Koszul algebras from graphs and hyperplane arrangements

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1 Introduction

This work was started as an attempt to apply theory from noncommutative graded algebra to questions about the holonomy algebra of a hyperplane arrangement. We soon realized these algebras and their deformations form a class of quadratic graded algebras that have not been studied much and are interesting to algebra, arrangement theory and combinatorics.

Let X be a topological space having homotopy type of a finite cell complex and $H_*(X)$ its homology coalgebra with coefficients in a field and comultiplication dual to the cup product. Then the holonomy Lie algebra G_X of X is the quotient of the free Lie algebra on $H_1(X)$ over the ideal generated by the image of the comultiplication $H_2(X) \to \Lambda^2(H_1(X))$. The universal enveloping algebra $U(X)$ of G_X is called the holonomy algebra of X.

Holonomy algebras were introduced to arrangement theory by T.Kohno in [15, 16]. If $\mathcal{A}$ is an arrangement over $\mathbb{C}$, that is a set $\{H_1, \ldots, H_n\}$ of linear hyperplanes in a linear space $\mathbb{C}^\ell$ then put $X = \mathbb{C}^\ell \backslash \bigcup_{i=1}^n H_i$ and $U(\mathcal{A}) = U(X)$. In [15], $U(\mathcal{A})$ is defined explicitly by generators and relations that can be read from the combinatorics of $\mathcal{A}$ (see Section 5). Recall that there is another graded algebra defined by the combinatorics of $\mathcal{A}$, the Orlik-Solomon algebra $A(\mathcal{A})$, [20]. A well known theorem of Brieskorn-Orlik-Solomon says $A(\mathcal{A})$ is isomorphic to $H^*(X, \mathbb{C})$.

In his papers, Kohno studied a complex, $\tilde{K}$, of free modules over $U(\mathcal{A})$ defined by $K_p = \mathrm{Hom}_{\mathbb{C}}(A(\mathcal{A})_p, U(\mathcal{A}))$ $(p = 0, 1, \ldots)$ and $K_{-1} = \mathbb{C}$ (the Aomoto-Kohno complex). He proved the acyclicity of this complex for certain classes of reflection arrangements. He also proved that if this complex is acyclic then the Lower Central Series (LCS) formula holds:

$$P(-t) = \prod_{n \geq 1} (1 - t^n)^{\phi_n},$$

where $P(t)$ is the Poincare polynomial of X and ϕ_n are the ranks of successive quotients in the lower central series of its fundamental group. The LCS formula was later extended to all supersolvable arrangements (equivalently fiber-type) by Falk and Randell [9] (see also [13]). The acyclicity of the complex for arbitrary supersolvable arrangements remained open in spite of attempts to prove it (e.g., see [12] and correction in [13]).

We begin our work with the simple but crucial observation that the algebra $U(\mathcal{A})$ is dual (in the sense of Koszul algebra theory) to the quadratic closure $\bar{A}(\mathcal{A})$ of $A(\mathcal{A})$. Notice that the algebras $U(\mathcal{A})$ and $A(\mathcal{A})$ are defined over an arbitrary field F and we substitute it for $\mathbb{C}$. Let T be the free graded F-algebra on a set of degree one generators $x_1, \ldots, x_n$. All of our graded F-algebras will be graded quotients of such a T. The Koszul dual of a quadratic graded F-algebra B is the quadratic graded F-algebra whose generating relations form an orthogonal complement in T_2 to the quadratic relations of B. This algebra is denoted $B^!$. The algebra B is said to be Koszul if $B^!$ is isomorphic to the cohomology ring of the trivial graded B-module $F = B/B_{>0}$. An immediate implication of our observation above is that the Aomoto-Kohno complex $\tilde{K}$ is never exact if $A(\mathcal{A})$ is not quadratic. Moreover, in the quadratic case the exactness of $\tilde{K}$ is equiva lent to $U(\mathcal{A})$ (equivalently $A(\mathcal{A})$) being a Koszul algebra.

To analyze the class of algebras $U(\mathcal{A})$ we use the idea of deformation theory, attempting to deform $U(\mathcal{A})$ into a simpler quadratic algebra. In particular, if there exists a monomial basis of $\bar{A}(\mathcal{A})$ such that the complementary set of monomials in the free algebra T forms a an ideal generated in degree 2, then there is a nice deformation of $U(\mathcal{A})$ into an algebra in a class of quadratic algebras we call graph algebras.

A graph algebra is an algebra given by a collection of relations of the form $x_i x_j - q_{i,j} x_j x_i = 0$, $q_{i,j} \in F^*$. Graph algebras seem, by themselves, to form an interesting class of quadratic algebras. These algebras (at least for $q_{i,j} = \pm 1$) have appeared in the literature under different names (see [6, 10, 11, 14]) but no general results about their Koszul properties are known to us. We prove all such algebras are Koszul.

Now the success of the deformation method depends on the existence of a good monomial basis for the Orlik-Solomon algebras $A(\mathcal{A})$, as mentioned above. This problem was studied by Bjorner and Ziegler in [5]. It was shown there every supersolvable arrangement has such a basis. Moreover, it is known that for supersolvable arrangements $\bar{A}(\mathcal{A}) = A(\mathcal{A})$. Using this, we give a deformation of $U(\mathcal{A})$, U_t, with the properties $U_t \cong U(\mathcal{A})$ for $t \neq 0$ and U_0 is a graph algebra. Now an application of a theorem of Drinfeld, [7], gives the Koszul property for $U(\mathcal{A})$ whenever $\mathcal{A}$ is a supersolvable arrangement.

Our presentation is outlined as follows. In section 2 we recall several equivalent definitions of the Koszul property and some basic results about

Koszul algebras. Section 3 defines graph algebras and proves such algebras are Koszul. We also recover here some interesting combinatorial information previously known from [10] and [11], see Corollary 3.11. Section 4 gives a generalization of the definitions and results of sections 3 to a larger class of algebras called generalized graph algebras. In section 5 we return to the study of arrangements. Here we recall the basic definitions of the algebras associated to an arrangement and prove the two main results: $A(\mathcal{A})$ must be quadratic for the complex $\tilde{K}$ to be exact and $U(\mathcal{A})$ is Koszul for all supersolvable arrangements. We conclude, in Section 6, with 3 examples of arrangements that are not supersolvable and which we analyze by more ad-hoc arguments. The last of these examples provides an open question.

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2 Koszul Algebras: Preliminaries

We collect in this section some of the basic results about Koszul Algebras, cf. [4], [3] and [19]. Let F be a field and V an n-dimensional vector space over F. We let $T = T(V)$ denote the full F-tensor algebra over V. Choosing a basis $x_1 \dots, x_n$ for V we can write $T \cong F\langle x_1, \dots, x_n\rangle$, the free F-algebra on $x_1, \dots, x_n$. We use the usual grading on T where $T_1 = V$ and $T_0 = F$. Fix an F-inner product on the space of 2-tensors $V \otimes V = T_2$. To ease the notation we will usually assume this is the standard inner product induced by the basis $x_1 \dots, x_n$.

Fix a homogeneous ideal I of T and let $U = U(I)$ be the graded algebra T/I. We may assume I contains no non-zero elements of degree 1 and we say that U is quadratic if I is generated, as an ideal, by its elements of degree 2. Since $T_1 \to U_1$ is an isomorphism, we identify these spaces and use $x_1, \dots, x_n$ to denote a basis of the space.

Definition 2.1 *Let $U = U(I)$ be a quadratic algebra. Let $I_2^!$ be the orthogonal complement to I_2 in $V \otimes V$ and $I^!$ the ideal of T generated by $I_2^!$. The quadratic algebra $U^! = U(I^!) = T/I^!$ is called the Koszul dual of U.*

We observe at once that $(U^!)^! = U$.

