Partially positive square root and Kuiper type theorem for operators on C*-Hilbert modules

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Abstract

Let us denote by $\mathrm{End}_A l_2(A)$ the Banach algebra of all bounded A-morphisms of Hilbert A-module $l_2(A)$, and by $\mathrm{End}^*_A l_2(A)$ the C^*-algebra of operators, admitting adjoint. Let GL and GL^* denote the corresponding groups of invertible operators. In the present paper we prove that GL is norm contractible for A being σ-unital. This completes the research of a number of authors during approximately 20 years.

1 Introduction

Let us denote by $\mathrm{End}_A l_2(A)$ the Banach algebra of all bounded A-homomorphisms of Hilbert A-module $l_2(A)$, and by $\mathrm{End}^*_A l_2(A)$ the C^*-algebra of operators, admitting adjoint. Let GL and GL^* denote the corresponding groups of invertible operators. The question about the contractibility of general linear groups is very important for K–theory to construct classifying spaces in terms of Fredholm operators. To this question a series of papers was devoted: [13, 5, 20, 14]. The author used these results to construct [19] the classifying spaces of K-theory $K^{p,q}(X;A)$ which arises in analytical approach to the Novikov Conjecture on higher signatures. In paper [3] J. Cuntz and N. Higson proved the contractibility of GL^* for A with strictly positive element (or, equivalent, with countable approximate unit = σ-unital). In the paper [21] (see also [23]) a simple proof of the theorem of Cuntz and Higson was given and the contractibility of GL was proved in some special cases, in particular, for A, being a subalgebra of algebra of compact operators in separable Hilbert space, and for $A = C_0(M)$, where M is a finite-dimensional manifold. The further results based on this approach were obtained by my students in [18] and [2].

In the present paper we obtain a sort of partially positive polar decomposition for GL. With its help we prove the following statement.

Theorem 1 *The group* GL *is norm contractible for any σ-unital algebra A.*

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2 Preliminary notes

It is known, that the set of invertible operators in a Banach space is open with the respect to the topology of a norm, while the set of bounded A-homomorphisms is closed in the set of all endomorphisms. Thus, GL is an open set in a Banach space. According to the Milnor theorem [12] such sets have the homotopy type of CW-complexes, and, therefore, by the theorem of Whitehead, strong and weak homotopy triviality are equivalent for them. Together with the contractibility of GL^* [3] this implies the following statement.

Lemma 2 *To prove the contractibility of* GL *it is sufficient to verify the following. Let* $f : S \to \mathrm{GL}$ *be a continuous map of a sphere of arbitrary dimension. Then* f *is homotopic to a map valued in* GL^*. □

Lemma 3 (a variant of the Atiyah theorem about small balls) [21, Lemma 2.2] *Let* $f : S \to \mathcal{G}$ *be a continuous map of a sphere of arbitrary finite dimension. Then* f *is homotopic to a map* f' *such that* $f'(S)$ *is a finite polyhedron in* $\mathrm{End}_A l_2(A)$*, laying in* GL *together with the homotopy.*

One more step from the original work of Kuiper [7] is universal. Let us denote orthogonal (A-Hilbert) sum by $\oplus$ and Banach module one by $\widetilde{\oplus}$.

Lemma 4 [21, Lemma 2.4] *Subset* $V \subset \mathrm{GL}$, *defined as*

$$V = \{g \in \mathrm{GL} \mid g|_{H'} = \mathrm{Id}_{H'},\ g(H_1) = H_1\},$$

where

$$l_2(A) = H' \oplus H_1, \qquad H' \cong H_1 \cong l_2(A),$$

is contractible in GL *to* $1 \in \mathrm{GL}$.

