The volume and the injectivity radius of a hyperbolic manifold

Alexander Reznikov

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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The main goal of this paper is to prove the following result.

Theorem *Let M be a compact Riemannian manifold of the negative curvature -1. If $n = \dim M \geq 6$, then the volume $\mathrm{Vol}(M)$ and the injectivity radius $i(M)$ satisfy*

$$i(M) \geq C_n(\mathrm{Vol}(M))^{-\left(1+\frac{6}{n-5}\right)}$$

One can deduce a weaker exponential estimate $i(M) \geq \exp\left(-C_n'\mathrm{Vol}(M)\right)$ from the argument of Gromov [3]. This exponential decay estimate remains true with the same proof if the curvature of M is bounded between two negative constants. We emphasize that our proof of the theorem above is valid only for hyperbolic manifolds (of constant curvature). It remains an appealing question to know if the polynomial estimate of the theorem is still true for manifolds of variable curvature.

The method of the proof relies on the precise estimate for volumes of so-called Margulis tubes, which constitute the "thin" part of a hyperbolic manifold. We show that as the length of the core geodesic decreases, all the widthes in orthogonal directions grow up so that one can estimate the $(n-1)$-volume of all orthogonal sections. It is interesting that the crucial for our argument property of the geodesic flow in hyperbolic manifolds is that one can control the divergence of geodesics from above, whereas in the volume-diameter theorem of Gromov [3] one needs a divergence estimate from below.

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1. Volume computations for Margulis tubes

We recall some now standard facts from the geometry of negatively curved manifolds. Let M be a compact manifold whose curvature satisfies $-1 \leq K(M) \leq 0$. Fix the Margulis constant ε_n and some $\varepsilon < \varepsilon_n$. Then the "thin" part of the manifold M, i.e. $\{x | i_x(M) < \varepsilon\}$ is a disjoint union of full tori, say $\mathcal{O}_p$, which are constructed as follows. There exists a closed geodesic, say γ_p , in $\mathcal{O}_p$, which is covered by some geodesic line in $\tilde{M}$. Next, for some deck transformation φ_p of $\tilde{M}$, which leaves γ_p invariant, one defines a tube $\tilde{\mathcal{O}}_p$ by $\left\{y \in \tilde{M}, \mathrm{dist}_{\tilde{M}}(y, \varphi_p^s y) < \varepsilon \quad \text{for some s}\right\}$. Then the Margulis tube $\mathcal{O}_p$ is by definition the quotient of $\tilde{\mathcal{O}}_p$ under the action of the cyclic group generated by φ_p .

We will adopt some notational conventions. We will call the length of the cone closed geodesic $\gamma_p = \tilde{\gamma}_p/\{\varphi_p^m\}$ the period of the Margulis tube and denote it l_p . Next, if $z \in \gamma_p$ and $Z \in T_zM$ such that Z is orthogonal to $\dot{\gamma}_p(z)$, we issue a geodesic from z along Z

and measure its part up to the first intersection with $\partial\mathcal{O}_p$. This length is called the width of $\mathcal{O}_p$ and denoted $w_p(z,Z)$. The crucial observation is contained in the following lemma.

Lemma 1 *The width and the period of a Margulis tube in M satisfy the relation*

$$\exp w_p(z,Z) \geq C_{n,\varepsilon} l^{\frac{-2}{n+1}} \cdot$$

for all z, Z .

Proof: Let ζ is the geodesic issued from z at the direction Z and let q of the first intersection of ζ with $\partial\mathcal{O}_p$. We lift z, ζ, q to $\tilde{M}$ and denote these $\tilde{z}, \tilde{\zeta}, \tilde{q}$. Since $\tilde{q} \in \partial\tilde{\mathcal{O}}_p$, we have $\mathrm{dist}_{\tilde{M}}(\tilde{q}, \varphi_p^s \tilde{q}) \geq \varepsilon$ for all $s \in \mathbf{Z}$. Let Φ be the (orthogonal) operator of the parallel transport from T_zM to $T_{\varphi_p(z)}M$ along ζ . Define $A : T_zM \to T_zM$ to be $A = \Phi^{-1} \circ \varphi_{p*}$. We claim $\mathrm{dist}_{\tilde{M}}(\tilde{q}, \varphi_p^s \tilde{q}) \leq C_n \exp w_p(z,Z) \cdot (sl_p + \rho(Z, A^s Z))$, where ρ stands for the spherical distance in the unit sphere of T_zM . Indeed, $\varphi_p^s \tilde{q}$ lies on the geodesic $\varphi_p^s \tilde{\zeta}$, whose initial point $\varphi_p^s \tilde{z}$ is at the distance sl_p from $\tilde{z}$ and the tangent vector is at the distance $\rho(Z, A^s Z)$ from the parallel transport of the tangent vector to $\tilde{\zeta}$. Next, fix some $N \geq 1$ and suppose for a moment that n is odd, so $(n-1)$ is even. Then A is a direct sum of $\frac{n-1}{2}$ rotations in planes, say by some angles θ_i . It follows immediately that for some $s \leq N$ one gets $\rho(Z, A^s Z) \leq C_n' \left(\frac{1}{N}\right)^{\frac{2}{n-1}}$, so for such s , $\mathrm{dist}_{\tilde{M}}(\tilde{q}, \varphi_p^s \tilde{q}) \leq C_n \exp w_p(z,Z) \left(Nl_p + C_n' \left(\frac{1}{N}\right)^{\frac{2}{n-1}}\right)$. Now we wish to minimize the expressioin in brackets, varying N. The elementary derivative computation gives $\min\limits_N \left(Nl_p + C_n' \left(\frac{1}{N}\right)^{\frac{2}{n-1}}\right) \leq C_n'' l^{\frac{2}{n+1}}$. So $\varepsilon \leq \min\limits_s \mathrm{dist}_{\tilde{M}}(\tilde{q}, \varphi_p^s \tilde{q}) \leq C_n''' \exp w_p(z,Z) l^{\frac{2}{n+1}}$ and, finally, $\exp w_p(z,Z) \geq C_{n,\varepsilon} l^{\frac{-2}{n+1}}$ as desired. Next, for n even ($n-1$ odd) the operator A is a direct sum of rotations and $\pm$ identity in $\mathbf{R}^1$ and the argument goes as above, completing the proof.

