EQUIVARIANT PRINCIPAL BUNDLES OVER SPHERES AND COHOMOGENEITY ONE MANIFOLDS

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Abstract. We classify $SO(n)$-equivariant principal bundles over S^n in terms of their isotropy representations over the north and south poles. This is an example of a general result classifying equivariant (Π, G)-bundles over cohomogeneity one manifolds.

1. Introduction

Let Π and G be Lie groups. A principal (Π, G)-bundle is a locally trivial, principal G-bundle $p\colon E \to X$ such that E and X are left Π-spaces. The projection map p is Π-equivariant and $\gamma(e \cdot g) = (\gamma e) \cdot g$, where $\gamma \in \Pi$ and $g \in G$ acts on $e \in E$ by the principal action. Equivariant principal bundles, and their natural generalizations, were studied by T. E. Stewart [24], T. tom Dieck [5], [6, I (8.7)], R. Lashof [15], [16] together with P. May [17] and G. Segal [18].

These authors study equivariant principal bundles by homotopy theoretic methods. There exists a classifying space $B(\Pi, G)$ for principal (Π, G)-bundles [5], so the classification of equivariant bundles in particular cases can be approached by studying the Π-equivariant homotopy type of $B(\Pi, G)$. If the structural group G of the bundle is *abelian*, then the main result of [18] states that equivariant bundles over a Π-space X are classified by the ordinary homotopy classes of maps $[X \times_\Pi E\Pi, BG]$. In practice, this program leads to an obstruction theory rather than a classification. See, however, the results of Lashof in the special cases where Π acts transitively [14] or semi-freely [16] on the base space X.

Another approach to equivariant principal bundles uses the "local" invariants arising from isotropy representations at singular points of (X, Π), together with equivariant gauge theory [1], [8], [9], [10]. By an isotropy representation at a Π-fixed point $x_0 \in X$ we mean the homomorphism $\alpha_{x_0}\colon \Pi \to G$ defined by the formula

$$\gamma \cdot e_0 = e_0 \cdot \alpha(\gamma)$$

where $e_0 \in p^{-1}(x_0)$. The homomorphism α is independent of the choice of e_0 up to conjugation in G. The relationship between the local invariants and the homotopy classification (in the form of a Localization Theorem ?) deserves further study.

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In this paper, we use the second approach for $\Pi = SO(n)$ acting in the standard way on $X = S^n$. In this concrete situation, we obtain a complete classification by relatively elementary geometric methods. It turns out that the local isotropy representations at the north and south poles of S^n explicitly determine the classification of $(SO(n), G)$ principal bundles over S^n (for short (n, G)-bundles). One surprising consequence is that the set $\mathcal{E}(n, G)$ of (n, G)-bundles is finite for $n \geq 3$. In contrast, the set of (non-equivariant) principal G-bundles over S^n is often infinite. A detailed statement of these results is given in the next section and their proofs, essentially self-contained, are explained in Sections 3 to 6. Several examples are given in Section 7. In Section 8 we show how these results fit into the more general setting of equivariant (Π, G)-bundles over certain Π-manifolds studied by K. Jänich [13] and E. Straume [25]. In particular, we obtain a classification of (Π, G)-bundles over cohomogeneity 1 manifolds.

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2. Statement of results

Let S^n be the n-dimensional sphere of radius 1 in $\mathbf{R}^{n+1}$. We consider the action on S^n of the group $SO(n)$, by orthogonal transformations fixing the poles $(0, \dots, 0, \pm 1)$.

Let G be a Lie group. We denote by $\mathcal{R}(n, G)$ the set of smooth homomorphisms from $SO(n)$ to G modulo the conjugations by elements of G. Unless specified, all maps between manifolds are smooth of class C^∞.

By a G-principal bundle η over S^n, we mean, as usual, a smooth map $p\colon E \to S^n$ from a manifold $E = E(\eta)$ and a free right action $E \times G \to E$ so that $p(z \cdot g) = p(z)$ with the standard local triviality condition. The isomorphism classes of G-bundles over S^n are in bijection with $\pi_{i-1}(G)/\pi_0(G)$, the quotient of the homotopy group $\pi_{n-1}(G)$ (based on the neutral element e of G) by the action of $\pi_0(G)$ induced by the conjugation of G on itself. The bijection associates to a bundle η the class $C(\eta) := [\partial(\mathrm{id}_{S^n})] \in \pi_{n-1}(G)/\pi_0(G)$, where $\partial\colon \pi_n(S^n) \to \pi_{n-1}(G)$ is the boundary operator in the homotopy exact sequence of η [23, Th. 18.5].

