A perturbative approach to the quantum elliptic Calogero-Sutherland model

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Abstract

We solve perturbatively the quantum elliptic Calogero-Sutherland model in the regime in which the quotient between the real and imaginary semiperiods of the Weierstrass $\mathcal{P}$ function is small.

The class of quantum and classical integrable systems known as Calogero-Sutherland models were first introduced by these authors in the seventies, [1, 2] and have since then attracted considerable interest, both for their intrinsic mathematical beauty and depth and for the numerous applications found, which range from condensed matter to supersymmetric Yang-Mills theory and strings/M-theory, see for instance [3, 4, 5]. These models, whose integrability stems from the fact that their Hamiltonians coincide with the Laplace-Beltrami operators on some symmetric spaces, can be formulated for an arbitrary number of particles. There are five possible interaction potentials: (a) $V(q) = q^{-2}$; (b) $V(q) = \sinh^{-2} q$; (c) $V(q) = \sin^{-2} q$; (d) $V(q) = \mathcal{P}(q)$, $\mathcal{P}$ being the elliptic Weierstrass function; and (e) $V(q) = q^{-2} + \omega^2 q^2$. In all cases, the particle coordinates enter in these potentials in combinations which are given by the roots of some simple Lie algebra, see [6] for details.

The most general among these systems is the elliptic one: all the other potentials arise as suitable infinite limits of one of both semiperiods of the $\mathcal{P}$-function. Nevertheless, to solve the quantum elliptic Calogero-Sutherland model is, even in the most simple cases, a difficult task. The elliptic problem for two particles and special values of the coupling constant was solved for Lamé more than one century ago in the course of his analysis of the stationary distribution of temperatures on an ellipsoid [7], and later in greater generality by Hermite [8]. Apart from this, one of the most successful results obtained so far is the exact solution for the case of three particles given in [9, 10]. However, the final formula for the eigenvalues is a very complicated expression involving transcendental functions, and it is therefore quite hard to grasp its content. In this letter, we will show that in some cases it is possible to take advantage of the solutions of the trigonometric Calogero-Sutherland model developed in [11, 12, 13] to give approximate solutions to the elliptic problem. These solutions are expressed by simple rational functions and the procedure for finding them is fairly elementary.

We begin by recalling some basic facts taken from [11, 12, 13]. The trigonometric quantum Calogero-Sutherland model of A_n-type describes the mutual interaction of $N = n + 1$ particles moving on the

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circle. The coordinates of these particles are q_j, $j=1,\ldots,N$, and the Schrödinger equation reads

$$\begin{aligned} H^{\rm trig}\Psi^\kappa &= E^{\rm trig}(\kappa)\Psi^\kappa \\ H^{\rm trig} = -\frac{1}{2}\Delta &+ \kappa(\kappa-1)\sum_{j<k}^N \sin^{-2}(q_j-q_k), \quad \Delta=\sum_{j=1}^N \frac{\partial^2}{\partial q_j^2}. \end{aligned} \tag{1}$$

The quantum eigenstates depend on a n-tuple of quantum numbers $\mathbf{m}=(m_1,m_2,\ldots,m_n)$

$$\begin{aligned} H^{\rm trig}\Psi^\kappa_{\mathbf{m}} &= E^{\rm trig}_{\mathbf{m}}(\kappa)\Psi^\kappa_{\mathbf{m}} \\ E^{\rm trig}_{\mathbf{m}}(\kappa) &= 2(\lambda+\kappa\rho,\lambda+\kappa\rho), \end{aligned} \tag{2}$$

where λ is the highest weight of the representation of A_n labelled by $\mathbf{m}$, i.e., $\lambda=\sum_{i=1}^n m_i\lambda_i$ with λ_i the fundamental weights of A_n, and ρ is the standard Weyl vector, $\rho=\frac{1}{2}\sum_{\alpha\in\mathcal{R}^+}\alpha$, with the sum extended over all the positive roots of A_n. The center-of-mass-frame eigenfunctions are of the form

$$\Psi^\kappa_{\mathbf{m}}(q_i)=\Big\{\prod_{j<k}^N \sin(q_j-q_k)\Big\}^\kappa P^\kappa_{\mathbf{m}}(z_i), \tag{3}$$

