VARIETIES WHICH CONTAIN MANY LINES

by

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In the present paper, the author will study varieties containing many lines and determine the structure of such varieties. As easy and simple examples, well-known are hypersurfaces of low degree in the projective space and $\mathbb{P}^r$-bundle embedded by very ample tautological line bundle. Our aim is to ask the reverse, namely, what is the structure of variety containing many lines? First let X be a projective variety and Λ a linear system which induces a closed embedding $\varphi_\Lambda : X \hookrightarrow \mathbb{P}^{\dim|\Lambda|}$. Then one defines as follows.

<u>Definition A</u> (X,Λ) is said to contain many lines, if for every point x in $\varphi_\Lambda(X)$, there is a line ℓ_x in $\mathbb{P}^{\dim|\Lambda|}$ contained in $\varphi_\Lambda(X)$.

Now we pose the following

<u>Problem.</u> <u>Under the above definition and the assumption describe the properties</u> of (X,Λ) <u>and classify</u> (X,Λ).

But without the assumption of suitable conditions, I think it is meaningless and loose to consider the classification of (X,Λ) simply. Therefore, for (X,Λ) we introduce a numerical quantity $\ell(X,\Lambda)$ which indicates the extent of the existence of lines on X. (See 1.3).

(B) In this paper let us assume that Λ is a complete linear system $|L|$ of a very ample line bundle L on X and (X,L) is written in place of $(X,|L|)$.

(C) Throughout this paper, (X,L) denotes a variety which contains many lines in the sense of A and B.

Then our Main Theorem is as follows.

Main Theorem. Let (X,L) be a smooth variety which contains many lines. Assume that $\ell(X,L) \leq 2$. In positive characteristic case moreover, assume that p is separable (see 1.3). Then we have

$\ell(X,L)$	(X,L)
0	$(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$
1	$(Q^n\ \mathcal{O}_{Q^n}(1))$, Q^n being a quadric hypersurface in $\mathbb{P}^{n+1}$
	$(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$, where E is a very ample vector bundle of rank n over a non-singular curve.
2	(Cubic hypersurface X, $\mathcal{O}_X(1)$)
	$(Q_1^{n+1} \cap Q_2^{n+1} (= X), \mathcal{O}_X(1))$
	$(Gr(4,1) \cap L^i (= X), \mathcal{O}_X(1))$ $(i = 6,7,8)$, where L^i is an i-dimensional linear subspace in $\mathbb{P}^9$ and $Gr(4,1)$ is

embedded in $\mathbb{P}^9$ by the Plücker embedding.

($\mathbb{P}(E)$; $\mathcal{O}_{\mathbb{P}(E)}(1)$ where E is a very ample vector bundle of rank rank n - 1 over a non-singular surface.

(Quadric hypersurface fiber space X,L)(See 2.1).

In the next place, let us talk about the method to get Main Theorem briefly, It is trivial to determine (X,L) with $\ell(X,L) = 0$ (Sublemma 1.7). Moreover we determine the surface with $\ell(X,L) = 1$ (Proposition 1.9) and 3-fold with $\ell(X,L) = 2$ (§ 5). For higher dimensional case, taking several hyperplane sections we observe the structure of (X,L).

In order to give more detailed explation, let us state the titles of each section.

§ 1. Definition of $\ell(X,L)$ and its properties.

§ 2. Varieties where hyperplane section is a quadric hypersurface fiber space over a curve.

§ 3. Varieties whose hyperplane section is $\mathbb{P}^r$-bundle and (X,L) with $\ell(X,L) = 1$.

§ 4. 3-folds with many quadric surfaces.

§ 5. Classification of (X,L) with $\ell(X,L) = 2$.

§ 6 3-folds with two fiber space structures. Section 2 and 3 show that a variety inherits a fiber structure from the ample divisor. In the characteristic zero case, there are many results by A. Sommese and T. Fujita (Proposition III in [13], Corollary 2.10 in [4] and [14]), which utilize Kodaira's vanishing Theorem and Lefschetz's Theorem.

On the other hand, in the characteristic p case, lacking the above two powerful theorems, we shall show the restricted results in § 3 and 4 by using Serre's Vanishing Theorem and the Lifting method.
In § 5, a big problem is to determine the structure of Fano 3-folds with special properties. Theorem 5.12 is important in order to obtain the classification of 3-folds with $\ell(X,L) = 2$ with more ease. Moreover, to get the final results, we make use of some results by T. Fujita [5] and Iskovskih [7]. In § 6 , we classify 3-folds (X,L) with two fiber structures whose fiber is a line. Unlike the characteristic zero case; we cannot obtain a complete solution of this problem in the positive characteristic case. We give a pathological example (6.11) pertaining to this problem in the positive characteritic case. Moreover, our result in § 6 is necessary to complete the proof in § 3.
Notation. Throughout this article, k is an algebraically closed field of characteristic $p(\geqq 0)$. Under a scheme we understand

a separated algebraic k-scheme. By a variety, we mean a reduced irreducible algebraic k-scheme. $\mathcal{O}_{\mathbb{P}^n}(1)$ is the line bundle corresponding to the divisor class of hyperplanes in the n-dimensional projective space $\mathbb{P}^n$. If X is isomorphic to $\mathbb{P}^{n_1}\times\ldots\times\mathbb{P}^{n_k}$, $\mathcal{O}(a_1,\ldots,a_k)$ denotes $\bigotimes_{i=1}^{k} pr_i \mathcal{O}_{\mathbb{P}^{n_i}}(a_i)$ where pr_i is the i-th projection from $\mathbb{P}^{n_1}\times\ldots\times\mathbb{P}^{n_k}$ to $\mathbb{P}^{n_i}$. This notation is often used where there is no danger of confusion. For a hypersurface X in $\mathbb{P}^{n+1}$, $\mathcal{O}_X(1)$ denotes the line bundle corresponding to a hyperplane section of X in $\mathbb{P}^{n+1}$, abbreviated by $\mathcal{O}(1)$. For a line bundle L on a variety, $|L|$ denotes a complete linear system. We use the terms vector bundle and locally free sheaf interchangeably. Gr(n,d) denotes the Grassmann variety parameterizing d-dimensional linear subspace of the n-dimensional space $\mathbb{P}^n$. $F\ell(n,1,0)$ is the flag variety $\{(x,y)\in Gr(n,1)\times\mathbb{P}^n \mid L_x \ni y$ with the line L_x on $\mathbb{P}^n$ corresponding to $x\}$. $\check{E}$ denotes the dual vector bundle of a vector bundle E. If T is a closed subscheme of S, then we use the notation $E|_T$ instead of i^*E where i is the natural immersion $i:T\hookrightarrow S$. Moreover, when T is a locally complete intersection in S, then $N_{T/S}$ denotes the normal bundle of T in S. $h^i(S,E)$ denotes $\dim H^i(S,E)$. When X is a non-singular variety, Ω^1_X (or, T_X) denotes the sheaf of differential 1-forms (or, the tangent sheaf, respectively) on X.

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§ 1. Definition of $\ell(X,L)$ and its properties.

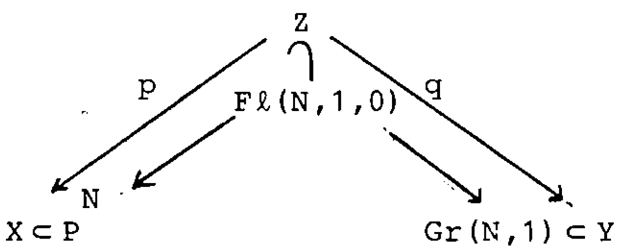
First let us begin with some remarks about definition A of (X,Λ) in the introduction.

Remark 1.1 For (X,Λ), let X be a subvariety embedded in $\mathbb{P}^{|\Lambda|}$ by Λ. Then if one projects X from a point outside X in $\mathbb{P}^{|\Lambda|}$, a line in X is projected to a line as well. Moreover there is a variety (X,L) which is biregularly projected from a point. But in this paper we consider only a complete linear system.

Remark 1.2 There is a variety X and finite many line bundles $L_1,\ldots,L_r$ $(r \geq 2)$ on X such that (X,L_i) contains many lines. For each (X,L) in the sense of A, we introduce an integer $\ell(X,L)$ which describes the quantity of lines on $\varphi_L(X)$. Let Y be a Hilbert scheme of lines in $\varphi_L(X)$. Then it is well-known that Y is a projective scheme. Let Y^i be an irreducible component of Y. Then for (X,L) we define $\ell(X,L)$ as follows:

$$(1.3) \qquad \ell(X,L) = \dim X - 1 - \max_i \dim (q(p^{-1}(x)) \cap Y_i)$$

where lines in Y_i fill up the whole space X, and p and q are projection as in the following diagram



and N denotes $\dim |L|$.

(1.4) Hereafter we use this diagram and projections p,q very often where we choose Y as a fixed irreducible component which gives rise to $\ell(X,L)$.

Remark 1.5 One can describe $\ell(X,L)$ in the following terms:

$$\ell(X,L) = \dim X - 1 - \max_{\ell} \dim H^0(\ell, N_{\ell/X} \otimes \mathcal{O}_{\ell}(-1))$$

where ℓ is a line and $N_{\ell/X}$ is generated by its global sections.

$\ell(X,L)$ takes the following numerical values.

Lemma 1.6 $0 \leq \ell(X,L) \leq \dim X - 1$

Proof. It is sufficient to show the left-hand inequality. For this purpose we prove the following Sublemma 1.7. If $\ell(X,L) \leq 0$, (X,L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{P^n}(1))$. Moreover $\ell(\mathbb{P}^n, \mathcal{O}_{P^n}(1)) = 0$.

Proof. Take a generic smooth point x in X and consider the tangent space T_x at x in $\mathbb{P}^{\dim|L|}$. Then T_x contains all lines passing through x , which is of, at least, n dimensional. Noting that T_x is an n-dimensional linear space in $\mathbb{P}^{\dim|L|}$, it is obvious that (X,L) is $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. The latter part is trivial.

We now deduce the relation between $\ell(X,L)$ and $\ell(H,L|_H)$ where H is a hyperplane section in $|L|$. Namely, we have

Proposition 1.8. Let (X,L) be a variety which contains many lines and H an effective divisor of $|L|$ which is reduced and irreducible. Assume that $\ell(X,L) \leq \dim X - 2$. Then $(H, L|_H)$ contains many lines and we have the inequality $\ell(X,L) \geq \ell(H,L|_H)$. Moreover if H is a general member of $|L|$, their equality holds.

Proof. The assumption means that for every point x in X, there is at least a one-dimensional family of lines passing through x, which makes a cone C in X with a vertex x. Hence any hyperplane H through x contains a generator of the cone C, which yields $\dim (C \cap H) \geq \dim C - 1$. Therefore we get the inequality. The latter part is trivial.

q.e.d.

To end this section, let us investigate the structure in the surface case.

Proposition 1.9. Let (X,L) be a smooth surface which contains many lines. Assume that $\ell(X,L) = 1$. Then there is a $\pi : X \rightarrow C$ P^1-bundle over a smooth curve C whose fiber is a line.

Proof. Our assumption yields a curve C (see the diagram (1.3)). Take its general line ℓ in X. Then we see easily that $N_{\ell/X}$ is trivial, which means that two general lines parameterized by Y does not intersect each other and $p(1.3)$ is a separable morphism. Hence $\deg p$ must be 1, which implies that p is an isomorphism since X is normal.

q.e.d

Corollary 1.10. Let (X,L) be as in Proposition 1.9. Assume that there are at least two lines through a general point in X. Then (X,L) is a quadric surface in $\mathbb{P}^3$. Consequently the Hilbert scheme of lines in the smooth quadric surface is $P^1 \coprod P^1$.

Proof. The proof of Proposition 1.9 implies that X has two fiberings $\pi_i X \to C_i$ such that every fiber of π_i is a line and C_i is a smooth curve, which gives rise to the morphism $\pi = (\pi_1, \pi_2): X \to C_1 \times C_2$. Easily we see that π is an isomorphism.

q.e.d.

§ 2. Varieties whose hyperplane section is a quadric hypersurface fiber space over a curve.

In this section we shall study an extension theory with respect to a morphism. Throughout this section we assume that char $k \neq 2$.

(2.1) Let (X,L) and $(\overline{X},\overline{L})$ be smooth varieties and X an ample divisor in $|\overline{L}|$ with $\dim X \geq 3$ and $N_{X/\overline{X}} = L$. We assume that $\varphi: X \to C$ is a fiber space over a non-singular curve C such that for every point (a general point) c, $(\varphi^{-1}(c), L|_{\varphi^{-1}(c)})$ is a quadric (irreducible quadric) hypersurface in $\mathbb{P}^n$. Hereafter we call such (X,L) a quadric hypersurface fiber space.

Then we have

Theorem 2. Let (X,L) and $(\overline{X},\overline{L})$ be as above. Then φ can be extended to a morphism $\overline{\varphi}: \overline{X} \to C$. (Lemma 2.6. and 2.7)

First we shall prove the following lemma.

Lemma 2.2 Let (X,L) and $(\overline{X},\overline{L})$ be as above (2.1). If $\dim \mathrm{Alb}\, X = 0$, C is isomorphic to $\mathbb{P}^1$ and $H^1(X,\mathcal{O}_X) = H^1(\overline{X},\mathcal{O}_{\overline{X}}) = 0$. Here Alb X denotes the Albanese variety of X.

For this lemma, we prepare several propositions.