A *$SO(n)$-equivariant principal G-bundle* ξ over S^n (or an *(n, G)-bundle* for short) is a G-principal bundle $\xi^\flat$ over S^n together with a left action $SO(n) \times E(\xi) \to E(\xi)$ commuting with the free right action of G and such that the projection to S^n is $SO(n)$-equivariant (we write $E(\xi)$ for $E(\xi^\flat)$). Two (n, G)-bundles ξ_1, ξ_2 are *isomorphic* if there exists a diffeomorphism $h : E(\xi_1) \to E(\xi_2)$ which is both $SO(n)$ and G-equivariant and which commutes with the projections to S^n. We will compute the set $\mathcal{E}(n, G)$ of isomorphism classes of (n, G)-bundles.

Let ξ be a (n, G)-bundle. Choose points $a, b \in E(\xi)$ such that $p(a) = (0, \dots, -1)$ and $p(b) = (0, \dots, 1)$. Let α, β be the maps from $SO(n)$ to G determined by the formulae $A \cdot a = a \cdot \alpha(A)$ and $A \cdot b = b \cdot \beta(A)$. We shall prove in Lemma 3.2 that α and β are smooth homomorphisms and that their class in $\mathcal{R}(n, G)$

depend only on $[\xi] \in \mathcal{E}(n,G)$. We call α and β the *isotropy representations* (associated to a and b). This defines a map $J\colon \mathcal{E}(n,G) \to \mathcal{R}(n,G) \times \mathcal{R}(n,G)$ by $J(\xi) := ([\alpha],[\beta])$. When $n=2$ and G is connected, $J(\xi)$ is a complete invariant which, in particular, determines the (non-equivariant) isomorphism class of $\xi^\flat$. More precisely, let $\psi\colon \mathcal{R}(2,G) \times \mathcal{R}(2,G) \to \pi_1(G)$ be the map determined by $\psi(\alpha,\beta)(z) := [\alpha(z)\beta(z)^{-1}]$ (one uses that $SO(2) \approx S^1$ and that ψ is well defined if G is connected).

Theorem A. *Suppose that G is connected Lie group. Then,*

(i) *the map $J\colon \mathcal{E}(2,G) \to \mathcal{R}(2,G) \times \mathcal{R}(2,G)$ is a bijection.*
(ii) *if $J(\xi) = ([\alpha],[\beta])$, then $\psi(\alpha,\beta) = C(\xi^\flat)$.*

We shall now generalize Theorem A for $n \geq 2$ or G any Lie group. In general, J is then neither injective nor surjective and $C(\xi^\flat)$ is not determined by $J(\xi)$. Consider $SO(n-1)$ as the subgroup of $SO(n)$ fixing the last coordinate. The restriction $[\mu] \mapsto [\mu\,|_{SO(n-1)}]$ gives a map $\mathrm{Res}: \mathcal{R}(n,G) \to \mathcal{R}(n-1,G)$. Denote by $\mathcal{R}(n,G) \times_{(n-1)} \mathcal{R}(n,G)$ the set of $([\alpha],[\beta]) \in \mathcal{R}(n,G) \times \mathcal{R}(n,G)$ such that $\mathrm{Res}\,[\alpha] = \mathrm{Res}\,[\beta]$. If $\varphi\colon H \to G$ is a group homomorphism, we denote by $Z_\varphi \subset G$ the centralizer of $\varphi(H)$.

