ON THE 2-COMPACT GROUP DI(4)

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Abstract. Besides the simple connected compact Lie groups there exists one further simple connected 2-compact group, constructed by Dwyer and Wilkerson, the group $DI(4)$. The mod-2 cohomology of the associated classifying space $BDI(4)$ realizes the rank 4 mod-2 Dickson invariants. We show that mod-2 cohomology determines the homotopy type of the space $BDI(4)$ and that the maximal torus normalizer determines the isomorphism type of $DI(4)$ as 2-compact group. We also calculate the set of homotopy classes of self maps of $BDI(4)$.

1. Introduction.

As introduced by Dwyer and Wilkerson in their influential paper [10], a p-compact group is a pair (X, BX) of spaces such that X is $\mathbb{F}_p$-finite, i.e. $H^*(X;\mathbb{F}_p)$ is a finite vector space, BX is pointed, both spaces are p-complete (in the sense of [2]) and the loop space ΩBX and X are homotopy equivalent. This is a homotopy theoretic generalization of the concept of compact Lie groups. For a connected compact Lie group G, the pair $(G_p^\wedge, BG_p^\wedge)$ together with the natural equivalence $\Omega BG_p^\wedge \simeq G_p^\wedge$ is a p-compact group, which, by abuse of notation, we also denote by G. For odd primes, there exist many connected p-compact groups which do not come from compact Lie groups, e.g. Sullivan showed that, if n divides $p-1$, the sphere S^{2n-1} gives rise to a connected p-compact group $((S^{2n-1})_p^\wedge, B(S^{2n-1})_p^\wedge)$ [27]. At the prime 2, Dwyer and Wilkerson constructed the only known example $DI(4)$ of a connected 2-compact group, which is not a Lie group [9]. The classifying space $BDI(4)$ realizes the rank 4 mod-2 Dickson invariants, namely the ring of invariants of the natural action of $Gl(4,\mathbb{F}_2)$ on the rank 4 polynomial algebra $H^*((B\mathbb{Z}/2)^4;\mathbb{F}_2)$. In fact, they conjectured that every connected 2-compact group is a product of a connected compact Lie group and copies of $DI(4)$. In this work we will approach this conjecture by showing that the 2-compact group $DI(4)$ satisfies some uniqueness results. We also show that this conjecture is at least rationally true (see Theorem 1.5)

p-compact groups behave astonishingly similar as compact Lie groups. Every p-compact group X has a maximal torus $T_X \to X$, a Weyl group W_X, a maximal torus normalizer $N_X \to X$ and a center $Z(X) \to X$, all with basically the same properties as in the case of compact Lie groups. And two p-compact groups X and

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Y are isomorphic if the classifying spaces BX and BY are homotopy equivalent. For further details about p-compact groups see the survey article [17] or the references mentioned there.

For p-compact groups, there are two classification conjectures around. A map between algebras over the Steenrod algebra is called a $\mathcal{K}$-map.

1.1 Conjecture.
(i) Let X be a simply connected or centerfree p-compact group. A p-complete space A is equivalent to BX if and only if there exists a $\mathcal{K}$-isomorphism $H^*(A;\mathbb{F}_p) \cong H^*(BX;\mathbb{F}_p)$.
(ii) Two connected p-compact groups X and Y are isomorphic, i.e. the classifying spaces BX and BY are equivalent, if and only if the maximal torus normalizer N_X and N_Y are isomorphic.

At the prime two, both conjectures are proved for special unitary groups [6] [22] [19], for the exceptional Lie group G_2 [28] and for the special orthogonal groups $SO(2n+1)$ [20] [25]. For the spinor groups $Spin(2n+1)$ only the second part is known [25].

We will show that both conjectures are also valid for the Dwyer-Wilkerson example $DI(4)$.

1.2 Theorem. *A 2-complete space A is homotopy equivalent to $BDI(4)$ if and only if $H^*(A;\mathbb{F}_2) \cong H^*(BDI(4);\mathbb{F}_2)$ as algebras over the Steenrod algebra.*

1.3 Theorem. *A connected 2-compact group X is isomorphic two $DI(4)$ if and only if the maximal torus normalizer N_X and $N_{DI(4)}$ are isomorphic.*

If $H^*(A;\mathbb{F}_2) \cong H^*((B\mathbb{Z}/2)^4;\mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$, standard methods show that ΩA is $\mathbb{F}_2$-finite (see [9]). The pair $(\Omega A, A)$ gives rise to a 2-compact group. The next proposition shows that Theorem 1.2 and Theorem 1.3 are equivalent statements.

1.4 Proposition. *For a 2-compact group X, the maximal torus normalizer N_X and $N_{DI(4)}$ are isomorphic if and only if there exists a $\mathcal{K}$-isomorphism $H^*(BX;\mathbb{F}_2) \cong H^*(BDI(4);\mathbb{F}_2)$.*

Every p-compact group X has an associated p-adic lattice $L_X := \pi_1(T_X)$ which carries an action of the Weyl group W_X. If X is connected, the representation $W_X \to Gl(L_X \otimes \mathbb{Q})$ represents W_X as a pseudo reflection group, i.e. W_X is finite and generated by elements fixing a hyperplane of codimension 1. The above representation gives the rational Weyl group data of X. We say that two p-compact groups X and Y have the same rational Weyl group data if the two representations $W_X \to Gl(L_X \otimes \mathbb{Q})$ and $W_Y \to Gl(L_Y \times \mathbb{Q})$ are weakly isomorphic. That is there exists abstract isomorphism $W_X \xrightarrow{\cong} W_Y$ such that $L_X \otimes \mathbb{Q}$ and $L_Y \otimes \mathbb{Q}$ are isomorphic as $\mathbb{Z}_p^\wedge[W_X]$-modules. The next theorem shows that the above mentioned conjecture of Dwyer and Wilkerson is rationally true.

1.5 Theorem. *Every connected 2-compact group X splits into a product $X \cong X_1 \times X_2$ such that X_1 is isomorphic to some copies of $DI(4)$ and X_2 has the same rational Weyl group data as a suitable connected compact Lie group G.*

Finally we characterize the homotopy classes of self maps of $BDI(4)$.

1.6 Theorem. *Let $i : T_{DI(4)} \subset DI(4)$ be a maximal torus of $DI(4)$. Then, the following conditions are equivalent.*
(i) f and g are homotopic.
(ii) The compositions $f \circ Bi$ and $g \circ Bi$ are homotopic.
(iii) $H^*(f;\mathbb{Z}_2^\wedge) \otimes \mathbb{Q} = H^*(g;\mathbb{Z}_2^\wedge) \otimes \mathbb{Q}$.

In fact, we will show that every self $f : BDI(4) \to BDI(4)$ is an Adams operation (see Section 6). That is there exists a $k_f \in \mathbb{Z}_2^\wedge$ such that f induces on $H^{2i}(BDI(4);\mathbb{Z}_2^\wedge) \otimes \mathbb{Q}$ multiplication by $(k_f)^i$. The 2-adic integer k_f is called the degree of f. It will turn out that k_f is either 0 or a 2-adic unit, that all these values can be realized and that k_f determines f up to homotopy (Theorem 6.5).