Definition 2.2 *Let $U = U(I)$ be a quadratic algebra and let ${}_UF$ be the trivial graded left U-module $U/U_{>0}$. The algebra U is said to be Koszul if ${}_UF$ admits a free graded resolution*

$$\cdots \to P^i \to \cdots \to P^1 \to P^0 \to {}_UF \to 0$$

such that P^i is generated by its component of degree i.

There are many equivalent ways of expressing this definition. The following theorem collects some of these variations. We denote by $E(U)$ the graded cohomology algebra $Ext_U^*({}_UF, {}_UF)$. For any graded F-vector space M we denote the Hilbert Series of M by $H(M,t) := \sum_n \dim_F(M_n)t^n$. The Koszul complex of U is the sequence

$$\cdots \to K_i \to \cdots \to K_1 \to K_0 \to {}_UF \to 0$$

of free (except ${}_UF$) left U-modules and their homomorphisms where $K_i = \mathrm{Hom}_F(U_i^!, U)$, and $d_i : K_i \to K_{i-1}$ is defined by $d_i f(a) = \sum_{k=1}^n f(x_i a)x_i$ for every $a \in U_{i-1}$.

Theorem 2.3 *Let $U = U(I)$ be a quadratic algebra. The following statements are all equivalent.*

(a) U is Koszul.

(b) $U^!$ is Koszul.

(c) $E(U)$ is a quadratic F-algebra generated as an algebra in degree 1.

(d) $E(U) \cong U^!$.

(e) The Koszul complex of U is acyclic.

(f) $H(U,t) \cdot H(E(U),-t) = 1$

The various equivalences of the theorem can be found in [3], [4], and [19]. Several more equivalent versions of the Koszul condition can also be found in these references.

Corollary 2.4 *If U is a Koszul algebra then $H(U,t) \cdot H(U^!,-t) = 1$.*

It is not known if the converse to this corollary is true.

3 Graph algebras

Let F be a field. Let Γ be an edge-labelled graph (without loops or multiple edges) on n vertices $1, 2, \ldots, n$ with a set E of edges. Each edge $\{i,j\}$ in E is labelled by a non-zero field element $q_{i,j}$. We associate two F-algebras to Γ. Recall $T = \oplus_{d \geq 0} T_d$, the free F-algebra on n generators $x_1, \ldots, x_n$, naturally graded. Define the ideal $I(\Gamma)$ of T

$$I(\Gamma) = (x_i x_j - q_{i,j} x_j x_i | \{i,j\} \in E, i < j).$$

One checks that

$$I(\Gamma)^! = I^!(\Gamma) = (x_i x_j, q_{k,l} x_k x_l + x_l x_k | \{i,j\} \notin E, \{k,l\} \in E, k < l).$$

Notice that $x_i^2 \in I^!(\Gamma)$ for every i. Then put

$$U(\Gamma) = T/I(\Gamma), \qquad A(\Gamma) = U^!(\Gamma) = T/I^!(\Gamma).$$

Clearly both algebras are quadratic and dual to each other. Let $\Lambda(V)$ denote the exterior algebra on V over F. We note that $A(\Gamma)$ is a deformation of the factor algebra $\Lambda(V)/(x_i x_j | \{i,j\} \notin E, i \neq j)$ and the two algebras have the same Hilbert series. In particular it follows, as for the exterior algebra, that $\dim_F A(\Gamma) \leq 2^n$.

Examples 3.1 *1. Let Γ be discrete, i.e., $E = \emptyset$. Then $U(\Gamma) = T$ and $A(\Gamma) = F \oplus V$ with zero multiplication on V.*

2. Let Γ be the complete graph K_n with all labels $q_{i,j} = 1$. Then $U(\Gamma) = F[x_1, \ldots, x_n]$ and $A(\Gamma) = \Lambda(V)$.

3. Let Γ be the complete bipartite graph $K_{k,l}$ $(k + l = n)$ and all labels $q_{k,l} = 1$. Then $U(\Gamma) = T' \otimes T''$ where T' and T'' are free algebras on k and l generators respectively. $A(\Gamma)$ is the exterior algebra on n generators with extra relations of products of generators for a fixed k-subset of generators and its complement. Equivalently, $A(\Gamma) = A(\Gamma') \underline{\otimes} A(\Gamma'')$ where Γ' and Γ'' are discrete graphs on k and l vertices respectively and $\underline{\otimes}$ is the operation Koszul dual to tensor multiplication (see [19]).

4. Let $n = 2$ and assume $(1,2) \in E$. Set $q = q_{1,2}$. Then the algebra $U(\Gamma)$ is usually denoted $F_q[x_1, x_2]$. This ring is often referred to as the quantum line as it can be realized as the twisted homogeneous coordinate ring of projective one-space. Alternatively one may think of this ring as the set of polynomials $F[x_1, x_2]$ with a new multiplicative structure, $\odot$, defined by $f(x_1, x_2) \odot g(x_1, x_2) = f(x_1, q^{-1}x_2) g(x_1, x_2)$ for all $f, g \in F[x_1, x_2]$.

In the rest of the paper we will apply to T the usual terminology from polynomial rings. For instance, each element a of T is the uniquely defined sum (with nonzero coefficients) of monomials. These monomials form the support $S(a)$ of a. If $S(a)$ has only two elements a is a binomial. Note that $S(a)$ does not involve the coefficients of the monomials in a.

The fact that graph algebras form a manageable class is based on the following simple observation.

Lemma 3.2 *Let $I = I(\Gamma)$. If $a \in I$ then $a = \sum a_i$ where each a_i is a binomial from I and $S(a) = \bigcup_i S(a_i)$.*

Proof. To write a given $a \in I$ as $a = \sum a_i$ where each a_i is a binomial in I is a triviality, since I is generated by binomials. Moreover, the containment $S(a) \subset \bigcup S(a_i)$ is clear. It is the opposite containment that is not automatically true. Among all possible representations $a = \sum a_i$, choose one with a minimal number of terms, say k.

Let us prove $S(a_k) \subset S(a)$. Suppose not. Choose a monomial $\mu \in S(a_k)$ with $\mu \notin S(a)$. For each i, $1 \leq i \leq k$, let r_i be the coefficient of μ in the binomial a_i (of course $r_i = 0$ unless μ is one of the two elements of $S(a_i)$). Since $\mu \notin S(a)$ we must have $\sum_i r_i = 0$. Define $b_i = a_i - r_i(r_k)^{-1}a_k$ for $1 \leq i \leq k-1$. By construction, each b_i is a binomial in I and $a = \sum_{i=1}^{k-1} b_i$. This contradicts the minimality of the representation $a = \sum_i a_i$ and proves the Lemma. □

Let a and b be elements of T. We say a and b belong to the same *projective coset of I*, or are *projectively congruent modulo I*, if there exists non-zero scalars λ and γ with $\lambda a - \gamma b \in I$.

Corollary 3.3 *Let I be as in Lemma 3.2. If $\mathcal{B}$ is a set of monomials of T, composed of exactly one element from each projective coset of I, then the image of $\mathcal{B}$ in $U = T/I$ under the standard projection is an F-basis of U.*

Proof. Clearly $\mathcal{B}$ generates U over F. The linear independence follows immediately from Lemma 3.2. □

Now we need to fix a specific monomial basis of $U = U(\Gamma)$. To do this we impose a total order on the monomials of T as follows. Let π be the standard projection $T \to F[x_1, \ldots, x_n]$. We order the monomials of $F[x_1, \ldots, x_n]$ first by degree and then by the inverse lexicographic order. Now if $\pi(\mu_1) \neq \pi(\mu_2)$ for monomials μ_1 and μ_2 from T we say that $\mu_1 < \mu_2$ whenever $\pi(\mu_1) < \pi(\mu_2)$. If, on the other hand, $\pi(\mu_1) = \pi(\mu_2)$, we use the inverse lexicographic order. Now we say that a monomial μ of T is *standard* if μ is minimal among all the monomials in the same projective coset of μ modulo I. We will identify standard monomials with their images in U. By corollary 3.3, standard monomials form an F-basis of U that we call the *standard basis*.