Theorem B. *Let G be any Lie group G. Then*

(i) *the image of J is $\mathcal{R}(n,G) \times_{(n-1)} \mathcal{R}(n,G)$.*
(ii) *let $\alpha,\beta\colon SO(n-1) \to G$ be two smooth homomorphisms such that $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)} =: \gamma$. Then $J^{-1}([\alpha],[\beta])$ is in bijection with with the set of double cosets $\pi_0(Z_\alpha)\backslash\pi_0(Z_\gamma)/\pi_0(Z_\beta)$.*

Remark 2.1. The compatibility statement in Part (i) of Theorem B was also observed by K. Grove and W. Ziller [7, Prop. 1.6]. In § 8, Theorem B is extended to a more general setting, to include equivariant principal bundles over "special" Π-manifolds in the sense of Jänich [13]. In particular this provides a classification of the equivariant bundles considered by Grove and Ziller.

Since $SO(1)$ is trivial, Theorem B reduces to Part (i) of Theorem A when $n=2$. To determine $C(\xi^\flat)$ as in Part (ii) of Theorem A, we must choose particular representatives of $[\alpha]$ and $[\beta]$ (in general, $J(\xi)$ does not determine $\xi^\flat$: see examples 7.2 and 7.5). An *isotropic lifting* for ξ is a smooth curve $\tilde{c}\colon [-1,1] \to E(\xi)$ lifting the meridian arc $c(t) = (0,\dots,\cos(\pi t/2),\sin(\pi t/2))$ and such that $B \cdot \tilde{c}(t) = \tilde{c}(t)\alpha(B)$ for all $B \in SO(n-1)$. Isotropic lifting always exist (see Lemma 3.5). Choosing $a := \tilde{c}(-1)$ and $b := \tilde{c}(-1)$ leads to isotropy representations $\alpha,\beta\colon SO(n) \to G$ such that $\alpha\,|_{SO(n-1)} = \beta\,|_{SO(n-1)}$. The map $\psi(\alpha,\beta)\colon SO(n) \to G$ constructed as in Theorem A then satisfies $\psi(\alpha,\beta)(AB) = \psi(\alpha,\beta)(A)$ when $B \in SO(n-1)$. It thus induces a map

$$\bar{\psi}(\alpha,\beta)\colon S^{n-1} \cong SO(n)/SO(n-1) \to G.$$

Note that $\bar{\psi}$ is well defined since α and β are actual homomorphisms and not conjugacy classes.

Proposition C. *Let ξ be a (n, G)-bundle. Let $\alpha, \beta\colon SO(n) \to G$ be the isotropy representation associated to the end points of an isotropic lifting. Then, $[\bar{\psi}(\alpha, \beta)] = C(\xi^\flat)$ in $\pi_{n-1}(G)/\pi_0(G)$.*

We shall prove two consequences of Theorem B and Proposition C which emphasize the constrast between the cases $n = 2$ and $n \geq 3$.

Proposition D. *Let η be a principal G-bundle over S^2 with G a non-trivial Lie group. Then there exist infinitely many $\xi \in \mathcal{E}(2, G)$ such that $\xi^\flat \cong \eta$.*

Proposition E. *For G a compact Lie group, the set $\mathcal{E}(n, G)$ is finite when $n \geq 3$.*

These results are proved in § 5, while the former sections are devoted to preliminary material. In Section 6, we determine which (n, G)-bundles come from an $SO(n + 1)$-equivariant bundles. Examples are given in § 7.

3. Preliminary constructions

3.1. *J is well defined.* This follows from the following lemma.

Lemma 3.2. *Let ξ be a (n, G)-bundle. Let $a, b \in E(\xi)$ such that $p(a) = (0, \ldots, -1)$ and $p(b) = (0, \ldots, 1)$. Let β, α be the maps from $SO(n)$ to G determined by the formulae $A \cdot a = a \cdot \alpha(A)$ and $A \cdot b = b \cdot \beta(A)$. Then α and β are smooth homomorphisms and their class in $\mathcal{R}(n, G)$ depends only on $[\xi] \in \mathcal{E}(n, G)$.*

Proof. Let $A, B \in SO(n)$. One has

$$\begin{aligned} a \cdot \alpha(BA) &= (BA) \cdot a = B \cdot (A \cdot a) = B \cdot (a \cdot \alpha(A)) = \\ &= (B \cdot a) \cdot \alpha(A) = a \cdot (\alpha(B)\alpha(A)). \end{aligned}$$