(2.3) Let us assume that $\dim X = 3$ and take a general smooth hyperplane section S in $|L|$ by virture of Beritini's

Theorem. Then we have

Proposition 2.3 S is a ruled surface.

Proof. By the assumption of X, we see that there are infinitely many smooth conics in S. Moreover we use

Sublemma 2.4 Let S be a non-singular surface in $\mathbb{P}^N$. Assume that there is an infinite set of non-singular rational curves whose degree is constant with respect to the hyperplane in $\mathbb{P}^N$. Then S is ruled.

Proof. The above infinite set yields an algebraic family $C = \{C_\lambda\}_{\lambda \in T}$ ($\subseteq S \times T$) with $\dim T \geq 1$ by the general theory of Hilbert scheme. Hence $C_\lambda^2 \geq 0$. On the other hand we know $C_\lambda \cdot K_S = -2 - C_\lambda^2 \leq -2$ by the adjunction formular applied to C_λ, which means $|m\, K_S| = \phi$ for every positive integer m. Then we use the Zariski Theorem [17].

q.e.d

The sublemma 2.4 immediately implies Proposition 2.3.

Proposition 2.5. For the above S, there is a canonical isomorphism Alb S $\overset{i}{\tilde{\rightarrow}}$ Alb C, induced by the restriction map $\varphi_{|S} : S \rightarrow C$. Moreover we have the commutativity of morphims: $\varphi_{|S} \cdot A_C = A_S \cdot i$ where A_V is canonical morphism: $V \rightarrow$ Alb V.

Proof. First of all assume that $C = \mathbb{P}^1$. Then Alb C and Alb S are both just a point, by virtue of the definition of the Albanese variety. Secondly assume that $C \neq \mathbb{P}^1$. Since every fiber of φ_S is a conic, we get $H^1(S\ O_S) \cong H^1(C,O_C)\ (\neq 0)$. Hence S is not rational by [17]. Therefore the commutativity is obvious.

q.e.d.

Proof of Lemma 2.2. By taking successive hyperplane sections of X, we get a sequence of smooth subvarieties $X = X_0 \supset X_1 \supset \ldots \supset X_{n-2}$ where $L_0 = L$, $L_i = L_{i-1}|_{X_i}$, $X_i \in |L_{i-1}|$ and $\dim X_i = n - i$. Then we get the following isomorphism: $\mathrm{Alb}\ X_{n-2} \cong \mathrm{Alb}\ X_{n-3} \cong \ldots \cong \mathrm{Alb}\ X$ by virtue of Theorem 5 of § 2 in VIII [10].

On the other hand $\dim \mathrm{Alb}\ X = 0$ and Proposition 2.5. means that $C = \mathbb{P}^1$ and $H^1(S,O_S) = 0$ with $S = X_{n-2}$.

Now we have

Claim. $H^1(X_{n-2}(= S), mL_{n-2}) = 0$ for any non-positive integer m.

Proof. Since L_{n-2} is very ample, take a curve X_{n-1} in $|L_{n-2}|$ and consider the exact sequence:

$$^*{}_{n-2}\quad 0 \to (m-1)L_{n-2} \to mL_{n-2} \to mL_{n-2}|_{X_{n-1}} \to 0\ .$$

We infer that $h^1(S,mL_{n-2})$ is a monotome-increasing function with respect to non-positive integers m. Therefore $h^1(S,O_S) = 0$ proves this claim.

Finally, to complete the proof of Lemma 2.2, we shall show $H^1(\overline{X},O_{\overline{X}}) = 0$. It suffices to prove the following:

If $H^1(X_i, mL_i) = 0$ $(i \leq n-3)$ for every non-positive integer m, so is $H^1(X_{i-1}, mL_{i-1}) = 0$. In the same way as in above claim, the vanishing of $H^i(X_i, mL_i)$ $(m \leq 0)$ gives rise to the surjection:

$$H^1(X_{i-1}, (m-1)L_{i-1}) \to H^1(X_{i-1}, mL_{i-1}) \to 0$$

by the exact sequence $*_{i-1}$. Hence Serre's vanishing theorem yields our desired result.

q.e.d of Lemma 2.2.

Lemma 2.6. *Let* (X,L) *and* $(\overline{X},\overline{L})$ *be as* (2.1). *Assume that* $C \neq \mathbb{P}^1$. *Then* $\varphi: X \to C$ *can be extended to a morphism* $\overline{\varphi}: \overline{X} \to C$ *and for every point* c *in* C, $(\varphi^{-1}(c), \overline{L}|_{\varphi^{-1}(c)}) \simeq (Q_{n-1}, \mathcal{O}_{Q_{n-1}}(1))$ *where* Q_{n-1} *is a* $(n-1)$-*dimensional quadric hypersurface in* $\mathbb{P}^n$ *and* $\mathcal{O}_{Q_{n-1}}(1) = \mathcal{O}_{\mathbb{P}^n}(1)|Q_{n-1}$.

Proof. First we have the following

Claim: Let Y *and* Z *be smooth varieties and* Y *an effective divisor in* Z. *We suppose that*

1) $\mathrm{Alb}\, Y \xrightarrow{\sim} \mathrm{Alb}\, Z \neq 0$.

2) Y *is numerically positive in* Z, *that is, for every curve* C *in* Z, $Y \cdot C > 0$.

Then we have the commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\ i\ } & Z \\ A_Y \downarrow & \circlearrowright & \downarrow A_Z \\ \mathrm{Alb}\, Y & \xrightarrow{\ \sim\ } & \mathrm{Alb}\, Z, \end{array}$$

and $j \circ A_Y(Y) = A_Z(Z) \circ i$

The proof is easy, hence we omit it. See the proof of Theorem 3 in [2].

Now for the proof of Lemma 2.6 we can suppose $\dim \mathrm{Alb}\, X > 0$ by Lemma 2.2.

First assume that $\dim X = 3$. Taking a general smooth surface S in $|L|$, we see that $j \cdot A_S(S) = A_X(X) \circ i$ by claim above and Lang's Theorem and it follows that $A_X(X) \simeq C$ by Proposition 2.4. Similarly we get the desired morphism $\overline{\varphi}: \overline{X} \to C$ induced by the albanese map $\overline{X} \to \mathrm{Alb}\, \overline{X}$. If $\dim X > 3$, we take a hyperplane section. Then in the same way as above, we can get a morphism $\overline{\varphi}: X \to C$ which is extended from $\varphi: X \to C$ inductively. Let us show that every fiber is a quadric hypersurface. There is a following sequence:

$$0 \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_{\overline{X}}(X) \longrightarrow N_{X/\overline{X}} \longrightarrow 0$$

Taking the direct image of $\overline{\varphi}$, we get

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \underset{\substack{\| \\ \overline{E}}}{\pi_* \mathcal{O}_X(X)} \longrightarrow \underset{\substack{\| \\ E}}{\pi_* N_{X/\overline{X}}} \longrightarrow R'\pi_* \mathcal{O}_{\overline{X}} \longrightarrow .$$

Then noting that $R^1 \pi_* \mathcal{O}_{\overline{X}} = 0$ and E is a vector bundle of rank $(n+1)$ over C, we see that $\overline{E}$ is a vector bundle of rank $(n+2)$ over C. Since X is a divisor of $\mathbb{P}(E)$, it is easy to check that X is linearly equivalent to $\mathcal{O}_{\mathbb{P}(E)}(2) \otimes \varphi^*L$ where L is a line bundle on C. Hence we see that $\overline{X}$ is linearly equivalent to $\mathcal{O}_{\mathbb{P}(\overline{E})}(2) \otimes \varphi^*M$ where M is a line bundle on C, which show that each fiber of $\overline{\varphi}:\overline{X} \longrightarrow C$ is a quadric hypersurface.

In the next place, we shall show the following

Lemma 2.7. Under the same condition as in (2.5), let us assume that $\dim X \geq 3$, and $C = \mathbb{P}^1$. Then for $\overline{X}$ the same conclusion holds as in (2.6).

Before coming to the proof of 2.7. we make a few preliminary remarks.

Taking the normal bundle of a general fiber $(= \varphi^{-1}(c))$ in $\overline{X}$ where $\varphi^{-1}(c)(=V)$ is a non-singular quadric hypersurface, we obtain the following exact sequence:

$$(2.7.0) \qquad 0 \longrightarrow N_{V/X}(=\mathcal{O}) \longrightarrow N_{V/\overline{X}} \longrightarrow N_{X/\overline{X}|V}(=M) \longrightarrow 0 .$$

Since X is a very ample divisor in $\overline{X}$, so is M. On the other hand, since we see easily that $H^1(V,\check{M}) = 0$, we get $N_{V/\overline{X}} = \mathcal{O} \oplus M$. Hence let T be an irreducible component of the Hilbert scheme of $\varphi^{-1}(c)(=V)$ and ω the universal scheme $(\subseteq X \times T)$ of T.

Then the first projection $p:\omega \to \bar{X}$ is surjective. Moreover remark that T is smooth at $[V]$ which denotes the point in T corresponding to V by virtue of $H^1(V,N_{V/\bar{X}}) = 0$ and that ω_t is a quadric hypersurface in $\mathbb{P}^n$ and C is naturally embedded in T.

Now to prove Lemma 2.7, we prepare several propositions. First of all, let us investigate the property of a quadric hypersurface U in $\bar{X}$ whose intersection number $U\cdot\omega_t$ with ω_t is zero. The next proposition is a useful oberservation.

Proposition 2.8. Let X and $\bar{X}$ be as in (2.1), and U an subvariety in $\bar{X}$ whose codimension in $\bar{X}$ is 2 and which is not in X. Assume that codim $(\omega_c \cap U) \leq 3$ with a point c in C. Then the order of $C(U)$ is one where $C(U)$ denotes the set $\{c \in C \mid U \cap \omega_c \neq \phi\}$.

Proof. Since X is ample in $\bar{X}$, $\omega_c \cap X$ is connected. The fiber structure of X immediately implies the proposition.

This gives rise to the following

Proposition 2.9. Using the above notations, let us consider the following three cases for U:

1) U is irreducible, reduced and is not contained in X.

2) U is reducible, namely, $U = P_1 \cup P_2$ and dim P_1 P_2 = n - 3 where P_i is an (n - 2)-linear space

and $n = \dim \overline{X}$.

3) U <u>is a double</u> (n - 2) <u>plane</u>.

<u>Then</u> C(U) <u>consists of one point in each of mentioned cases.</u>

<u>Proof.</u> First consider the case 2). Moreover assume that neither P_1 non P_2 is contained in X. By Proposition 2.8., we see that $C(P_2)$ is a point c_i in C. On the other hand, recalling that $P_i \cong \mathbb{P}^{n-2}$ and $P_1 \cap P_2 \cong \mathbb{P}^{n-3}$ we get $c_1 = c_2$. Secondly, assume that only P_1 is contained in X. Since any morphism from $\mathbb{P}^{n-2}$ to a curve is a constant map, we see P_1 is in a fiber of $\varphi: X \to C$. Hence for this case, we get the desired result. Finally assume that both P_i are contained in X. Easily we obtain the same result. Hence we complete the proof of case 2). Case 1) and 3) are trivial.

q.e.d.

By virtue of the above investigation, we divide Lemma 2.7 to two cases:

(2.7.1) For every member $\omega_t (t \notin C)$, the order of $C(\omega_t)$ is one.

(2.7.2) There is an irreducible $\omega_t (t \notin C)$ in T which is contained in X and which meets $\omega_c (c \in C)$.

<u>Proof of Lemma</u> 2.7 in the case (2.7.1),
for every element c in $C(=P^1)$, we define
$D_c = \cup\{\omega_t \mid \omega_t \cap \omega_c \neq \phi\}$ such that D_c has a reduced structure.
It is obvious that $\{D_c\}_{c \in P^1}$ induces a pencil $\{\tilde{D}_c : c \in P^1\}$

of divisors in $\overline{X}$. Moreover, we see that $\tilde{D}_c \cap \tilde{D}_{c'} = \phi$ for generic elements c, c' in $\mathbb{P}^1$ by our construction of D_c and Proposition 2.9. Hence we can take a reduced and irreducible divisor (=D) and get the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow [D] \longrightarrow N_D(=\mathcal{O}_D) \longrightarrow 0 .$$

Noting that $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$, the complete linear system of [D] gives rise to a morphism $\overline{\varphi}: \overline{X} \longrightarrow P^1$, which extends the morphism φ.

q.e.d.

Next, let us consider the case (2.7.2). The conclusion in this case is as follows

<u>Proposition</u> 2.10. <u>The case</u> 2.7.2 <u>does not occur</u>. <u>Hereafter</u>, till the end of this section, we shall be concerned only with this case. First let us show

(2.11) <u>Step 1.</u> dim X = 3 <u>and</u> ω_t <u>is a non-singular quadric surface (Hereafter we write simply</u> Y <u>in place of this</u> ω_t). <u>Hence for every point</u> c <u>in</u> C, $\varphi^{-1}(c) \cap Y$ <u>corresponds to a fiber</u> $(\cong P^1)$ <u>of</u> $Y(\cong P^1 \times P^1)$.

