_1^*,u_1^*) is that point on the curve $ru = -1$ which is an inflection point with vertical tangent for one profile curve, (see Section II), and consider the one-parameter family of stable pendent drops for Problem B as the volume is increased from zero. Through an initial range of volumes $0 < V < V_1(\bar{r})$ the profile curves will be convex and the drops will develop a bulge. At $V_1(\bar{r})$ the profile curve will develop an inflection point at the edge of the drop. With increasing volume the drops lose convexity but before the limit of stability is reached pendent drops possessing both a neck an bulge will evolve. [See Figure 6]

(b) For $\bar{r} > \beta_1$ where $J_1(\beta_1) = 0$ the drop $u \equiv 0$ is unstable for Problem B due to non-symmetric perturbations. For $\bar{r} < \beta_1$ the drop $u \equiv 0$ is stable and with increasing volume the profile curves for the family of stable pendent drops will develop an inflection point before the maximum volume is attained.

(c) For any radius $\bar{r}$, drop height increases monotonically throughout the range of stability.

The result (a) of this theorem was observed by A.K. Chesters [4] in the case of small drops with narrow necks.

We will now prove two theorems on strong minimizing properties of stable pendent drops for Problem B. The first theorem restricts comparison with other symmetric pendent drops while the second considers general comparisons. The situation is clarified by the example of a symmetricly stable pendent drop of reentrant type where the angle of inclination on the profile curve becomes negative. Such a drop might be symmetricly stable but is always unstable with respect to general perturbations. The theorems take their nicest form if we assume that the envelope Γ_B has the smoothness properties discussed earlier. If this were not true the statements of the theorems would have to be adjusted accordingly.

<u>Theorem 5.4.</u>: Let (r_1,V_1) be a point in the stable control set $B(O_S')$ for Problem B. Let Σ be any axially symmetric surface whose generating curve C is a rectifiable curve $(r,u) = (f(t),g(t))$ $0 \leq t \leq t_1$ with $f(0) = 0$ and $f(t)$ positive for $0 < t \leq t_1$ and $(f(t_1),g(t_1)) = (r_1,0)$ so that the boundary of Σ is the circle $r = r_1$ in the plane $u \equiv 0$. Suppose that the curve $(f(t),V(t))$ $0 < t \leq t_1$ in the (r,V)-plane lies inside $B(O_S')$ connecting $(0,0)$ with (r_1,V_1). There is a unique stable curve $r = r(s,\kappa_1)$, $u = u(s,\kappa_1)$ $0 \leq s \leq s_1$ such that $r(s_1,\kappa_1)=r_1$ $V(s_1,\kappa_1) = V_1$. Let $(r(s_1,\kappa_1),u(s_1,\kappa_1)) = (r_1,u_1)$. Let Σ_1 be the extremal surface whose generating curve C_1 is $r = r(s,\kappa_1)$, $u = u(s,\kappa_1) - u_1$, $0 \leq s \leq s_1$. Thus Σ and Σ_1 have the same boundary and enclose the same volume. We claim that $E_0(\Sigma) \geq E_0(\Sigma_1)$ with equality only if $\Sigma = \Sigma_1$.

Proof: The proof is an application of the Weierstrass technique for parametric integrals as presented in [2]. If the generating curve of some surface Σ is $(r(t),u(t))$ $0 \leq t \leq T$ then our relevant integrals are

$$E_0(\Sigma) = \int_0^T F(r,u,\dot{r},\dot{u})dt$$

where $F(r,u,\dot{r},\dot{u}) = \pi[2r\sqrt{\dot{r}^2+\dot{u}^2} + r^2u\dot{u}]$

$$V(\Sigma) = \int_0^T G(r,u,\dot{r},\dot{u})dt, \quad G(r,u,\dot{r},\dot{u}) = \pi r^2\dot{u} .$$

Consider the surface Σ in the statement of the theorem with generating curve C given by $(r,u) = (f(t),g(t))$ $0 \leq t \leq t_1$ and assume for the moment that the curve is continuously differentiable. For $0 \leq \bar{t} \leq t_1$ let $V(\bar{t})$ be the volume integral evaluated between 0 and $\bar{t}$ and let $\tilde{C} = \{(f(t),g(t),V(t)),\ 0 \leq t \leq t_1\}$ be the lift of C into the (r,u,V)-space. The curve connects $(0,u_0,0)$ to $(r_1,0,V_1)$ and its projection into the (r,V)-plane lies inside $B(O_S')$. We construct a one parameter family $C(\bar{t})$, $0 \leq \bar{t} \leq t_1$ of generating curves for symmetric surfaces $\Sigma(\bar{t})$ all with the same boundary and enclosing the same volume such that $\Sigma(0) = \Sigma$ and $\Sigma(t_1) = \Sigma_1$.

For $0 < \bar{t} < t$, the curve C determines a point $(\bar{r},\bar{u},\bar{V}) = (f(\bar{t}),g(\bar{t}),V(\bar{t}))$ with projection $(\bar{r},\bar{V})$ in $B(O_S')$. There is a unique extremal $r = r(s,\bar{\kappa})$, $u = u(s,\bar{\kappa})$, $0 \leq s \leq \bar{s}$ with $r(\bar{s},\bar{\kappa}) = \bar{r}$, $V(\bar{s},\bar{\kappa}) = \bar{V}$. Suppose $u(\bar{s},\bar{\kappa}) = \tilde{u}$. The generating curve $C(\bar{t})$ is

$$C(\bar{t}) = \begin{cases} r = r(s,\bar{\kappa}),\ u = u(s,\bar{\kappa}) - \tilde{u} + \bar{u},\ 0 \leq s \leq \bar{s} \\ \\ r = f(t),\ u = g(t),\ \bar{t} \leq t \leq t_1 \end{cases} .$$