Therefore, α and similarly, β, are homomorphism. They are smooth because the action of $SO(n)$ is smooth. If a' is another choice instead of a, there exists $g \in G$ such that $a' = a \cdot g$ and one has

$$a \cdot (g\alpha'(A)) = a' \cdot \alpha'(A) = A \cdot a' = A \cdot a \cdot g = a \cdot (\alpha(A)g), \tag{3.3}$$

whence $\alpha'(A) = g^{-1}\alpha(A)g$. This proves that the class of (α, β) in $\mathcal{R}(n, G)$ does not depend on the choice of a and b. Now, if $h\colon E(\xi) \xrightarrow{\approx} E(\xi')$ is a $(SO(n), G)$-equivariant diffeomorphism over the identity of S^n, then, by choosing $a' := h(a)$ and $b' := h(b)$, one has $(\alpha', \beta') = (\alpha, \beta)$. The proof of Lemma 3.2 is then complete. □

3.4. *Isotropic liftings.* Let $I := [-1, 1]$ and $c\colon I \to S^n$ be the parametrisation of the meridian arc $c(t) = (0, \ldots, \cos(\pi t/2), \sin(\pi t/2))$. Let $\tilde{c}\colon I \to E = E(\xi)$ be a (smooth) lifting of c. As $c(t)$ is fixed by $SO(n-1)$, one has $B \cdot \tilde{c}(t) = \tilde{c}(t) \cdot \alpha_t(B)$, for $B \in SO(n-1)$. As in the proof of Lemma 3.2, one checks that this gives a smooth path α_t $(t \in I)$ of homomorphisms from $SO(n - 1)$ to G, which depends on the lifting $\tilde{c}$. Call $\tilde{c}$ *isotropic* if α_t is constant: $\alpha_t(B) = \alpha(B)$ for all $B \in SO(n - 1)$.

Lemma 3.5. *Any (n, G)-bundle admits an isotropic lifting.*

We shall make use of connections on (n, G)-bundles which are $SO(n)$-invariant. These can be obtained by averaging any connection (see [1, p. 522]), since the space of connections is an affine space. If a curve $u(t)$ in $E(\xi)$ is horizontal for a $SO(n)$-invariant connection, then $u(t) \cdot g$ and $A \cdot u(t)$ are horizontal. Lemma 3.5 then follows from the following

Lemma 3.6. *Let ξ be a (n, G)-bundle endowed with a $SO(n)$-invariant connection. Then, any lifting $\tilde{c}$ of c which is horizontal is isotropic.*

Proof. If $\tilde{c}$ is an horizontal lifting , so are $B \cdot \tilde{c}$ and $\tilde{c} \cdot \alpha(B)$ for $B \in SO(n-1)$. As $B \cdot \tilde{c}(-1) = \tilde{c}(-1) \cdot \alpha(B)$, one has $B \cdot \tilde{c}(t) = \tilde{c}(t) \cdot \alpha(B)$ for all $t \in I$. □

3.7. *The (n, G)-bundles $\xi_{\alpha,\beta}$.* If X is a topological space, the unreduced suspension ΣX is

$$\Sigma X := I \times X \big/ \{(-1, x) \sim (-1, x') \text{ and } (1, x) \sim (1, x'),\ \forall\ x, x' \in X\}.$$

We donote by $C_{-}X$ the image of $[-1, 1) \times X$ in ΣX and by $C_{+}X$ those of $(-1, 1] \times X$.

Let (α, β) be a pair of smooth homomorphisms from $SO(n)$ to G. Define the space $\widehat{E}_{\alpha,\beta}$ by

$$\begin{aligned} \widehat{E}_{\alpha,\beta} \quad := \quad I \times SO(n) \times G \big/ \{&(-1, A', g) \sim (-1, A, \alpha(A^{-1}A')\, g) \text{ and} \\ &(1, A', g) \sim (1, A, \beta(A^{-1}A')\, g),\ \forall A \in SO(n)\}. \end{aligned}$$

The space $\widehat{E}_{\alpha,\beta}$ admits an obvious free right action of G and a map $p\colon \widehat{E}_{\alpha,\beta} \to \Sigma SO(n)$. This makes a principal G-bundle over $\Sigma SO(n)$; indeed, trivializations on $C_{\pm}SO(n)$ are given by

$$\begin{array}{lllll} \hat{\varphi}_{-}\colon & [t, A, g] & \mapsto & ([t, A], \alpha(A)\, g) & \text{if} \quad -1 \leq t < 1 \\ \hat{\varphi}_{+}\colon & [t, A, g] & \mapsto & ([t, A], \beta(A)\, g) & \text{if} \quad -1 < t \leq 1. \end{array} \tag{3.8}$$