The strategy for the proofs of Theorem 1.2 and Theorem 1.3 is clear and known to the experts. It uses the construction of the classifying space $BDI(4)$ of Dwyer and Wilkerson. They constructed $BDi(4)$ as a homotopy colimit of a diagram based on the Rector category of $H^*((B\mathbb{Z}/2)^4;\mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$, which is equivalent to the Quillen category of $DI(4)$, namely the category of elementary abelian 2-subgroups of $DI(4)$. One of the spaces in that diagram is the classifying space $BSpin(7)_2^\wedge$ of the 2-compact group $Spin(7)$, which plays a major role in the construction. The missing gap for completing the proof is a uniqueness result for $Spin(7)$ which is provided in [25]. Having this at hand the idea of the proof is basically straightforward. We show that a 2-compact group X satisfying Proposition 1.4 has the same decomposition diagram as $DI(4)$; i.e. BX is the completion of the homotopy colimit of a diagram which, as a diagram of spaces, is equivalent to the diagram used for the construction of $BDI(4)$.

The paper is organized as follows. In Section 2 we recall the construction of $BDI(4)$ of Dwyer and Wilkerson. Section 3 contains the proof of Proposition 1.4, Section 4 the proof of the theorems 1.2 and 1.3, Section 5 the proof of theorem 1.5 and the last section the proof of Theorem 1.6.

We will use the language of p-compact groups all over the places. For references we refer the reader to the survey article [17] or the references mentioned there.

Mostly, cohomology is taken with $\mathbb{F}_2$ as coefficients. And $H^*_{\mathbb{Q}_2^\wedge}(-)$ denotes the cohomology groups $H^*(-;\mathbb{Z}_2^\wedge) \otimes \mathbb{Q}$.

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2. The construction of $DI(4)$.

In this section we recall the construction of $DI(4)$ and other material from [9]. The general linear group $Gl(4,\mathbb{F}_2)$ acts on the 4-dimensional $\mathbb{F}_2$-vector space $E_4 = (\mathbb{F}_2)^4$ and therefore on the algebra of formal polynomial functions $\mathbb{F}_2[E_4]$ on E_4 with coefficients in $\mathbb{F}_2$. That is the symmetric part of the tensor algebra of the dual of E. We can identify this algebra with the algebra $H^*(BE_4;\mathbb{F}_2)$. The ring of invariants $H^*(BE_4)^{Gl(4,\mathbb{F}_2)} \cong \mathbb{F}_2[c_8,c_{12},c_{14},c_{15}]$ carries an action of the Steenrod algebra $\mathcal{A}_2$. The indices denote the degree of the classes. The action is determined by $Sq^4(c_8) = c_{12}$, $Sq^2(c_{12}) = c_{14}$ and $Sq^1(c_{14}) = c_{15}$.

The Bockstein spectral sequence collapses at the E_2-page and shows that $H^*_{\mathbb{Q}}(BDI(4);\mathbb{Z}_p^\wedge) \cong \mathbb{Q}_p^\wedge[x_8,x_{12},x_{28}]$. This is also true for any space A such that

$H^*(A) \cong H^*(BE)^{Gl(4,\mathbb{F}_2)}$.

$DI(4)$ is a 2-compact group of rank 3 with maximal torus $T_{DI(4)} \simeq (S^1)^3 \to DI(4)$. The Weyl group $W_{DI(4)} \cong \mathbb{Z}/2 \times Gl(3,\mathbb{F}_2)$ arises as the pseudo reflection group No. 24 in the Clark-Ewing list of irreducible rational pseudo reflection groups [3]. The representation

$$W_{DI(4)} \cong \mathbb{Z}/2 \times Gl(3,\mathbb{F}_2) \to Gl(L_{DI(4)}) \cong Gl(3,\mathbb{Z}_2^\wedge)$$

maps the generator of $\mathbb{Z}/2$ on $-I$ where I is the identity. The composition $W_{DI(4)} \to Gl(3,\mathbb{Z}_2^\wedge) \to Gl(3,\mathbb{F}_2)$ is the projection on the second factor [9, Theorem 4.1]. In particular $L_{DI(4)}/2 := L_{DI(4)} \otimes \mathbb{F}_2$ is an irreducible $W_{DI(4)}$-module.

2.1 Lemma. *Every $W_{DI(4)}$-lattice of $L_{DI(4)} \otimes \mathbb{Q}$ is isomorphic to $L_{DI(4)}$ as $\mathbb{Z}_p^\wedge[W_{DI(4)}]$-module.*

Proof. Let $L \subset L_{DI(4)} \otimes \mathbb{Q}$ be a $W_{DI(4)}$-lattice. There exists a $W_{DI(4)}$-equivariant monomorphism $L \to L_{DI(4)}$ such that the cokernel K is a torsion group and such that rank of $K/2 := K \otimes \mathbb{F}_2$ is smaller than 3. Tensoring with $\mathbb{Z}/2$ yields an exact sequence

$$0 \to Tor(K,\mathbb{Z}/2) \to L/2 \to L_{DI(4)}/2 \to K/2 \to 0$$

of $W_{DI(4)}$-modules. Since $L_{DI(4)}/2$ is irreducible, $K/2 = 0$ as well as $K = 0$ and $L \cong L_{DI(4)}$. $\square$

Next we describe the Quillen category of $DI(4)$. Since $H^*(BDI(4);\mathbb{F}_2) \cong H^*(BE_4;\mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$ every elementary abelian 2-subgroup $E \subset DI(4)$ is subconjugated to E_4 and every pair of subgroups $E, E' \subset E_4$ of the same dimension are conjugate in $DI(4)$. Essentially, for $i = 1,..,4$, there is exactly one elementary abelian 2-subgroup $E_i \subset DI(4)$ of dimension i. The Quillen category is equivalent the full subcategory of the following shape

$$\begin{tikzcd} E_1 \arrow[loop above, "\{1\}"] \arrow[r, Rightarrow] & E_2 \arrow[loop above, "Gl(2,\mathbb{F}_2)"] \arrow[r, Rightarrow] & E_3 \arrow[loop above, "Gl(3,\mathbb{F}_2)"] \arrow[r, Rightarrow] & E_4 \arrow[loop above, "Gl(4,\mathbb{F}_2)"] \end{tikzcd}$$

where the groups above the objects denote the self maps and where $\Longrightarrow$ denotes the set of morphisms from E_i to E_{i+1}.

We have $C_{DI(4)}(E_4) = E_4$. And $C_{DI(4)}(E_3) \cong T_{DI(4)} \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts on $T_{DI(4)}$ via an Adams operation ψ^{-1} of degree -1, i.e. ψ^{-1} induces on $\pi_2(BT_{DI(4)})$ multiplication by -1. The group E_3 is contained in $T_{DI(4)}$ and therefore toral in $DI(4)$ as well as E_2 and E_1. The induced action of $Gl(3,\mathbb{F}_2)$ on $T_{DI(4)}$ is given by the restriction of the representation $Gl(3,\mathbb{F}_2) \to Gl(3,\mathbb{Z}_2^\wedge)$ given by the action of the 2-adic Weyl group representation $W_{DI(4)} \to Gl(3,\mathbb{Z}_2^\wedge)$ on the second component. Moreover, $C_{DI(4)}(E_2) \cong SU(2)^3/\Delta =: Q$, where Δ is the image of the diagonal embedding of $\mathbb{Z}/2 \to SU(2)^3$. And $C_{DI(4)}(E_1) \cong Spin(7)$.