$C(\bar{t})$ is a piecewise smooth curve whose lower part is a stable extremal replacing part of the original curve while the upper part is unchanged. The lower portion is an extremal for the functional $E_0 + \bar{\lambda} V$ where $\bar{\lambda} = \tilde{u} - \bar{u}$ is the Lagrange multiplier. We set $E_0(t) \equiv E_0[\Sigma(t)]$ and $V(t) = V(\Sigma(t))$, so that

$$E_0(\bar{t}) = \int_0^{\bar{s}} F(r,u,r_s,u_s)ds + \int_{\bar{t}}^{t_1} F(f,g,\dot{f},\dot{g})dt$$

where in the first integral we use the functions

$r = r(s,\bar{\kappa},\bar{\lambda}) = r(s,\bar{\kappa})$, $u = u(s,\bar{\kappa},\bar{\lambda}) = u(s,\bar{\kappa}) - \bar{\lambda}$, $\bar{\lambda} = \tilde{u} - \bar{u}$.

The key observation is that from our assumptions it follows that $\bar{s}$, $\bar{\kappa}$ and $\bar{\lambda}$ are differentiable functions of $\bar{t}$. We may compute dE_0/dt .

$$\frac{dE_0}{d\bar{t}} = \frac{dE_0}{d\bar{t}} + \bar{\lambda}\frac{dV}{d\bar{t}} = - E(\bar{r},\bar{u},\bar{r}_s,\bar{u}_s,f'(\bar{t}),g'(\bar{t}))$$

where E is the Weierstrass E-function for the parametric integral $H = F + \bar{\lambda}G$.

$$E(\bar{r},\bar{u},\bar{r}_s,\bar{u}_s,f'(\bar{t}),g'(\bar{t}))$$
$$= [H_{r_s}(\bar{r},\bar{u},f'(\bar{t}),g'(\bar{t})) - H_{r_s}(\bar{r},\bar{u},\bar{r}_s,\bar{u}_s)]f'(\bar{t})$$
$$+ [H_{u_s}(\bar{r},\bar{u},f'(\bar{t}),g'(\bar{t})) - H_{u_s}(\bar{r},\bar{u},\bar{r}_s,\bar{u}_s)]g'(\bar{t}) .$$

A direct calculation now gives

$$E_0'(\bar{t}) = -2\pi\bar{r}(1 - \cos\bar{\theta})\sqrt{f'(\bar{t})^2+g'(\bar{t})^2}$$

where $\bar{\theta}$ is the angle between the tangent vectors $(\bar{r}_s,\bar{u}_s)$ and $(f'(\bar{t}),g'(\bar{t}))$. Therefore

$$E_0(\Sigma)-E_0(\Sigma_1) = \int_0^{t_1} 2\pi f(t)[1-\cos\theta(t)]\sqrt{f'(t)^2+g'(t)^2}\,dt .$$

This formula clearly can be extended to the space of rectifiable curves. The theorem follows directly from the formula.

For more general configurations we have the following result.

<u>Theorem 5.5.</u>: Let X be a connected open set in R^3 which lies below the horizontal plane $u = 0$ and where $\bar{X} \cap \{u = 0\}$ is a disk B_{r_1} with center the origin and of radius r_1 . Suppose $\partial X = B_{r_1} \cup \Sigma$ where Σ is an oriented surface with boundary $\partial\Sigma = C_{r_1} = \partial B_{r_1}$. Suppose $V(X) = V_1$. Let X^* be the symmetrization of X about the vertical axis, namely $X^* \cap \{u = \bar{u}\}$ is a disk $B_{\bar{r}}$ whose area is equal to the area of $X \cap \{u = \bar{u}\}$. Suppose that $\partial X^* = B_{r_1} \cup \Sigma^*$ where Σ^* is a symmetric surface lying below the plane $u = 0$ and $\partial\Sigma^* = C_{r_1}$. Let the generating curve of Σ^* be $C^* = \{(r,u) = (f(t),g(t)),\ 0 \leq t \leq t_1\}$ and suppose that the

symmetrized configuration satisfies the conditions of Theorem 5.4. In particular we have determined an extremal curve C_1 given by $r = r(s,\kappa_1)$, $u = u(s,\kappa_1) - u_1$, $0 \leq s \leq s_1$ with $r(s_1,\kappa_1) = r_1$, $V(s_1,\kappa_1) = V_1$. Let Σ_1 be the surface generated by C_1. We then have $V(\Sigma) = V(\Sigma_1) = V_1$ and $E_0(\Sigma) \geq E_0(\Sigma_1)$ with equality only if $\Sigma = \Sigma_1$.

Proof: The proof follows directly from two well-known properties concerning symmetrization, namely
$V(X^*) = V(X)$ and $A(\Sigma^*) \leq A(\Sigma)$ with equality only if $\Sigma = \Sigma^*$. From this latter statement we find that $E_0(\Sigma^*) \leq E_0(\Sigma)$ as well. We now can apply Theorem 5.4.

VI. Stability for Problem C (Drop from Horizontal Plate)

The control parameters are now the angle of inclination ψ and the volume V, giving

$$6.1)\quad C(s,\kappa) = (\psi(s,\kappa),V(s,\kappa))$$

for the mapping from the parameter space to the control space. As before we let O'' be the set of all points (s,κ) where the derivative $DC(s,\kappa)$ is invertible. We observe that $C(s,0) = C(0,\kappa) = (0,0)$ so that O'' does not meet either of the coordinate axes. For any $\kappa > 0$ there is a value $s_C = s_C(\kappa)$ such that $C(s,\kappa)$ is invertible for $0 < s < s_C(\kappa)$ and singular at $s = s_C(\kappa)$. We let $O''_S \subset O_S$ be the set of points (s,κ)

where $\kappa > 0$ and $0 < s < s_C(\kappa)$.