where the $z_i, i=1,2,\ldots,n$, variables are the elementary symmetric functions of $x_j=e^{2iq_j}$,

$$z_p=\sum_{j_1<j_2<\ldots<j_p}^N x_{j_1}x_{j_2}\ldots x_{j_p}, \tag{4}$$

and $P^\kappa_{\mathbf{m}}$ are the generalized Gegenbauer polynomials related to A_n. Some properties of these polynomials, as well as explicit examples, can be found in [11, 12, 13, 14, 15]. We mention, in particular, that each product of the form $z_i\,P_{\mathbf{m}}$ can be decomposed as a linear combination of Gegenbauer polynomials which mimics the structure of the Clebsch-Gordan series for the irreducible representations of $SU(n)$. Here we only quote two of these recurrence relations which are specially relevant for what follows:

$$\begin{aligned} z_1 P^\kappa_{\mathbf{m}} &= \sum_{j=1}^N c^\kappa_{j,\mathbf{m}} P^\kappa_{\mathbf{m}+\mu_j} \\ z_n P^\kappa_{\mathbf{m}} &= \sum_{j=1}^N \tilde{c}^\kappa_{j,\mathbf{m}} P^\kappa_{\mathbf{m}-\mu_j}; \end{aligned} \tag{5}$$

in these formulas μ_j is the n-tuple whose i-th element is $\delta_{i,j}-\delta_{i,j-1}$ and $c^\kappa_{j,\mathbf{m}}$, $\tilde{c}^\kappa_{j,\mathbf{m}}$ are some coefficients which can be obtained by known algorithms, see [16, 17, 18]. Note that as $z_n=z_1^\dagger$ both recurrence relations are simply related.

The elliptic model related to A_n has the same structure. The Schrödinger equation is $H^{\rm ell}\Phi^\kappa = E^{\rm ell}(\kappa)\Phi^\kappa$, the Hamiltonian being

$$H^{\rm ell}=-\frac{1}{2}\Delta+\kappa(\kappa-1)\sum_{j<k}^N \mathcal{P}(q_j-q_k;\omega_1,\omega_2), \tag{6}$$

where $\mathcal{P}(z;\omega_1,\omega_2)$ is the Weierstrass elliptic function with semiperiods chosen to be $\omega_1=\frac{\pi}{2}$ and ω_2 an imaginary number. This ensures that the $\mathcal{P}$ function will take real values on the real axis. For $|\omega_2|\ll\omega_1$, the Weierstrass function can be expanded in the parameter $g=e^{-4|\omega_2|}$:

$$\mathcal{P}(z;\frac{\pi}{2},\frac{\ln g}{4i})=\sin^{-2}z-\frac{1}{3}+8\sum_{k=1}^\infty \frac{kg^k}{1-g^k}(1-\cos 2kz); \tag{7}$$

that is, $\mathcal{P}$ is represented by the explicit power series

$$\mathcal{P}(z;\frac{\pi}{2},\frac{\ln g}{4i})=\sin^{-2}z-\frac{1}{3}+\sum_{p=1}^{\infty}g^{p}V_{p}(z), \tag{8}$$

with

$$V_{p}(z)=8\sum_{h\in D_{p}}h(1-\cos 2hz), \tag{9}$$

D_p being the set of natural divisors of p, i.e., $D_p=\{h\in\mathbf{N}\,|\,p/h\in\mathbf{N}\}$. Therefore,

$$H^{\rm ell}=H^{\rm trig}-\frac{1}{6}\kappa(\kappa-1)N(N-1)+\kappa(\kappa-1)\sum_{p=1}^{\infty}g^{p}\Big(\sum_{j<k}^{N}V_{p}(q_{j}-q_{k})\Big) \tag{10}$$

and a perturbative treatment of the elliptic problem becomes feasible. The first order term in that expansion is

$$\kappa(\kappa-1)g\sum_{j<k}^{N}V_{1}(q_{j}-q_{k})=8g\kappa(\kappa-1)\sum_{j<k}^{N}(1-\cos 2(q_{j}-q_{k}))=4g\kappa(\kappa-1)(N^{2}-z_{1}z_{n}), \tag{11}$$