The decomposition diagram of $BDI(4)$ given by the centralizers of the elementary abelian subgroups of $DI(4)$ [8] [11] has the form

$$\begin{tikzcd} BSpin(7) \arrow[loop above, "\{1\}"] & BQ \arrow[l, Leftarrow] \arrow[loop above, "Gl(2,\mathbb{F}_2)"] & B(TDI(4) \rtimes \mathbb{Z}/2) \arrow[l, Leftarrow] \arrow[loop above, "Gl(3,\mathbb{F}_2)"] & E_4 \arrow[l, Leftarrow] \arrow[loop above, "Gl(4,\mathbb{F}_2)"] \end{tikzcd} .$$

2.2 Remark. We do need some more detailed information about the subgroup $Spin(7) \subset DI(4)$. Let $E_1 \to E_4$ be the inclusion of the first coordinate. The isotropy subgroup $\Gamma := Iso(E_1) \subset Gl(4, \mathbb{F}_2)$ is the subgroup of all matrices of the form $\begin{pmatrix} 1 & * \\ 0 & A \end{pmatrix}$ such that $A \in Gl(3, \mathbb{F}_2)$. The classifying space $BC_{DI(4)}(E_1) \simeq BSpin(7)^\wedge_2$ has mod-2 cohomology

$$H^*(BSpin(7); \mathbb{F}_2) \cong H^*(BC_{DI(4)}(E_1); \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^\Gamma .$$

In particular, $H^*(BDI(4); \mathbb{F}_2) \to H^*(BSpin(7); \mathbb{F}_2)$ is a monomorphism. Moreover, since E_1 is subconjugated to $T_{DI(4)}$, $Spin(7) \subset DI(4)$ is a subgroup of maximal rank and $W_{Spin(7)} \subset W_{DI(4)}$. The Euler characteristic $\chi(DI(4)/Spin(7))$ of the homotopy fiber of $BSpin(7) \to BDI(4)$ is the index of $[W_{DI(4)} : W_{Spin(7)}]$ of the Weyl groups. The order of $W_{DI(4)}$ is 336, the order of $W_{Spin(7)}$ is 48 and $\chi(DI(4)/Spin(7)) = 7$. We explained this calculation in detail since, unfortunately, there is is misprint in [9, Theorem 1.8]. This calculation always works for subgroups of p-compact groups of maximal rank [11] [18].

The maximal torus normalizer $N_{DI(4)}$ contains the subgroup $T_{DI(4)} \rtimes \mathbb{Z}/2 \cong C_{DI(4)}(E_3)$. And therefore a unique subgroup $E_4 \subset T_{DI(4)} \rtimes \mathbb{Z}/2 \subset N_{DI(4)}$. Since $Spin(7) \cong C_{DI(4)}(E_1)$ and since $E_1 \subset T_{DI(4)}$, the Weyl group $W_{Spin(7)} \subset W_{DI(4)}$ is the isotropy subgroup of E_1 for the $W_{DI(4)}$-action on $T_{DI(4)}$ [11].

We finish this section with a proposition necessary for later purpose.

2.3 Proposition. *For any elementary abelian 2-group E, the map $[BE, BE_4] \to [BE, BDI(4)]$ factors through the bijection $[BE, BE_4]/Gl(4, \mathbb{F}_2) \to [BE, BDI(4)]$.*

Proof. Since $H^*(BDI(4); \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$, this statement is well known. It follows from facts about Lannes' T-functor, namely that $H^*(map(BE, BX); \mathbb{F}_2) \cong T^E(H^*(BDI(4); \mathbb{F}_2))$ and that

$$T^E(H^*(BE_4; \mathbb{F}_2)^{Gl(4,\mathbb{F}_2)} \cong (T^E(H^*(BE_4; \mathbb{F}_2)))^{Gl(4,\mathbb{F}_2)} \cong H^*(map(BE, BE_4); \mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$$

[15] [10] (for the latter sequence of isomorphisms see also [9]). □

3. The proof of Proposition 1.4.

First we assume that X is a 2-compact group, such that

$$H^*(BX; \mathbb{F}_2) \cong H^*(BDI(4); \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^{Gl(4,\mathbb{F}_2)} .$$

3.1 Lemma.
(i) Every elementary abelian 2-subgroup $E \subset X$ is subconjugated to $E_4 \subset X$.
(ii) The 2-compact group X is connected, centerfree and simple. In particular, the rational Weyl group representation $W_X \to Gl(L_X \otimes \mathbb{Q})$ is irreducible.
(iii) X and $DI(4)$ have the same rational Weyl group data.

Proof. The $\mathcal{K}$-map $H^*(BX;\mathbb{F}_2) \to H^*(BE_4;\mathbb{F}_2)$ can be realized by a topological map $BE_4 \to BX$ [15] which makes $E_4 \subset X$ to a subgroup of X. Moreover, since $H^*(BE;\mathbb{F}_2)$ is an injective object in the category $\mathcal{K}$ of algebras over the Steenrod algebra, every map $BE \to BX$ factors through BE_4. This proves (i).

For p-compact groups, the T-functor calculates the cohomology of classifying spaces of centralizers of elementary abelian p-subgroups [10]. Since the T-functor commutes with taking invariants, a central subgroup $E \subset E_4 \subset X$ has to be fixed under the action of $Gl(4,\mathbb{F}_2)$. There are no nontrivial fixed points, and therefore X is centerfree.

As a connected centerfree 2-compact group, X splits into a product of simple centerfree 2-compact groups [12]. A non trivial splitting of X would establish a non trivial splitting of $H^*(BX;\mathbb{F}_2)$ into a tensor product. But such a splitting does not exist for $H^*(BE_4;\mathbb{F}_2)^{Gl(4,\mathbb{F}_2)}$. Hence, X is simple. Since, for p-compact groups, simple just means that the associated rational Weyl group is irreducible [12] [23], the last fact of (ii) is obvious.

As already mentioned in Section 2, the $\mathcal{A}_2$-action on $H^*(BX;\mathbb{F}_2)$ implies that $H^*_{\mathbb{Q}_2^\wedge}(BX) \cong \mathbb{Q}_2^\wedge[y_8, y_{12}, y_{28}] \cong H^*_{\mathbb{Q}_2^\wedge}(BDI(4))$. Since $H^*_{\mathbb{Q}_2^\wedge}(BX) \cong H^*_{\mathbb{Q}_2^\wedge}(BT_X)^{W_X}$ [10], the 2-compact groups X and $DI(4)$ have the same rank and the Weyl groups W_X and $W_{DI(4)}$ the same degrees. Looking at the classification of all irreducible rational pseudo reflection groups [3] shows that both Weyl groups are weakly isomorphic. That is that X and $DI(4)$ have the same rational Weyl group data. □

Proof of Proposition 1.4 (first half): Let X be a 2-compact group, such that $H^*(X;\mathbb{F}_2) \cong H^*(BDI(4);\mathbb{F}_2)$. We want to show that both have isomorphic maximal torus normalizer. By Lemma 3.1, we can identify W_X and $W_{DI(4)}$ via the rational Weyl group data. Then L_X and $L_{DI(4)}$ are $W_{DI(4)}$-lattices of $L_{DI(4)} \otimes \mathbb{Q}$. By Lemma 2.1, both lattices are isomorphic. That is we can identify BT_X and $BT_{DI(4)}$ as $W_{DI(4)}$-spaces. Since for every connected centerfree p-compact group the maximal torus normalizer splits [10] we have $BN_X \simeq B(T_X \rtimes W_X) \simeq B(T_{DI(4)} \rtimes W_{DI(4)}) \simeq BN_{DI(4)}$ which is the claim. □

Now we prove the other half of Proposition 1.4, which is nothing but Part (iv) of the following proposition.

3.3 Proposition. *For a 2-compact group X, such that N_X and $N_{DI(4)}$ are isomorphic, the following holds:*
(i) The 2-compact group X is connected.
(ii) There exists a monomorphism $Spin(7) \to X$ such that $Spin(7)$ is a subgroup of maximal rank.
(iii) The Euler Q $\chi(X/Spin(7))$ is odd.
(iv) The map $H^*(BSpin(7);\mathbb{F}_2) \to H^*(BX;\mathbb{F}_2)$ is a monomorphism.