The change of trivializations is

$$\hat{\varphi}_{-} \circ \hat{\varphi}_{+}^{-1}([t, A], g) = ([t, A], \alpha(A)\beta(A)^{-1} g). \tag{3.9}$$

Now, suppose that $\alpha\,_{|SO(n-1)} = \beta\,_{|SO(n-1)}$. Form the space $E_{\alpha,\beta}$ as the quotient

$$E_{\alpha,\beta} := \widehat{E}_{\alpha,\beta} \big/ \{[t, AB, g] \sim [t, A, \alpha(B) g],\ \forall\, B \in SO(n-1)\}.$$

Let $\varepsilon\colon SO(n) \to S^{n-1}$ be the map which associates to a matrix its last column; it is also the projection $\varepsilon\colon SO(n) \to SO(n)/SO(n-1) \cong S^{n-1}$. There is an map $p\colon E_{\alpha,\beta} \to \Sigma S^{n-1}$ given by $p([t, A, g]) = [t, \varepsilon(A)]$ and a free G-action given by $[t, A, g] \cdot g_1 := [t, A, g g_1]$ As above, we check that this defines a G-principal bundle over ΣS^{n-1}; the trivializations $\hat{\varphi}_{\pm}$ descend to trivializations $\check{\varphi}_{\pm}$ over $C_{\pm}S^{n-1}$.

The map $[t, A] \mapsto A \cdot c(t)$ descends to a homeomorphism $f\colon \Sigma S^{n-1} \xrightarrow{\approx} S^n$. By replacing p by $f \circ p$, we obtain a (topological) principal G-bundle

$$\xi_{\alpha,\beta}\colon E_{\alpha,\beta} \xrightarrow{p} S^n.$$

Let $S^n_\pm$ be the punctured spheres $S^n_\pm := f(C_\pm S^{n-1})$. The trivializations given by the compositions

$$\varphi_\pm \colon p^{-1}(S^n_\pm) \xrightarrow{\check\varphi_\pm} C_\pm S^{n-1} \times G \xrightarrow{f\times \mathrm{id}} S^n_\pm \times G \tag{3.10}$$

are homeomorphisms from $p^{-1}(S^n_\pm)$ onto manifolds. The change of trivialization is a diffeomorphism, being obtained by conjugating that of (3.9) by f. Therefore, $\varphi_\pm$ produce a smooth manifold structure on $E_{\alpha,\beta}$. The map p and the G action are smooth. One checks that the map

$$SO(n) \times \widehat{E}_{\alpha,\beta} \to \widehat{E}_{\alpha,\beta} \quad \text{given by} \quad C\cdot[t,A,g] := [t, CA, g]$$

descends to a smooth $SO(n)$-action on $E_{\alpha,\beta}$ which makes $\xi_{\alpha,\beta}$ a (n,G)-bundle.

3.11. *Proof of Part (i) of Theorem B.* Let ξ be a (n,G)-bundle. By Lemma 3.5 there exists an isotropic lifting $\tilde c\colon I \to E(\xi)$ of c. Choosing $a := \tilde c(-1)$ and $b := \tilde c(1)$ produces a representative (α,β) of $J(\xi)$ with $\alpha_{|SO(n-1)} = \beta_{|SO(n-1)}$. Therefore, the image of J is contained in $\mathcal{R}(n,G) \times_{(n-1)} \mathcal{R}(n,G)$.