<u>Proof</u>. <u>Since</u> φ_Y <u>is a surjective map from</u> Y <u>to</u> $\mathbb{P}^1$, <u>it is easy to see that</u> Y <u>is isomorphic to</u> Proj $k[X_0, \ldots, X_{n-1}]/F$ $(=Q_{n-2})$ where $F = \sum_{i=0}^{3} X_i^2$ and $n = \dim \overline{X} (\geq 4)$. Then Pic $Q_{n-2} \cong \mathbb{Z} \oplus \mathbb{Z}$ (See Ex. II. 6.5 in [7]). Moreover it is easy to check that every fiber of φ_Y is $\mathbb{P}^{n-3}$. Hence it follows

that every fiber $\varphi^{-1}(c)$ is an (n-2)-dimensional quadric hypersurface and contains a $\mathbb{P}^{n-3}$ as an ample divisor. Now assume that $\dim X \geq 4$. Noting that a general fiber $\varphi^{-1}(c)$ is smooth, it is known that the Picard group of such a fiber $\varphi^{-1}(c)$ is $\mathbf{Z}$ if $n \geq 5$, which yields a contradiction to the fact that $\varphi^{-1}(c)$ is a projective space by Corollary 3.11 in [11]. Hence we see that $\dim X = 3$ and ω is a non-singular quadric surface. The last part is trivial.

(2.12) <u>Step 2</u> $N_Y = \mathcal{O}(1\ 1)$ <u>where</u> N_Y <u>is the restriction of</u> $N_{X/\overline{X}}$ <u>on</u> Y.

<u>Proof.</u> By virtue of Proposition 5 (Kleiman [9]), $N_Y{}^2 = (N_{X/\overline{X}}, N_{X/\overline{X}}, \omega)_X = (\overline{L}, \overline{L}, \omega)_{\overline{X}}$ where $\overline{L}$ is a line bundle of a divisor X in $\overline{X}$. Since Y and $\varphi^{-1}(c)$ belong to the same algebraic family (see Lemma 2.7) and $\varphi^{-1}(c)$ is a quadric surface, we get $N_Y{}^2 = 2$. On the other hand N_Y is an ample line bundle on a smooth quadratic surface, which completes this step.

(2.13) <u>Step 3</u> $N_{Y/X}$ <u>is a trivial line bundle.</u>

<u>Proof</u>. We have the following exact sequence:

$$0 \to N_{Y/X}(= \mathcal{O}(a,b)) \to N_{Y/\overline{X}} \to N_Y(=\mathcal{O}(1,1)) \to 0.$$

Assume that a or b is negative. Then $h^0(\omega, N_{Y/\overline{X}}) \leq h^0(P^1 \times P^1, \mathcal{O}(1,1)) = 4$, which means that $\dim T \leq 4$ because ω is contained in T.

On the other hand we remark that M in (2,7.0) is $\mathcal{O}(1.1)$ by $M^2 = 2$, $H^1(\mathcal{O}^{-1}(c), \mathcal{O} \oplus M) = 0$ and, therefore, dim $T = 5$. Hence we see that a and b are non-negative. Since $H^1(Y, N_{Y/X} \oplus N_Y^\vee) = 0$, we get $N_{Y/\overline{X}} = \mathcal{O}(a,b) \oplus \mathcal{O}(1.1)$. Noting that $H^1(\omega, N_{Y/\overline{X}}) = 0$ and, therefore, T is smooth at Y, we see that $a = b = 0$, by the computation of $h^0(Y, N_{Y/\overline{X}})$.

(2.14) <u>Step 4</u> X <u>is isomorphic to</u> $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

<u>Proof.</u> As for the line bundle $N_{Y/X}$ we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow [Y] \longrightarrow \mathcal{O}_Y \longrightarrow 0 .$$

Noting $H^1(X; 0_X) = 0$, we get a morphism $\alpha : X \longrightarrow \mathbb{P}^1$ whose general fiber is a non-singular quadric surface. On the other hand, there is originally the morphism $\varphi \quad X \rightarrow \mathbb{P}^1$ by our assumption, which gives rise to a morphism $(\alpha,\varphi) : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. By Step 1, for every point c in C $\varphi^{-1}(c) \cap Y \cong \mathbb{P}^1$. We now

<u>Claim:</u> Every $\varphi^{-1}(c)$ <u>is a non-singular quadric surface</u>.

<u>Proof.</u> Notice that every $\varphi^{-1}(c)$ is a quadric surface. Since we know that every curve on a singular cone intersects other, it does not happen that $\varphi^{-1}(c)$ is a singular cone. Similarly it does not occur that $\varphi^{-1}(c)$ is reducible.

The above claim says that the intersection of a fiber of φ and a fiber of α is a line. Hence $(\alpha,\varphi) : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$

which is a Proj of vector bundle E of rank 2 on $\mathbb{P}^1 \times \mathbb{P}^1$. We see easily that $\mathbb{P}^1 \times \mathbb{P}^1$ itself is an irreducible component of the Hilbert scheme of a general fiber with respect to φ because this fiber is a line. Moreover the universal scheme over $\mathbb{P}^1 \times \mathbb{P}^1$ is X itself by construction. Finally to complete this step we have the following

<u>Claim</u> E <u>is isomorphic to</u> $L \oplus L$ <u>for a line bundle</u> L <u>on</u> $\mathbb{P}^1 \times \mathbb{P}^1$.

<u>Proof.</u> Let pr_i be a i-th projection from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. We must again note that fibres of φ and α are non-singular quadric surfaces. Hence for every point c in $\mathbb{P}^1$, $\mathbb{P}(E)|_{\varphi_c}$, is $\mathbb{P}^1 \times \mathbb{P}^1$ $(\varphi_c = \varphi^{-1}(c))$, which implies that $E|_{\varphi_c}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a_c) \oplus \mathcal{O}_{\mathbb{P}^1}(a_c)$ with an integer a_c. On the other hand $c_1(E)$ is $\mathcal{O}(s,t)$, therefore $c_1(E)|_{\varphi_c} = \mathcal{O}(t)$ and $2a_c = t$, which follows that a_c is independent of c. Applying the base change theorem, we see that $E \otimes \mathcal{O}(0,\frac{t}{2})$ is isomorphic to pr_1^*L where L is a vector bundle on $\mathbb{P}^1$. Next, taking another projection, we see that E is a direct sum of copies of the same line bundle, which completes this claim and Step 4 at the same time. In the next step, let us assume that the characteristic of the base field is zero.

(2.15) <u>Step 5.</u> <u>For the characteristic zero case, there is no</u> 4-<u>fold which contains</u> $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 (=Y)$ <u>as an ample divisor.</u>

<u>Proof</u>. Assume that there exists 4-fold (=X) enjoying the above property. Let p_i be the i-th projection: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$.

Then there is a surjective morphism $q_i: X \longrightarrow \mathbb{P}^1$ extending p_i, by virtue of Proposition III in [14]. Hence we get a surjection $(q_1, q_2, q_3): X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. which means that rank Pic $X \geq 4$. This contradicts the Lefschetz theorem. (For example, see § 1 in [4])

q.e.d.

This yields Proposition in the zero characteristic case. We now turn to the case of positive characteristic. In the next step, we shall prove that $(\overline{X}, X)$ has a lifting $(\overline{\mathfrak{X}}, \mathfrak{X})$ and $\mathfrak{X}_K$ is isomorphic to $\mathbb{P}^1_K \times \mathbb{P}^1_K \times \mathbb{P}^1_K$. Then it is straight-forward to get a contradiction by virtue of the previous step 5. For this purpose, we gather some notation. Given an algebraically closed field k with char $k = p > 0$, let $W(k)$ denote the ring of Witt vectors. This is a discrete valuation ring with the residue field k, the maximal ideal is (p) and the quotient field K is of characteristic zero. S denotes Spec $W(k)$. Let c be the closed point of S. Let X be a non-singular variety defined over k. Then X is said to be liftable if there is an S-scheme $f: \mathfrak{X} \to S$ such that $f^{-1}(c)$ is a given k-scheme X. Then $\mathfrak{X}$ is called a lift of X. Similarly for a given divisor D on X, the pair $(\mathfrak{X}, D)$ is called a lift of (X, D) if there is a lift $f: \mathfrak{X} \to S$ not X and there is an effective Cartier divisor D on $\mathfrak{X}$ whose restriction to X is D.

As for a criterion for a pair $(\overline{X}, X)$ as above to have a lift $(\overline{\mathfrak{X}}, \mathfrak{X})$, we have the following

Theorem 2.15. *Assume that*

1) $H^i(X, \mathcal{O}_X) = H^i(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ *for* $i = 1,2$

2) $H^1(\overline{X}, N) = 0$, *where* N *is line bundle corresponding to a divisor* X *in* $\overline{X}$.

3) $H^2(\overline{X}, T_{\overline{X}}) = 0$.

Then $(\overline{X}, X)$ *has a lifting.*

For the proof, see Lemma 1 in [3] and Theorem (A1) in [5].

(2.16) *Step* 6. The $(\overline{X}, X)$ in question has a lifting $(\overline{\mathfrak{X}}, \mathfrak{X})$.

Proof. Here, $\mathcal{O}(a_1, a_2, a_3)$ denotes a line bundle $\prod_{i=1}^{3} pr_i^{*}, \mathcal{O}_{\mathbb{P}^1}(a_i)$ where $pr_i : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the i-th projection. Notice, that N_X is $\mathcal{O}(1,1,b)$ with $b > 0$ where N_X is the restriction of N to X . First we have the following:

Claim $H^i(X, aN_X) = 0$ *for any integer* a *and* $i = 1,2$. *Therefore* $H^i(\overline{X}, aN) = 0$ *under the above condition.*

Proof. By the Künneth Formula, the former is trivial. For the latter case, we use the exact sequence:

$$0 \longrightarrow (a - 1)N \longrightarrow aN \longrightarrow aN_X \longrightarrow 0$$

which induces the following

$$\longrightarrow H^1(\overline{X}, (a-1)N) \longrightarrow H^1(\overline{X}, aN) \longrightarrow H^1(X, aN_X)$$
$$\longrightarrow H^2(\overline{X}, (a-1)N) \longrightarrow H^2(\overline{X}, aN)$$

Hence $\dim H^1(\overline{X}, aN)$ is monotone decreasing with respect to a, which implies $H^1(\overline{X},aN) = 0$ by Serre duality and Serre's vanishing theorem for ample line bundles. In the same way, we see that $H^2(\overline{X},aN) = 0$.

Finally, to show that $H^2(\overline{X},T_{\overline{X}})$ vanishes, we use the following:

$$0 \to T_{\overline{X}} \otimes (a-1)N \to T_{\overline{X}} \otimes aN \to T_{\overline{X}} \otimes aN|_X \to 0$$

and $0 \to T_X \otimes aN_X \to T_{\overline{X}} \otimes aN|_X \to (a+1)N_X \to 0$.

where N_X denote $N|_X$.

It suffices to show that $H^2(T_X \otimes aN_X)$ vanishes.

But since $T_X \otimes aN_X = (a+2, a+2, ab+2)$ the vanishing is easy by the Künneth formula.

2.17 <u>Step 7</u> $\mathfrak{X}_K$ <u>is isomorphic to</u> $\mathbb{P}^1_K \times \mathbb{P}^1_K \times \mathbb{P}^1_K$ <u>where</u> $\mathfrak{X}_K$ <u>denotes</u> $\mathfrak{X} \times_S \operatorname{Spec} K$.

<u>Proof.</u> It is well-known that $\operatorname{Pic} \mathfrak{X}_K \cong \operatorname{Pic} \mathfrak{X} \cong \operatorname{Pic} X$ by virtue of section 6 in [5]. Hence take a line bundle L^i on $\mathfrak{X}$ such that $L^i \times_S \operatorname{Spec} k$ $(= L^i_k)$ is isomorphic to a line bundle $(0,1,0)$ on $\mathbb{P}^1_k \times \overset{i\text{-th}}{\mathbb{P}^1_k} \times \mathbb{P}^1_k$ $(=X)$. About the notation. Then we have

<u>Claim</u> $H^j(\mathfrak{X}_K, L^i_K) = 0$ <u>for</u> $j = 1,2,3$ <u>where</u> $L^i_K = L^i \times_S \operatorname{Spec} K$. Therefore $\dim H^0(\mathfrak{X}_K, L^i_K) = 2$.

<u>Proof.</u> It is trivial that $H^j(\mathfrak{X}_k, L^i_k) = 0$ for $j \geq 1$. Hence by the semi-continuity of the cohomology of L^i as for

$\varphi:\mathfrak{X} \longrightarrow S$, the former ia obvious. Moreover by the flatness of φ, we get $\chi(X_K, L_K^i) = \chi(X_k, L_k^i)$, which means the latter.

q.e.d.

The above claim implies the following:

$\varphi_* L^i$ is a torsion free sheaf on S. Since S is a discrete valuation ring, $\varphi_* L^i$ is a free module of rank 2. Hence take a basis s_1, s_2 of $\varphi_* L^i$,which gives rise to an S-rational map Φ^i from $\mathfrak{X}$ to $\mathbb{P}^1_S$. It is straight -forward to see that Φ^i is a morphism, because $\Phi^i \times_S k$ is equal to pr_i. Hence we can construct a morphism $\sigma: X_K \to \mathbb{P}^1_K \times \mathbb{P}^1_K \times \mathbb{P}^1_K$ as the fiber product of the Φ^i_K $(i = 1,2,3)$. Noting that 1 is the intersection number of three line bundles $(1,0,0) \cdot (0,1,0) \cdot (0,0,1)$ in X, we know $1 = L_K^1 \cdot L_K^2 \cdot L_K^3$ in X_K . Hence it is easy to check that σ is a finite birational morphism, and therefore, by Zariski Main Theorem, an isomorphism.

q.e.d. Step 7

Now at last we show Lemma 2.10.
$\mathfrak{X}_{\overline{K}}$ and $\overline{\mathfrak{X}}_{\overline{K}}$ are defined over an algebraically closed field $\overline{K}$ of characteristic 0 and $\mathfrak{X}_{\overline{K}}$ is isomorphic to $\mathbb{P}^1_{\overline{K}} \times \mathbb{P}^1_{\overline{K}} \times \mathbb{P}^1_{\overline{K}}$ and ample in $\overline{\mathfrak{X}}_{\overline{K}}$, which yields a contradiction to Step 5.