and thus, first order perturbation theory gives

$$E_{\mathbf{m}}^{\rm ell}(\kappa)=E_{\mathbf{m}}^{\rm trig}(\kappa)-\frac{1}{6}\kappa(\kappa-1)N(N-1)+\delta_{1}E_{\mathbf{m}}(\kappa)+o(g^{2}) \tag{12}$$

$$\delta_{1}E_{\mathbf{m}}(\kappa)=4g\kappa(\kappa-1)\frac{\langle\Psi_{\mathbf{m}}^{\kappa}|N^{2}-z_{1}z_{n}|\Psi_{\mathbf{m}}^{\kappa}\rangle}{\langle\Psi_{\mathbf{m}}^{\kappa}|\Psi_{\mathbf{m}}^{\kappa}\rangle}. \tag{13}$$

The recurrence relations for the trigonometric case allow an easy evaluation of the energy correction: it follows from (5) that

$$z_{1}z_{n}P_{\mathbf{m}}^{\kappa}=a_{\mathbf{m}}^{\kappa}P_{\mathbf{m}}^{\kappa}+\cdots, \tag{14}$$

where the dots stand for terms proportional to polynomials other than $P_{\mathbf{m}}^{\kappa}$ and

$$a_{\mathbf{m}}^{\kappa}=\sum_{j=1}^{N}\tilde{c}_{j,\mathbf{m}}^{\kappa}c_{j,\mathbf{m}-\mu_{\mathbf{j}}}^{\kappa}\,. \tag{15}$$

The orthogonality properties of the system of generalized Gegenbauer polynomials guarantee that these terms do not contribute to (13), and we come to the simple result

$$\delta_{1}E_{\mathbf{m}}(\kappa)=4g\kappa(\kappa-1)\left[N^{2}-a_{\mathbf{m}}^{\kappa}\right]. \tag{16}$$

We can use the explicit expression of the coeffients $c_{j,\mathbf{m}}^{\kappa}$, $\tilde{c}_{j,\mathbf{m}}^{\kappa}$ given in [13, 14] to write this correction for the A_1, A_2 and A_3 cases, i.e., for two, three and four particles, respectively.

- A_1 case: Here $\mathbf{m}=(m)$, $N=2$ and

$$\begin{array}{ll} c_{1,m}^{\kappa}=1 & c_{2,m}^{\kappa}=c_{m}(\kappa)\\ \tilde{c}_{1,m}^{\kappa}=c_{m}(\kappa) & \tilde{c}_{2,m}^{\kappa}=1 \end{array} \tag{17}$$

with

$$c_{m}(\kappa)=\frac{m(m-1+2\kappa)}{(m+\kappa)(m-1+\kappa)}. \tag{18}$$

Therefore,

$$\delta_1 E_m(\kappa) = 8g\kappa(\kappa-1)\left[1+\frac{\kappa(\kappa-1)}{(m+1+\kappa)(m-1+\kappa)}\right]. \tag{19}$$

- A_2 case: Here $\mathbf{m}=(m,n),\ N=3$ and

$$\begin{array}{lll} c^\kappa_{1,(m,n)}=1 & c^\kappa_{2,(m,n)}=c_m(\kappa) & c^\kappa_{3,(m,n)}=a_{m,n}(\kappa) \\ \tilde c^\kappa_{1,(m,n)}=a_{n,m}(\kappa) & \tilde c^\kappa_{2,(m,n)}=c_n(\kappa) & \tilde c^\kappa_{3,(m,n)}=1 \end{array} \tag{20}$$

with

$$a_{m,n}(\kappa)=\frac{n(m+n+\kappa)(n-1+2\kappa)(m+n-1+3\kappa)}{(n+\kappa)(n-1+\kappa)(m+n+2\kappa)(m+n-1+2\kappa)}. \tag{21}$$

Therefore,

$$\begin{aligned} \delta_1 E_{m,n}(\kappa) &= 24g\kappa(\kappa-1)+8g\kappa^2(\kappa-1)^2\,\frac{3\kappa^2+3(m+n)\kappa+m^2+n^2+mn-3}{(m+1+\kappa)(m-1+\kappa)(n+1+\kappa)} \\ &\times \frac{2\kappa^2+(3m+3n+1)\kappa+m^2+n^2+mn-1}{(n-1+\kappa)(m+n+1+2\kappa)(m+n-1+2\kappa)}. \end{aligned} \tag{22}$$