Another useful feature of the ideal $I(\Gamma)$ is the ease with which we can check if two monomials of T are in the same projective coset. For each pair (i, j) such that $1 \leq i < j \leq n$ denote by $\bar{\pi}_{i,j}$ the F-algebra homomorphism from T to $F\langle x_i, x_j\rangle$ defined by evaluation of x_k at 1 for every $k \neq i$ or j. For each pair $(i, j) \notin E$ we get an induced epimorphism $\pi_{i,j} : U(\Gamma) \to F\langle x_i, x_j\rangle$ and for each pair $(i, j) \in E$ we get an induced epimorphism $\pi_{i,j} : U(\Gamma) \to F_q[x_i, x_j]$ (where $q = q_{i,j}$).

Lemma 3.4 *Two monomials m_1 and m_2 of T are in the same projective coset modulo $I(\Gamma)$ if and only if $\pi_{ij}(m_1)$ and $\pi_{ij}(m_2)$ are projectively congruent modulo 0 for every pair (i, j) , $1 \leq i < j \leq n$.*

Proof. The "only if-part" follows from $\pi_{ij}(m_1) = \pi_{ij}(m_2)$ for every generator $g = c_1m_1 + c_2m_2$ $(c_1, c_2 \in F)$ of $I(\Gamma)$ and every pair (i, j). Now suppose $\pi_{ij}(m_1)$ and $\pi_{ij}(m_2)$ are projectively congruent modulo 0 for every pair (i, j) but m_1 and m_2 are not projectively congruent modulo $I(\Gamma)$. Changing the monomials in their projective congruency classes one can assume they are both standard monomials. Write the monomials as $m_1 = a_1x_ib$ and $m_2 = a_2x_jb$ where a_1, a_2, and b are some monomials of T and $i \neq j$. One can assume that $i > j$. Then applying the condition for (i,j) one can represent $a_1 = ax_jx_{i_1}\cdots x_{i_k}x_ib$ for some monomial a and some integers $k, i_1, \ldots, i_k$. Applying the condition for the pairs j, i_r $(r = 1, \ldots, k)$ and (i,j) one sees that all these pairs belong to E, whence m_1 is projectively congruent to the monomial $m_3 = ax_{i_1}\cdots x_{i_k}x_ix_jb$. Since $m_3 < m_1$ this contradicts the assumption that m_1 is standard. □

We now consider certain subalgebras of the algebra $U(\Gamma)$. Our first goal is to prove these subalgebras are quadratic. Let J be a subset of the vertices of the graph Γ and write Γ_J for the complete subgraph of Γ with vertices J. The graph Γ_J also inherits the edge labels from Γ.

Lemma 3.5 *Let $U_J(\Gamma)$ be the F-subalgebra of $U(\Gamma)$ generated by the set $\{x_i|i \in J\}$. Then the canonical epimorphism $\rho : U(\Gamma_J) \to U_J(\Gamma)$ is an isomorphism. In particular, $U_J(\Gamma)$ is a quadratic algebra.*

Proof. Let T_J be the subalgebra of T generated by $\{x_i|i \in J\}$. It suffices to prove: $T_J \cap I(\Gamma)$ is generated, as an ideal of T_J, by its elements of degree two, i.e. $T_J \cap I(\Gamma) = I(\Gamma_J)$. By Lemma 3.2, it suffices to consider binomials in $T_J \cap I(\Gamma)$. But then by Lemma 3.4, applied to $I(\Gamma_J)$, it is clear that any such binomial must be in $I(\Gamma_J)$. □

With this Lemma in hand we will now identify $U(\Gamma_J)$ and $U_J(\Gamma)$. Now we must analyze $U(\Gamma)$ as right module over the subalgebras $U(\Gamma_J)$.

Lemma 3.6 *The algebra $U(\Gamma)$ is free as a right $U(\Gamma_J)$-module for every subset J of the vertices.*

Proof. Reordering if necessary, we may assume $J = \{1, \ldots, k\}$ for some k, $1 \leq k \leq n-1$, (the case $k = n$ is trivial). Let $J' = \{1, \ldots, k, k+1\}$. Assume $k < n-1$. By downward induction on k, we may assume $U(\Gamma)$ is free as a right $U(\Gamma_{J'})$ module. By induction on n, we may assume $U(\Gamma_{J'})$ is a free right $U(\Gamma_J)$-module. Transitivity then tells us $U(\Gamma)$ is free as a right $U(\Gamma_J)$-module.

It remains only to get the first induction started, i.e. we may assume $k = n-1$. Let $\mathcal{B}$ be the set $\{1\}$ union with the set of all standard monomials in T of the form ax_n, where a is some monomial. We claim the image of $\mathcal{B}$ in $U(\Gamma)$ is a basis for $U(\Gamma)$ as a right $U(\Gamma_J)$-module. We begin with the following observation: the ordering on the monomials in T_J defined by the graph Γ_J is exactly the same as the ordering inherited from the ordering on the monomials in T. From this we can immediately conclude the following: if $ax_n \in \mathcal{B}$ and b is a monomial in T_J, then ax_nb is a standard monomial in T if and only if b is a standard monomial in T_J. The claim now follows immediately from Corollary 3.3 applied to both $U(\Gamma)$ and $U(\Gamma_J)$. $\square$

Let $F(\Gamma)$ denote the trivial one-dimensional graded $U(\Gamma)$-module, concentrated in degree 0. If J is any subset of the indices of Γ we define the $U(\Gamma_J)$-module $F(\Gamma_J)$ similarly. Observe that the graded $U(\Gamma)$-module $U(\Gamma)\bigotimes_{U(\Gamma_J)} F(\Gamma_J)$ is isomorphic to $U(\Gamma)/(\sum_{i\in J} U(\Gamma)x_i)$. We need one last technical Lemma before our main theorem of the section. Recall the usual notation for the shift operation on graded objects. For any graded object M, $M[n]$ is the graded object defined by $M[n]_k = M_{k+n}$.

Lemma 3.7 *Set $J = \{1,\ldots,n-1\}$ and $C = \{i|\{i,n\} \in E\}$. Let K be the kernel of the left $U(\Gamma)$-module epimorphism $U(\Gamma)\bigotimes_{U(\Gamma_J)} F(\Gamma_J) \to F(\Gamma)$. Then $K \cong \big(U(\Gamma)\bigotimes_{U(\Gamma_C)} F(\Gamma_C)\big)[1]$.*

Proof. The module K is graded and cyclic and generated in degree one by the tensor $x_n \otimes 1_F$. We need only compute the annihilator of this vector. Let m be any standard monomial in T. Then there are two exclusive possibilities for mx_n, either mx_n is also a standard monomial or mx_n is not a standard monomial, in which case mx_n is projectively equivalent modulo $I(\Gamma)$ to some monomial $m'x_j$ where $j \neq n$. In the latter case, by Lemma 3.4, $\pi_{j,n}(mx_n)$ and $\pi_{j,n}(m'x_j)$ must be projectively equivalent modulo 0 and this can only happen if $j \in C$. We have shown that either mx_n is standard or $m \in \sum_{j\in C} Tx_j$. Let $a \in U(\Gamma)$ be in the annihilator of $x_n \otimes 1_F$. Write $a = \sum_\alpha a_\alpha \bar{m}_\alpha$ where $a_\alpha \in F$ and $\bar{m}_\alpha$ is the image in $U(\Gamma)$ of a standard monomial m_α in T. Then $ax_n = \sum_\alpha a_\alpha m_\alpha x_n$ is in $U(\Gamma_J)^+$. Since the images of the standard monomials form an F-basis, this can only happen if $a_\alpha = 0$ for every α such that $m_\alpha x_n$ is a standard monomial. Thus $a \in \sum_{j\in C} U(\Gamma)x_j$ and the annihilator of $x_n \otimes 1_F$ is therefore exactly $\sum_{j\in C} U(\Gamma)x_j$. This proves the Lemma. $\square$

Theorem 3.8 *The algebra $U(\Gamma)$ is Koszul for every edge labelled graph Γ.*

Proof. We assume that Γ has n vertices, labelled $\{1,\ldots,n\}$ and a set of edges E. We proceed by induction on n, the case $n = 1$ being trivial. Let $J = \{1,\ldots,n-1\}$ and let $C = \{i|(i,n) \in E\}$. By Lemma 3.5 and 3.6, the graph algebras $U(\Gamma_J)$ and $U(\Gamma_C)$ are quadratic algebras that can be identified with their images in $U(\Gamma)$. By induction, the algebras $U(\Gamma_J)$ and $U(\Gamma_C)$ are Koszul. To expedite notation, let $U = U(\Gamma)$, $R = U(\Gamma_J)$ and $S = U(\Gamma_C)$.