Lemma 5 [21, Lemma 2.5] *Subset* $W \subset \mathrm{GL}$, *defined as*

$$W = \{g \in \mathrm{GL} \mid g|_{H'} = \mathrm{Id}_{H'}\},$$

where

$$l_2(A) = H' \oplus H_1, \qquad H' \cong H_1 \cong l_2(A),$$

is contractible inside GL *to*

$$V = \{g \in \mathrm{GL} \mid g|_{H'} = \mathrm{Id}_{H'},\ g(H_1) = H_1\}.$$

3 Technical lemmas

Suppose, $\mathcal{K}_A$ is the C^*-algebra of A-compact operators on $l_2(A)$, $\mathbf{LM}(\mathcal{K}_A) \cong \mathrm{End}_A l_2(A)$ is the algebra of the left multipliers, $\mathbf{M}(\mathcal{K}_A) \cong \mathrm{End}^*_A l_2(A)$ is the C^*-algebra of multipliers and $\mathbf{QM}(\mathcal{K}_A) \cong \mathrm{Hom}_A(l_2(A), l_2(A)')$ is the space of quasi-multipliers (see [1, 4, 6, 9, 15, 16, 17, 22, 23]).

Suppose, α is a strictly positive element in a σ-unital algebra A, $\alpha_i := \varphi_i(\alpha)$ is a countable approximate unit, where φ_i is defined by the formula

$$\varphi_i(t) := \begin{cases} 0, & \text{if } t \in [0, \frac{1}{i}], \\ 1, & \text{if } t \in [\frac{1}{i-1}, \infty), \\ \text{linear}, & \text{if } t \in [\frac{1}{i}, \frac{1}{i-1}], \end{cases} \tag{1}$$

Suppose,

$$\beta_i := \psi_i(\alpha), \qquad \psi_i(t) := \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } t \in [\frac{1}{i}, \infty), \\ \text{linear}, & \text{if } t \in [0, \frac{1}{i}], \end{cases} \tag{2}$$

Then β_i is a strictly positive element in A and

$$\alpha_i \beta_i = \alpha_i \alpha_{i+k} = \alpha_i, \qquad k = 1, 2, \ldots \tag{3}$$

Put $\omega_i := (\alpha_i - \alpha_{i-1})^{1/2}$ for $i \geq 3$ and $\omega_2 = \alpha_2^{1/2}$, so

$$\omega_j \alpha_i = \alpha_i \omega_j = 0, \quad j = i+2, i+3, \ldots, \qquad \omega_j \alpha_i = \alpha_i \omega_j = \omega_j, \quad j = 1, \ldots, i-1. \tag{4}$$

Since there is no unit in A, the notion of "standard base" $\{e_i\}$ of module $l_2(A)$ makes no sense. Nevertheless, the elements $e_i\gamma$ are well defined for any $\gamma \in A$, namely,

$$e_i \gamma := (0, \ldots, 0, \gamma, 0, \ldots), \qquad \gamma \text{ at } i\text{-th place.}$$

Let us denote the correspondent orthoprojections on these one-dimensional submodules A_i by q_i, and $p_i := q_1 + \ldots + q_i$, $L_i := \operatorname{Im} p_i$.

We say that a linear mapping F is *topologically injective,* if there is a constant $c_0 > 0$ such that $\|Fx\| \geq c_0 \|x\|$.

Lemma 6 [21] *The injection $J : l_2(A) \to l_2(A)$ under the formula*

$$(a_1, a_2, \ldots) \mapsto \sum_j \sum_i v_{ij} a_j, \qquad \langle v_{ij}, v_{ij} \rangle = \omega_i^2, \qquad v_{ij} \in M_{k(i,j)},$$

$$l_2(A) = M_1 \oplus M_2 \oplus \ldots, \qquad M_r = \{\, (0, \ldots, 0, a_{s(r)}, \ldots, a_{s(r+1)-1}, 0, \ldots) \,\},$$
$$\{k(1,1); k(1,2), k(2,1); k(1,3), k(2,2), k(3,1); \ldots\} = \{1, 2, \ldots\},$$

leaves the inner product unchanged and admits an adjoint. In particular, the image is defined by a selfadjoint projection of the form JJ^*.

We have the following evident lemma.

Lemma 7 *Let $U \in \mathrm{GL}^*$ be a unitary operator. For any $x \in l_2(A)$ and any $\varepsilon > 0$ there exists a number k_0, such that for any $\gamma \in A$, $\|\gamma\| \leq 1$, and any $k > k_0$*

$$\|\langle U e_k \gamma, x \rangle\| < \varepsilon.$$

Let us denote as obvious $\operatorname{Re} a := \frac{a + a*}{2} \in A$ for $a \in A$.