Finally in this section let us give an important

Lemma 2.18. *Let (Y,L) and $(\overline{Y},\overline{L})$ be a smooth 3-fold and a 4-fold respectively and let Y a divisor in $\overline{Y}$. Assume Y*

is isomorphic to $S_1 \times_C S_2$ _where_ S_i _is a_ $\mathbb{P}^1$-_bundle over a curve_ C _and_ $\mathbb{L}|_{\varphi^{-1}(c)} = \mathcal{O}(1,1)$ _with the canonical projection_ $\varphi: S_1 \times_C S_2 \to C$. _Then_ Y _is not ample in_ $\overline{Y}$.

Proof. If Y were ample, Lemma 2.6 and 2.7 would yield a morphism $\overline{\varphi}: \overline{Y} \longrightarrow C$ extending φ. Then since every fiber of $\overline{\varphi}$ is a quadric hypersurface we can use the latter part of the proof in Proposition 4.10 in [4].

§ 3 Varieties whose hyperplane section is a P^n-bundle.

In this section we work over an arbitrary characteristic field. What we shall prove is as follows:

Theorem 3.1. Let (X,L) be an $(m+n)$-dimensional smooth variety containing many lines, $(\bar{X},\bar{L})$ and $(m+n+1)$-dimensional polarized smooth variety such that $\bar{L}$ is a very ample line bundle on $\bar{X}$. Assume that there is a vector bundle E of rank $(n+1)$ over an m-dimensional variety S such that (X,L) is isomorphic to $(\mathbb{P}(E),\mathcal{O}_{P(E)}(1))$ and that X is a member of $|\bar{L}|$, $N_{X/\bar{X}} = L$ and $n+1 \geq m$. Then, unless X is a quadric surface, there is a vector bundle $\bar{E}$ of rank $n+2$ over S enjoying the following exact sequence:

$$0 \longrightarrow \mathcal{O} \longrightarrow \bar{E} \xrightarrow{\varphi} E \longrightarrow 0,$$

where X is contained in $\bar{X}$ $(\cong \mathbb{P}(\bar{E}))$ via φ. If X is a quadric surface, $(\bar{X},\bar{L})$ is isomorphic to $(Q_3,\mathcal{O}_{Q_3}(1))$.

Hereafter for our proof we consider $\bar{X}$ to be embedded in $\mathbb{P}^{N+1}$ by the line bundle $\bar{L}$, and X to be a hyperplane section of $\bar{X}$ in P^{N+1}.

Moreover we consider three cases separately:

α) $n \geq 2$

β) $n = 1$ and $m = 2$

γ) $n = 1$ and $m = 1$.

Case α)

Step α.1. $(\overline{X},\overline{L})$ contains many lines.

Proof. Take a line ℓ in a fiber $\pi^{-1}(s)$ $(= X_s)$ for a point s in S where π is a canonical projection $\mathbb{P}(E) \to S$. Then we have the following exact sequence:

$$0 \longrightarrow N_{\ell/X_s} \longrightarrow N_{\ell/X} \longrightarrow N_{X_s/X}|_{\ell} \longrightarrow 0$$

which means that $N_{\ell/X} = \mathcal{O}^m_{P^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n-1}$.

Moreover we get the exact sequence:

$$(3.1) \quad 0 \longrightarrow N_{\ell/X} \longrightarrow N_{\ell/\overline{X}} \longrightarrow N_{X/\overline{X}}|_{\ell} \longrightarrow 0 ,$$

which implies $N_{\ell/\overline{X}} = \mathcal{O}^m_{P^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^n$.

Hence we obtain step 1 by the computation of $h^i(\ell,N_{\ell/X})$ and $h^i(\ell,N_{\ell/\overline{X}})$.

q.e.d.

Remark. In step 1, take an irreducible component T (or, $\overline{T}$) of Hilbert scheme of lines in X (or, $\overline{X}$, resp.) containing a line ℓ in a fiber of π. Note that T (or, $\overline{T}$) can be naturally considered to be a smooth subvariety in $Gr(N,1)$ (or, a variety in $Gr(N+1,1)$ which is smooth on S) where $Gr(N,1)$ is a canonically embedded in $Gr(N+1,1)$.

Step α.2. For every fiber of π $(=\mathbb{P}^n_\pi)$, there is a unique $(n+1)$-linear space in $\overline{X}$ containing $\mathbb{P}^n_\pi$. Hence $\overline{X}$ is

<u>fulled up by these</u> $(n+1)$-<u>linear spaces.</u>

<u>Proof.</u> First of all, for a subset B in P^{N+1} and a point $\mathfrak{y}$ in B, $B_{\mathfrak{y}}$ denotes the set of lines in B passing through $\mathfrak{y}$. Now take a line ℓ of a fiber $\mathbb{P}^n_{\pi}$ and a point x in ℓ. P^N_x and X_x are naturally a hyperplane section $(= P^{N-1})$ and an $(n-1)$-dimensional linear subspace in $(P^{N+1})_x (= P^N)$ respectively. Moreover by considering the exact sequence $(3,1) \otimes \mathcal{O}_{\ell}(-x)$, X_x is an irreducible component of $\overline{X}_x \cap (\mathbb{P}^N)_x$ and it is an $(n-1)$-dimensional linear space with multiplicity 1. Taking account of $\overline{X}_x \not\subset \mathbb{P}^{N-1}$, $\overline{X}_x$ has an n-dimensional linear space in $\mathbb{P}^{N+1}_x$ as its component which yields an $(n+1)$-dimensional linear space in $\overline{X}$ containing $\mathbb{P}^n_{\pi}$.

q.e.d.

Now fix an $(n+1)$-linear space R in $\overline{X}$ and take an irreducible component $\overline{U}$ of Hilbert scheme containing R in $\overline{X}$. Let $\mathfrak{U}$ be the universal scheme in $\overline{U} \times \overline{X}$ where $p_{\overline{U}}$ (or $p_{\overline{X}}$) is the canonical projection $\mathfrak{U} \longrightarrow \overline{U}$ (or, $\mathfrak{U} \to \overline{X}$, resp.). Remark that S itself is a Hilbert scheme of a fiber of π and X itself its universal scheme. Then taking an inverse image $p_{\overline{X}}^{-1}(X)$, we get the morphism $\varphi : \overline{U} \longrightarrow S$ by the universality of Hilbert scheme $\overline{U}$, which follows that φ is an isomorphism by Step α.2 and Zariski Main Theorem. Hence $p_{\overline{X}} : \mathfrak{U} \longrightarrow \overline{X}$ is birational, and when we put $W = \{x \in \overline{X} \mid \dim p_{\overline{X}}^{-1}(x) \geq 1\}$, W is at most a finitely many set because X is an ample divisor on $\overline{X}$ and $p_{\overline{X}}: p_{\overline{X}}^{-1}(X) \to X$ is an isomorphism. Now we shall show that W is empty. Assume that $n \geq m$ and W is not empty. That W is a finite subset means that there are two $(n+1)$-linear spaces: R_1, R_2 in $\overline{X}$ such that $\dim (R_1 \cap R_2) = 0$, which immediately yields a contradiction because $\dim R_1 + \dim R_2 - \dim \overline{X} \geq n - m + 1 \geq 1$. Secondly assume that $n + 1 = m$ and W is a finitely many set. Then we see that every $(n+1)$-dimensional linear space R_u induced by U intersects at, one point with each other. Moreover there is unique point $\mathfrak{z}$ in $\overline{X} - X$ contained in every R_u because of the flatness of $p_u: \mathfrak{U} \to U$ and finiteness of the set W. Now take a tangent space $T_{\mathfrak{z}}$ of $\overline{X}$ at $\mathfrak{z}$ in P^{N+1}. Clearly $\overline{X}$ is contained in $T_{\mathfrak{z}}$ $(\cong \mathbb{P}^{n+m+1})$ which follows that $\overline{X} \cong \mathbb{P}^{n+1}$. This is absurd. Therefore W is empty, which follows that $p_{\overline{X}}: \mathfrak{U} \to \overline{X}$ is an isomorphism. This is a desired fact.

q.e.d.

Case β) In this case, taking account of the above proof, we divide to two cases. We maintain the notations $\overline{\mathfrak{T}}$, Gr(N,1) in Remark.

(β.1) S is the only irreducible component in $\overline{T} \cap Gr(N,1)$ whose lines fill up the whole space X.

(β.2) Otherwise, namely, there is another line passing through a general point x in X besides a fiber of π, which yields another irreducible component S_2 of Hilbert scheme of another lines whose lines move X.

Hereafter we shall determine the structure of $\overline{X}$ in case of (β,1)

(β.1) If one reads the proof of step (α.1) and (α.2) carefully, these say the fact that there is unique 2-linear space R with multiplicity 1 containing a general line ℓ in X. Now maintain notations $\mathfrak{U}$, U in Step α.1. Then for every point of u in U, $p_U^{-1}(w))$ is isomorphic to a 2-linear space, because $p_U^{-1}(u)$ for a general point u in U is a 2-linear space. Simultaneously we see that there is a point u in U whose 2-linear space contains a fiber of π (=line). Therefore we have

<u>Claim</u> (β.6). <u>For every fiber</u> F <u>of</u> π, <u>put as</u> U(F) <u>the subset</u> $\{u \in U \mid R_u \supset F\}$ <u>where</u> R_u <u>is a 2-plane corresponding to</u> u. <u>Then for a general fiber</u> F, U(F) <u>is one point with multiplicity 1.</u> <u>Moreover for every</u> F, U(F) <u>is a finite set.</u>

<u>Proof</u>. The former part is already proved. For the latter part assume that U(F) is an infinite set. One-dimensioanl irreducible curve C in U(F) yields a divisor D in $\overline{X}$. Hence dim $(D \cap X) = 2$.

On the other hand, R (= 2-linear space) is not contained in X, therefore $R \cap X = F$. Hence $D \cap X = \ell$ which is absurd.

q.e.d.

Continued proof.

The above claim induces a finite birational morphism $\varphi: U \to S$ by the same way as in (α.2). Hence φ is an isomorphism. The remainder is entirely same as in case α. Hence we finish the case (β.1).

(β,2) By virtue of Th. 6.3., X is one of the following

(β.2.1) $X \cong \mathbb{P}(T_{\mathbb{P}^2})$

(β.2.2) $X \cong S_1 \times_C S_2$ where S_i is a $\mathbb{P}^1$-bundle over a smooth curve C.

Hereafter we shall show that above two cases do not occur.

Case (β,2.1) At first we show that $(\overline{X},L)$ has the following property: $\Delta(\overline{X},L) = 1$, $g(\overline{X},L) = 1$ and $d(\overline{X},L) = 6$ in the sense of Fujita [5].

The Second and last part are tivial. For the first, we must check the following: $H^1(X, mN_{X/\overline{X}}) = 0$ for any integer m, which implies that $H^1(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ by virtue of Serre's vanishing Theorem.

Hence we get $\Delta(\overline{X},L) = 1$. Consequently we see that $(\overline{X},\overline{L})$ is isomorphic to $(\mathbb{P}^2 \times \mathbb{P}^2, p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1))$ where $p_i: \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ is the i-th proposition by virtue of [5], which implies that two pieces of family of lines in $\mathbb{P}(T_{\mathbb{P}^2})$ are induced by two pieces of family of planes in $\mathbb{P}^2 \times \mathbb{P}^2$. This is a contradiction.

Case (β.2,2) This case does not occur by virtue of Lemma 2.18.

Case γ) First assume that x is a quadric surface. Then by virtue of Corollary 1.10, there is only one line going through a general point. In the same way as in (β.1) above, we get a desired result.

q.e.d.

Gathering the above observations of case α), β) and γ) we complete a proof of Theorem 3.1.

Finally we shall give a

(3.2) <u>Proof of Main Theorem in the case:</u> $\ell(x\mathbb{L}) = 1$.

Let (x,L) be a smooth n-fold $(n \geq 3)$ which contains many lines in any characteristic case. Assume that $\ell(x,L) = 1$. Then (x,L) is isomorphic to

1) (Quadric hypersurface Q in $\mathbb{P}^{n+1}, \mathcal{O}_Q(1)$)
2) (Proj of a very ample vector bundle E of rank n over a smooth curve, $\mathcal{O}_{\mathbb{P}(E)}(1)$)

<u>Proof</u>. In the same way as the proof in Lemma 2.2 we take a sequence of smooth subvarieties $X = X_0 \supset X_1 \supset \ldots \supset X_{n-1}$ with $\dim X_i = n - i$. Then Proposition 1.8 gives rise to the fact that $\ell(X_i, L_i) = 1$. Hence by Proposition 1.9 and Corollary 7.10, Theorem 3.1, we get the desired result.

q.e.d.

§ 4. 3-fold with many quadric surfaces

In the present section we shall study the structure of 3-fold (X,L) with many quadric surfaces, which is defined as follows: Let L be a very ample line bundle on X and φ_L the corresponding closed immersion of L. We assume the existence of an irreducible quadric surface in $\varphi_L(X)$ in $\mathbb{P}^{\dim|L|}$ passing through a general point in $\varphi_L(X)$. Hereafter throughout this section (X,L) denotes the above.