(v) The subgroup $E_4 \subset N_{DI(4)} \subset X$ is self centralizing and

$$H^*(BX;\mathbb{F}_2) \cong H^*(BE_4;\mathbb{F}_2)^{\overline{W}_X} ,$$

where $\overline{W}_X$ is the Weyl group of $E_4 \subset X$.
(vi) $H^*(BX;\mathbb{F}_2) \cong H^*(BDI(4);\mathbb{F}_2)$ as algebras over the Steenrod algebra.

Here, the Weyl group $\overline{W}_X$ is the group of all elements $w \in Gl(4,\mathbb{F}_2)$ such that the inclusion $i_X : E_4 \to X$ and the composition $i_x w$ are conjugate in X.

Proof. By [10], for any p-compact group Y, the map $W_Y = \pi_0(N_Y) \to \pi_0(Y)$ is an epimorphism and the kernel is the Weyl group of the component Y_0 of the unit. Hence, there exists an epimorphism $W_X = \mathbb{Z}/2 \times Gl(3,\mathbb{F}_2) \to \pi_0(X)$. The target is a 2-group and the kernel W_0 is the Weyl group of X_0. In particular, since X and X_0 have both rank 3, the rational representation, $W_0 \to Gl(3,\mathbb{Q}_2^\wedge)$ represents W' as a pseudo reflection group. The order of W_0 divides $336 = 2^4 \cdot 3 \cdot 7$ which is the order of $W_{DI(4)}$. We have to figure out the pseudo reflection groups such that the product of the degrees is divisible by 3 and 7. Only the group No. 24 satisfies this property, which is the $W_{DI(4)}$. Hence, $\pi_0(X)$ is trivial and X connected. This proves part (i).

The maximal torus normalizer $N_X \cong N_{DI(4)}$ contains a unique subgroup $E_4 \subset N_{DI(4)}$ (Remark 2.2). Let $E_1 \subset E_4$ be the inclusion of the first coordinate. Since the Weyl group of $C_X(E_1)$ and of the component of the unit $C_X(E_1)_0$ can be read off from the action of W_X on T_X [11, 7.6], the centralizers $C_X(E_1)$ and $C_{DI(4)}(E_1)$ have isomorphic maximal torus normalizer as well as $C_X(E_1)_0$ and $C_{DI(4)}(E_1)_0$. Since $Spin(7) \cong C_{DI(4)}(E_1)$ is connected, $C_X(E_1)$ and $C_X(E)_0$ have isomorphic Weyl groups. Hence, $C_X(E_1)$ is connected and has the same maximal torus normalizer as $Spin(7)$. By [25], this implies that both are isomorphic, which proves part (ii).

The subgroup $Spin(7) \subset X$ is of maximal rank. As explained in Remark 2.2, the Euler characteristic of $X/Spin(7)$ equals the Weyl group index $[W_X : W_{Spin(7)}] = [W_{DI(4)} : W_{Spin(7)}]$ which is odd (Remark 2.2). This proves Part (iii).

There exists a transfer $H^*(BSpin(7);\mathbb{F}_2) \to H^*(BX;\mathbb{F}_2)$ such that the composition with the restriction map $H^*(BX;\mathbb{F}_2) \to H^*(BSpin(7);\mathbb{F}_2)$ is multiplication by $\chi(X/Spin(7)$ [4]. Since this is an odd number, the composition is the identity and the $\mathcal{K}$-map of (iv) is a monomorphism as desired.

Since E_4 centralizes E_1, we have $E_4 \subset Spin(7)$. And since $E_1 \subset E_4$, we have $C_X(E_4) \cong C_{Spin(7)}(E_4) \cong E_4$. The last identity follows because $E_4 \subset Spin(7)$ is self centralizing. Since $H^*(BX;\mathbb{F}_2) \to H^*(BE_4;\mathbb{F}_2)$ is the Adams-Wilkerson embedding [1], a combination of [7, 3.6], [29, Theorem II] and of properties of the T-functor [15] shows that $H^*(BX;\mathbb{F}_2) \cong H^*(BE_4,\mathbb{F}_2)^{\overline{W}_X}$, which is Part (v).

Finally we have to prove part (vi). By construction $\Gamma \subset \overline{W}_X \subset Gl(4,\mathbb{F}_2)$ where, as explained in Section 2, Γ is the Weyl group of $E_4 \subset Spin(7)$. The first inclusion is proper. Otherwise, the map $BSpin(7)_2^\wedge \to BX$ induces an isomorphism in mod-2 cohomology and is an equivalence. This is a contradiction, since $Spin(7)$ and X have non isomorphic maximal torus normalizer, and are therefore non isomorphic [18].

Lemma 3.4 below shows that $\Gamma \subset Gl(4,\mathbb{F}_2)$ is a maximal subgroup. This implies that $\overline{W}_X = Gl(4,\mathbb{F}_2)$ and that $H^*(BX;\mathbb{F}_2) \cong H^*(BDI(4);\mathbb{F}_2)$. This proves the last part and finishes the proof. $\square$

The stabilizer subgroup of a subspace $U' \subset U := (\mathbb{F}_p)^n$ is the the subgroup of $Gl(n,\mathbb{F}_p)$ of all elements which map U' onto itself. For $p = 2$ and for a 1-dimensional subspace, the isotropy subgroup and the stabilizer subgroup are equal.

3.4 Lemma. *For any prime p, the stabilizer subgroup $\Gamma \subset Gl(n,\mathbb{F}_p)$ of any 1-dimensional subspace of U is a maximal subgroup.*

Proof. We can assume that Γ is the stabilizer subgroup of the subspace generated by the first standard basis vector e_1. We write every element of $B \in Gl(n,\mathbb{F}_p)$ in the form $B = \begin{pmatrix} \mu & c \\ b & M \end{pmatrix}$, where $\mu \in \mathbb{F}_p$, b and c have $n-1$ components and where M is a $(n-1)\times(n-1)$ matrix. For elements of Γ, we have $\mu \neq 0$, $b = 0$ and $M \in Gl(n-1,\mathbb{F}_p)$.

Let $\Gamma \subsetneqq \Gamma' \subseteqq Gl(n,\mathbb{F}_p)$. We have to show that $\Gamma' = Gl(n,\mathbb{F}_p)$. Since $\Gamma \setminus Gl(n,\mathbb{F}_p)/ \cong U' := U \setminus \{0\}$, the quotient $\Gamma' \setminus Gl(n,\mathbb{F}_p)$ is a quotient of U'. Therefore, it is sufficient to show that Γ' acts transitively on U'.

Let $B = \begin{pmatrix} \mu & c \\ b & M \end{pmatrix} \notin \Gamma$; i.e. $b \neq 0$. Multiplying B by elements of Γ from the left shows that, for every vector $u \in U$, there exists a matrix $B' \in \Gamma'$ such that the first column (μ', b') equals u. Thus, the Γ'-orbit of e_1 contains all vectors of U' and Γ' acts transitively on U'. $\square$

4. Proof of the theorems 1.2 and 1.3.

The Quillen category of a p-compact group X is given by purely algebraic data and depends only on the mod-p cohomology of the classifying space BX. For compact lie groups this is shown in [8] and for p-compact groups the same proof works. This shows that two p-compact groups X and Y with $H^*(BX;\mathbb{F}_p) \cong H^*(BY;\mathbb{F}_p)$ have equivalent Quillen categories. And this equivalence only depends on the chosen $\mathcal{K}$-isomorphism between the mod-p cohomology algebras of the classifying spaces. Moreover, for any elementary abelian p-subgroup $E \subset X$ the mod-p cohomology $H^*(BC_X(E);\mathbb{F}_p)$ can be calculated with the help of Lannes' T-functor and also depends only on the algebraic object $H^*(BX;\mathbb{F}_p)$.