Lemma 8 *Suppose, $a, b, c, d \in A$. Then*

$$\mathrm{Re}\,(a^*cb + b^*da) \geq -(\|c\| + \|d\|)(a^*a + b^*b).$$

Proof:

$$\mathrm{Re}\,(a^*cb + b^*da) = a^*\frac{c+d^*}{2}b + b^*\frac{c^*+d}{2}a =$$

$$= \left(\|c^*+d\|^{1/2}\,a + \frac{c+d^*}{2\|c^*+d\|^{1/2}}\,b\right)^* \left(\|c^*+d\|^{1/2}\,a + \frac{c+d^*}{2\|c^*+d\|^{1/2}}\,b\right) -$$

$$-\|c^*+d\|\,a^*a - b^*\frac{(c+d^*)^*(c+d^*)}{4\|c^*+d\|}\,b \geq$$

$$\geq -(\|c\| + \|d\|)\,(a^*a - b^*b).$$

□

Lemma 9 *Let x and y be elements of $l_2(A)$. Then*

$$\mathrm{Re}\,\langle x, y\rangle \leq \frac{1}{2}\,(\langle x, x\rangle + \langle y, y\rangle).$$

Proof: Passing to coordinates $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$, it is sufficient to verify

$$\mathrm{Re}\,(a^*b) \leq \frac{1}{2}(a^*a + b^*b), \qquad a, b \in A.$$

Since $0 \leq (a-b)^*(a-b) = a^*a + b^*b - 2\mathrm{Re}\,(a^*b)$, we are done. □

Lemma 10 *Let $S, T : l_2(A) \to l_2(A)$ be bounded A-operators. Suppose,*

$$c\,\langle x, x\rangle < \mathrm{Re}\,\langle Sx, Tx\rangle$$

for some $x \in l_2(A)$. Then

$$\frac{c^2}{\|T\|^2}\,\langle x, x\rangle < \langle Sx, Sx\rangle.$$

Proof: By Lemma 9

$$\begin{aligned} c\langle x, x\rangle &< \mathrm{Re}\,\langle Sx, Tx\rangle = \mathrm{Re}\,\left\langle \frac{\|T\|}{\sqrt{c}}Sx, \frac{\sqrt{c}}{\|T\|}Tx \right\rangle \leq \\ &\leq \frac{1}{2}\left(\frac{\|T\|^2}{c}\langle Sx, Sx\rangle + c\langle x, x\rangle\right), \end{aligned}$$

$$\frac{c^2}{\|T\|^2}\langle x, x\rangle < \langle Sx, Sx\rangle.$$

□

Lemma 11 (see, e.g. [10]) *Let L be a closed subspace of a Banach space X, which is not equal to X. Then for any $\varepsilon > 0$ there exists $y \in X$ such that $\|y\| = 1$ and $\|x - y\| > 1 - \varepsilon$ for any $x \in L$.*

Lemma 12 *Suppose, $l_2(A) = H_0 \oplus H_1$, $V \in \mathrm{GL}^*$, $V(H_0) \perp V(H_1)$, $F \in \mathrm{GL}$, and for any $x \in H_0$ one has*

$$\mathrm{Re}\,\langle Vx, Fx\rangle \geq c \cdot \langle x, x\rangle.$$

Then the linear homotopy

$$V(x,t) := \begin{cases} (1-t)F(x) + tV(x) & \text{if } x \in H_0, \\ V(x) & \text{if } x \in H_1, \end{cases}$$

$t \in [0,1]$, is in GL.

Proof: Let p be the orthogonal projection on $V(H_0)$. We have to verify that

$$p|_{((1-t)F+tV)(H_0)} : ((1-t)F + tV)(H_0) \to V(H_0)$$

is an isomorphism. This will complete the proof.