Our main goal in this section is to show

Theorem 4 Let (X,L) be as above. Then (X,L) is isomorphic to one of the following:

1) X is a quadric surface fiber space over a smooth curve (2.1)
2) $(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1,1))$.
3) (smooth quadric hypersurface Q in $\mathbb{P}^4$, $\mathcal{O}_Q(1)$)

(4.1) Now take a general irreducible quadric surface Q in $\varphi_L(X)$ and consider the Hilbert scheme of Q in $\varphi_L(X)$. Then there is an irreducible component T of the Hilbert scheme containing Q such that $\dim T \geq 1$. Moreover let $\omega = \{\omega_t\}$ be the universal scheme of $T (\subseteq X \times T)$ where $\omega_t = \omega \cap \{t\} \times T$.

(4.2) Assume that there is a general point (r,s) in $T \times T$ such that $\omega_r \cap \omega_s = \phi$, which is maintained until (4.6).

After (4.7) we shall discuss the case of $\omega_r \cap \omega_s \neq \phi$. Put as N the normal bundle of Q in X. To observe the structure of X, we divide to two cases:

a) There is a point t such that ω_t is smooth.

b) There is no point t such that ω_t is smooth.

Lemma 4.2.a. For the case a), dim T = 1 and N is trivial.

Proof. Put $N(=N_{Q/X}) = \mathcal{O}(c,d)$ $(Q \cong \mathbb{P}^1 \times \mathbb{P}^1)$.

The assumption of (4.2) means that c and d are non-negative. On the other hand if either c or d is positive, then $\dim H^0(Q,N) \geq 2$ and $H^1(Q,N) = 0$, which follows there are two points r,s such that ω_s properly. This contradicts (4.2)

q.e.d.

(4.2.b) For the case (b), we have the same result as in case of (a).

Proof. Picard group of an irreducible singular quadric surface Q is isomorphic to $\mathbb{Z}L$ whre L is the line bundle corresponding to a hyperplane section. Remarking that $h^0(Q,\mathcal{O}_Q(c)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(c))$ and $h^1(Q,\mathcal{O}_Q(c)) = 0$ for every integer c , one can get the same result as in case a)

q.e.d.

(4.3) Assume that $H^1(X,\mathcal{O}) = 0$.

(4.3.a) For the case a) , we have the following exact sequence:

$$0 \longrightarrow \mathcal{O} \longrightarrow [Q] \longrightarrow N(=\mathcal{O}) \longrightarrow 0 ,$$

which gives rise to the exact sequence of cohomologies

$$0 \longrightarrow H^0(\mathcal{O}) \longrightarrow H^0([Q]) \longrightarrow H^0(N) \longrightarrow 0$$

where $[Q]$ is the line bundle corresponding to the divisor Q. It yields the morphism $\varphi: X \longrightarrow \mathbb{P}^1$.

(4.3.b) For b), we get a morphism $\varphi: X \longrightarrow \mathbb{P}^1$ entirely in the same way. By the construction, a general fiber is irreducible, hence, smooth, which contradicts our assumption b).

Summerising the above results,

<u>Lemma</u> 4.4. <u>Let</u> (X,L) <u>be a smooth</u> 3-<u>fold. Assume</u> (X,L) <u>has many quadric surface, general such surface is irreducible and there is a point</u> (r,s) <u>in</u> $T \times T$ <u>such that</u> $\omega_r \cap \omega_s = \phi$. <u>Moreover we assume</u> $H^1(X, \mathcal{O}_X) = 0$. <u>Then there is a morphism</u> $\varphi: X \rightarrow \mathbb{P}^1$ whose general fiber is smooth quadric.

On the next place let us assume

(4.5) $\quad H^1(X, \mathcal{O}_X) \neq 0.$

Then we shall show

<u>Lemma</u> 4.6 <u>Let</u> (X,L) <u>be a smooth</u> 3-<u>fold satisfying the same condition as in Lemma 4.4 except</u> $H^1(X, \mathcal{O}_X) = 0$. <u>If</u> $H^1(X, \mathcal{O}_X) \neq 0$, <u>the same conclusion holds.</u>

For the purpose we prepare the several steps.

(4.6.1) Step 1. Let S be a generic smooth member in $|L|$. Then S is ruled.

Proof. Noting that an irreducible quadric surface has at most one singular point and its general hyperplane section is a smooth conic, we get Step 1 by virtue of sublemma 2.4.

(4.6.2) Step 2. $H^1(S,m[S]|_S) = 0$ for $m < 0$ where $[S]$ is the line bundle in X corresponding to a divisor S.

Proof. By step 1, S is ruled. In the set $\{\omega_t \cap S(=C_t) \mid t \in T\}$ there is an infinitely many elements T_0 and T such that C_t and $C_{t'}$ have no common point for $t \neq t'$ in T_0. Noting that C_t is a smooth conic, S has a fiber structure $\varphi : S \longrightarrow C$ where C_t $(t \in T_0)$ is a fiber of φ. Hence Leray spectral sequence yields the isomorphism $H^1(S,m[S]_{|S}) \cong H^0(C,R^1\varphi_*(m[S]|_S))$ for $m< 0$ because $H^0(f,m[S]|_f) = 0$, where f is a fiber of φ. Now take a general element C' in $|[S]|_S|$. The morphism $\varphi : S \rightarrow C$ induces a double covering φ' $C' \longrightarrow C$ where φ' is the restrict of φ to C'. We have the following exact sequence:

$$0 \longrightarrow \mathcal{O}(-C') \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{C'} \longrightarrow 0 \quad ,$$

hence, $0 \longrightarrow (m-1)[S]_{|S} \longrightarrow m[S]_{|S} \longrightarrow m[S]_{|C'} \longrightarrow 0 \qquad *_m$

For $m < 0$, take a direct image of the above exact sequence by φ. Then we get

$$0 \longrightarrow \varphi_*(m[S]|_{C'}) \longrightarrow R'\varphi_*((m-1)[S]|_S) \longrightarrow \underset{\displaystyle E_m}{\underset{\|}{R'\varphi_*(m[S]|_S)}} \rightarrow 0.$$

Hence, taking the long exact sequence of above exact and noting that $-m[S]|_{C'}$ is ample for $m < 0$, we see that $h^0(C,E_m)$ is a monotone-increasing function with respect to negative integers m. Now we have

<u>Claim</u> $H^0(C_1,E_{-1}) = 0.$

In fact, consider the direct image of $*_0$. Then we get the following:

$$\underset{\displaystyle 0}{\underset{\|}{\varphi_*\mathcal{O}(-C')}} \longrightarrow \underset{\displaystyle \mathcal{O}_C}{\underset{\|}{\varphi_*\mathcal{O}_S}} \longrightarrow \varphi_*\mathcal{O}_{C'} \longrightarrow \underset{\displaystyle E_{-1}}{\underset{\|}{R'\varphi_*\ \mathcal{O}(-C')}} \longrightarrow \underset{\displaystyle 0}{\underset{\|}{R'\varphi_*\mathcal{O}_S}}$$

On the other hand, since $\varphi'\ C' \rightarrow C$ is a double covering, we see that $\varphi_*\mathcal{O}_{C'} \cong \mathcal{O}_C \oplus R'\varphi_*\mathcal{O}(-C')$ by char $k \neq 2$ and trace map of φ'. Hence we get a desired result.

Step 2 is shown immediately by the monotone-increasing property and the above claim.

q.e.d. of Step 2

(4.6.3) <u>Step 3</u> $H^1(S,\mathcal{O}_S)\ (\cong H^1(C,\mathcal{O}_C)) \neq 0$

<u>Proof.</u> Consider the exact sequence:

$$0 \longrightarrow (m-1)[S] \longrightarrow m[S] \longrightarrow m[S]|_S \longrightarrow 0.$$

Step 2 says that $H^1((m-1)[S]) \longrightarrow H^1(m[S])$ is surjective for any negative integer m, hence, which infers that $H^1(S,m[S])$ vanishes for $m < 0$ by Serre's vanishing theorem. Therefore we can show Step 3 by the exact sequence:

$$0 \longrightarrow [-S] \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_S \longrightarrow 0 \quad ,$$

and by our assumption that $H^1(X,\mathcal{O}_X) \neq 0$. It is trivial that $H^1(S,\mathcal{O}_S)$ is isomorphic to $H^1(0,\mathcal{O}_C)$.

q.e.d of Step 3.

Finally to complete the proof Lemma 4.6, we take an Albanese variety as for $S \hookrightarrow X$.

$$\begin{array}{ccc} S & \overset{i}{\hookrightarrow} & X \\ \alpha\downarrow & & \downarrow\beta \\ \mathrm{Alb}\ S & \underset{a}{\longrightarrow} & \mathrm{Alb}\ X\ . \end{array}$$

At first we must remark that a is isomorphic by virtue of Theorem 5 of § 2 in VIII [10]. Moreover α factorizes the product of the morphism $\varphi: S \longrightarrow C$ and a closed immersion $i: C \hookrightarrow \mathrm{Alb}\ S$. Hence noting the claim in Lemma 2.5, we see easily that $\beta(c)$ is C and a general fiber of β is a quadric surface.

<u>Remark.</u> In characteristic zero case, we can obtain Lemma 4.6 more easily. We have only to take the albanese map of X,

$i : X \to \mathrm{Alb}\, X$. Then we know that any quadric surface collapses one point by i, which induces lemma 4.6.

Let us maintain the condition and notations in (4.1) Now we assume that

(4.7) (X, L) is not isomorphic to $(\mathbb{P}^3, \mathcal{O}(1))$ and there is a general point (r,s) in $T \times T$

such that $\omega_r \cap \omega_s \neq \phi$.

Since ω_r is a quadric surface in $\mathbb{P}^3$, let $[\omega_r]$ be the smallest linear subspace containing ω_r. For almost all ω_r, $[\omega_r]$ is of 3-dimensional. Hence for two general ω_r, ω_s, noting that $[\omega_r] \neq [\omega_s]$ by $X \neq \mathbb{P}^3$ and taking account of the fact that $[\omega_r] \cap \omega_s \supset \omega_r \cap \omega_s$, $\omega_r \cap \omega_s$ is one of the following

(4.8) $\omega_r \cap \omega_s$ is a line

(4.9) $\omega_r \cap \omega_s$ is a conic which may be singular, including a double line.

From now on we shall show that

Lemma 4.10 X in (4.8) is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$, and X in (4.9) is a quadric hypersurface in $\mathbb{P}^4$.

In the first place let us investigate (4.8). Then we shall divide to two cases

(4.8.1) every member ω_t $(t \in T)$ is singular.

(4.8.2) a general member ω_t is smooth.

(4.8.1) Both ω_r and ω_s are singular cone. Hence, if $\omega_r \cap \omega_s$ contain a vertex of either cone, $\omega_r \cap \omega_s$ must be singular at this vertex. On the other hand, a line on a cone must go through a vertex. It is absurd.

(4.8.2) For a smooth member ω_r, we can take another line ℓ on ω_r such that $\omega_r \cap \omega_s \cap \ell \neq \omega_r$. Then we have the following exact sequence:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_{\ell/\omega_r} & \longrightarrow & N_{\ell/X} & \longrightarrow & N_{\omega_r/X|\ell} & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
 & & \mathcal{O} & & & & \mathcal{O}(1) & &
\end{array}
$$

which follows that $N_{\ell/X} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore by virtue of we see that $\ell(X,L) = 1$, which means that X is one of a quadric hypersurface Q^3 in $\mathbb{P}^4$ and $\mathbb{P}^2$-bundle over curve C. But the former does not occur because the intersection of two quadric surfaces in Q^3 is conic.

In the next place, when $\pi : X \to C$ denotes an P^2-bundle over C, we see that C is $\mathbb{P}^1$ because ω_r is a smooth quadric surface and $\pi|_{\omega_r} : \omega_r \longrightarrow C$ is surjective. Hence X is described as Proj of vector bundle $\mathcal{O} \oplus \mathcal{O}(a) \oplus \mathcal{O}(b)$ on P^1. It is easy to check that $H^1(X, \mathcal{O}_X) = 0$ by Leray's spectral sequence.

Now there is the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow [\omega_r] \longrightarrow N\omega_{r/X} \ (= \mathcal{O}(1,0)) \longrightarrow 0$$

hence we get

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0([\omega_r]) \longrightarrow H^0(\mathcal{O}(1,0)) \longrightarrow 0$$

which gives rise to a morphism $g:X \to \mathbb{P}^2$.
By the construction of g, a general fiber of g is a line. Hence the morphism $(\pi,g):X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ is a birational morphism by virtue of the property of π and g. On the other hand, the structure of Chow ring as for X and $\mathbb{P}^1 \times \mathbb{P}^2$ is equal to each other, which means that (π,g) is finite, therefore is isomorphism.

Finally, let us consider the case (4,9). If we can show that $\dim |L| = 4$, it is straight-forward to see that (X,L) is a quadric hypersurface in $\mathbb{P}^4$.