Let **Top** denote the category of topological spaces. Let $\mathbf{A}_p(X)$ denote the Quillen category of the p-compact group X and let

$$\widetilde{BC}_X : \mathbf{A}_p(X) \to \mathbf{Top}$$

denote the functor which maps E on the space $BC_X(E)$ and describes the Dwyer-Wilkerson decomposition of BX [8] [11]. We can collect the above considerations in the following proposition.

4.1 Proposition. *Let X and Y be two p-compact groups. Let $\phi : H^*(BY;\mathbb{F}_p) \xrightarrow{\cong} H^*(BX;\mathbb{F}_p)$ be a $\mathcal{K}$-isomorphism. Then the following holds:*

(i) There exists an equivalence $\phi_{\mathbf{A}} : \mathbf{A}_p(X) \to \mathbf{A}_p(Y)$ between the Quillen categories of X and Y only depending on ϕ.
(ii) For each elementary abelian subgroup $E \subset X$ there exists an $\mathcal{K}$-isomorphism $\psi_E : H^*(BC_Y(\phi_{\mathbf{A}}(E)); \mathbb{F}_p) \xrightarrow{\cong} H^*(BC_X(E); \mathbb{F}_p)$ which fit together to a natural transformation $\Psi : H^*(\widetilde{BC}_Y \circ \phi_{\mathbf{A}}; \mathbb{F}_p) \to H^*(\widetilde{BC}_X; \mathbb{F}_p)$.

If we identify the Quillen categories of X and Y, then the proposition says that the Dwyer-Wilkerson decompositions of both p-compact groups are identical in mod-p cohomology. In fact, for $Y = DI(4)$, we will show that these decomposition diagrams are equivalent on the level of diagrams of topological spaces. This will be done in three steps. First we show that $\widetilde{BC}_X$ is an H^*-clone of $\widetilde{BC}_{DI(4)}$ in the sense of [13] (see below). In the second step we show that both diagrams are equal inside the homotopy category of spaces and in the last step we show that, up to homotopy of diagrams, there is only one lift to the category **Top**. The last two steps are lifting problems. In our situation, for both lifting problems, there exists an obstruction theory, in the first case due to Dwyer and Wilkerson [13] and in the latter case due to Dwyer and Kan [5]. Fortunately, in [9] is shown that all these obstruction groups vanish. In particular, only the first step needs some work. And this is the place where we do need a uniqueness result for $Spin(7)$.

We have to recall what H^*-clones are. Let $\Phi, \Psi : \mathbf{C} \to \mathbf{HoTop}$ be two functors from a discrete category into the homotopy category of **Top**. Following [13] we say that Ψ is a H^*-clone of ϕ, if for each object c of $\mathbf{C}$ there exists an equivalence $s_c : \phi(c) \to \Psi(c)$ such that the isomorphisms $H^*(s_c; \mathbb{F}_p)$ establish a natural equivalence $H^*(\Psi; \mathbb{F}_p) \to H^*(\Phi; \mathbb{F}_p)$. A lift of Ψ is a functor $\hat{\Psi} : \mathbf{C} \to \mathbf{HoTop}$ such that $H^*(\hat{\Psi}; \mathbb{F}_p)$ is naturally equivalent to Ψ. And a lift of $\hat{\Psi}$ to **Top** is a functor $\widetilde{\Psi} : \mathbf{C} \to \mathbf{Top}$ such that composition of $\widetilde{\Psi}$ with $\mathbf{Top} \to \mathbf{HoTop}$ is naturally equivalent to $\hat{\Psi}$.

In the rest of this section, we will prove the following theorem which, because of Proposition 1.4, implies Theorem 1.2 as well as Theorem 1.3.

4.3 Theorem. *Let X be a 2-compact group, such that X and $DI(4)$ have isomorphic maximal torus normalizer and such that $H^*(BX; \mathbb{F}_2) \cong H^*(BDI(4); \mathbb{F}_2)$. Then, X and $DI(4)$ are isomorphic as 2-compact groups.*

The proof of this theorem will occupy the rest of this section. From now on and throughout this section X denotes a 2-compact group which satisfies the assumption of the theorem. By Proposition 4.1, we can and we will identify the Quillen categories $\mathbf{A}_p(X) \cong \mathbf{A}_p(DI(4)) =: \mathbf{A}$. We can assume that X is connected (Lemma 3.1). The composition $H^*(BX; \mathbb{F}_2) \cong H^*(BE_4; \mathbb{F}_2)^{Gl(4,\mathbb{F}_2)} \to H^*(BE_4; \mathbb{F}_2)$ establishes a map $BE_4 \to BX$ [15] which gives rise to a monomorphism $i_X : E_4 \to X$ of 2-compact groups.

4.4 Lemma. *There exists a monomorphism $Spin(7) \to X$ such that the diagram*

$$\begin{tikzcd} H^*(BX;\mathbb{F}_2) \arrow[rr, "\cong"] \arrow[dr] & & H^*(BDI(4);\mathbb{F}_2) \arrow[dl] \\ & H^*(BSpin(7);\mathbb{F}_2) & \end{tikzcd}$$

commutes.

Proof. Let $T := T_{DI(4)}$ denote the maximal torus and $N := N_{DI(4)}$ the maximal torus normalizer of $DI(4)$. We have a chain $E_4 \subset T \rtimes \mathbb{Z}/2 \subset N \subset X$ of subgroups and $E_1 \subset E_4$ is actually contained in T. By Proposition 3.3, $C_X(E_1) \cong Spin(7)$ which establishes the monomorphism $f : Spin(7) \to X$. And f induces a monomorphism $f^* : H^*(BX;\mathbb{F}_2) \to H^*(BSpin(7);\mathbb{F}_2)$ in mod-2 cohomology (Proposition 3.3).

Let $i_{Spin} : E_4 \subset Spin(7)$ be the standard embedding and let $j_X := f \circ i_{Spin(7)} : E_4 \to X$ be the composition. Then, $Bj_X^* : H^*(BX;\mathbb{F}_2) \to H^*(BE_4;\mathbb{F}_2)$ is a monomorphism and there exists an isomorphism $\rho : E_4 \to E_4$ such that the diagram

$$\begin{tikzcd} & H^*(BX;\mathbb{F}_2) \arrow[dl, "Bi_X^*"'] \arrow[dr, "Bj_X^*"] & \\ H^*(BE_4;\mathbb{F}_2) \arrow[rr, "B\rho^*"] & & H^*(BE_4;\mathbb{F}_2) \end{tikzcd}$$

commutes.