Lemma 6.1: The set O''_S is an open simply connected set bounded on the left by the line $s = 0$, on the bottom by $\kappa = 0$ and on the right by a curve γ_C which is the graph of an analytic function $s = s_C(\kappa)$ where limit $s_C(\kappa)$ is zero as κ becomes infinite.

Definition 6.1: For a given profile curve $(r(s,\bar{\kappa}),u(s,\bar{\kappa}))$ $s \geq 0$, the volume constrained focal point for problem C is the point (r_C,u_C) on the curve where

$$r_C = r(s_C(\kappa),\kappa),\ u_C = u(s_C(\kappa),\kappa) .$$

If a profile curve is to generate a stable configuration it is necessary that the angle of inclination ψ be non-negative along the segment of the profile curve generating the drop. Otherwise the drop would intersect the face and would also fail to be stable due to non-symmetric perturbations. This eliminates re-entrant drops from consideration. Therefore we let $\tilde{O}''_S$ be the set of points $(\bar{s},\bar{\kappa})$ in O''_S such that the angle of inclination $\psi(s,\bar{\kappa})$ is positive $0 < s < \bar{s}$. We allow the possibility that $\psi(\bar{s},\bar{\kappa}) = 0$.

Theorem 6.1: [11] The profile curve segment corresponding to any member of $\tilde{O}''_S$ generates a stable configuration for Problem C. If $(\bar{s},\bar{\kappa})$ lies outside the closure of $\tilde{O}''_S$ then the generated drop is unstable.

In other words, let (r_C, u_C) be the volume constrained focal point for Problem C on some curve where we suppose that this focal point comes before the point where the angle of inclination is zero. If $(\bar{r}, \bar{u})$ is a point on the curve prior to (r_C, u_C) then the corresponding pendent drop is stable for Problem C while if it comes after (r_C, u_C) the resulting drop will be unstable.

<u>Theorem 6.2</u> [11]: The volume constrained focal point (r_C, u_C) on a given profile curve lies between the first and second inflection points. It comes ahead of the volume constrained conjugate point (r_B, u_B) for Problem B if the angle of inclination at (r_B, u_B) is positive. The two points coincide if the angle of inclination at (r_B, u_B) is zero.

We consider the set $C(\tilde{O}_S'')$ in the control space, the (ψ, V)-plane. From our definition of $\tilde{O}_S''$ it follows that $C(\tilde{O}'')$ lies in the first quadrant and is bounded by the axes $\psi = 0$, $V = 0$ and $\Gamma_C = C(\gamma_C)$. As in Problem B, Γ_C is the envelope of the family of curves $(\psi(s,\kappa), V(s,\kappa))$. By Theorem 6.2 each curve of the family will touch the envelope at a point where $\psi_S(s,\kappa)$ is negative. Therefore in a neighborhood of the touching point each of these curves may be expressed non-parametricly $V = h(\psi,\kappa)$. The envelope is determined by the condition $h_\kappa(\psi,\kappa) = 0$. It will be a smooth curve of $h_{\kappa\kappa}(\psi,\kappa) \neq 0$. If the angle ψ is positive then $dV/d\psi = h_\psi(\psi,\kappa) = V_s/\psi_s$ will be negative. Where it is smooth the envelope Γ_C will be the graph of a decreasing function.

Conjecture: That part of the envelope Γ_C which lies in the first quadrant of the (ψ,V)-plane is the graph of a smooth function $V = V(\psi)$, $0 \le \psi \le \pi$ with $V'(0) = 0$, $V'(\psi)$ negative for $0 < \psi < \pi$, and $V'(\pi) = 0$.

Computer calculations support the conjecture. If true then (as in Problem B) the map C would be a diffeomorphism of $\tilde{O}_S''$ onto its image, and the intersection of a vertical line $\psi = \bar{\psi}$ with $C(\tilde{O}_S'')$ would be a connected interval. This would imply that as we move vertically along the line $\psi = \bar{\psi}$ from $(\bar{\psi},0)$ to $(\bar{\psi},V_{Max})$ in the control space, we would pass through the entire family of stable pendent drops for Problem C with $\psi = \bar{\psi}$. If Γ_C were not smooth then there might exist stable drops not accessible by this procedure. [See Figure 7]

The following theorem identifies those stable pendent drops which are accessible from drops of small volume.

Theorem 6.3 [11](a) For any angle of contract $\bar{\psi}$, $0 < \bar{\psi} < \pi$ stable drop of small volume are convex and resemble spherical caps. These drops are generated by profile curves whose tip is at u_0 where u_0 is increasing from $-\infty$. At a certain positive volume V_1 depending on $\bar{\psi}$, the profile curve for the drop will develop an inflection point at its edge. This drop is stable. As the volume is increased, further stable pendent drops are formed and the inflection point on the profile curve will move to its interior. The drops have lost convexity. With increasing volume the limit of stability will be reached before a second inflection point appears. With increasing volume the area of contact of the drop with the horizontal plate will initially increase but will start to decrease before the limit of stability is reached.

(b) If $\bar{\psi} = 0$ then all profile curves corresponding to drops of positive volume contain an inflection point. Drops of small volume correspond to small negative values of the drop tip, u_0 . As u_0 is decreased stable pendent drops of increasing volume are formed. The drop generated by that profile curve which possesses a simultaneous vertical tangent and inflection point ($u_0 = u_0^* \cong -2.5678$) is unstable for $\bar{\psi} = 0$. Computer results indicate that the drop of maximum volume occurs with $u_0 \cong -1.6$ with $V_M \cong 18.4$. Furthermore as the volume increases the area of contact of the drop with the plate decreases monotonically.