Let us start from the topological injectivity. For $x \in H_0$ we have

$$c\langle x,x\rangle < \mathrm{Re}\,\langle Fx, Vx\rangle = (1-t)\mathrm{Re}\,\langle V(x,t), Vx\rangle - t\langle Vx, Vx\rangle,$$

$$\left(c + \frac{t}{\|V^{-1}\|^2}\right)\langle x,x\rangle < (1-t)\mathrm{Re}\,\langle V(x,t), Vx\rangle,$$

$$c\langle x,x\rangle < \mathrm{Re}\,\langle V(x,t), Vx\rangle.$$

Let us remark that

$$\langle V(x,t), Vx\rangle = \langle pV(x,t), Vx\rangle.$$

By Lemma 10

$$\frac{c^2}{\|V\|^2}\langle x,x\rangle < \langle pV(x,t), pV(x,t)\rangle.$$

Topological injectivity is verified.

Hence, the image is a closed subspace of $V(H_0)$. Suppose, it is not equal to $V(H_0)$. Then we can apply Lemma 11 and find an element $y = Vz \in V(H_0)$ such that $\|y\| = 1$ and $\|x - y\| > 1 - \varepsilon$ for any x from our closed subspace. In particular, we can take ε such that

$$(1-\varepsilon)^2 > 1 - \frac{c^2\,\|V^{-1}\|^2}{(\|F\| + \|V\|)^2}, \tag{5}$$

and $x = \tau pV(z,t)$ with

$$\tau = \frac{c}{(\|F\| + \|V\|)^2}. \tag{6}$$

We have

$$\begin{aligned}
\langle x-y, x-y\rangle &\leq \tau^2(\|F\| + \|V\|)^2\langle z,z\rangle - 2c\tau\langle z,z\rangle + \langle y,y\rangle = \\
&= \tau^2(\|F\| + \|V\|)^2\langle V^{-1}y, V^{-1}y\rangle - 2c\tau\langle V^{-1}y, V^{-1}y\rangle + \langle y,y\rangle \leq \\
&\leq [\,\tau^2((\|F\| + \|V\|)\,\|V^{-1}\|)^2 - 2c\tau\|V^{-1}\|^2 + 1]\,\langle y,y\rangle = \\
&= [\,(\tau(\|F\| + \|V\|)^2 - 2c)\tau\|V^{-1}\|^2 + 1]\,\langle y,y\rangle = \\
&= \left[1 - \frac{c^2\,\|V^{-1}\|^2}{(\|F\| + \|V\|)^2}\right]\langle y,y\rangle.
\end{aligned}$$

Hence,
$$1-\frac{c^2\,\|V^{-1}\|^2}{(\|F\|+\|V\|)^2}<(1-\varepsilon)^2<\|x-y\|^2\le 1-\frac{c^2\,\|V^{-1}\|^2}{(\|F\|+\|V\|)^2},$$
and we obtain a contradiction. □

Lemma 13 *Let p and q be two orthogonal projections in some Hilbert module $\mathcal{M}$. Suppose, $g := q|_{\operatorname{Im} p} : \operatorname{Im} p \to \operatorname{Im} q$ is an isomorphism. Then $ig^{-1}q$ is adjointable where $i : \operatorname{Im} p \subset \mathcal{M}$ is the inclusion.*

Proof: This follows immediately from the self-adjointness of p, i, and q, and the equality $g = q \circ i$. □

Lemma 14 *Suppose, $H_0 \oplus H_1 = l_2(A)$, $H_0 \cong H_1 \cong l_2(A)$, $F \in \mathrm{GL}$ and $\mathrm{GL} \ni \widetilde{F} = \begin{cases} F & \text{on} \quad H_0, \\ F' & \text{on} \quad H_1, \end{cases}$ are continuous families. Then they are homotopic each other.*