- A_3 case: Here $\mathbf{m}=(m,l,n),\ N=4$ and

$$\begin{array}{lllll} c^\kappa_{1,(m,l,n)}=1 & c^\kappa_{2,(m,l,n)}=c_m(\kappa) & c^\kappa_{3,(m,l,n)}=a_{m,l}(\kappa) & & c_{4,(m,l,n)}=d_{m,l,n}(\kappa) \\ \tilde c^\kappa_{1,(m,l,n)}=d_{n,l,m}(\kappa) & \tilde c^\kappa_{2,(m,l,n)}=a_{n,l}(\kappa) & \tilde c^\kappa_{3,(m,l,n)}=c_n(\kappa) & \&\tilde c^\kappa_{4,(m,l,n)}=1 & \end{array} \tag{23}$$

with

$$d_{m,l,n}(\kappa)=\frac{n(l+n+\kappa)(n-1+2\kappa)(m+l+n+2\kappa)(l+n-1+3\kappa)(m+l+n-1+4\kappa)}{(n+\kappa)(n-1+\kappa)(l+n+2\kappa)(l+n-1+2\kappa)(m+l+n+3\kappa)(m+l+n-1+3\kappa)}. \tag{24}$$

Therefore,

$$\begin{aligned} \delta_1 E_{m,l,n}(\kappa) &= 4g\kappa(\kappa-1) \\ &\times \left[16-\frac{n(l+1)(l+m+1+\kappa)(l+2\kappa)(n-1+2\kappa)(l+m+3\kappa)}{(l+\kappa)(l+1+\kappa)(n+\kappa)(n-1+\kappa)(l+m+2\kappa)(l+m+1+2\kappa)}\right. \\ &- \frac{(n+1)(l+n+1+\kappa)(n+2\kappa)(l+m+n+1+2\kappa)(l+n+3\kappa)(l+m+n+4\kappa)}{(n+\kappa)(n+1+\kappa)(l+n+2\kappa)(l+n+1+2\kappa)(l+m+n+3\kappa)(l+m+n+1+3\kappa)} \\ &- \frac{m(l+m+\kappa)(m-1+2\kappa)(l+m+n+2\kappa)(l+m-1+3\kappa)(l+m+n-1+4\kappa)}{(m+\kappa)(m-1+\kappa)(l+m+2\kappa)(l+m-1+2\kappa)(l+m+n+3\kappa)(l+m+n-1+3\kappa)} \\ &- \left.\frac{l(m+1)(l+n+\kappa)(m+2\kappa)(l-1+2\kappa)(l+n-1+3\kappa)}{(l+\kappa)(l-1+\kappa)(m+\kappa)(m+1+\kappa)(l+n+2\kappa)(l+n-1+2\kappa)}\right]. \end{aligned} \tag{25}$$

This expression becomes particularly simple when only one quantum number is non-vanishing:

$$\begin{aligned} \delta_1 E_{m,0,0}(\kappa) &= 24g\kappa(\kappa-1)\left[2+\frac{4\kappa^3+(4m-2)\kappa^2+(m^2-2)\kappa}{(m-1+\kappa)(1+2\kappa)(m+1+3\kappa)}\right] \\ \delta_1 E_{0,l,0}(\kappa) &= 16g\kappa(\kappa-1)\left[3+\frac{3\kappa^3+4l\kappa^2+(l^2-3)\kappa}{(1+\kappa)(l-1+\kappa)(l+1+3\kappa)}\right] \\ \delta_1 E_{0,0,n}(\kappa) &= 24g\kappa(\kappa-1)\left[2+\frac{4\kappa^3+(4n-2)\kappa^2+(n^2-2)\kappa}{(n-1+\kappa)(1+2\kappa)(n+1+3\kappa)}\right]. \end{aligned} \tag{26}$$

The extension of the perturbative approach to higher orders is straightforward. There is only a new ingredient: due to the contribution of intermediate states, the norms of the unperturbed eigenfunctions enter explicitly in the corrections to the energies. Fortunately, these norms are known [18, 19]. Apart from this, the procedure to follow is analogous to that used in first order and the keypoint is that the recurrence relations will save us from doing all the difficult integrals. We will analyse only the simplest example, that is, second order perturbation theory for the A_1 case.