Conversely, a class $P \in \mathcal{R}(n,G) \times_{(n-1)} \mathcal{R}(n,G)$ is represented by a pair (α,β) with $\alpha_{|SO(n-1)} = \beta_{|SO(n-1)}$. Let $\mathbf{1}$ be the identity matrix in $SO(n)$ and e be the unit element of G. Computing $J(\xi_{\alpha,\beta})$ with the points $a := [-1,\mathbf{1},e]$ and $b := [1,\mathbf{1},e]$ in $E_{\alpha,\beta}$ shows that $J(\xi_{\alpha,\beta}) = P$. □

4. The map $\tilde J_\gamma$

Let $\gamma\colon SO(n-1) \to G$ be a smooth homomorphism. Define a set $\mathcal{R}_\gamma(n,G)$ as follows: an element of $\mathcal{R}_\gamma(n,G)$ is represented by a pair (α,β) of smooth homomorphisms from $SO(n)$ to G such that $\alpha_{|SO(n-1)} = \beta_{|SO(n-1)} = \gamma$. Two pairs (α_1,β_1) and (α_2,β_2) represent the same element of $\mathcal{R}_\gamma(n,G)$ if there is a smooth path $\{g_t \mid t \in [-1,1]\}$ in the centralizer Z_γ of $\gamma(SO(n-1))$ such that $\alpha_2(A) = g_{-1}\alpha_1(A)g_{-1}^{-1}$ and $\beta_2(A) = g_1\beta_1(A)g_1^{-1}$. There is an obvious map $j\colon \mathcal{R}_\gamma(n,G) \to \mathcal{R}(n,G)\times_{(n-1)}\mathcal{R}(n,G)$.

Part (i) of Theorem B, already proven in (3.11), permits us to define a map $\bar J\colon \mathcal{E}(n,G) \to \mathcal{R}(n-1,G)$ by $\bar J(\xi) := \mathrm{Res}\,[\alpha] = \mathrm{Res}\,[\beta]$. We shall now compute the preimage $\bar J^{-1}([\gamma])$.

Proposition 4.1. *Let $\gamma\colon SO(n-1) \to G$ be a smooth homomorphism. Then there exists a bijection $\tilde J_\gamma\colon \bar J^{-1}([\gamma]) \xrightarrow{\approx} \mathcal{R}_\gamma(n,G)$ such that $j\circ \tilde J_\gamma = J$.*

The proof divides into several steps.

4.2. *Definition of $\tilde J_\gamma$.* Let ξ be a (n,G)-bundle with $\bar J(\xi) = [\gamma]$. Choose, using Lemma 3.5, an isotropic lifting $\tilde c_0\colon I \to E(\xi)$ of c. As $\bar J(\xi) = [\gamma]$, the constant path $\alpha^0_t\colon SO(n-1) \to G$ associated to $\tilde c_0$ is conjugated to γ: there exists $g \in G$ such that $\alpha^0_t(B) = g\gamma(B)g^{-1}$. Let $\tilde c := \tilde c_0 \cdot g$. Like in Equation (3.3), one checks that $\tilde c$ is γ-*isotropic*, i.e. $\alpha_t = \gamma$. Choosing $a := \tilde c(-1)$ and $b := \tilde c(1)$ then produces a pair (α,β) of smooth homomorphisms from $SO(n)$ to G which represents a class $\tilde J_\gamma(\xi)$ in $\mathcal{R}_\gamma(n,G)$.

To see that $\tilde J_\gamma$ is thus well defined, let $\tilde c'$ be another γ-isotropic lifting of c. The smooth path $t \mapsto g_t \in G$ defined by $\tilde c'(t) = \tilde c(t)\cdot g_t$ then satisfies, for all $B \in SO(n-1)$:

$$\gamma(B) = \alpha'_t(B) = g_t^{-1}\alpha_t(B) g_t = g_t^{-1}\gamma(B) g_t.$$

Therefore, $g_t \in Z_\gamma$. One has $\alpha'(A) = g_{-1}^{-1}\alpha_t(A)g_{-1}$ and $\beta'(A) = g_1^{-1}\beta_t(A)g_1$, for all $A \in SO(n)$, which proves that $\tilde J_\gamma(\xi)$ does not depend on the choice of a γ-isotropic lifting.

Now, if $h\colon E(\xi) \xrightarrow{\approx} E(\xi')$ is a $(SO(n), G)$-equivariant diffeomorphism over the identity of S^n and $\tilde c\colon I \to E(\xi)$ is a γ-isotropic lifting for ξ, then $c' := h\circ\tilde c$ is a γ-isotropic lifting for ξ' giving $(\alpha', \beta') = (\alpha, \beta)$. This proves that $\tilde J_\gamma$ is well defined.