Let K be the kernel of the canonical epimorphism $f : U\bigotimes_R F(\Gamma_J) \to F(\Gamma)$. By Lemma 3.7, $K \cong (U \bigotimes_S F(\Gamma_C))[1]$.

Since the algebra S is Koszul, the module $F(\Gamma_C)$ has a free resolution $P_S^* \to F(\Gamma_C)$ with each $P_S^m \cong S[m]^{k_m}$, $k_m = \dim_F(S^!_m)$. Since U_S is a free module we may tensor this resolution by U and shift in every degree by 1 to get a free U-resolution $P_U^* \to K$ where $P_U^m \cong \left(U \otimes_S S[m]^{k_m}\right)[1] \cong U[m+1]^{k_m}$.

Similarly, since the algebra R is Koszul, the module $F(\Gamma_J)$ has a free resolution $P_R^* \to F(\Gamma_J)$ with each $P_R^m \cong R[m]^{l_m}$, $l_m = \dim_F(R_m^!)$. Since U_R is a free module we may tensor this resolution by U to get a free U-resolution $Q_U^* \to (U \otimes_R F(\Gamma_J))$ where $Q_U^m \cong (U \otimes_R R[m]^{l_m}) \cong U[m]^{l_m}$.

All that is left is to apply the algebraic mapping cone, [18], page 46, to the short exact sequence

$$0 \to K \to U \bigotimes_R F(\Gamma_J) \to F(\Gamma) \to 0.$$

We obtain a free U-resolution $N_U^* \to F(\Gamma)$ where

$$N_U^m = P_U^{m-1} \bigoplus Q_U^m \cong U[m]^{(k_{m-1}+l_m)}.$$

By Definition 2.2 , U is Koszul. $\square$

Corollary 3.9 *The finite dimensional algebra $A(\Gamma)$ is Koszul.*

We note the Hilbert series of $U(\Gamma)$ is now easily computed from the graph Γ. Let $X = X(\Gamma)$ be the simplicial complex on $\{1, \ldots, n\}$ defined as follows: the set $K = \{i_1, \ldots, i_p\}$ is a $(p-1)$-simplex in X if and only if the unlabeled subgraph Γ_K of Γ is a complete graph (X is called the flag-complex of Γ, cf. [22]). Notice that Γ itself is the 2-skeleton of X. Let J and C be as in the proof of Theorem 3.8 and let X_J and X_C be the corresponding simplicial complexes. We notice that X_J is the subcomplex of X on the vertices $1, \ldots, n-1$ and X_C is the link of the vertex n in X. For any simplicial complex X, let $F_p(X)$ be the number of $(p-1)$-simplices of X ($F_0(X) = 1$). We denote the Euler characteristic polynomial of X by $E(X,t)$, i.e. $E(X,t) = \sum_i (-1)^i F_i(X) t^i$.

Corollary 3.10 $H(A(\Gamma), -t) = E(X,t)$.

Proof. We proceed by induction on n, the number of vertices in Γ. Inductively we have $H(A(\Gamma_C), -t) = E(X_C, t)$ and $H(A(\Gamma_J), t) = E(X_J, t)$. It therefore suffices to prove

$$H(A(\Gamma), t) = H(A(\Gamma_J), t) + tH(A(\Gamma_C), t).$$

The algebras $A(\Gamma)$, $A(\Gamma_J)$ and $A(\Gamma_C)$ are the Koszul duals of the Koszul algebras $U(\Gamma)$, $U(\Gamma_J)$ and $U(\Gamma_C)$ respectively. Using the notation from the proof of Theorem 3.8 we have $H(A(\Gamma_J), t) = \sum_i l_i t^i$ and $H(A(\Gamma_C), t) =$

$\sum_i m_i t^i$. Furthermore, the free resolution $N_U^* \to F(\Gamma)$ constructed in the proof must be a minimal resolution. Thus the equation above follows from the last equation in the the proof of Theorem 3.8.

□

The polynomial $E(X, t)$ considered as an invariant of Γ is called dependence polynomial of Γ (see [11]). The next corollary and some combinatorial implications are contained in [10, 11] (cf. also [22]).

Corollary 3.11 $H(U(\Gamma), t)E(X, t) = 1$.

Corollary 3.12 *$gldim(U(\Gamma)) \leq n$ with equality if and only if Γ is the complete graph on n vertices.*

Proof. The global dimension of $U(\Gamma)$ is the projective dimension of the trivial module F. By the Koszul property this is the maximal k for which $A(\Gamma)_k \neq 0$. But $A(\Gamma)_{n+1} = 0$ and $A(\Gamma)_n \neq 0$ if and only if there are no relations of the form $x_i x_j$, $i \neq j$ in the ideal $I^!$. □

4 Generalized Graph Algebras

In this section we extend Theorem 3.8 to a class of algebras containing the graph algebras. Our proof that the algebras in this larger class are Koszul is nearly the same as Theorem 3.8.

Let Γ be an edge labelled graph on n vertices, exactly as in the previous section. Let K be a subset of the vertices. To the pair Γ, K we associate one algebra $U(\Gamma, K) = U(\Gamma)/I_K$ where I_K is the ideal of $U(\Gamma)$ generated by elements x_k^2 for $k \in K$.

Fix a graph Γ and a subset K of its vertices. Let J be any other subset of the vertices. As before, Γ_J is the edge-labelled subgraph on the vertices in J and we set $K_J = K \cap J$.

Lemma 4.1 *Let Γ and K be as above. Let $SB = SB(\Gamma)$ denote the set of standard monomials in the free algebra T with respect to $I(\Gamma)$. Let $SB(K) = SB(\Gamma, K)$ be all those monomials $m \in SB$ whose image in $U(\Gamma, K)$ is not zero. Then the images of the elements in $SB(K)$ form an F-basis for $U(\Gamma, K)$.*

Proof. Recall that the images of the elements of SB form an F-basis for the algebra $U(\Gamma)$. Denote by $I(K)$ the two-sided ideal of T generated by the elements x_k^2 for $k \in K$. Then $U(\Gamma, K) \cong T/(I(\Gamma) + I(K))$. Notice that if

$a \in I(K)$ and $m \in Supp(a)$ then $m \in I(K)$. From this it is immediate that the elements of $SB(K)$ remain linearly independent in $U(\Gamma, K)$. $\square$

Let J be a subset of the vertices of Γ and, as before, let T_J be the free subalgebra of T generated by the x_j, $j \in J$. In the proof of Lemma 3.5 it was shown that $T_J \cap I(\Gamma) = I(\Gamma_J)$. This in turn implies the following: if m is a monomial in T_J then m is a standard monomial in T_J with respect to $I(\Gamma_J)$ if and only if m is a standard monomial in T with respect to Γ, i.e. $SB(\Gamma) \cap T_J = SB(\Gamma_J)$. Clearly a monomial from $SB(\Gamma) \cap T_J$ will vanish in $U(\Gamma_J, K_J)$ if and only if it vanishes in $U(\Gamma, K)$. Combining this with Lemma 4.1 we have proved the following.