(c) For any angle of contact drop height increases monotonically with volume throughout the range of stability. [See Figure 8]

Remark: For example, if the angle of contact were $\psi = \pi/2$, it would follow that with increasing volume and before the limit of stability is reached, pendent drops containing both a neck and a bulge will appear. [See Figure 9]

We now use the Weierstrass technique coupled with a symmetrization argument to prove a strong minimization property for the stable pendent drops of Problem C. We state the theorems with the assumption that the envelope Γ_C is smooth. This makes the statement cleaner than it otherwise would be. Our first result is restricted to comparisons with axially symmetric drops.

Theorem 6.4: Let (ψ_1, V_1) be a point in $C(\tilde{O}''_S)$. By our assumptions there is exactly one extremal curve $(r(s,\kappa_1),u(s,\kappa_1))$ $0 \le s \le s_1$ such that $(s,\kappa_1) \in \tilde{O}''_S$ for $0 < s \le s_1$ and with $(\psi(s_1,\kappa_1),V(s_1,\kappa_1)) = (\psi_1,V_1)$. Let Σ_1 be the surface whose generating curve C_1 is $\{r(s,\kappa_1),u(s,\kappa_1)-u_1\}$ $0 \le s \le s_1$ where $u_1 = u(s_1,\kappa_1)$. Thus Σ_1 meets the plate $\{u = 0\}$ on a circle C_{r_1} of radius r_1 with constant angle of contact ψ_1 and enclosed volume V_1 . Let Σ be an axially symmetric comparison surface with generating curve $C : \{(r,u) = (f(t),g(t)),\ 0 \le t \le t_1\}$ with $f(0) = 0$, $f(t)$ positive for $t > 0$, $(f(t_1),g(t_1)) = (\tilde{r},0)$, and $V(t_1) = V_1$. Suppose further that the curve $(f(t),V(t))$ $0 \le t \le t_1$ lies in $B(O'_S)$ and that $(f(t_1),V(t_1))$ is in the set $B(\tilde{O}''_S)$. Let $E_{\psi_1}(\Sigma)$ be the energy

$$6.2) \qquad E_{\psi_1}(\Sigma) = E_0(\Sigma) - (\cos\psi_1)|G|$$

where $|G|$ is the wetted area for the drop. We claim that $E_{\psi_1}(\Sigma) \ge E_{\psi_1}(\Sigma_1)$ with equality only if $\Sigma = \Sigma_1$.

Proof: The curve $(f(t),V(t))$ corresponding to Σ lies in $B(O'_S)$ ending up at the point $(\tilde{r},V_1) = (f(t_1),V(t_1)) \in B(\tilde{O}''_S)$. This point determines a unique stable pendent drop $\tilde{\Sigma}$ for Problem B with generating curve $\tilde{C}$ meeting the plate $(u = 0)$ in a circle of radius $\tilde{r}$ and enclosing volume V_1 . It follows from Theorem 5.4 that $E_0(\Sigma) \ge E_0(\tilde{\Sigma})$ with equality only if $\Sigma = \tilde{\Sigma}$. Since the wetted area is the same for both surfaces it follows that $E_{\psi_1}(\Sigma) \ge E_{\psi_1}(\tilde{\Sigma})$. It remains to show that $E_{\psi_1}(\tilde{\Sigma}) \ge E_{\psi_1}(\Sigma_1)$.

Now $(\tilde{r},V_1)$ lies on $B(\tilde{O}''_S)$ which means that $\tilde{\Sigma}$ determines a point $(\tilde{\psi},V_1)$ in $C(\tilde{O}''_S)$. Consider the family of drops as we move along the line $V = V_1$ in $C(\tilde{O}''_S)$ from ψ_1 to $\tilde{\psi}$. Each point (ψ,V_1) determines a symmetric drop $\Sigma(\psi)$ which is an extremal for Problem C. Let $E_0(\psi), V(\psi) = V_1$ and $|G(\psi)|$ denote the energy, volume and wetted area respectively of the drop $\Sigma(\psi)$. Since the volume stays constant we find

$$6.3) \qquad \frac{dE_0(\psi)}{d\psi} - (\cos\psi)\frac{d|G(\psi)|}{d\psi} = 0 .$$

Now $dG/d\psi = (2\pi r)(dr/d\psi)$ and one can show that $dr/d\psi$ is negative. This gives us

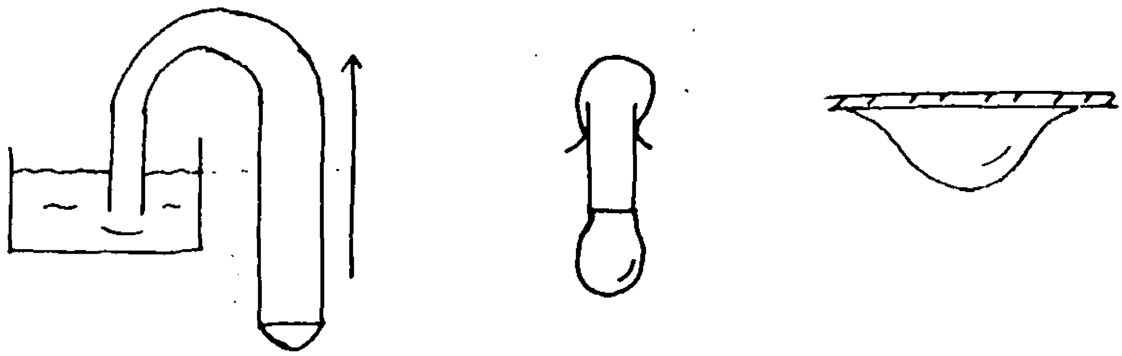
$$6.4) \qquad dE_{\psi_1}(\psi)/d\psi = (\cos\psi - \cos\psi_1)(2\pi r)(dr/d\psi) .$$

If $\psi_1 \geq 0$ and $0 \leq \psi \leq \pi$ then we find this derivative to be positive for $\psi > \psi_1$ and negative for $\psi < \psi_1$. Thus the energy $E_{\psi_1}(\psi)$ has a minimum at ψ_1 which gives us Σ_1 .