Proof: Let us consider the continuous family $G = \begin{cases} \mathrm{Id} & \text{on } F(H_0) \\ F'F^{-1} & \text{on } F(H_1) \end{cases}$ in GL, so $\widetilde{F} = GF$. Then there is a linear homotopy $G \sim \widetilde{G}$ in GL, where
$$\widetilde{G} = \begin{cases} \mathrm{Id} & \text{on } F(H_0), \\ p_F F' F^{-1} & \text{on } F(H_1), \end{cases}$$
and p_F is the projection onto $F(H_1)$ along $F(H_0)$ (cf. [21, Lemma 2.5] and Lemma 5). One can complete the proof in the same way as in [21, Lemma 2.5] by verifying $\widetilde{G} \sim \mathrm{Id}$. □

Lemma 15 *Let $F \in K \subset \mathrm{GL}$, where K is a finite polyhedron. Suppose, $l_2(A) = H' \oplus H_0'$, $l_2(A) \cong H' \cong H_0'$, and there exists a continuous mapping $F \mapsto D_F$, where*

1) *$D_F : H_0' \to l_2(A)$ is an adjointable injection;*

2) *$(\operatorname{Im} D_F)^\perp \cong l_2(A)$.*

Then there exists a continuous mapping $F \mapsto V_F \in \mathrm{GL}$ such that

(i) *$V_F|_{H_0'} = D_F$;*

(ii) *$V_F(H') = (D_F(H_0'))^\perp$;*

(iii) *$V_F \in \mathrm{GL}^*$, i.e. admits an adjoint.*

Proof: Denote by P_F the orthoprojection on $\operatorname{Im} D_F$. Let us choose a sufficiently small subdivision of K such that $P_{F_2}|_{\operatorname{Im} D_{F_1}} : \operatorname{Im} D_{F_1} \to \operatorname{Im} D_{F_2}$ is an isomorphism for any F_1 and F_2 from one simplex. Then the same is true for $1 - P_F$. Denote by
$$\gamma_{F_1,F_2} := (1-P_{F_2})|_{\operatorname{Im}(1-P_{F_1})} : \operatorname{Im}(1-P_{F_1}) \cong \operatorname{Im}(1-P_{F_2})$$

the corresponding isomorphism.

Let us define the operators V_F by the induction over the dimension of skeletons. Each time we will define V_F by (i) and (ii) and in such a way that $V'_F := V_F|_{H'} : H' \to (D_F(H'_0))^\perp = (1-P_F)(l_2(A))$ is adjointable. In fact, we only need to define by induction this family V'_F. It is well known that two Banach-isomorphic countably generated Hilbert A-modules are unitary isomorphic (see, e.g. [4] or [11]). Hence we can define $F \mapsto V'_F$ over 0-skeleton. Suppose, we have defined the mapping over the boundary $\partial\sigma$ of an n-simplex σ with vertices $F_0, \dots, F_n$ and we would like to extend it up to the whole simplex. Let us consider a continuous mapping

$$\mu : F \mapsto (1-P_{F_0})V'_F = \gamma(F,F_0)V'_F, \qquad F \in \partial\sigma,$$

valued in the space $\mathrm{GL}^*(H',(1-P_{F_0})(l_2(A)))$ of adjointable invertible operators $H' \to (1-P_{F_0})(l_2(A))$. The last space is contractible by Cuntz and Higson theorem [3] (see also [21]). Hence we have an extension $\widetilde{\mu} : \sigma \to \mathrm{GL}^*(H',(1-P_{F_0})(l_2(A)))$ of μ. Let us define V'_F for $F \in \sigma$ by the formula

$$V'_F = (\gamma(F,F_0))^{-1}\widetilde{\mu}(F).$$

By Lemma 13 this operator is adjointable. This gives the step of induction. □

4 Proof of the main theorem

Let us choose a small subdivision of the polyhedron K, s.t.