Since $H^*(BX;\mathbb{F}_2)$ is isomorphic to the Dickson invariants, the map Bj_X^* also maps $H^*(BX;\mathbb{F}_2)$ onto the Dickson invariants. The same is true for the composition $H^*(BDI(4);\mathbb{F}_2) \to H^*(BSpin(7);\mathbb{F}_2) \xrightarrow{Bi_{Spin}^*} H^*(BE_4;\mathbb{F}_2)$. Because of the degrees of the polynomial generators, $H^*(BDI(4);\mathbb{F}_2)$ has only one automorphism and there exists only one isomorphism between the mod-2 cohomology algebras of BX and $BDI(4)$. This shows that there exists a diagram as desired. $\square$

4.5 Proposition. *The two functors $\widetilde{BC}_X, \widetilde{BC}_{DI(4)}$: $\mathbf{A} \to \mathbf{Top}$ are H^*-clones.*

Proof. For the elementary abelian 2-subgroups $E_i \subset E_4 \subset Spin(7) \subset X$, $i = 1,..,4$, we have to ways to calculate the mod-2 cohomology of $BC_X(E_i)$. As explained in the beginning of this section we can use Lannes' T-functor [15]. Then, $H^*(BC_X(E_i);\mathbb{F}_2) = T^{E_i}(H^*(BX;\mathbb{F}_2);\phi_i)$, where ϕ_i denotes the map induced by the composition $BE_i \to BE_4 \to BX$. On the other hand, since $E_1 \subset E_i$ for all i, the inclusion $C_{Spin(7)}(E_i) \to C_X(E_i)$ is an isomorphism of 2-compact groups. The same holds if we replace by X by $DI(4)$.

Now, for each E_i, we define s_{E_i} to be the composition

$$BC_{DI(4)}(E_i) \cong BC_{Spin(7)}(E_i) \cong BC_X(E_i) \ .$$

By construction, the collection of these maps establishes a natural transformation $H^*(\widetilde{BC}_X;\mathbb{F}_2) \to H^*(\widetilde{BC}_{DI(4)};\mathbb{F}_2)$ which shows that $\widetilde{BC}_X$ and $\widetilde{BC}_{DI(4)}$ are H^*-clones. $\square$

4.6. Proof of Theorem 4.3 By the last lemma we know that $\widetilde{BC}_X$ and $\widetilde{BC}_{DI(4)}$ are H^*-clones. We have to analyze, up to equivalence, how many lifts to **HoTop** do exist for $H^*(\widetilde{BC}_{DI(4)})$. Since the diagram is given as the diagram of classifying spaces of centralizers of elementary abelian subgroups, the diagram is centric in the sense of [13] and [5]. Basically this means that

$$map(BC_{DI(4)}(E_i), BC_{DI(4)}(E_i))_{id} \rightarrow map(BC_{DI(4)}(E_i), BC_{DI(4)}(E_j))_{Bi}$$

is an equivalence for any $E_j \subset E_i$ where $i : C_{DI(4)}(E_i) \rightarrow C_{DI(4)}(E_j)$ denotes the induced inclusion [13]. As shown in [13], there exists an obstruction theory for the above question . The obstruction groups are given by the first higher derived functor of the inverse limit functor of particular functors $\Phi_i : \mathbf{A} \rightarrow \mathbf{A}b$ with values in the category of abelian groups. On objects, $\Phi_i(E_i) = \pi_i(BZ(C_{DI(4)}(E_i))$ where $Z(C_{DI(4)}(E_i))$ denotes the center of $C_{DI(4)}(E_i)$ [13].

For the number of lifts of the composition $\mathbf{A} \xrightarrow{\widetilde{BC}_{DI(4)}} \mathbf{Top} \rightarrow \mathbf{HoTop}$ to **Top** up to natural equivalences, there also exists an obstruction theory. Again, the obstruction groups are given by the higher derived functors of the inverse limit functor of the same functors Φ_i [5].

In [9] all these higher derived functors are proven to vanish. This shows that $\widetilde{BC}_X$ and $\widetilde{BC}_{DI(4)}$ are equivalent as functors taking values in **HoTop** as well as as functors taking values in **Top**. In particular, both functors $\widetilde{BC}_X$ and $\widetilde{BC}_{DI(4)}$ have homotopy equivalent homotopy colimits. And, since the homotopy colimits are mod-2 equivalent to BX respectively to $BDI(4)$, this proves that X and $DI(4)$ are isomorphic as 2-compact groups. □

5. Proof of Theorem 1.5.

We start directly with th proof. For any connected 2-compact group X there exists a short exact sequence $Z \rightarrow X \rightarrow \overline{X}$ of 2-compact groups respectively a fibration $BZ \rightarrow BX \rightarrow B\overline{X}$ for the associated classifying spaces, such that $Z \subset X$ is a central subgroup, in particular a product of a finite 2-group and a torus, and such that $\overline{X}$ is centerfree [18]. In fact, the fibration is principal and has a classifying map $B\overline{X} \rightarrow B^2Z$. By [12] $\overline{X} \cong \prod_i Y_i$ is congruent to a finite product of simple centerfree 2-compact groups Y_i. In particular, for all i, the rational Weyl group representation $W_{Y_i} \rightarrow Gl(L_{Y_i} \otimes \mathbb{Q})$ is irreducible. (Actually, all this is true for any prime.) If Y_i has the same rational Weyl group data as $DI(4)$, then, as the proof of Proposition 1.4 (first half) shows (see Section 3), Y_i and $DI(4)$ have isomorphic maximal torus normalizer and are isomorphic by Theorem 1.3.

Let X_1 be the product of all factors Y_i with the same rational Weyl group data as $DI(4)$ and $\overline{X}_2$ the product of all the others. At the prime 2, the rational Weyl group representation of $DI(4)$ is the only irreducible pseudo reflection group over $\mathbb{Q}_2^\wedge$ which is not weakly isomorphic to the 2-adic rational Weyl group representation of a simple connected compact Lie group. This follows by checking the Clark-Ewing list [3]. Therefore $\overline{X}_2$ has the same rational Weyl group data as a suitable connected compact Lie group, and X_1 is congruent to a product of copies of $DI(4)$.

Since $BDI(4)$ is 3-connected, the composition $BX_1 \rightarrow BX_1 \times B\overline{X}_2 \rightarrow B^2Z$ is null homotopic and $BX \simeq BX_1 \times BX_2$ where BX_2 is the fiber of the composition

$B\overline{X}_2 \to BX_1 \times B\overline{X}_2 \to B^2Z$. Since BX_2 fits into a fibration $BZ \to BX_2 \to B\overline{X}_2$, we can think of it as the classifying space of a 2-compact group X_2 . And, up to some trivial summands, BX_2 has the same rational Weyl group data as $B\overline{X}$. In particular, BX_2 has the same rational Weyl group data as a suitable connected compact Lie group. This proves Theorem 1.5. $\square$

6. Self maps of $BDI(4)$.

In this section we want to prove Theorem 1.6. We use the notation of the previous chapters.

6.1 Remark. Every elementary abelian 2-subgroup $E \subset DI(4)$ of rank i, $1 \leq i \leq 4$, is conjugate to the inclusion $E_i \subset E_4 \subset DI(4)$. We denote by $j_i : C_i := C_{DI(4)}(E_i) \to DI(4)$ the inclusion of the centralizer of E_i. Let $\phi : C_i \to DI(4)$ be a monomorphism. Since $E_4 \subset C_i$, there exists an homomorphism $\rho : E_4 \to E_4$ such that the diagram

$$\begin{array}{ccc} E_4 & \xrightarrow{\subset} & C_i \\ {\scriptstyle\rho}\big\downarrow & & {\scriptstyle\phi}\big\downarrow \\ E_4 & \xrightarrow{\subset} & DI(4) \end{array}$$

commutes up to conjugation (Proposition 2.3). Actually, ρ is a monomorphism, because ϕ is one [10], and therefore an isomorphism. Since the Weyl group of $E_4 \subset DI(4)$ is $Gl(4;\mathbb{F}_2)$ we can assume that $\rho = id$ and that the restriction $\phi|_{E_i}$ is the inclusion $E_i \subset DI(4)$. Passing to centralizers establishes a monomorphism $\phi' : C_i \cong C_{C_i}(E_i) \to C_{DI(4)}(E_i) = C_i$. In fact, this is an isomorphism [18].