Therefore assume that $\dim |L| \geq 5$. Then 4 is the dimension of the smallest linear space which contains two generic quadric surface ω_r, ω_s in $\mathbb{P}^{\dim|L|}$ because $\omega_r \cap \omega_s$ is a conic. Hence we can find a divisor $H_\omega = \omega_r + \omega_s + D'$ in $|L|$ where D' is an effective divisor or empty. Now take a line ℓ on ω_r. Then we see that the intersection number $\ell \cdot \omega_r$ and $\ell \cdot \omega_s$ is 1 respectively, which implies $H_\omega \cdot \ell \geq 2$. On the other hand $L \cdot \ell = 1$ was our assumption, which is absurd.

q.e.d

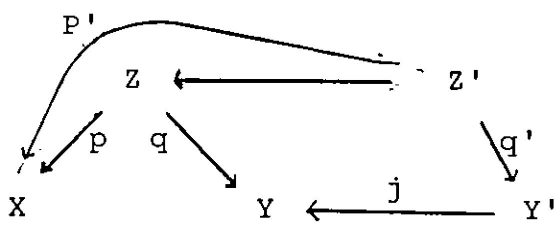
Gathering Lemma 4.4, 4.6 and 4.8, we complete the proof of Theorem 4.

§ 5 Classification of (X,L) with $\ell(X,L) = 2$

Throughout this section, we assume that $p:Z \longrightarrow X$ is separable (see (1.3)) and $\ell(X,L) = 2$.

(5.1) First we suppose $\dim X = 3$, therefore $\dim Y = 2$. Remark that Z is isomorphic to Proj of the vector bundle E where $E = U(N,1)|_Y$ and $U(n,1)$ is the universal bundle of rank 2 over $Gr(n,1)$.

Now let Y' be a desingularisation of Y, Z' the fiber product $Z \times_Y Y'$, which is smooth.



(5.2) Take a general point y in Y. Then we can assume that Y is smooth at y. Let ℓ_y be a line in X corresponding to y.

Then we have

Proposition 5.3. Let $\ell(=\ell_y)$ be a line in X as above. Then $N_{\ell/N} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.

Proof. The separable morphism p induces a generally surjective homomorphism of normal bundles: $k:N_{q^{-1}(y)/Z} \longrightarrow N_{\ell/X}$. Noting that the first part of two normal bundles in $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and the latter $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with integers a, b, we see that a and b are non-negative. On the other hand $\ell(X,L) = 2$ means that $h^0(\ell, N_{\ell/X}) = 2$, namely, $a = b = 0$.

q.e.d.

The above proposition immediately gives rise to

Corollary 5.4. *Under the above diagram in* (5.1), *let* R *be the ramification divisor in* Z' *with respect to* p', *which may be empty. Then there is an effective divisor* C *in* Y' *which is possibly empty, such that* $R = q'^{-1}(C)$. *Moreover we have* $p'^*(K_X + 2L) = q'^*(\det j^*E + K_{Y'} - \mathcal{O}_{Y'}(C))$.

Proof. The former part is obvious by virtue of Corollary 5.4. As for the latter part, consider the following exact sequence,

$$0 \longrightarrow q'^*\Omega_{Y'} \longrightarrow \Omega_{Z'} \longrightarrow \Omega_* \longrightarrow 0$$

$$0 \longrightarrow T \otimes \Omega_* \longrightarrow q'^*j^*E \longrightarrow T \longrightarrow 0,$$

where Ω_* is the relative canonical bundle with respect to q', and T the tautological line bundle of j^*E. Noting that $p'^*L = T$ and $p'^*K_X = K_{Z'} + \mathcal{O}_{Z'}(R)$, we get the desired results.

q.e.d.

Now at first we shall determine the structure of (X,L) with $\deg p = 1$.

Proposition 5.5. *Let* (X,L) *be a smooth* 3-*fold with* $\ell(X,L) = 2$.

Assume that deg $p = 1$. Then (X,L) is isomorphic to $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$, where $E(=Q(N,1)|_Y)$ is a very ample vector bundle on Y.

Proof. Put X_0 as $\{x \in X \mid \dim p^{-1}(x) \geq 1\}$ and assume that X_0 is non-empty. Since X is smooth we see that $Z - p^{-1}(X_0) \longrightarrow X - X_0$ is isomorphic and $\dim X_0 = 1$. By virtue of Proposition 5.2, it follows that $p^{-1}(X_0) = q^{-1}C'$ where C' is a closed subscheme in Y and $\dim C' = 1$. Because if $\dim C' = 0$, $p:Z \rightarrow X$ is an isomorphism by the definition of Hilbert scheme and Zariski Main Theorem as for p. Noting that ⊛ $\dim p^{-1}X_0 > \dim X_0$ and every fiber of q can be transformed a line in X by p, ⊛ contradicts the definition of Hilbert scheme.

q.e.d.

Remark 5.5.1 Let F be a locally free sheaf over a smooth projective M. Then obviously $\mathbb{P}(F)$ is smooth. On the contrary let F' be a torsion free sheaf over a smooth projective surface S. Assume $\mathbb{P}(F')$ is smooth. Then we have the following result: F' is locally free. Because $\mathbb{P}(F')$ is embedded in a high-dimensional projective space $(=\mathbb{P}^N)$ such that every fiber of $\pi:\mathbb{P}(F') \longrightarrow S$ is a linear space in $\mathbb{P}^N$. Taking a general hyperplane section of $\mathbb{P}(F')$ in $\mathbb{P}^N$ sucessively, we reduce this problem to the case of rank $F' = 2$. Finally by Proposition 5.5. we get the result: F' is locally free.

In the next place, we assume that $\deg p \geq 2$. First for every line ℓ in X, we define a reduced closed subscheme in X as follows.

(5.6) $S(\ell) :\underset{\text{def}}{=}$ union of lines L_y which intersect the line ℓ where L_y is the line corresponding to an element y in Y. (Such L_y is said to be a line in Y).

Noting that $S(\ell)$ is equal to $pa^{-1}qp^{-1}(\ell)$ as a set in (5.1), we can give $S(\ell)$ a reduced algebraic structure. We see easily that

(5.7) $S(\ell)$ is a 2-dimensional closed subscheme in X and there exists an open subset Y_0 in Y enjoying the following: For every line ℓ in Y_0, there is an irreducible 2-dimensional component $A(\ell)$ of $S(\ell)$ such that $A(\ell)$ contains a line in Y passing through every point in ℓ.

Under the above notations, we consider two cases.

(5.7.1) For every point y in Y_0, $p^*A(\ell_y)$ is contained in $q^*\mathrm{Pic}\, Y$.

(5.7.2) There is a line ℓ in Y_0, such that $p^*A(\ell)$ is not contained in $q^*\mathrm{Pic}\, Y$.

Then we can show for the first case $\deg p = 2$ and X is a quadric surface fiber space. In the second case, we infer that X is a Fano 3-fold.

For the purpose we need several propositions.

<u>Proposition</u> 5.8. <u>Under the condition of</u> deg $p \geq 2$, <u>let</u> $\mathfrak{D} = \{D_\lambda \mid \lambda \in \Lambda\}$ be a set which consists of infinitely many <u>irreducible effective divisors in</u> (X,L). <u>Assume that</u> $\ell(X,L) = 2$ <u>and</u> p^*D_λ <u>is contained in</u> $q^*\mathrm{Pic}\ Y$ <u>for every element in</u> $\mathfrak{D}$.

<u>Then the following holds:</u>

1) <u>Set the subset:</u> $\{D_\lambda | D_\lambda$ <u>is a plane as for</u> $(X,L)\}$ <u>in</u> $\mathfrak{D}$ <u>as</u> M. <u>Then</u> M <u>is a finite set.</u>

2) <u>Except a finite many elements in</u> $\mathfrak{D}$, $p^{-1}D_\lambda$ <u>consists of two irreducible components</u> E.

<u>Therefore</u> (X,L) <u>is a quadric surface fiber space.</u>

<u>Proof</u>. First of all, we have

<u>Claim: If</u> (X,L) <u>has infinite many planes in</u> $\mathbb{P}^{\dim|L|}$ <u>as for the immersion induced by the complete linear system</u> $|L|$, $\ell(X,L) = 0$ <u>or</u> 1.

<u>Proof</u>. The above assumption gives rise to an algebra family of planes $(\mathfrak{X},T)$ with $\dim T \geq 1$ and $\mathfrak{X} \subset X \times T$ by taking a Hilbert scheme of planes, which infers that there is a plane P in X such that $N_{P/X} = \mathcal{O}_{\mathbb{P}^2}(a)$ and $a \geq 0$. This yields the following exact sequence of normal bundles: for a line ℓ in X

$$0 \longrightarrow \underset{\mathcal{O}_\ell(1)}{\underset{\|}{N_{\ell/P}}} \longrightarrow N_{\ell/X} \longrightarrow \underset{\mathcal{O}_\ell(a)}{\underset{\|}{N_{P/X|\ell}}} \longrightarrow 0 \quad .$$

Hence we get $\ell(X,L) = 0$ or 1.

q.e.d.

The above claim induces 1).

Secondly, noting that if a surface is generated by $r(\geq 3)$ pieces of 1-dimensional families of lines, then such a surface is a plane, 2) is obvious by 1) and the separability of p. Finally, since we know that only a quadric surface has 2-pieces of 1-dimensional families of lines, the last part is trivial by our assumption of $\ell(X,L) = 2$.

q.e.d.

The above immediately yields

Corollary 5.9 Under the notations in (5.6) and (5.7), assume the condition (5.7.1). Then $\deg p = 2$ and (X,L) is a quadric surface fiber space.

In the next place, let us determine the structure of (X,L) in the case (5.7.2).

The following proposition (5.10) is a key for main theorem, which states a criterion for a given line bundle to be numerically equivalent to zero. Here we consider this terminology on a complete algebraic scheme which is not necessarily smooth. Hence let us recall it by Kleiman's paper [9].

Let V be a complete algebraic scheme over k and M an invertible sheaf on V. We call M numerically trivial and write $M \approx 0$ if $(M \cdot C)_V = 0$ for all closed integral curve C in V. Then he shows that

Proposition (4, Corollary 1 [9])

Let $f: V' \longrightarrow V$ be a morphism between algebraic complete schemes, M an invertible sheaf on V and $M' = f^*M$. Then

(i) $M \approx 0 \Rightarrow M' \approx 0$, *and conversely*.

(ii) $M \approx 0 \Leftarrow M' \approx 0$, *if f is surjective*.

Now we get

Proposition 5.10. *Under the diagram in (5.1), let D be an irreducible divisor in X, and D_1, D_2 irreducible Weil divisors in Z and let ℓ be a line sitting on D in X. Now we assume that*

1) *D_1 and D_2 are irreducible components of $p^{-1}D$ in Z,*
2) *qD_1 is a curve in Y and $qD_2 = Y$, and ,*
3) *Two curve ℓ_1 and ℓ_2 are irreducible components of $p^{-1}\ell$ such that $q(\ell_1)$ is one point in Y and $q(\ell_2)$ a curve. At last we assume that there are two line bundles L and M on X and Y respectively, such that $p^*L = q^*M$. Then L and M are numerically trivial.*

This proposition immediately provides us with

Corollary 5.11. Under the condition (5.7.2), $K_X + 2L$ is numerically trivial.

Proof of Proposition 5.10. Consider the restriction of the line bundle on $\ell_1 : p^*L|_{\ell_1} = q^*M|_{\ell_1}$. Since $q^*M|_{\ell_1} = q^*(M|_{q(\ell_1)})$ by the commutativity and $q(\ell_1)$ is one point, we get $L|_\ell = \mathcal{O}_\ell$ because the restricted map $p|_{\ell_1} : \ell_1 \longrightarrow \ell$ is an isomorphism. Next, taking $p^*L|_{\ell_2} = q^*M|_{\ell_2}$, the left-hand side is trivial, which implies that $M|_{q(\ell_2)} \approx 0$, because of the above proposition. Thirdly, taking $p^*L|_{q^{-1}C} = q^*M|_{q^{-1}C}$ $(= q^*(M|_C))$ with $q\ell_2 = C$, we see that the right-hand side is numerically trivial, and, hence so is $L_{|D}$ with $D = p(q^{-1}C)$ by virtue of the above proposition. Entirely in the same way, assumption 1) yields our proposition.

q.e.d.

Proof of Corollary 5.11. As a divisor D in Proposition 5.10. take $A(\ell)$ satisfying the condition (5.7.2). Obviously $A(\ell)$ gives two divisor D_1, D_2 enjoying the condition in Proposition 5.10. In the same way we have ℓ, ℓ_1 and ℓ_2 . Hence we can show this Corollary by Proposition 5.10.

q.e.d.

The next theorem is important to determine the structure of the variety in (5.7.1) and at the same time it enable us to observe (5.7.2) easily.

Theorem 5.12. Let (X,L) be a smooth variety with $\ell(X,L) = 2$. Assume that $\dim |L| \geq 7$ and p is a separable morphism. Then

(X,L) *is one of* $\mathbb{P}^1$*-bundle over a surface and a quadric surface fibre space over a curve.*

Let us assume that (X,L) is not a $\mathbb{P}^1$-bundle, namely deg $p \geq 2$. Then to show this theorem, by Proposition 5.8, it suffices to show that there are infinitely many divisors $\{D_\lambda\}$ in X such that $p^*D_\lambda \in q^*\mathrm{Pic}\, Y$.

For this purpose we need several facts.