$$\mathrm{diam}(\text{any simplex}) < \delta = \frac{1}{4\,C^3}, \tag{7}$$

where

$$\|F\| < C, \quad \|F^{-1}\| < C \qquad \forall \quad F \in K. \tag{8}$$

For any F let us denote its inverse by G. Let $\mathbf{k} \in \mathcal{K}_A$ be an operator with dense image and without kernel. For any invertible G one can define invertible element by the following formula $(\mathbf{k}^*(G)^*G\mathbf{k})^{1/2}\mathbf{k}^{-1}$ on the dense set $\mathbf{k}(l_2(A))$. This slightly generalize [1] in the present particular case. For this "positive square root" we have the corresponding "polar decomposition", namely the following unitary operator is defined:

$$U_F := (\mathbf{k}^*(G)^*G\mathbf{k})^{1/2}\mathbf{k}^{-1}F. \tag{9}$$

Let us define for each vertex F_j an operator $\mathbf{k}_j \in \mathcal{K}_A$ in the following way. We put

$$\mathbf{k}_j|_{F_j(A_k)} := F_j\left(\frac{\beta_k}{2^k}\right)(F_j)^{-1}, \tag{10}$$

where β_k are defined by (2). So,

$$\mathbf{k}_jF_je_k\alpha_k = F_je_k\alpha_k\frac{1}{2^k}. \tag{11}$$

Let

$$U_j := (\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j \tag{12}$$

be the corresponding unitary.

Now we will chose a subsequence $k(s)$ of k in the following way. Put

$$\rho := \frac{1}{8\,C^3}.$$

Let $k(1) = 1$. Suppose, $k(1), \dots, k(m)$ are chosen in a way such that for any $s_0 = 2, \dots, m$, $s = 1, \dots, s_0 - 1$, $j = 1, \dots, N$

$$\| \langle U_j e_{k(s_0)}\alpha_{k(s_0)}, F_j e_{k(s)}\alpha_{k(s)} \rangle \| < \rho \cdot \frac{1}{(s_0 - 1) \cdot 2^{s_0 - 1}},$$

and

$$\| \langle U_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(s_0)}\alpha_{k(s_0)} \rangle \| < \rho \cdot \frac{1}{(s_0 - 1) \cdot 2^{s_0 - 1}}.$$

By Lemma 7 we can choose a large number $k(m+1)$ such that

$$\| \langle U_j e_{k(m+1)}\alpha_{k(m+1)}, F_j e_{k(s)}\alpha_{k(s)} \rangle \| < \rho \cdot \frac{1}{m \cdot 2^m}, \qquad s = 1, \dots, m, \quad j = 1, \dots, N, \tag{13}$$

and (taking into account (11)

$$\begin{aligned}
& \| \langle U_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(m+1)}\alpha_{k(m+1)} \rangle \| = \\
= \; & \| \langle (\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(m+1)}\alpha_{k(m+1)} \rangle \| \\
= \; & \left\| \left\langle (\mathbf{k}_j)^{-1}F_j e_{k(s)}\alpha_{k(s)}, (\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j e_{k(m+1)}\alpha_{k(m+1)}\frac{1}{2^{k(m+1)}} \right\rangle \right\| \\
= \; & \left\| \left\langle F_j e_{k(s)}\alpha_{k(s)} \cdot 2^{k(s)}, U_j e_{k(m+1)}\alpha_{k(m+1)}\frac{1}{2^{k(m+1)}} \right\rangle \right\| \\
\leq \; & \left\| \left\langle F_j e_{k(s)}\alpha_{k(s)}, U_j e_{k(m+1)}\alpha_{k(m+1)} \right\rangle \right\| \\
< \; & \rho \cdot \frac{1}{m \cdot 2^m}, \qquad s = 1, \dots, m, \quad j = 1, \dots, N,
\end{aligned} \tag{14}$$

Let $H_0 \cong l_2(A)$ be a submodule with orthogonal complement, constructed as in Lemma 6, by vectors of the form $e_{i_{k(s)}}\alpha_{k(s)}\omega_\mu$. Of course, the "positive square root" is not positive, but nevertheless we can prove

Lemma 16 *For $x \in H_0$ one has*

$$\mathrm{Re}\,\langle (\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j x, F_j x \rangle \geq \frac{1}{2\,C^3}\,\langle x, x \rangle \geq \frac{1}{2\,C^5}\,\langle F_j x, F_j x \rangle. \tag{15}$$