<u>Theorem 6.5</u>: Let Σ_1 be the stable pendent drop for Problem C corresponding to the point (ψ_1,V_1) in $C(\tilde{O}''_S)$. Let X be an open set in R^3 lying below the plane $(u = 0)$ with $\bar{X} \cap \{u = 0\} = G$. Suppose the boundary of X , $\partial X = \Sigma \cup G$ where Σ is a smooth surface with boundary $\partial\Sigma = \partial G$. Let X^* be the symmetrization of X about the u-axis and suppose that X^* satisfies the hypotheses of Theorem 6.4. Then $E_{\psi_1}(\Sigma) \geq E_{\psi_1}(\Sigma_1)$ with equality only if $\Sigma = \Sigma_1$.

Proof: The proof is exactly as in the discussion of Theorem 5.4. Symmetrization decreases area while preserving the volume, wetted area, and gravitational potential.

Figure 1: Three Different Experiments



Problem A (the Siphon)

Problem B (the Medicine Dropper)

Problem C (Drop from Ceiling)

Figure 2: Possible Drop Configurations

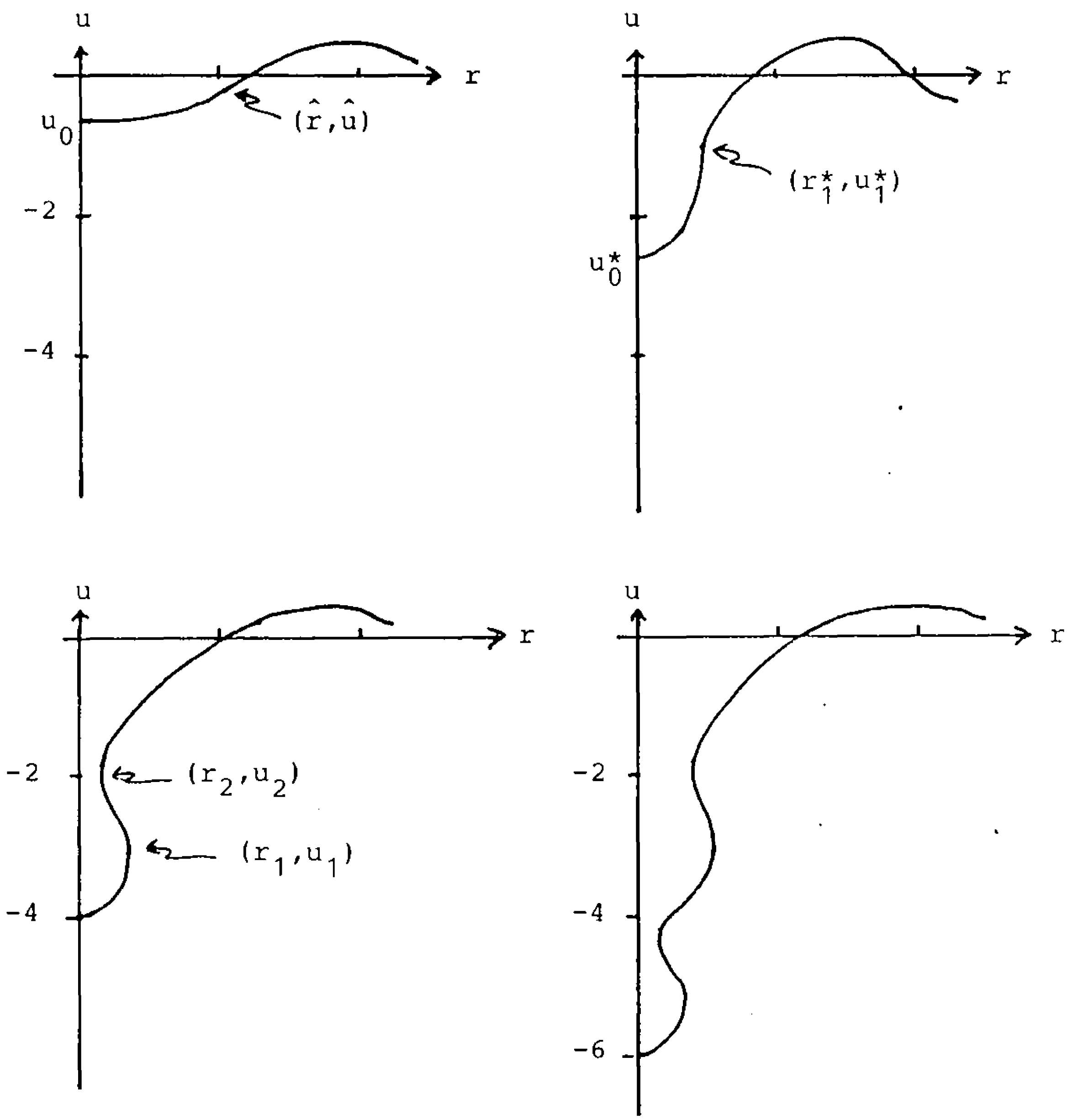


Figure 3. The Envelope Γ_A .