(5.13) Choose a general line ℓ in Y such that $N_{\ell/X} = \mathcal{O} \oplus \mathcal{O}$ and let $\sigma : X^1 \longrightarrow X$ be the blowing up with the line ℓ in X as the center. Then φ factors the product $\sigma^{-1} \cdot \Psi$ of two rational maps.

$$\begin{array}{ccccc} \sigma^{-1}(\ell) \subset X^1 & & \xrightarrow{\ \Psi\ } & & \\ \big\downarrow \sigma & & & & W \subseteq \mathbb{P}^{N-2} \\ \ell \subset X & & \xrightarrow{\ \varphi\ } & & \\ \cap & & & & \\ \mathbb{P}^N & & & & \end{array}$$

Here φ is a rational map corresponding to a linear system $|L - \ell|$, W an image of X via φ and Ψ is a morphism corresponding to a linear system $|H^1|$ where H^1 denotes $\sigma^*L - \sigma^{-1}\ell$ and it is base point free. Then we obtain

Proposition 5.14. *Under the above notations* (5.13), *assume that* $H^3 \geq 4$ *and* $N \geq 5$. *Then* dim $W = 3$ *and* $\varphi(\ell)$ *is a plane or a quadric surface.*

Proof. Noting that $N_{\ell/X} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell$, we see that

$$H'^3 = (\sigma^*L)^3 - 3(\sigma^*L)^2\sigma^{-1}\ell + 3\sigma^*L(\sigma^{-1}\ell)^2 - (\sigma^{-1}(\ell))^3$$

$$= L^3 + 3(\sigma^*L)(-\sigma^{-1}(\ell) + c_1(N_{\ell/X})f) + c_1(N_{\ell/X})$$

$$= L^3 - 3\sigma^*(L\cdot\ell) = L^3 - 3$$, where f is a fiber of a P^1-bundle $:\sigma^{-1}(\ell) \to \ell$ and $c_1(*)$ is a 1 st Chern class of $*$. Similarly we get $H'^2\cdot\sigma^{-1}|\ell| = 2$. (See [6]). These yields our proposition.

q.e.d.

(5.15) We must remark that each line intersecting the line ℓ in above proposition collapses a point by φ on $\varphi(\ell)$.

Proposition 5.14. Under the above notation (5.13), let us assume that $L^3 \geq 4$ and dim $|L| \geq 6$. Then for a general line ℓ in Y_0 (see (5.7)) such that $N_{\ell/X} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell$, there exists an effective divisor H_0 in $|L|$ such that supp $H_0 \supset A(\ell)$ in (5.7). Moreover assume that dim $|H| \geq 7$. Then there exists an effective divisor H_1 in $|L|$ such that besides $A(\ell)$, H_1 has a component D not contained in $S(\ell)$.

Proof. For a former part, we can take a hyperplane S containing $\varphi(\ell)$ (= a plane or a quadric surface) in P^{N-2} $(N - 2 \geq 4)$ by virtue of (5.14). Therefore Remark 5.13 implies that the divisor in $|L|$ corresponding to S is a desired one. For the remainder, since there is at least one-dimensional family of hyperplanes in $\mathbb{P}^{N-2}$ containing $\varphi(\ell)$, we can choose a hyperplane S' passing through a point in W except $\varphi(\ell)$.

The corresponding divisor H_1 in $|L|$ is what we look for.

q.e.d.

<u>Remark 5.16.</u> In the above proof when we fix a general point x in X for a given line ℓ we can take a divisor H_1 in the above sense passing through x. We write this H_1 as D_ℓ.

<u>Proof of Theorem 5.12</u>

Now let us observe the property of $A(\ell)$ and D_ℓ. Noting that the intersection number of L and a line ℓ is one and that D_ℓ and $A(\ell)$ are irreducible component of a divisor in $|L|$, we see that $D_\ell \cdot \ell = 0$ or $A(\ell) \cdot \ell = 0$. On the other hand it is well known that $\mathrm{Pic}\ Z$ is isomorphic to $\mathbf{Z}p^*L \oplus q^*\mathrm{Pic}\ Y$. Hence we see that either $p^{-1}D_\ell$ or $p^{-1}A(\ell)$ is contained in $q^*\mathrm{Pic}\ Y$. Hence we get Theorem 5.12.

q.e.d.

By virtue of Theorem 5.12, we have only the study the structure of (X,L) with $\dim |L| \leq 6$.

(5.17) If $\dim |L| = 4$, then we see eaily that (X,L) is a cubic hypersurface in $\mathbb{P}^4$.

To consider the other cases, we need the following

<u>Lemma 5.18.</u> <u>Let</u> (X,L) <u>be a smooth</u> 3-<u>fold with</u> $\ell(X,L) = 2$. <u>Assume that</u> $K_X + 2L$ <u>is numerically trivial. Then</u>

$H^i(X,tL) = 0$ <u>for</u> $i = 1,2$ <u>and every integer</u> t.

<u>Proof.</u> Take a smooth member H in $|L|$. Since $K_H = K_X + L_{|H}$, we see that $-K_H$ is ample, which implies that H is a Del Pezzo surface. It is known that Picard group of such surface is torsion free, hence we get $L|_H = -K_H$. Moreover we know $H^1(H,tK_H) = 0$ for every integer t [See III, Theorem 1 in [16]]. Hence considering the following sequence:

$$\oplus_t \quad 0 \longrightarrow (t-1)L \longrightarrow tL \longrightarrow tL|_H \longrightarrow 0 .$$

We obtain a surjective morphism: $H^1(X,(t-1)L) \longrightarrow H^1(X,tL) \longrightarrow 0$ and an injective morphism $0 \longrightarrow H^2(X,(t-1)L) \longrightarrow H^2(X,tL)$ for every integer t. Serre duality and Serre's vanishing theorem yields our desired fact.

q.e.d.

(5.19) Let us consider the case of $\dim |L| = 5$. Considering the above exact sequence $\oplus_1$ we get $\dim|-K_H| = 4$ and, therefore, $K_H^2 = 4$ by Riemann Roch Theorem. H is known to be a complete intersection of two quadric hypersurface in $\mathbb{P}^4$. Hence X is a complete intersection of two quadric hypersurface in $\mathbb{P}^5$ by virtue of Proposition 3.8 in [11].

(5.20) Let us consider the case: $\dim |L| = 6$, namely $K_H^2 = 5$. But the proof (6.5) by Iskovskih [8] is valid for our case, as it is, even in the positive characteristic case. As other reference, see [5]. By virtue of Lemma 5.18, $\Delta(X,L)$, in the meaning by Fujita, is 1.

(5.21) Proof of Main Theorem (in the case of $\ell(X,L) = 2$)

Finally let us study the structure of an $n(\geq 4)$ dimensional smooth variety (X,L) with $\ell(X,L) = 2$. We take a series of subvarieties $X = X_0 \supset X_1 \supset \dots \supset X_{n-3}$ such that X_i is a smooth member of L_i $(= L_{i-1}|_{X_i})$, $X_0 = X$ and $L_0 = L$. Then by virtue of Proposition 1.8, (X_{n-3}, L_{n-3}) is a 3-fold which contains many lines and $\ell(X_{n-3}, L_{n-3}) = 2$. Therefore we have five cases (5.5), (5.9), (5.17), (5.20) and (5.21). If (X_{n-3}, L_{n-3}) is as in (5.5), we infer that (X_n, L) is isomorphism to $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ where E is a very ample vector bundle over a smooth surface by using Theorem 3.1 inductively. For the (5.9) , we can get a desired result by virtue of Theorem 2 in the same way as above. For (5.20) (5.21), we can check that $H^1(X, \mathcal{O}_X) = 0$ entirely in the same way as in Lemma 5.18. Hence it is straightforward to see that (X,L) is a Del Pezzo manifold whose degree is 4 and 5 respectively in the meaning by T. Fujity by using (5.7.3), (5.7.5) in [5]. Hence by virtue of Theorem b, we have a desired result.

§6. 3-folds with two fiber space structures

In this section we shall study the structure of 3-fold with two families of lines, each of which fill up the whole space X.

Let (X,L) be a smooth 3-fold which contains many lines.

(6.1) Assume that there are two irreducible components S_1, S_2 of Hilbert scheme of lines in (X,L) whose lines fill up X.

(6.2) Moreover, we assume that the canonical morphism $p_i : Z_i \longrightarrow X$ is separable (see 1.3) where Z_i is the universal space of S_i.

Then we have the following

Theorem 6.3. Let (X,L) be a smooth 3-fold and let us maintain the assumption (6.1) and 6.2). Then (X,L) is one of the following:

1) $\mathbf{P}^1 \times \mathbf{P}^2$
2) $S_1 \times_C S_2$, where S_i is a $\mathbf{P}^1$-bundle over a smooth curve C.
3) $\mathbf{P}(T_{\mathbf{P}^2})$

To show this, we need several propositions.

(6.4) Remark. If the characteristic of k is zero, (6.2) holds automatically.

By virtue of our results (= Main Theorem) in 3-dimentional case), there is the following possibility as for $\dim S_i$ $(i=1,2)$

(6.5)

$\dim S_1$	$(\geq)$	$\dim S_2$
3		3
3		2
2		2

But we immediatly have

Claim. The case of $\dim S_i = 3$ does not occur.

Proof. $\dim S_i = 3$ means that X is a $\mathbf{P}^2$-bundle over a smooth curve C_i by (3.2). On the other hand since every morphism from $\mathbf{P}^2$ to a curve is constant, this case is absurd.

q.e.d.

By the same reason, there exists no (X,L) which is both $\mathbf{P}^2$-bundle over a curve and a quadric surface fiber space over a curve. (abbreviates qsfs often)

(6.6) Hence we shall investigate (X,L) with following two fiber structures:

X is a

α) $\mathbb{P}^1$-bundle and $\mathbb{P}^2$-bundle

β) $\mathbb{P}^1$-bundle and $\mathbb{P}^1$-bundle

γ) $\mathbb{P}^1$-bundle and qsfs

δ) qsfs and qsfs

Then we have

<u>Lemma</u>.

α) $\mathbb{P}^1 \times \mathbb{P}^2$

β) $S_1 \times_C S_2$, or $\mathbb{P}(T_{\mathbb{P}^2})$

<u>where</u> S_i <u>is a</u> $\mathbb{P}^1$-<u>bundle over a curve</u> C .

γ), δ) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(6.7) Now we would like to study 3-fold with two kinds of fibre structure in the case α, β, γ, δ (6.6). For the purpose we shall determine the structure under a little weaker condition, namely, not assuming the projective space fiber bundle in the Zariski topology. Precisely speaking, let X be a smooth 3-fold which is a $\mathbb{P}^1$-bundle $p : X \longrightarrow S$ over a smooth surface S in the étale topology. Moreover as the second fiber structure we consider three cases as follows $(i = \alpha, \beta, \gamma)$: $q_i : X_i (\cong X) \longrightarrow T_i$ is a fiber space, where q_α is a $\mathbb{P}^2$-bundle over a non-singular curve T, q_β a $\mathbb{P}^1$-bundle over a smooth surface T_β and q_β is a quadric surface fiber space over T_γ . Note that

q_α and q_β are bundle maps in the ētale topology, and the concept of lines is not assumed. On the other hand, remark that the case δ is shown in (2.14).

<u>Theorem 6.7.</u> <u>Let us maintain above notation</u> (6.7) <u>in any characteristic case.</u> <u>Then the same conclusion as in Lemma</u> 6.6 <u>holds</u> .

For the purpose, the following is useful to determine whether S_i is of negative Kodaira dimension. The first is a well-known

<u>Proposition 6.8.</u> <u>Let us consider the following exact sequence of vector bundles:</u> $0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$. <u>Then</u> $S^m(E)\,(=F_m)$ <u>has a sequence of subbundles:</u>

$$F_0 = \subset F_1 \subset \ldots \subset F_m$$

<u>where</u> $F_{i+1}/F_i = S^{m-i}(E_1) \otimes S^i(E_2) \quad (1 \leq i \leq m)$.

Next, let us consider the following exact sequence of vector bundles.

$0 \longrightarrow D_i \xrightarrow{k_i} E \longrightarrow F_i \longrightarrow 0$ for $i = 1,2$. Then we have

<u>Proposition 6.9.</u> <u>Under the above notation</u>, <u>asssume that for</u> $i = 1,2$ and $j \geq i$, $H^0(Z, S^{m-j}(D_j) \otimes S^j(F_i)) = 0$ <u>and</u> $D_1 \otimes D_2 \xrightarrow{k_1+k_2} E$ <u>is generally injective.</u> <u>Then we have</u>

$H^0(Z, S^m(D_i)) = 0$ for $i = 1,2$.

Proof is easy.

As α) and β) yield the following exact sequence:

$$(1) \qquad 0 \longrightarrow p^*\Omega_S^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_p^1 \longrightarrow 0$$

$$(2)_i \qquad 0 \longrightarrow q_i^*\Omega_{T_i}^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_{q_i}^1 \longrightarrow 0$$

where Ω_p and Ω_q are the relative canonical bundles with respect to p and q_i .

Using Proposition 6.8 and 6.9, let us start the proof of case α), β), γ) .

Case α). By virtue of Main Theorem in [Sa2] , it suffices to show that S is $\mathbb{P}^2$ and C is $\mathbb{P}^1$. In characteristic zero case, it is obvious because the smooth surface which is dominated by $\mathbb{P}^2$ is also $\mathbb{P}^2$.

Secondly we assume that char k is positive, taking the second exterior of (1) and $(2)_\alpha$.

$$(1)' \qquad 0 \longrightarrow p^*K_S \longrightarrow \overset{2}{\Lambda}\Omega_X^1 \longrightarrow p^*\Omega_S^1 \otimes \Omega_p \longrightarrow 0$$

$$(2)'_\alpha \qquad 0 \longrightarrow q^*\Omega_T^1 \otimes \Omega_q \longrightarrow \overset{2}{\Lambda}\Omega_X^1 \longrightarrow \overset{2}{\Lambda}\Omega_q \longrightarrow 0 ,$$

where $q_\alpha = q$ and $T_\alpha = T$.