Proof: By (11) one has

$$\langle (\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(s)}\alpha_{k(s)} \rangle =$$

$$= 2^{k(s)}\langle (\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(s)}\alpha_{k(s)} \rangle \geq$$

$$\geq 2^{k(s)} \left\langle \frac{1}{C}(\mathbf{k}_j^*\mathbf{k}_j)^{1/2} F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(s)}\alpha_{k(s)} \right\rangle \geq$$

$$\geq 2^{k(s)} \mathrm{Re} \left\langle \frac{1}{C}\mathbf{k}_j F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(s)}\alpha_{k(s)} \right\rangle =$$

$$= \frac{1}{C} \left\langle F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(s)}\alpha_{k(s)} \right\rangle .$$

An arbitrary element $x \in H_0$ has the form

$$x = \sum_s e_{k(s)}\alpha_{k(s)}\gamma_s = \sum_s e_{k(s)}\gamma_s,$$

$\langle x, x\rangle = \sum_s \gamma_s^*\gamma_s$, this series is norm convergent (ω_μ "participate" in γ_s). Then by Lemma 8 and by (13) and (14)

$$\begin{aligned}
& \mathrm{Re}\,\langle(\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j x, F_j x\rangle = \\
= \;& \mathrm{Re} \sum_{s,t} \gamma_s^* \langle(\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(t)}\alpha_{k(t)}\rangle\gamma_t \geq \\
\geq \;& \sum_s \frac{1}{C}\gamma_s^* \left\langle F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(s)}\alpha_{k(s)} \right\rangle \gamma_s - \\
& - \sum_{s<t} (\|\langle(\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j e_{k(s)}\alpha_{k(s)}, F_j e_{k(t)}\alpha_{k(t)}\rangle\| \\
& \qquad + \|\langle(\mathbf{k}_j^*(G_j)^*G_j\mathbf{k}_j)^{1/2}(\mathbf{k}_j)^{-1}F_j e_{k(t)}\alpha_{k(t)}, F_j e_{k(s)}\alpha_{k(s)}\rangle\|)(\gamma_s^*\gamma_s + \gamma_t^*\gamma_t) \geq \\
\geq \;& \frac{1}{C}\,\frac{1}{C^2} \sum_s \gamma_s^*\gamma_s - \sum_t \sum_{s=1}^{t-1} \left(2\rho \cdot \frac{1}{t\cdot 2^t}\right)(\gamma_s^*\gamma_s + \gamma_t^*\gamma_t) \geq \\
\geq \;& \frac{1}{C^3} \sum_s \gamma_s^*\gamma_s - \sum_t \left(2\rho \cdot \frac{1}{t\cdot 2^t}\right) \sum_{s=1}^{\infty} \gamma_s^*\gamma_s - \sum_t \left(2\rho\cdot\frac{1}{2^t}\right)\gamma_t^*\gamma_t \geq \\
\geq \;& \left(\frac{1}{C^3} - 4\rho\right) \sum_s \gamma_s^*\gamma_s = \frac{1}{2\,C^3} \sum_s \gamma_s^*\gamma_s = \frac{1}{2\,C^3}\langle x, x\rangle \geq \frac{1}{2\,C^5}\langle F_j' x, F_j' x\rangle.
\end{aligned}$$

□

Let us define the following continuous family of operators $D_F : H_0 \to l_2(A)$ extending simplex wise linearly $U_j|_{H_0}$ to the whole complex K:

$$D_F(x) = \sum_{i=1}^n t_i U_{j(i)}(x), \quad \text{if } F = \sum_{i=1}^n t_i F_{j(i)} \in \text{ simplex}\,\langle F_{j(1)}, \ldots, F_{j(n)}\rangle, \quad x \in H_0, \quad (16)$$

$n = \dim K = \dim S^n$. It is evident, that these operators are adjointable.

Lemma 17 1) *The operators D_F are topological injections.*

2) *There exist a decomposition $H_0 = H_0' \oplus H_0''$ and a continuous family V_F of adjointable operators such that*

$$V_F|_{H_0'} = D_F|_{H_0'}, \qquad V_F(H_0') \perp V_F((H_0')^\perp).$$

3) *There exists a constant* c' *such that*

$$\mathrm{Re}\,\langle V_F(x), Fx\rangle \geq c'\langle x, x\rangle, \qquad x \in H_0'.$$

Proof: Let us find a constant c' such that

$$\mathrm{Re}\,\langle D_F(x), Fx\rangle \geq c'\langle x, x\rangle, \qquad x \in H_0.$$

Then 3) will be a particular case of this inequality and 1) will follow from it by Lemma 10.