Lemma 4.2 *Let Γ and K be as above. Fix a subset, J, of the vertices of Γ. Let U_J be the subalgebra of $U(\Gamma, K)$ generated by the x_j for $j \in J$. Then U_J is a quadratic algebra isomorphic to $U(\Gamma_J, K_J)$.*

We will identify the subalgebra U_J of $U(\Gamma, K)$ with $U(\Gamma_J, K_J)$.

Lemma 4.3 *The algebra $U(\Gamma, K)$ is free as a right $U(\Gamma_J, K_J)$-module.*

Proof. The proof of this lemma is exactly the same as the proof of Lemma 3.6 with the following exception. The set $\mathcal{B}$ should now consist of $\{1\}$ union with the set of all monomials of the form ax_n which are in $SB(\Gamma, K)$.

We need now the analog of Lemma 3.7. We let $F(\Gamma, K)$ be the trivial graded $U(\Gamma, K)$-module. If there is little chance of confusion we write simply F.

Lemma 4.4 *Let $J = \{1, \ldots, n-1\}$ and $C = \{i | (i, n) \in E\}$ (note: $n \notin C$). Let Ker be the kernel of the left $U(\Gamma, K)$-module epimorphism $U(\Gamma, K) \otimes_{U(\Gamma_J, K_J)} F \to F$.*

(a) Assume $n \notin K$. Then $Ker \cong \left(U(\Gamma, K) \otimes_{U(\Gamma_C, K_C)} F\right)[1]$.

(b) Assume $K = \{1, \ldots, n\}$. Let $C' = C \cup \{n\}$. Then $Ker \cong \left(U(\Gamma, K) \otimes_{U(\Gamma_{C'}, K_{C'})} F\right)[1]$.

Proof. (a) Assume $n \notin K$. As in Lemma 3.7, it suffices to prove that the annihilator of $x_n \otimes 1_F$ is $\sum_{i \in C} U(\Gamma, K)x_i$. The proof is exactly the same as Lemma 3.7 once we make the following observation: for a monomial m in $SB(\Gamma, K)$, the image of mx_n in $U(\Gamma, K)$ cannot be zero. This is because $n \notin K$.

(b) Assume $K = \{1, \ldots, n\}$. Now for a monomial m in $SB(\Gamma, K)$, the image of mx_n in $U(\Gamma, K)$ can be zero, but only if m is projectively equivalent modulo $I(\Gamma)$ to a monomial $m'x_n$. This shows, as in the proof of Lemma 3.7 that the annihilator of $x_n \otimes 1_F$ in Ker is exactly $\sum_{i \in C'} U(\Gamma, K)x_i$. $\square$

Theorem 4.5 *The algebra $U(\Gamma, K)$ is Koszul for every edge labelled graph Γ and every subset, K, of its vertices.*

Proof. Assume Γ has n vertices, $\{1, \ldots, n\}$ and use induction on n. The proof proceeds exactly as the proof of Thoerem 3.8 with the following replacements. Lemma 4.2 is used in place of Lemma 3.5. Lemma 4.3 replaces Lemma 3.6. Lemma 3.7 is replaced by part (a) or part (b) of Lemma 4.4 depending on whether $n \notin K$ or $K = \{1, \ldots, n\}$ respectively. $\square$

5 Holonomy algebras of arrangements

In this section we are concerned with certain quadratic algebras related to an arrangement of hyperplanes.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a set of $(\ell - 1)$-dimensional linear subspaces of an ℓ-dimensional linear space V over a field F. We fix linear functionals α_i such that $\ker \alpha_i = H_i$ and call a subset of $\mathcal{A}$ independent if the respective set of functionals is linearly independent. The collection of minimal dependent subsets of $\mathcal{A}$ (*circuits*) forms a matroid $\mathcal{M}$. From the point of view of matroid theory $\mathcal{A}$ is a representation of $\mathcal{M}$ over F. In fact most constructions in this section depend just on $\mathcal{M}$ and not on its representation $\mathcal{A}$.

Associated with $\mathcal{A}$, is the well-known Orlik-Solomon algebra $A(\mathcal{A}) = A$ [20]. A slightly unusual definition of A is as follows. Recall from the previous section that T is the free F-algebra on generators $\{x_1, \ldots, x_n\}$ and denote by $J(\mathcal{A}) = J$ the ideal of T generated by $x_i^2, x_i x_j + x_j x_i$ for every $1 \leq i < j \leq n$, and

$$\sum_{j=1}^{k} (-1)^{j-1} x_{i_1} \cdots x_{i_{j-1}} x_{i_{j+1}} \cdots x_{i_k}$$

for all dependent subsets $\{H_{i_1}, \ldots, H_{i_k}\}$ of $\mathcal{A}$. Then $A = T/J$. One can easily see that $A_p = 0$ for $p > \ell$.

The algebra A is not necessarily quadratic. First of all for it to be quadratic $\mathcal{A}$ must be formal, i.e., all the linear relations among the functionals of H_i should be linearly generated by relations among triples of them. Indeed if $\mathcal{A}$ is not formal then there exists a formal arrangement $\bar{\mathcal{A}}$ such that $J(\bar{\mathcal{A}}) \subset J(\mathcal{A})$ and $J(\bar{\mathcal{A}})_2 = J(\mathcal{A})_2$ but $J(\bar{\mathcal{A}}) \neq J(\mathcal{A})$ (cf. [23]). But even formal arrangements do not in general produce a quadratic algebra A. A necessary condition for a formal arrangement to have a quadratic Orlik-Solomon algebra is contained in [8]. It is proved there in particular that for all the reflection arrangements of type D_k ($k \geq 3$) the algebras A are not quadratic.

It is easy to construct a quadratic algebra that is in a way the quadratic closure of A. This is the algebra $\bar{A} = \bar{A}(\mathcal{A}) = T/\bar{J}$ where $\bar{J}$ is the ideal of

T generated by J_2. Notice that $\bar{A}$ is a finite-dimensional algebra since it is a factor of the exterior algebra on n generators. It is also graded since $\bar{J}$ is homogeneous but unlike A it can have graded components of degree higher than ℓ. Another algebra associated with $\mathcal{A}$ is the algebra $U = U(\mathcal{A})$ studied by Aomoto and Kohno [1, 15, 16] that is the universal enveloping algebra of the holonomy Lie algebra of the complement of $\bigcup H_i$ in V (for $F = \mathbb{C}$). The explicit discription of U (over an arbitrary field F) is as follows. Let $I(\mathcal{A}) = I$ be the ideal of T generated by $[x_i, \sum_{j\in X} x_j]$ for every i and every maximal $X \subset \{1, \ldots, n\}$ such that $i \in X$ and $\bigcap_{j\in X} H_j$ has codimension 2 in V. Here we put $[a, b] = ab - ba$ for $a, b \in T$. Then $U = T/I$. The following simple observation has initiated this work.

Lemma 5.1 *For every arrangement $\mathcal{A}$ we have $U(\mathcal{A}) = (\bar{A}(\mathcal{A}))^!$.*

Now we define a complex K_* of free left U-modules (the Aomoto-Kohno complex). For every $p \geq 0$ put $K_p = \mathrm{Hom}_F(A_p, U)$ and define $d_p : K_p \to K_{p-1}$ ($p = 1, \ldots, \ell$) via

$$d_p f(a) = \sum_{i=1}^{n} f(x_i a) x_i$$

for every $f \in \mathrm{Hom}_F(A_p, U)$ and $a \in A_{p-1}$. Clearly $\mathrm{Im} d_1 = U_+$ whence K_* can be augmented on the right by the canonical map $d_0 : U \to_U F$. Exactly in the same manner one can construct a complex $\bar{K}_*$ using $\bar{A}$ instead of A.