We shall check $H^0(X, S^{\cdot}(\overset{2}{\Lambda}\Omega_X^1)) = 0$ for every positive integer m, hence, $H^0(X, p^*K_S^{\otimes m}) = H^0(S, K_S^{\otimes m}) = 0$. Restricting $(2)'_\alpha$ on a fiber $(= f)$ of q, we get

$$0 \longrightarrow \Omega_{\mathbb{P}^2} \longrightarrow \overset{2}{\Lambda}\Omega_X^1|_f \longrightarrow K_{\mathbb{P}^2} \longrightarrow 0 .$$

Therefore Proposition 6.8 says that $H^0(X, S^m(\overset{2}{\Lambda}\Omega_X^1))$ vanishes for $m > 0$, which implies that S is ruled. Since $\mathbb{P}^2$ dominates S, $\mathrm{Pic}\, S \cong \mathbb{Z}$, which means that S is $\mathbb{P}^2$.

q.e.d. of α)

<u>Case</u> β) This case is already shown in characteristic zero case in [13]. Hence we assume that $\mathrm{char}\, k$ is positive. In the same way as in case α) take the exterior of (1) and (2):

$$(2)' \qquad 0 \longrightarrow q^*K_T \longrightarrow \overset{2}{\Lambda}\Omega_X^1 \longrightarrow q^*\Omega_T^1 \otimes \Omega_q \longrightarrow 0$$

where $q_\beta = q$, and $T_\beta = T$.

Restricting $q^*\Omega_T^1 \otimes \Omega_q$ to a fiber of $q (\cong \mathbb{P}^1)$, we see easily that $H^0(X, S^r(q^*K_T) \otimes S^t(q^*\Omega_T^1 \otimes \Omega_q)) = 0$ for $r \geq 0$ and $t > 0$, which yields $H^0(X, p^*K_S^{\otimes m}) = 0$ for every positive integer m, by virtue of Proposition 6.9. Hence S is ruled, similarly T is ruled. Moreover we have a

<u>Claim</u>. <u>Both</u> S <u>and</u> T <u>are a geometrically ruled surface or</u> $\mathbb{P}^2$.

<u>Proof.</u> Taking a general smooth curve C in T, we see

that the restricted map $p : q^{-1}(C) \longrightarrow S$ is surjective. Therefore 1 or 2 is the rank of the first cohomology group of S modulo numerical equivalence. Hence in the former case S and T are $\mathbb{P}^2$, and in the latter case they are geometrically ruled.

Now if S and T are $\mathbb{P}^2$, we know X is isomorphic to $\mathbb{P}(T_{\mathbb{P}^2})$ by virtue of Main Theorem A in [13]. Secondly in the case of a geometrically ruled surface, we see that X is isomorphic to $S \times_C T$ where S and T are $\mathbb{P}^1$-bundle over a smooth curve C. The proof is completely the same as the proof of Theorem B in [13].

Case γ). First let us determine the structure of S, namely $\mathbb{P}^1 \times \mathbb{P}^1$. There is the following exact sequence:

$$(1)' \quad 0 \longrightarrow p^*K_S \longrightarrow \overset{2}{\Lambda}\Omega_X^1 \longrightarrow p^*\Omega_S \otimes \Omega_p \longrightarrow 0 .$$

On the other hand, since a general fiber of q_γ is a smooth quadric surface Q, there is another exact sequence:

$$\circledast \quad 0 \longrightarrow \check{N} = 0 \longrightarrow \Omega^1_{X|Q} \longrightarrow \Omega^1_Q \longrightarrow 0 .$$

Hence taking 2nd exterior of $\circledast$, we get $0 \longrightarrow \Omega_Q^1 \longrightarrow \overset{2}{\Lambda}\Omega_X^1|_Q \longrightarrow K_Q \longrightarrow 0$, which implies that $H^0(Q, S^m(\overset{2}{\Lambda}\Omega_X^1|_Q)) = 0$ for every positive integer m. This yields $H^0(X, p^*K_S^{\otimes m}) = 0$ for $m > 0$, hence,

$H^0(S, K_S^{\otimes m}) = 0$, namely S is ruled. Moreover since S is dominated by $\mathbb{P}^1 \times \mathbb{P}^1 (= Q)$, we see that S is $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$, which provides us with the fact that X is a Proj of a vector bundle E of rank 2 over S (see [1].

(γ.1) First we shall show that S is not $\mathbb{P}^2$.

Proof. Assume that S is $\mathbb{P}^2$. Let $\ell_\lambda (\lambda \in \check{\mathbb{P}}^2)$ be a line in $\mathbb{P}^2$ where $\check{\mathbb{P}}^2$ denotes the dual space of $\mathbb{P}^2$ and let us denote $p^{-1}(\ell_\lambda) \cap q^{-1}(a)$ by $D_{a,\lambda}$. Now fix a geometrically ruled surface : $p : p^{-1}(\ell_\lambda) \longrightarrow \ell_\lambda$. For two points a, a' in $C (= \mathbb{P}^1)$, $D_{a,\lambda} \cap D_{a',\lambda'} = \phi$, which means the existence of an irreducible curve C whose self intersection is zero because $H^1(F_n, \mathcal{O}_{F_n}) = 0$ with $F_n = p^{-1}(\ell_\lambda)$. By virtue of Proposition 3.6 [13], we see that $p^{-1}(\ell_\lambda)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore the vector bundle E over $\mathbb{P}^2$ is uniform, which implies that E is a direct sum of the same two line bundles by virtue of [12]. Hence X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$. This easily contradicts our assumption that $q : X \longrightarrow C (= \mathbb{P}^1)$ is a quadric surface fiber space.

(γ.2) Assume that S is a smooth quadric surface.

Let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 (= S) \longrightarrow \mathbb{P}^1$ be the i-th projection and $p_i : X_a (= p^{-1}(p_i^{-1}(a))) \longrightarrow p_i^{-1}(a)$ a $\mathbb{P}^1$-bundle. Now considering $X_a \cap q^{-1}(b)$ $(= D_{a,b})$, we have $D_{a,b} \cap D_{a,b'} = \phi$. Since $H^1(X_a, \mathcal{O}_{X_a}) = 0$, there is an irreducible curve

C in X_a such that $C^2 = 0$, which means that X_a is $\mathbb{P}^1 \times \mathbb{P}^1$ similarly in $(\gamma.1)$. Since $X = \mathbb{P}(E)$, we see that for every point a in $\mathbb{P}^1$, $E|_{f(a)}$ is $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(m)$ with $f(a) = p_1^{-1}(a)$ and it is independent of the choice of a in $\mathbb{P}^1$ by the existence of the 1st Chern class of E. By virtue of Base change theorem, we see that $E = p_2{}^*\mathcal{O}_{\mathbb{P}^1}(m) \otimes p_1{}^*(\mathcal{O}_{\mathbb{P}^1}{}^*(\mathcal{O}_{\mathbb{P}^1}(s) \oplus \mathcal{O}_{\mathbb{P}^1}(t))$. As for the second projection p_2, taking the same procedure, we know $E = p_1{}^*\mathcal{O}_{\mathbb{P}^1}(n) \otimes p_2{}^*(\mathcal{O}_{\mathbb{P}^1}(u) \oplus \mathcal{O}_{\mathbb{P}^1}(v))$. Hence we get $s = t = n$ and $u = v = m$, which means X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

q.e.d. of Theorem 6.7.

6.10 Remark. Theorem 6.7 gives a complete proof for the generalization of Theorem B [13] in any characteristic case.

Theorem 6.7 is applied to show theorem in §3. To enable the application, let us state a useful proposition telling us when p_i becomes a separable morphism.

(6.11) Let (X,L) and $(\bar{X},\bar{L})$ be smooth projective 3-fold and 4-fold respectively which contain many lines, and let X be a member of $|\bar{L}|$ with $N_{X/\bar{X}} = L$. Moreover, let (X,L) be as in (6.1) and let us assume that S_1 is a surface which yields a P^1-bundle : $X \longrightarrow S_1$. Then taking a fiber $(= \ell)$ of $\pi : X \longrightarrow S$, there is an exact sequence:

$$0 \longrightarrow N_{\ell/X} \longrightarrow N_{\ell/X} \longrightarrow N_{X/\bar{X}}|_{\ell} \longrightarrow 0$$

hence, we get $N_{\ell/\bar{X}} = \mathcal{O}_\ell^{\,2} \oplus \mathcal{O}_\ell(1)$, which implies the irreducible component T of Hilbert scheme of ℓ in $\bar{X}$. Put $N + 1 = \dim |\bar{L}|$. Then T (or, S_i) can be naturally considered as a closed subscheme in $Gr(N+1,1)$ (or, $Gr(N,1)$, resp.) under the canonically embedding: $Gr(N,1) \hookrightarrow Gr(N+1,1)$. (see §3). Here we obtain the following

<u>Proposition 6.12.</u> <u>Let</u> (X,L), $(\bar{X},\bar{L})$, S_i, T <u>and</u> p_1 <u>be as in</u> (6.1) and (6.11).
<u>Assume that</u> S_1 <u>and</u> S_2 <u>are irreducible components in</u> $Gr(N,1) \cap T$ $(\subseteq Gr(N+1,1))$. <u>Then</u> S_2 <u>is a surface and</u> p_2 <u>is separable.</u>

<u>Proof.</u> Taking a line L corresponding to a point in S_2, there is the following exact sequence as for normal bundles

$$0 \longrightarrow N_{L/X} \longrightarrow N_{L/\bar{X}} \longrightarrow N_{X/\bar{X}}|L \longrightarrow 0 .$$

By virtue of our assumption, T is the irreducible component of Hilbert scheme of ℓ in X whose Hilbert polynomial is $\chi(N_{\ell/\bar{X}}(n)) = 3n+4$. Since L and ℓ are contained in T, we see that $(N_{L/X}(n)) = 2n+2$ hence, $\det N_{L/X} = 0$. Now we have a

Claim. There is a smooth hyperplane section H in X containing ℓ. Therefore $N_{L/X}$ is $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.

Proof. Assume X is contained in $\mathbf{P}^N$. Then Bertini's Theorem says that a general hyperplane section containing ℓ is smooth outside ℓ. The dimension of such hyperplane section is $N-2$. On the other hand, $N-4$ is the dimension of hyperplane section H where $H \supset \ell$ and H is singular at a fixed point in ℓ. Hence we showed the former part. In the next place we have the following

$$0 \longrightarrow N_{L/H} \longrightarrow N_{L/X} \longrightarrow N_{H/X}|_L \ (= \mathcal{O}(1)) \longrightarrow 0 .$$

Hence we get $N_{L/X} = \mathcal{O}(-1)$, which means the latter.

By the above Claim, we see $H^1(L, N_{L/X}) = 0$, which gives us the separability of p_2.

Finally let us give a pathological example (X,L) which contains my line.

<u>6.11. Remark.</u> Let $p(>0)$ be the characteristic of the field k.

Put X as $\{(x:y:z)\times(a:b:c)\in \mathbf{P}_1^2\times\mathbf{P}_2^2 \mid x^q a + y^q b + z^q c = 0\}$ with $q = p^m$. Let $r : X \longrightarrow \mathbf{P}_1^2$ be the first projection and $s : X \longrightarrow \mathbf{P}_2^2$ the second projection.
Then $(X, \mathcal{O}(1,1)|_X)$ is the following properties:

1) It is a smooth 3-fold which contains my lines.

2) r is a $\mathbf{P}^1$-bundle and s an inseparable morphism whose fiber is $\mathbf{P}^1$ set-theoretically. Every line in X is a fiber of p or q.

3) Hence r and s yield the Hilbert scheme of lines in (X,L). Let f_r and f_s be the fiber $(\cong \mathbf{P}^1)$ or r and s respectively, set-theoretically. Then $N_{f_r/X} = \mathcal{O} \oplus \mathcal{O}$ and $N_{f_s/X} = \mathcal{O}(1) \oplus \mathcal{O}(-p^m)$.

<u>Proof.</u> Let us show the latest part only. X is isomorphic to $\mathbf{P}(\varphi^* T_{\mathbf{P}^2})$ where φ is the m-power of a Frobenius map $\mathbf{P}^2 \longrightarrow \mathbf{P}^2$. Then r means the canonical projection $: \mathbf{P}(\varphi^* T_{\mathbf{P}^2}) \longrightarrow \mathbf{P}^2{}_1$. Now take a line $\ell\ (\cong \mathbf{P}^1)$ induced by s. Then, noting that $r|_\ell$ is a closed immersion from ℓ to $\mathbf{P}_1^2$, there is the following exact sequence :

$$0 \longrightarrow N_{\ell/R} \longrightarrow N_{\ell/X} \longrightarrow N_{R/X|\ell} \longrightarrow 0$$

where R means $r^{-1}(r(\ell)) = F_q$ (= Hirzebruch Surface), which follows that

$$0 \longrightarrow \mathcal{O}(-p^m) \longrightarrow N_{\ell/X} \longrightarrow \mathcal{O}(1) \longrightarrow 0 .$$

Since s yields a 2-dimension surface as an irreducible component of Hilbert scheme, we have $h^0(\ell, N_{\ell/X}) \geq 2$, which implies $N_{\ell/X} = \mathcal{O}(1) \oplus \mathcal{O}(-p^m)$.

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