Suppose, F_j and F_k are some vertices of one simplex. Then (as in Lemma 10) by Lemma 9

$$\mathrm{Re}\,\langle U_j(x), (F_k - F_j)x\rangle = \mathrm{Re}\,\left\langle \|F_k - F_j\|^{1/2} U_j(x), \frac{1}{\|F_k - F_j\|^{1/2}}(F_k - F_j)x\right\rangle \leq$$

$$\leq \frac{1}{2}\left(\|F_k - F_j\|\,\langle U_j(x), U_j(x)\rangle + \frac{1}{\|F_k - F_j\|}\,\langle (F_k - F_j)x, (F_k - F_j)x\rangle\right) \leq$$

$$\leq \|F_k - F_j\|\,\langle x, x\rangle$$

and by Lemma 16

$$\mathrm{Re}\,\langle U_j(x), F_k x\rangle = \mathrm{Re}\,\langle U_j(x), F_j x\rangle + \mathrm{Re}\,\langle U_j(x), (F_k - F_j)x\rangle \geq$$

$$\geq \mathrm{Re}\,\langle U_j(x), F_j x\rangle - \|F_k - F_j\|\,\langle x, x\rangle \geq \left(\frac{1}{2\,C^3} - \delta\right)\,\langle x, x\rangle = \frac{1}{4\,C^3}\,\langle x, x\rangle.$$

and

Hence, for $x \in H_0$ we have

$$\begin{aligned}\mathrm{Re}\,\langle D_F(x), Fx\rangle &= \mathrm{Re}\,\langle \sum_{i=1}^n t_i U_{j(i)}(x), \sum_{i=1}^n t_i F_{j(i)}\rangle = \sum_{i,m=1}^n t_i t_m \mathrm{Re}\,\langle U_{j(i)}(x), F_{j(m)}\rangle \geq \\ &\geq \sum_{i,m=1}^n t_i t_m \frac{1}{4\,C^3}\,\langle x, x\rangle = \frac{1}{4\,C^3}\,\langle x, x\rangle =: c'\,\langle x, x\rangle.\end{aligned}$$

So, 1) and 3) are verified.

To construct a family as in 2) we would like to apply Lemma 15. For this purpose we have only to pass to any submodule $H_0' \subset H_0$, $H_0 = H_0' \oplus (H_0')^\perp$, $H_0' \cong (H_0')^\perp$. Then Condition 2) of Lemma 15 holds, namely $(\mathrm{Im}\, D_F|_{H_0'})^\perp \cong l_2(A)$. Indeed, this is a countable generated Hilbert A-module (as direct summand in $l_2(A)$) and by the Kasparov stabilization theorem [6]

$$(\mathrm{Im}\, D_F|_{H_0'})^\perp \cong D_F(H_0'') \oplus (D_F(H_0))^\perp \cong l_2(A) \oplus (D_F(H_0))^\perp \cong l_2(A).$$

This completes the proof. □

Definition 18 Let us define a homotopy R_t,

$$R_t(F) = R(F, t), \qquad R : K \times [0, 1] \to \mathrm{End}_A(l_2(A))$$

by the formula

$$R_t(F) := \begin{cases} (1-t)F + V_F & \text{on } H_0', \\ V_F & \text{on } (H_0')^\perp. \end{cases}$$

Due to linearity in t and continuity of V_F this family is continuous.

By 3) of Lemma 17 we can apply Lemma 12 to obtain that $R_t(F) \in \mathrm{GL}$. The family R_1 is in GL^*. Hence, it is contractible by Cuntz and Higson theorem [3]. To complete the proof of main Theorem 1 it remains to verify that the families $R_0(F)$ and F are homotopic to each other. But this fact is just Lemma 14.

$\square$

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