Stable Configurations for Problem A.

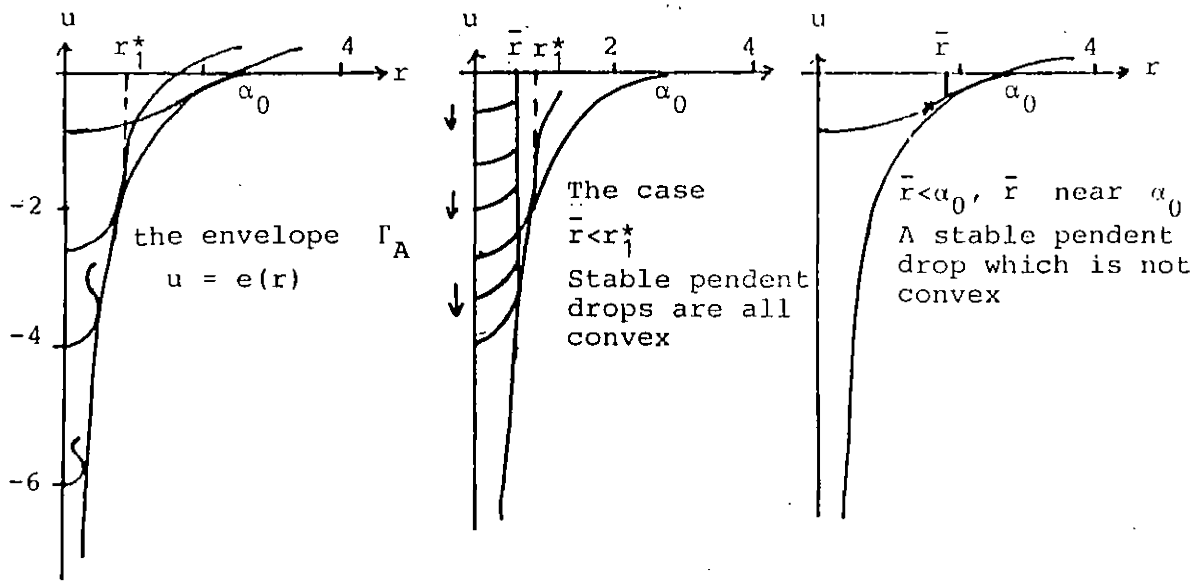


Figure 4. Location of Volume-constrained conjugate point

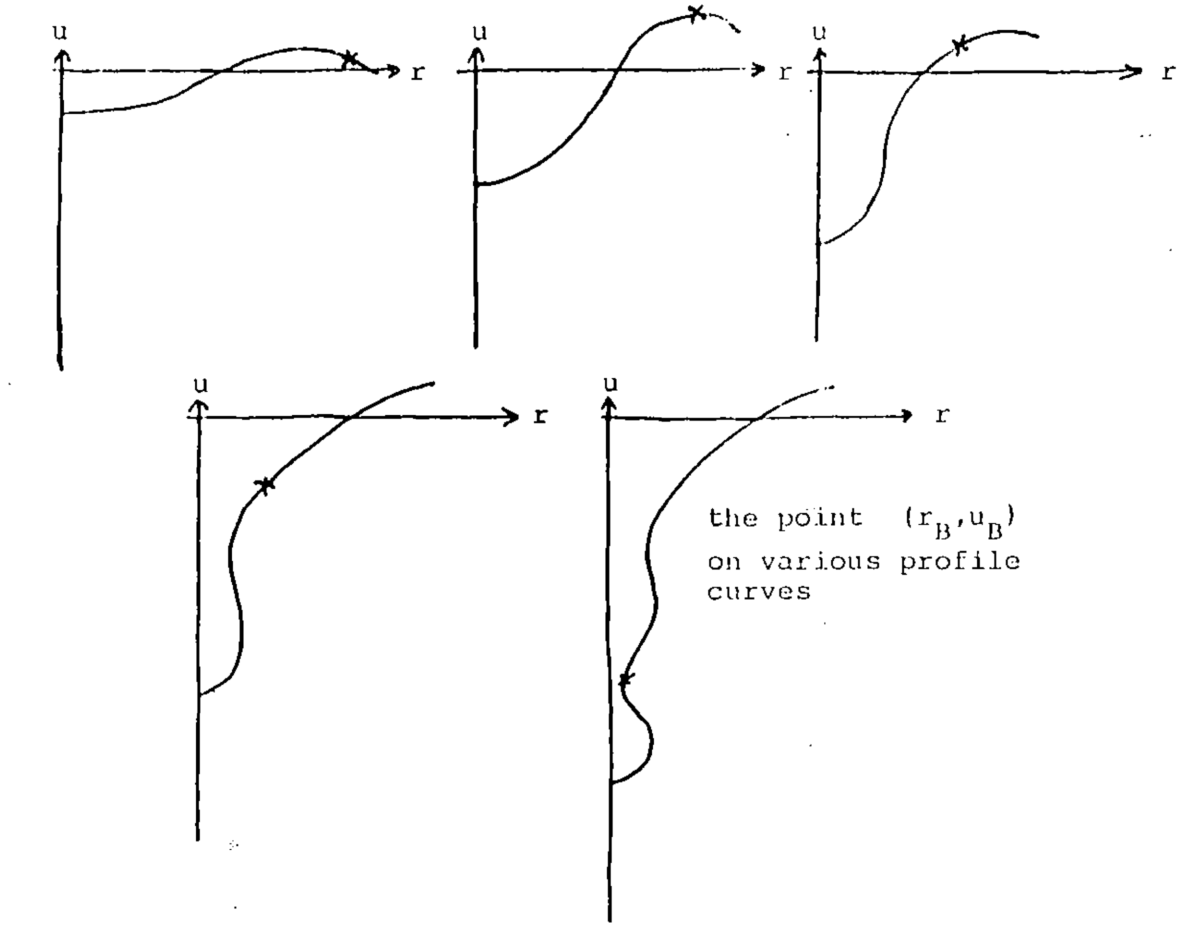


Figure 5. The Stability Region for Problem B with the Envelope Γ_B