Because $E_i \subset C_i$ is central there exists a homomorphism $\mu : C_i \times E_i \to C_i$. And, by construction, $B\phi' : BC_i \to map(BE_i, BDI(4))_{B\phi|_{BE_i}} \simeq BC_i$ is the adjoint of the composition $BC_i \times BE_i \xrightarrow{B\mu} BC_i \xrightarrow{B\phi} BDI(4)$ and an equivalence. If $i = 1$ or 2, then C_i is connected and $E_i \subset C_i$ is subconjugated to the maximal torus of C_i.

All this together establishes the assumptions made in Proposition 4.6 of [24]. The conclusion we can draw is the following. If $B\psi : BC_i \to BDI(4)$ is another map such that $H^*_{\mathbb{Q}}(B\phi) = H^*_{\mathbb{Q}}(B\psi)$, then $B\phi$ and $B\psi$ are homotopic.

6.2 Lemma. *A map $g : BSpin(7)^\wedge_2 \to BDI(4)$ is either homotopic to the constant map or there exists a self equivalence $h : BSpin(7)^\wedge_2 \to BSpin(7)^\wedge_2$ such that $Bj \circ h \simeq g$.*

Proof. Let $\phi : Spin(7) \to DI(4)$ be the underlying homomorphism such that $g \simeq B\phi$. The homomorphism ϕ is either conjugate to the constant map or the kernel of ϕ is a central subgroup of $Spin(7)$ [21], hence a subgroup of $\mathbb{Z}/2$. If the kernel is the trivial group, then ϕ is a monomorphism. If the kernel equals $\mathbb{Z}/2$, then ϕ factors through a monomorphism $\overline{\phi} : SO(7) = Spin(7)/(\mathbb{Z}/2) \to DI(4)$ [21]. This gives a contradiction, since $SO(7)$ contains an elementary abelian 2-group of rank 6 and $DI(4)$ does not. Therefore, ϕ is either the constant map or a monomorphism. In the latter case, the existence of h follows from Remark 6.1. $\square$

6.3 Proposition. *Let $f : BDI(4) \to BDI(4)$ be a self map. Then the following holds:*
(i) The map f is either homotopic to the constant map or an equivalence.
(ii) There exists a self map $h : BSpin(7)^\wedge_2 \to BSpin(7)^\wedge_2$ such that the diagram

$$\begin{array}{ccc} BSpin(7)^\wedge_2 & \xrightarrow{h} & BSpin(7)^\wedge_2 \\ {\scriptstyle Bj}\downarrow & & {\scriptstyle Bj}\downarrow \\ BDI(4) & \xrightarrow{f} & BDI(4) \end{array}$$

commutes up to homotopy.

Proof. If $f \circ Bj$ is null homotopic we can choose the constant map fro h. Hence, the second part is a consequence of Lemma 6.2.

The map h is either null homotopic or an equivalence [14]. If h is an equivalence, then f induces a monomorphism in mod-2 cohomology, therefore an isomorphism, and is an equivalence.

Since the Weyl group index $[W_{DI(4)} : W_{Spin(7)}]$ is odd, $Spin(7)$ and $DI(4)$ have the same 2-Sylow subgroup $P \subset Spin(7) \subset DI(4)$. thus, if h is null homotopic, the restriction $f|_{BP}$ is null homotopic and so is f itself [23] $\square$

For a p-compact group X, we call a self map $f : BX \to BX$ an Adams operation of degree $k \in \mathbb{Z}^\wedge_p$ if, for any class $x \in H^{2i}_{\mathbb{Q}^\wedge_p}(BX)$, $f^*(x) = k^i x$. By $\mathbb{Z}^{\wedge *}_2$ we denote the units of $\mathbb{Z}^\wedge_2$.

6.4 Corollary. *Every self equivalence of $BDI(4)$ is an Adams operation whose degree is a 2-adic unit.*

Proof. . Every self equivalence of $BSpin(7)^\wedge_2$ has this property [14]. Since $H^*_{\mathbb{Q}^\wedge_2}(BDI(4)) \to H^*_{\mathbb{Q}^\wedge_2}(BSpin(7))$ is a monomorphism, the statement is a consequence of Proposition 6.3. $\square$

Considering the constant map as an Adams operation of degree 0, the above corollary establishes a map

$$D : [BDI(4), BDI(4)] \to \{0\} \coprod \mathbb{Z}^{\wedge *}_2 \ .$$

6.5 Theorem. *The map $D : [BDI(4), BDI(4)] \to \{0\} \coprod \mathbb{Z}^{\wedge *}_2$ is a bijection.*

We postpone the proof of Theorem 6.5 to the end of this section.

6.6. Proof of Theorem 1.6. If, for two maps $f, g : BDI(4) \to BDI(4)$, the restrictions to $BT_{DI(4)}$ are homotopic, then both induce the same map in rational cohomology, are Adams operations of the same degree and therefore homotopic by Theorem 6.5. $\square$

As in Section 4 we denote by $\mathbf{A} := \mathbf{A}_p(DI(4))$ the Quillen category of $DI(4)$ and by $\widetilde{BC} := \widetilde{BC}_{DI(4)} : \mathbf{A} \to \mathbf{Top}$ the functor describing the Dwyer-Wilkerson decomposition. Since hocolim $\widetilde{BC} := \operatorname{hocolim}_{\mathbf{A}} \widetilde{BC} \to BDI(4)$ induces an equivalence after 2-adic completion we get an equivalence

$$map(BDI(4), BDI(4)) \simeq \text{holim } map(\widetilde{BC}, BDI(4)) \ .$$

The equivalence establishes a map

$$[BDI(4), BDI(4)] \to \lim_{\overleftarrow{\mathbf{A}}} \ [\widetilde{BC}, BDI(4)] \ .$$

For $\phi \in \lim_{\overleftarrow{\mathbf{A}}} \ [\widetilde{BC}, BDI(4)]$ and an object $E \subset X$ of $\mathbf{A}$ we denote by $map(BC(E), BDI(4))_\phi$ the component determined by ϕ and by

$$map(BDI(4), BDI(4))_\phi := \operatorname{holim} \ map(\widetilde{BC}, BDI(4))_\phi$$

the union of all components whose image is ϕ.

The homotopy groups of the last space can be calculated with the Bousfield-Kan spectral sequence [2]. The E^2-term has the form

$$E^2_{i,j} \cong \lim_{\overleftarrow{\mathbf{A}}}{}^i \ \pi_j(map(\tilde{BC}, BDI(4))_\phi =: \lim_{\overleftarrow{\mathbf{A}}}{}^i \ \Pi_j(\phi) \ .$$

If ϕ is the image of the constant map, then, for all E_i,

$$map(BC_{DI(4)}(E_i), BDI(4))_\phi \simeq BDI(4)$$

and the functor $\Pi_j(\phi)$ is the constant functor. If ϕ is the image of a self equivalence $f : BDI(4) \to BDI(4)$, then composition with f induces a natural equivalence

$$\Pi_j := \Pi_j(id) \xrightarrow{\cong} \Pi_j(\phi) \ .$$

Since for any abelian p-compact group A and any p-compact group X

$$BZ(C_X(A)) \simeq map(BC_X(A), BC_X(A))_{id} \simeq map(BC_X(A), BX)_{Bi}$$

where i denotes the inclusion of the centralizer [11] [24], the functor Π_j is naturally equivalent to the functor Φ_j considered in Section 4.