The natural question about K_* is whether this complex is exact. Kohno proved the exactness for the reflection arrangements of types A_k in [16] and claimed it for types C_k, D_k, G_2, and $I_2(p)$ in his unpublished but often cited paper [17]. Since the arrangements of the first two types are supersolvable the result for them also follows from the main theorem of this section. The following proposition shows that it cannot be true for D_k ($k > 3$).

Proposition 5.2 *If K_* is exact then A is quadratic.*

Proof. Suppose that A is not quadratic and $\bar{A}_i = A_i$ for $i = 0, 1, \ldots, p-1$ while $\bar{A}_p \neq A_p$. If K_* is not exact in some dimension less than $p-1$ then the result is proved. Suppose that K_* is exact in all dimensions less than $p-1$. It suffices to prove that K_* is not exact in K_{p-1}.

Notice that $\bar{K}_* = K_*$ up to dimension $p-1$. Suppose that K_* is exact in dimension $p-1$, i.e., $\mathrm{Im} d_p = \ker d_{p-1}$. Denote by $\bar{d}$ the differential $\bar{K}_p \to \bar{K}_{p-1}$ of $\bar{K}_*$. Since $\bar{J} \subset J$ we have a surjective graded homomorphism $\bar{A} \to A$ that allows us to view K_p as a subspace of $\bar{K}_p$. Besides $\bar{d}_{|K_p} = d_p$. Thus the exactness assumption implies that

$$\mathrm{Im} \bar{d} = \mathrm{Im} d_p. \quad (1)$$

On the other hand, since $\bar{A}_p \neq A_p$ there exists a nonzero map $f : \bar{A}_p \to F = U_0$ such that $f \notin K_p$. Notice that $\deg f = 0$ and (1) implies that there exists $g \in K_p$ such that $f - g \in \ker d_p$. But it is easy to see from definition of d_p that $\ker d_p$ cannot have nonzero elements of degree 0. Thus $f = g$ which contradicts the choice of f. The contradiction completes the proof. $\square$

Now we focus our attention on the complex $\bar{K}_*$. It is clear from definition and Lemma 5.1 that this complex is the usual Koszul complex for the algebra U whence this complex is exact if and only if U is Kosul (Theorem 2.3). In the rest of the section we prove that U is Koszul for supersolvable arrangements.

There are many different ways to characterize supersolvable arrangements (cf. [20]). The best suitable definition for our goal is the one given by Bjorner and Zieglerin [5]. First let us recall that for any (ordered) arangement $\mathcal{A}$ one can exhibit a specific monomial F-basis of $A = A(\mathcal{A})$ called the broken circuit basis. A circuit is a sequence of hyperplanes such that their functionals form a minimal dependent set. A broken circuit is a sequence $(H_{i_1}, \ldots, H_{i_p})$ such that $i_1 < \cdots < i_p$ and $(H_{i_1}, \ldots, H_{i_p}, H_j)$ is a circuit for some $j > i_p$. Now the broken circuit basis is formed by the set of monomials $x_{i_1} \cdots x_{i_p}$ such that $i_1 < \cdots < i_p$ and the respective sequence of hyperplanes does not contain any broken circuit. Finally an arrangement $\mathcal{A}$ is supersolvable if every minimal broken circuit consists of two hyperplanes.

It follows from [8, 9] that for a supersolvable arrangement the algebra A is quadratic, i.e., $\bar{A} = A$. Since the proof there involves rational homotopy theory we give a direct elementary proof below.

Lemma 5.3 *If $\mathcal{A}$ is supersolvable then $A = A(\mathcal{A})$ is quadratic.*

Proof. Define the F-linear map $d : T \to T$ via

$$d(x_{i_1} \cdots x_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} x_{i_1} \cdots x_{i_{j-1}} x_{i_{j+1}} \cdots x_{i_p}.$$

Clearly $d^2 = 0$ and $d(ab) = (da)b + (-1)^p a db$ for $a \in T_p$ and $b \in T$. Since all generators of $\bar{J}$ are annihilated by d the ideal $\bar{J}$ is invariant with respect to d. Recall from the begining of this section that only generators of J of degree different from 2 have form $d(\mu_S)$ where $\mu_S = x_{i_1} \cdots x_{i_p}$ for a dependent sequence $S = (H_{i_1}, \ldots, H_{i_p})$ $(p > 3)$.

Suppose now that A is not quadratic, i.e., $\bar{J} \neq J$. Then the previous paragraph implies that there exists a monomial $\mu = \mu_S \notin \bar{J}$ while the sequence S is dependent. Without any loss of generality we can assume that μ is maximal in the reverse lexicographic order among all the monomials of degree p with these properties. Clearly some subsequence of S is a broken circuit. Since $\mathcal{A}$ is supersolvable there exist i_r and i_s with $1 \leq r < s \leq p$

and such that $S_0 = (i_r, i_s, u)$ is a circuit for some u with $u > i_s$. Since S_0 is dependent we have

$$\mu_0 = x_{i_r} x_{i_s} x_u = (d\mu_0)x_u \in J_2 \subset \bar{J}.$$

If H_u is an element of S then μ_0 divides μ modulo $\bar{J}$ whence $\mu \in \bar{J}$. That is a contradiction. Suppose that u is not among i_k ($k = 1, \ldots, p$). Then consider two other monomials μ' and μ'' substituting x_u for x_{i_r} and x_{i_s} respectively. Notice that $\mu', \mu'' > \mu$ in the reverse lexicographic order and the respective sequences of hyperplanes are still dependent. Thus by choice of μ we have $\mu', \mu'' \in \bar{J}$. But, modulo $\bar{J}$, the monomial μ is a linear combination (with coefficients ± 1) of μ' and μ'' whence again $\mu \in \bar{J}$. This contradiction completes the proof. □

Now we want to deform U to a graph algebra. To do this, define the graph $\Gamma = \Gamma(\mathcal{A})$ on the vertices $\{1, 2, \ldots, n\}$ whose edges are exactly those 2-sets $\{i, j\}$ for which (H_i, H_j) is a broken circuit. Label every edge by 1. Then put $\tilde{A} = \tilde{A}(\mathcal{A}) = A(\Gamma)$. For every $\lambda \in F$ put $A_\lambda = A_\lambda(\mathcal{A}) = \Lambda / J_\lambda$ where Λ is the exterior algebra on n generators (as above) and J_λ is its ideal generated by the relations

$$x_i x_j - \lambda^{k-j} x_i x_k + \lambda^{k-i} x_j x_k$$

for every 3-circuit (H_i, H_j, H_k) with $i < j < k$. Let us sum up obvious properties of these algebras.

Lemma 5.4 *(i)* $A_1 = A, A_0 = \tilde{A}$.

(ii) For every $\lambda \neq 0$ the algebra homomorphism defined by $x_i \mapsto \lambda^i x_i$ is an isomorphism of A onto A_λ.

Our goal is to apply the Drinfeld theorem [7] to the family A_λ (cf. also [21]). First we need the following definition. If N is a natural number we call a quadratic algebra N-Koszul if its Koszul complex is exact in the first N terms from the right.

The formal distinction of our case from the main theorem of [7] is that the parameter λ is not real but belongs to the field F. However using the Zariski topology it easy to obtain the following form of the theorem.