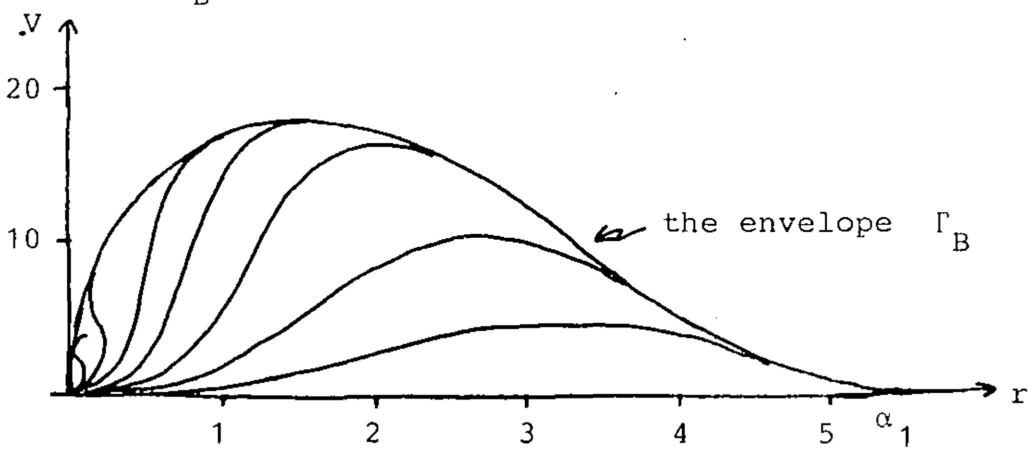


Figure 6. Drop Formation for Problem B.

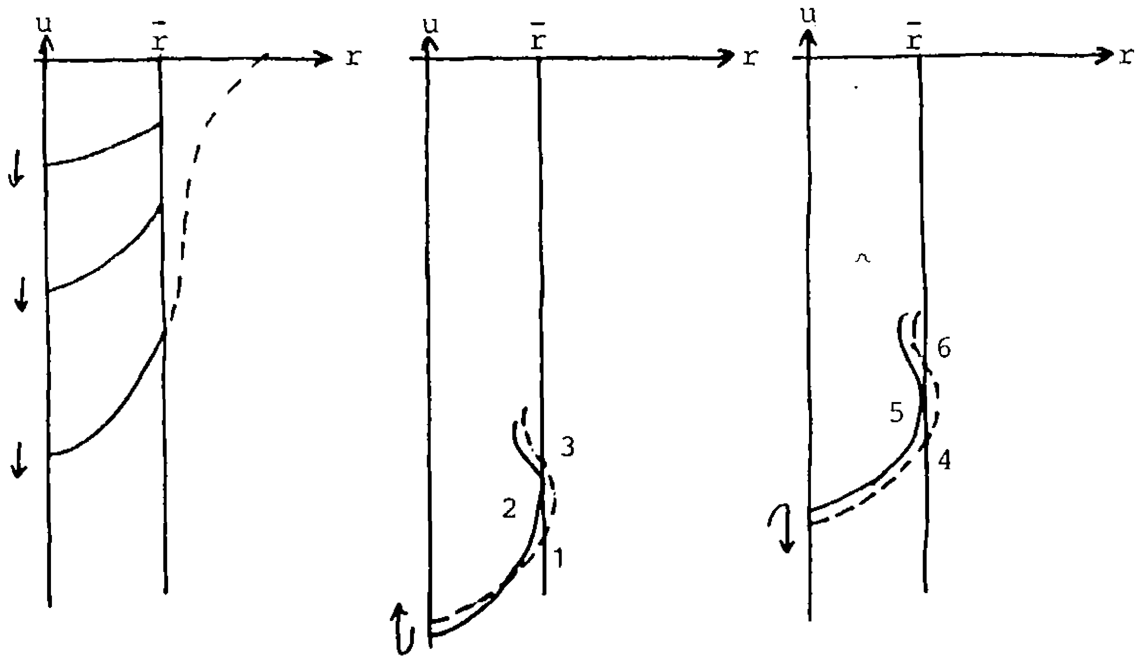


Figure 7. The Stability Region for Problem C with the envelope Γ_C.