The second order contribution to the Hamiltonian $H^{\rm ell}$ (10)

$$\begin{aligned}\kappa(\kappa-1)g^2\sum_{j<k}^{N} V_2(q_j-q_k) &= 8\kappa(\kappa-1)g^2\sum_{j<k}^{N}[3-\cos 2(q_j-q_k)-2\cos 4(q_j-q_k)] \\ &= 4\kappa(\kappa-1)g^2[3N^2-z_1z_n-2(z_1^2-2z_2)(z_n^2-2z_{n-1})]\end{aligned} \tag{27}$$

for the A_1 case gives, taking $z_0=1$,

$$E_m^{\rm ell}(\kappa)=E_m^{\rm trig}(\kappa)-\frac{1}{6}\kappa(\kappa-1)N(N-1)+\delta_1\,E_m(\kappa)+\delta_2\,E_m(\kappa)+o(g^3) \tag{28}$$

with

$$\begin{aligned}\delta_2\,E_m(\kappa) &= 4g^2\kappa(\kappa-1)\frac{\langle\Psi_m^\kappa|4+7z_1^2-2z_1^4|\Psi_m^\kappa\rangle}{\langle\Psi_m^\kappa|\Psi_m^\kappa\rangle} \\ &+ \frac{16g^2\kappa^2(\kappa-1)^2}{\langle\Psi_m^\kappa|\Psi_m^\kappa\rangle}\sum_{n\neq m}\frac{1}{\langle\Psi_n^\kappa|\Psi_n^\kappa\rangle}\frac{|\langle\Psi_n^\kappa|z_1^2|\Psi_m^\kappa\rangle|^2}{E_m^{\rm trig}(\kappa)-E_n^{\rm trig}(\kappa)}.\end{aligned} \tag{29}$$

The recurrence relations for the Gegenbauer polynomials related to A_1 (see (5) and (17)) also hold for the eigenfunctions Ψ_m^κ of the trigonometric problem, i.e., $z_1\Psi_m^\kappa=\Psi_{m+1}^\kappa+c_m\Psi_{m-1}^\kappa$, from which and the Hermitian character of z_1 (for A_1) it follows that

$$\langle\Psi_m^\kappa|\Psi_m^\kappa\rangle=c_m\langle\Psi_{m-1}^\kappa|\Psi_{m-1}^\kappa\rangle, \tag{30}$$

a very useful relation which allows to express the equation (29) in terms of the coefficients c_m only:

$$\begin{aligned}\delta_2E_m(\kappa) &= 4g^2\kappa(\kappa-1)[4+7(c_m+c_{m+1})-2(c_m+c_{m+1})^2-2c_{m+2}c_{m+1}-2c_mc_{m-1}] \\ &+ 16g^2\kappa^2(\kappa-1)^2\left\{\frac{c_{m+2}c_{m+1}}{E_m^{\rm trig}(\kappa)-E_{m+2}^{\rm trig}(\kappa)}+\frac{c_mc_{m-1}}{E_m^{\rm trig}(\kappa)-E_{m-2}^{\rm trig}(\kappa)}\right\}.\end{aligned} \tag{31}$$

Finally, using (18) and the expression $E_m^{\rm trig}=m^2+2\kappa m-\kappa^2$ for the energy levels, (31) can be evaluated quite easily, obtaining the simple result

$$\begin{aligned}&\delta_2E_m(\kappa)=8g^2\kappa^2(\kappa-1)^2\left\{3-\frac{\kappa^2-(10m+6)\kappa-5m^2+8}{[(m+\kappa)^2-4][(m+\kappa)^2-1]}\right\} \\ &+\frac{4g^2\kappa^2(\kappa-1)^2}{m+\kappa}\left\{\frac{m(m-1)(m-1+2\kappa)(m-2+2\kappa)}{(m-1+\kappa)^3(m-2+\kappa)}-\frac{(m+1)(m+2)(m+2\kappa)(m+1+2\kappa)}{(m+1+\kappa)^3(m+2+\kappa)}\right\}.\end{aligned} \tag{32}$$

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