Theorem 5.5 *Suppose that for every $\lambda \in F$ we have a quadratic algebra A_λ whose quadratic relations depend on λ polynomially. Suppose that* $\dim(A_\lambda)_i$ *does not depend on λ for $i = 1, 2$, and 3 and A_0 is Koszul. Then for every natural number N there exists a Zariski open subset $W_N \subset F$ containing 0 and such that for every $\lambda \in W_N$ the algebra A_λ is N-Koszul.*

Now we are ready to prove the main result of this section

Theorem 5.6 *If an arrangement $\mathcal{A}$ is supersolvable then the algebras $A(\mathcal{A})$ and $U(\mathcal{A})$ are Koszul.*

Proof. It suffices to prove the statement for $A(\mathcal{A})$. Let us check the conditions of Theorem 5.5. Due to Theorem 3.8 $A_0 = \tilde{A}$ is Koszul. Consider the ideal I_0 of Λ defining this algebra. The monomials of Λ contained in this ideal are those containing submonomials corresponding to broken circuits of length 2. Since $\mathcal{A}$ is supersolvable this is equivalent to containing any broken circuits. Thus due to the broken circuit basis theorem $H(\tilde{A},t) = H(A,t) = H(A_\lambda, t)$ for every λ.

Now extend F to an infinite field if necessary. It follows from theorem 5.5 that for every positive integer N there exists $\lambda_N \neq 0$ such that A_{λ_N} is N-Koszul. Now part (ii) of Lemma 5.4 implies that A is Koszul. □

Corollary 5.7 *If an arrangement $\mathcal{A}$ is supersolvable then its Aomoto-Kohno complex is exact.*

This result would have followed from a theorem in [12] but that theorem is false (see [13] for corrections). The following corollary was first proved in [9] using rational homotopy theory.

Corollary 5.8 *If an arrangement $\mathcal{A}$ is supersolvable then*
$H(U(\mathcal{A}),t)H(A(\mathcal{A}),-t) = 1.$

6 Examples

In this section we consider three examples of non-supersolvable arrangements (or rather matroids). The first two arrangements have non-quadratic algebras A. The last example is quadratic. No example of a non-supersolvable arrangement with a Koszul algebra U is known to us. The last two examples are as close to that as we can find. To simplify notation we identify an arrangement with the respective set of linear functionals.

Example 6.1 *Let $charF \neq 2$ and $\mathcal{A} = \{x, y, z, x+y, x+z, y+z\}$.*

This is a formal arrangement whose algebra A is non-quadratic (and is the smallest such). $H(A,t) = 1 + 6t + 12t^2 + 7t^3$, $H(\bar{A},t) = 1 + 6t + 12t^2 + 8t^3 + t^4$. The algebra U is not Koszul since some of the coefficients of the series $1/(1 - 6t + 12t^2 - 8t^3 + t^4)$ are negative, contradicting Corollary 2.4. (Note: the first negative coefficient occurs at t^{13}).

Example 6.2 *Again $charF \neq 2$ and $\mathcal{A} = \{z, x+y, x-y, x+z, x-z, y+z, y-z\}$. This a representation of the celebrated non-Fano matroid.*

The algebra A is again non-quadratic. $H(A,t) = 1 + 7t + 15t^2 + 9t^3$, $H(\bar{A},t) = 1+7t+15t^2+10t^3+t^4$. The series $1/(1-7t+15t^2-10t^3+t^4)$ has all positive coefficients since the denominator has four positive real roots. There exist at least two different kinds of deformations $A(\lambda)$ of $\bar{A}$ (i.e., $A(1) = \bar{A}$) such that $H(A(\lambda),t) = H(\bar{A},t)$ for all $\lambda \in F$. For the first kind all algebras $A(\lambda)$ with $\lambda \neq 0$ are isomorphic to $\bar{A}$ but $A(0)$ is not Koszul. For the second kind $A(0)$ is Koszul but $A(\lambda)$ are not isomorphic to $\bar{A}$ anymore.

To describe the deformations of the second kind more explicitly notice that $\bar{A}$ is the quotient of the exterior algebra with generators $x_1, \ldots, x_7$ (in the given order of the functionals) over the ideal generated by the six elements $R_1 = x_1x_4 - x_1x_5 + x_4x_5, R_2 = x_1x_6 - x_1x_7 + x_6x_7, R_3 = x_2x_4 - x_2x_7 + x_4x_7, R_4 = x_2x_5 - x_2x_6 + x_5x_6, R_5 = x_3x_4 - x_3x_6 + x_4x_6, R_6 = x_3x_5 - x_3x_7 + x_5x_7$. Considering the dual algebra $U = \bar{A}^!$ one notices that $z = x_1 + \cdots + x_7$ is a central element. (A similar fact is true for any arrangement). Thus changing the generators to $x_1, \ldots, x_6, z$ one can make the identification $U = W \otimes F[z]$ where W is the subalgebra of U generated by $x_1, \ldots, x_6$. (In fact, W is the holonomy algebra of the affine arrangement induced in the hyperplane $x_7 = 1$.) Since U is Koszul if and only if W is Koszul we will focus on W. The algebra $B = W^!$ is the quotient of the exterior algebra on generators $x_1, \ldots, x_6$ over the ideal generated by the relations R_1, R_4, R_5 and also by $R_2' = x_1x_6, R_3' = x_2x_4$, and $R_6' = x_3x_5$. Notice $H(B,t) = H(\bar{A},t)/(1+t) = 1 + 6t + 9t^2 + t^3$. For every $\lambda \in F$ define $B(\lambda)$ by the last 3 relations and by $R_1(\lambda) = x_1x_4 - \lambda x_1x_5 + \lambda x_4x_5, R_4(\lambda) = x_2x_5 - \lambda x_2x_6 + \lambda x_5x_6, R_5(\lambda) = \lambda x_3x_4 - x_3x_6 + x_4x_6$. Clearly $B(1) = B$. A staightforward computation shows that $H(B(\lambda),t)$ does not depend on λ.

Now we want to prove that the algebra $B(0)$ is Koszul. This is equivalent to its dual algebra $W(0) = B(0)^!$ being Koszul. Notice that $W(0)$ has the 9 defining relations:

$$[x_1,x_2],[x_1,x_3],[x_1,x_5],[x_2,x_3],[x_2,x_6],[x_3,x_4],[x_4,x_5],$$

$$[x_5,x_6],[x_3+x_4,x_6].$$

The Koszul complex of $W(0)$ has the form

$$0 \to K_3 \to K_2 \to K_1 \to K_0 \to F \to 0$$

and it is exact (as for every algebra) in terms K_0 and K_1. So to prove that it is exact it suffices to prove that the kernel of the map $\delta_2 : K_2 \to K_1$ is generated in degree 1. The map δ_2 can be represented by the following matrix

$$M = \begin{pmatrix} x_2 & -x_1 & 0 & 0 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 & 0 & 0 \\ x_5 & 0 & 0 & 0 & -x_1 & 0 \\ 0 & x_3 & -x_2 & 0 & 0 & 0 \\ 0 & x_6 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & x_4 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & x_5 & -x_4 & 0 \\ 0 & 0 & 0 & 0 & x_6 & -x_5 \\ 0 & 0 & x_6 & x_6 & 0 & -x_3 - x_4 \end{pmatrix}$$

that acts on the row-vectors from $K_2 = W(0)^9$ via the right multiplication. Denote the rows of M by $r_1, \ldots, r_9$. Then a row-vector $a = (a_1, \ldots, a_9) \in K_2$ belongs to the kernel of δ_2 if and only if the vector

$$\sum_{i=1}^{9} a_i r_i = 0 \qquad (*)$$

(in $K_1 = W(0)^6$).

Now we need a lemma.

Lemma 6.3 *Let $x, y, z \in W(0)$.*

(i) If $xx_1 + yx_4 + zx_6 = 0$ then $x = y = z = 0$.

(ii) If $xx_1 + yx_3 + zx_6 = 0$ then there exists $u \in W(0)$ such that $x = ux_3, y = -ux_1$, and $z = 0$.