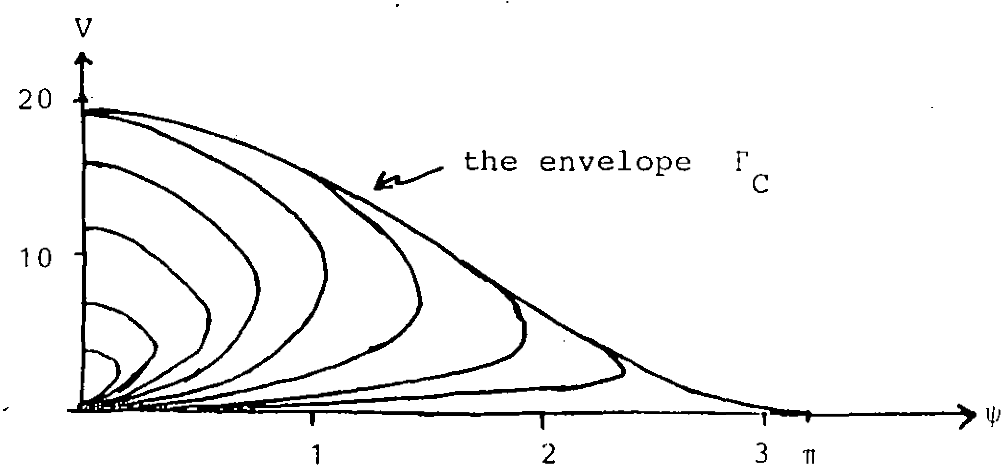


Figure 8. Drop Formation for Problem C

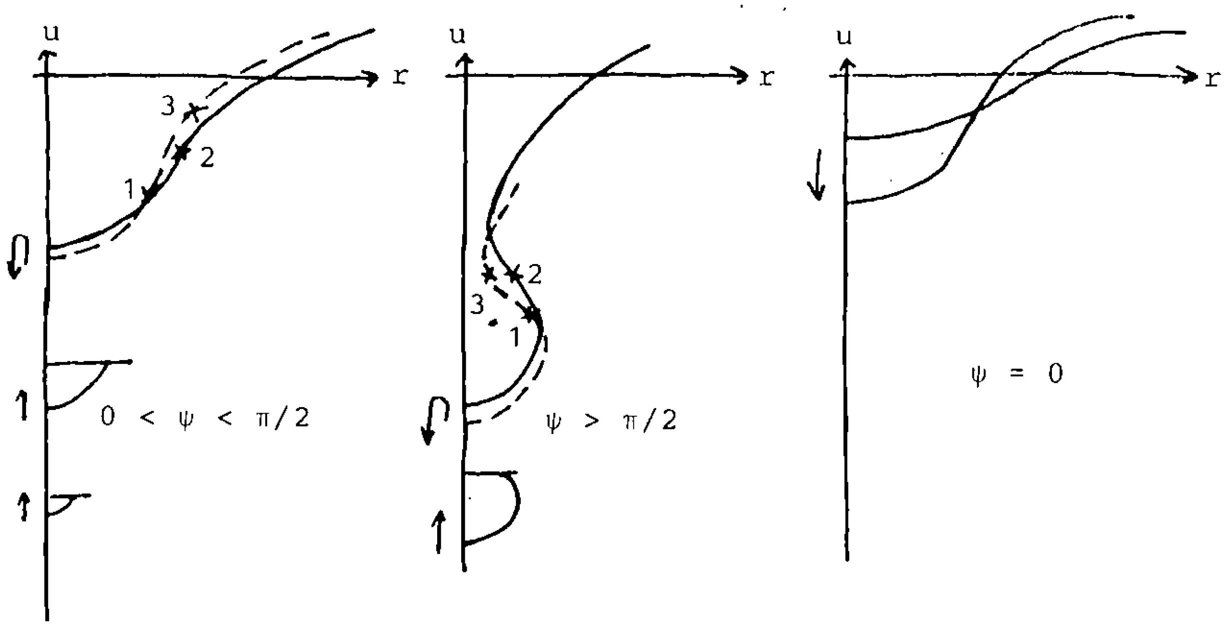
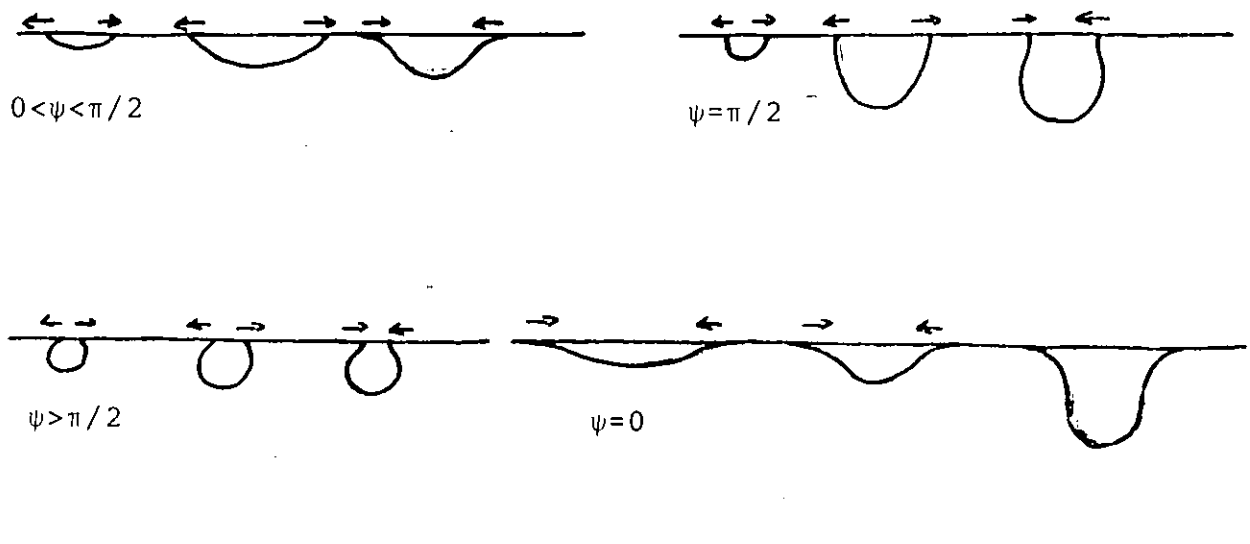


Figure 9. Stable Drops of Increasing Volume



The arrows indicate motion of drop on wetted plate. The only unstable drop is the third one when $\psi = 0$.

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