From now on *we assume M to be hyperbolic* , i.e. $K(M) \equiv -1$. For $\tilde{z}$ in $\tilde{\gamma}_p$ let $\tilde{H}_p(z)$ be the totally geodesic hyperplane containing z and orthogonal to $\tilde{\gamma}_p$. We have $\mathrm{Vol}_{n-1}\left(\tilde{H}_p(z) \cap \tilde{\mathcal{O}}_p\right) \geq C_n \min\limits_z \exp((n-2)w_p(z,Z))$. So by the lemma 1 we get $\mathrm{Vol}(\mathcal{O}_p) \geq C_n l_p \left(C_{n,\varepsilon} \cdot l_p^{-\frac{2}{n+1}}\right)^{n-2} = C_{n,\varepsilon}' l_p^{\frac{5-n}{n+1}}$ and if $n \geq 6$, then $l_p \geq C_{n,\varepsilon}''(\mathrm{Vol}(\mathcal{O}_p))^{-(1+\frac{6}{n-5})} \geq C_{n,\varepsilon}''(\mathrm{Vol}(M))^{-(1+\frac{6}{n-5})}$. Now we take $\varepsilon = \frac{\varepsilon_n}{2}$, then $i(M) = \min\limits_p l_p$ by the properties of Margulis tubes (see [8], for example), and the theorem follows.

2. Applications and Discussion

The theorem above combined with the Cheeger finiteness results implies for hyperbolic manifolds in dimensions ≥ 6 the finiteness theorem of Gromov [3]: there are no more than a finite number of differential structures of compact manifolds, admitting a hyperbolic metric with the volume bounded from above by a constant V. Moreover, it is immediate to show the existence of a simplicial complex, homotopically equivalent to M with no more than $C_n(\mathrm{Vol}(M))^{\frac{2n^2+n}{n-5}}$ vertices, so the Betti numbers $b_k(M)$ satisfy the polynomial by $\mathrm{Vol}(M)$ estimate $b_k(M) \leq C_{n,k}(\mathrm{Vol}(M))^{\frac{2n^2+n}{n-5} \cdot k}$. A stronger growth estimate was stated in Gromov [4], see Ballmann, Gromov and Schroeder [1] for a proof.

We already have mentioned that the proof given above uses very essentially the hyperbolicity of M. For manifolds with the negative curvature bounded as $-K \leq K(M) \leq -k < 0$, one derives an estimate $\mathrm{Vol}(\mathcal{O}_p) \geq C_{n,k,K}|\log l_p|$ instead of our lemma 1, using the argument of Gromov [3]. This gives an exponential by $\mathrm{Vol}(M)$ decay of $i(M)$. We do not know to which extent the dimension restriction $(n \geq 6)$ is important. One could expect that for $n \geq 4$ the polynomial by $\mathrm{Vol}(M)$ estimate for $i(M)$ still holds. Of course, this completely breaks down for $n = 3$.

Finally, we wish to specify the relation of the volume estimates of Margulis tubes to the isoperimetrical inequalities in M. Recall that the classical isoperimetrical inequality $\mathrm{Vol}_n(D) \leq C_n(\mathrm{Vol}_{n-1}(\partial D))^{n/(n-1)}$ holds for domains in the hyperbolic n-space. Here C_n can be made the standard "euclidean" constant for $n = 2,3,4$ (see [2], [5] and [7]). Next, in compact hyperbolic manifolds M, the linear isoperimetrical inequality $\mathrm{Vol}(D) \leq C_n' \; \mathrm{Vol}_{n-1}(\partial D)$ holds by Schoen [8]. Moreover, the classical isoperimetrical inequality holds for $n \geq 4$. The proof is parallel to [8] and uses the diameter-volume inequality of Gromov. But for the three-dimensional hyperbolic manifoolds the classical isoperimetrical inequality breaks down (yet the linear one is still true), and the counterexamples are provided by Margulis tubes. We refer to [6] for the topological consequences of these inequalities.

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Department of Mathematics

Hebrew University

Giv'at Ram, Jerusalem

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