In both cases the higher derived limits all vanish [9], the Bousfield spectral sequence collapses and is concentrated in a singular line. This allows to prove the following proposition:

6.7 Proposition.
(i) The map $[BDI(4), BDI(4)] \to \lim_{\overleftarrow{\mathbf{A}}} \ [\widetilde{BC}, BDI(4)]$ *is a bijection*

(ii) For all non trivial self maps $f : BDI(4) \to BDI(4)$*, the mapping space* $map(BDI(4), BDI(4))_f$ *is contractible, and* $map(BDI(4), BDI(4))_{const} \simeq BDI(4)$.

Proof. The first part follows from the above considerations. They also show that, for all non trivial self maps $f : BDI(4) \to BDI(4)$, $map(BDI(4), BDI(4))_{id} \simeq map(BDI(4), BDI(4))_f$. The first mapping space is equivalent to the classifying space of the center of $DI(4)$ [11], which is trivial [9]. The component of the constant map is calculated in [11]. □

Proof of Theorem 6.5. Let $\mathbf{A}' \subset \mathbf{A}$ denote the full subcategory whose objects are elementary abelian 2-subgroups of $DI(4)$ of rank less or equal to 2. Then, the results of [26] say that the second arrow of

$$[BDI(4), BDI(4)] \xrightarrow{\cong} \lim_{\overleftarrow{\mathbf{A}}} [\widetilde{BC}, BDI(4)] \to \lim_{\overleftarrow{\mathbf{A}'}} [\widetilde{BC}, BDI(4)]$$

is an isomorphism. The first isomorphism follows from Proposition 6.7. We denote by $j_E : C_{DI(4)}(E) \to DI(4)$ the inclusion of the centralizer of E.

We first show that D is an injection. The degree of a self map of $BDI(4)$ is determined by the the induced map in rational cohomology. Let $f, g : BDI(4) \to BDI(4)$ be two self maps. Since a self map is null homotopic if and only if it induces the trivial map in rational cohomology [16], the map f is null homotopic if and only if g is. Moreover, if f is not null homotopic it is an equivalence as well as g. The compositions $f \circ Bj_E$ and $g \circ Bj_E$ induce the same map in rational cohomology. If the rank of E is less or equal to 2, then Remark 6.1 shows that both compositions are actually homotopic. By Proposition 6.6, this implies that $f \simeq g$. Hence, D is an injection.

The degree 0 is realized by the constant map. For $k \in \mathbb{Z}_2^{\wedge *}$ and $E \subset DI(4)$ of rank ≤ 2, the centralizer $C_{DI(4)}(E)$ is the completion of a connected compact Lie group. There exists an Adams operation $f'_E : BC_{DI(4)}(E) \to BC_{DI(4)}(E)$ of degree k [14]. Let $f_E := Bj_E \circ f'_E : BC_{DI(4)}(E) \to BDI(4)$ denote the composition. We want to show that the tuple $(f_E)_{E \in \mathbf{A}'}$ gives an element of $\lim_{\overleftarrow{\mathbf{A}'}} [\widetilde{BC}, BDI(4)]$. That is that for every morphism $E_0 \to E_1$ the composition $BC(E_1) \to BC(E_0) \xrightarrow{f_{E_0}} BDI(4)$ is homotopic to f_{E_1}. By construction, the composition and f_{E_1} induce the same map in rational cohomology. Remark 6.1 implies that both maps are homotopic. Therefore, $(f_E)_{E \in \mathbf{A}'} \in \lim_{\overleftarrow{\mathbf{A}'}} [\widetilde{BC}, BDI(4)]$ and D is surjective (Proposition 6.7), which finishes the proof. $\square$

References.

[1] J.F. Adams & C.W. Wilkerson, *Finite H–spaces and algebras over the Steenrod algebra*, Ann. Math. **111** (1980), 95-143.

[2] A. Bousfield & D. Kan, *Homotopy limits, completion and localisations*, SLNM 304, Springer Verlag, 1972.

[3] A. Clark & J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. **50** (1974), 425-434.

[4] W.G. Dwyer, *Transfer maps for fibrations*, Math. Proc. Camb. Philos. Soc. **120** (1996), 221-235.

[5] W.G. Dwyer & D. Kan, *Centric maps and realization of diagrams in the homotopy category*, Proc. AMS **114** (1992), 575-584.

[8] W.G. Dwyer & C.W. Wilkerson, *A cohomology decomposition theorem*, Topology **31** (1992), 433-443.

[9] W.G. Dwyer & C.W. Wilkerson, *A new finite loop space at the prime two*, J. AMS **6** (1993), 37-63.

[10] W.G. Dwyer & C.W. Wilkerson, *Homotopy fixed–point methods for Lie groups and finite loop spaces*, Ann. Math. (2) **139** (1994), 395-442.

[11] W.G. Dwyer & C.W. Wilkerson, *The center of a p-compact group*, The Cech Contennial. Contemporary Math. **181** (1995), 119-157.

[12] W.G. Dwyer & C.W. Wilkerson, *Product splittings of p–compact groups*, Fund. Math. **147** (1995), 279-300.

[13] W.G. Dwyer & C.W. Wilkerson, *Diagrams up to cohomology*, Preprint.

[14] S. Jackowski, J. McClure & R. Oliver, *Self homotopy equivalences of classifying spaces of connected compact Lie groups*, Fund. Math. **147** (1995), 99-126.

[15] J. Lannes, *Sur les espaces fonctionelles dont la source est la classifiant d'un p–groupe abélien élemementaires*, IHES **75** (1992), 135-244.

[16] J.M. Møller, *Rational isomorphism of p-compact groups*, Topology **35** (1996), 201-225.

[17] J.M. Møller, *Homotopy Lie groups*, Bull. Amer. Math. Soc. (N.S.) **32** (1995), 413-428.

[18] J.M. Møller & D. Notbohm, *Centers and finite coverings of finite loop spaces*, J. reine u. angew. Math. **456** (1994), 99-133.

[19] J.M. Møller & D. Notbohm, *Connected finite loop spaces with maximal tori*, Trans. AMS **350** (1998), 3483-3504.

[20] *Homotopieeindeutigkeit von Produkten orthogonaler Gruppen*, Thesis, Göttingen 1996 or ftp://hopf.math.purdue.edu/pub/morgenroth/BOn.dvi.

[21] D. Notbohm, *Kernels of maps between classifying spaces*, Isreal J. Math. **87** (1994), 243-256.

[22] D. Notbohm, *Homotopy uniqueness of classifying spaces of compact connected Lie groups at primes dividing the order of the Weyl group*, Topology **33** (1994), 271-330.

[23] D. Notbohm, *Unstable splittings of classifying spaces of p–compact groups*, to appear Quart. J. Math. Oxford.

[24] D. Notbohm, *Topological realzation of a family of pseudo reflection groups*, Fundam. Math. **155** (1998), 1-31.

[25] D. Notbohm, *A uniqueness result for orthogonal groups as 2-compact groups*, Preprint, available at ftp://hopf.math.purdue.edu/pub/Notbohm/bdi4.dvi.

[26] R. Oliver, *Higher limits via Steinberg representations*, Comm. in Algebra **22** (1994), 1381–1402.

[27] D. Sullivan, *Geometric Topology, Part I: Localisations, periodicity and Galois theory*, Mimeographed notes, MIT (1970).

[28] A. Viruel, *Homotopy uniqueness of BG_2*, Manuscr. Math. **95** (1998), 471-497.

[29] Rings of invariants and insepaprable forms of algebras over the Steenrod algebra, Preprint.

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