Proof. We can assume that x, y, z are homogeneous of a common degree d and apply induction on d. If $d = 0$ the result is obvious. Suppose that $d > 0$ and prove (i). The condition implies that $b = (x, 0, 0, y, 0, z)$ belongs to the kernel of $\delta_1 : K_1 \to K_0$. Thus there exist elements $b_1, \ldots, b_9$ of $W(0)_{d-1}$ such that $b = \sum_{i=1}^{9} b_i r_i$ in $W(0)^6$. In particular

$$-b_1 x_1 + b_4 x_3 + b_5 x_6 = 0,$$

$$b_8 x_6 - b_7 x_4 - b_3 x_1 = 0,$$

$$-b_2 x_1 - b_4 x_2 + b_6 x_4 + b_9 x_6 = 0.$$

Using (i) and (ii) for $d-1$ we obtain from first two equalities $b_3 = b_7 = b_8 = 0$, $b_1 = ux_3$, $b_4 = ux_1$, and $b_5 = 0$ for some $u \in W(0)$. Substituting this into the third equality we have

$$-(b_2 + ux_2)x_1 + b_6 x_4 + b_9 x_6 = 0.$$

Using again (i) for $d - 1$ and the absense of zero divisors in $W(0)$ (see for example [2]) we have $b_1 = ux_3$, $b_2 = -ux_2$, $b_4 = ux_1$, and the other b_i vanish.

Computing vector b we obtain $x = y = z = 0$. The proof of (ii) is similar. $\square$

Now we can finish the computation of the kernel of δ_2. Using Lemma 6.3 we have from (*) $a_3 = a_5 = a_7 = a_8 = 0$ and $a_1 = ux_3$, $a_4 = ux_1$ for some $u \in W(0)$. Then using the absense of zero divisors again we obtain $a_2 = -ux_2$ and $a_6 = a_9 = 0$. Thus $a = u(x_3, -x_2, 0, x_1, 0, \ldots, 0)$ which completes the proof.

Example 6.4 *For this example it is more instructive to describe the matroid itself. This matroid is known in geometry as the plane of order 3. It can be given on 9 elements* $\{1, 2, \ldots, 9\}$ *as the collection of 3-circuits* $\mathcal{X} = \{\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}, \{1,5,9\}, \{2.6.7\}, \{3,4,8\}, \{1,6,8\}, \{2,4,9\}, \{3,5,7\}\}$ *and all the 4-sets are dependent. It can be represented over any field having a primitive cubic root of 1 (see [5]).*

Let us first prove that for this matroid A is quadratic. Notice that every two elements i and j uniquely define a third one $k = \phi(i,j)$ such that $\{i,j,k\}$ is a circuit. Now fix an arbitrary 4-circuit $S = \{i_1, i_2, i_3, i_4\}$ (with the natural order) and consider six elements $k_{r,s} = \phi(i_r, i_s)$ $(1 \le r < s \le 4)$. Since S is a circuit none of $k_{r,s}$ belongs to S. Thus there exists a partition of S in two pairs (without any loss of generality $\{i_1, i_2\}$ and $\{i_3, i_4\}$) such that $k_{1,2} = k = k_{3,4}$. Now one easily checks that

$$R_S = R_{X_1}(x_{i_3} - x_{i_4}) + (x_{i_1} - x_{i_2})R_{X_2}$$

in T where $X_1 = (i_1, i_2, k)$, $X_2 = (k, x_3, x_4)$ and R_Z is the element of J corresponding to an ordered circuit Y. This implies that $R_S \in \bar{J}$ and A is quadratic.

We have $H(A,t) = (1+t)(1+4t)^2$, in particular all coefficients of the series $1/H(A,-t)$ are positive. Using the rooted complex RC constructed in [5], Example 4.1(4), it is easy to exhibit a family of quadratic algebras $A(\lambda)$ $(\lambda \in F)$ such that $A(1) = A$, $A(0) = A(RC_1)$, and $H(A(\lambda), t)$ does not depend on λ. To be more explicit we need to recall that the root complex is defined on the elements of the matroid and its 1-skeleton includes exactly two 2-subsets of each $X \in \mathcal{X}$. The ommitted 2-subsets, written in the order of elements of $\mathcal{X}$ above, are $\{1,3\}, \{4,6\}, \{7,9\}, \{1,7\}, \{2,8\}, \{3,6\}, \{1,9\}, \{2,6\}, \{4,8\}, \{6,8\}, \{2,4\}, \{3,5\}$. Now $A(\lambda)$ is the quotient of the exterior algebra with nine generators $x_1, \ldots, x_9$ over the ideal generated by the relations $R_X(\lambda)$ $(X \in \mathcal{X})$ where for $X = (i,j,k)$ with the ommitted subset, say $\{i,j\}$,

$$R_X(\lambda) = x_i x_j - \lambda x_i x_k + \lambda x_j x_k.$$

It is not hard to prove that A is not isomorphic to $A(\lambda)$ with $\lambda \ne 1$. We suspect that A is Koszul but cannot prove this.

References

[1] Aomoto, K., Poincaré series of the holonomy Lie algebra attached to a configuration of lines, preprint, Nagoya Univ., 1984.

[2] Aubry, M., Lamaire, J.-M., Zero divisors in enveloping algebras of graded Lie algebras, J. Pure Appl. Algebra **38** (1985), 159-166.

[3] Backelin, J. and Fröberg, R., Koszul algebras, Veronese subrings and rings with linear resolutions, Rev. Roumaine Math. Pures Appl. **30** (1985), 85-97.

[4] Beilinson, A., Ginzburg, V., and Soergel, W., Koszul duality patterns in representation theory, preprint.

[5] Bjorner, A. and Ziegler, G., Broken circuit complexes: factorizations and generalizations, Combinatorial Theory, Series B **51** (1991), 96-126.

[6] Cartier, P. and Foata, D., Problèmes combinatoires de commutation et réarrangements, Lecture Notes in Mathematics **85**, Springer-Verlag, Berlin, 1969.

[7] Drinfeld, V., On quadratic quasi-commutational relations in quasi-classical limit, Selecta Math. Sovetica **11** (1992), 316-326.

[8] Falk, M., The minimal model of the complement of an arrangement of hyperplanes, Transactions of AMS **309** (1988), 543-556.

[9] Falk, M. and Randell, R., The lower central series of a fiber-type arrangement, Invent. Math. **82** (1985), 77-88.

[10] Fisher, D.C., The number of words of length n in a free "semi-abelian" monoid, AMS Monthly **96** (1989), 610-614.

[11] Fisher, D.C. and Solow, A.E., Dependence polynomials, Discrete Math. **82** (1990), 251-258.

[12] Jambu, M., Algèbre d'holonomie de Lie et certaines fibrations topologiques, C.R. Acad. Sci. Paris Serie I Math. **306** (1988), 479-482.

[13] Jambu, M., Fiber-type arrangements and factorization properties, Advances in Math. **80** (1990), 1-21.

[14] Kim, K., Makar-Limanov, L. and Roush, F., Graph monoids, Semigroup Forum **25** (1982), 1-7.

[15] Kohno, T., On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of surfaces, Nagoya Math. J. **92** (1983), 21-37.

[16] Kohno, T., Série de Poincaré-Koszul associée aux groupes de tresses pures, Invent. Math. **82** (1985), 57-75.

[17] Kohno, T., Poincaré series of the Malcev completion of generalized pure braid groups, preprint, IHES, 1985.

[18] MacLane, S., Homolgy, Springer Verlag, Berlin, New York, 1975.

[19] Manin, Yu., Quantum groups and non-commutative geometry, CRM, Univ. de Montreal, Montreal, 1988.

[20] Orlik, P. and Terao, H., Arrangements of hyperplanes, Springer -Verlag, Berlin, 1992.

[21] Polishchuk, A., and Positselski, L., Quadratic algebras, preprint, 1994.

[22] Stanley, R., Graph Colorings and Related Symmetric Functions: Ideas and Applications, preprint, 1995.

[23] Yuzvinsky, S., The first two obstructions to the freeness of arrangements, Transactions of AMS **335** (1993), 231-244.