4.3. *Surjectivity of $\tilde J_\gamma$.* Let (α, β) represent a class P in $\mathcal{R}_\gamma(n, G)$. One checks that $\tilde J_\gamma(\xi_{\alpha,\beta}) = P$, using that the path $t \mapsto [t, \mathbf{1}, e]$ is a γ-isotropic lifting for $\xi_{\alpha,\beta}$.

4.4. *Injectivity of $\tilde J_\gamma$.* Let $\gamma\colon SO(n-1) \to G$ be a smooth homomorphism. and let ξ be a (n, G)-bundle with $\bar J(\xi) = [\gamma]$. There exists $a \in E(\xi)$ with $p(a) = (0, \dots, -1)$ and $B \cdot a = a \cdot \gamma(B)$ for all $B \in SO(n-1)$. Choose a $SO(n)$-invariant connection on ξ and let $\tilde c$ an horizontal lifting of c with $\tilde c(-1) = a$. By Lemma 3.6, $\tilde c$ is γ-isotropic. If $\beta\colon SO(n) \to G$ is defined by $A\cdot\tilde c(1) = \tilde c(1)\cdot\beta(A)$, then (α, β) represents $\tilde J_\gamma(\xi)$.

Consider the map $\widehat\lambda\colon \widehat E_{\alpha,\beta} \to E(\xi)$ given by

$$\widehat\lambda([t, A, g]) := A\cdot\tilde c(t)\cdot g.$$

The map $\widehat\lambda$ descends to a continuous map $\lambda\colon E_{\alpha,\beta} \to E(\xi)$ which is both $SO(n)$ and G-equivariant and which covers the identity of S^n. Therefore ξ and $\xi_{\alpha,\beta}$ are isomorphic as topological (n, G)-bundles. What remains to prove is that λ is a diffeomorphism, which is only non-trivial around the fibers $E_\pm$ above the north and south poles.

The connection on ξ provides a smooth trivialization of ξ restricted to the punctured sphere S^n_- (see 3.7) in the following way. Consider the map $s_-: p^{-1}(S^n_-) \to E_-$ assigning to z the end point in E_- of the horizontal path through z above the meridian arc through $p(z)$. Define the G-equivariant map $\sigma_-: p^{-1}(S^n_-) \to G$ by $s_-(z) = a\cdot\sigma_-(z)$. The required trivialization $\tau_-: p^{-1}(S^n_-) \to S^n_- \times G$ is $\tau_-(z) := (p(z), \sigma_-(z))$.

Take the trivialization φ_- for $\xi_{\alpha,\beta}$ defined in Equations (3.10) of (3.7). As $\tilde c$ is horizontal, one has

$$\tau_-\circ\lambda\circ\varphi_-^{-1}(x, g) = (x, g).$$

This, and the same for E_+, prove that λ is a diffeomorphism. We have thus established that if $\tilde J_\gamma(\xi)$ is represented by (α, β) then the (n, G)-bundle ξ is isomorphic to $\xi_{\alpha,\beta}$, which proves the injectivity of $\tilde J_\gamma$.

The proof of Proposition 4.1 is now complete. □

5. Proof of the main results

This section contains the proofs of the results stated in § 2.

5.1. *Proof of Theorem B.* Part (i) has already been proven in (3.11). We shall now prove Part (ii). Let $\alpha, \beta\colon SO(n-1) \to G$ be two smooth homomorphisms such that $\alpha_{|SO(n-1)} = \beta_{|SO(n-1)} = \gamma$. The pair (α, β) defines a class $[\alpha, \beta] \in \mathcal{R}_\gamma(n, G)$. The group $Z_\gamma \times Z_\gamma$ acts on $\mathcal{R}_\gamma(n, G)$ by

$$(g, h) \cdot [\alpha, \beta] := [g\alpha g^{-1}, h\beta h^{-1}].$$

The set $J^{-1}([\alpha], [\beta]) \subset \bar{J}^{-1}([\gamma])$ is, via the bijection $\tilde{J}_\gamma\colon \bar{J}^{-1}([\gamma]) \xrightarrow{\approx} \mathcal{R}_\gamma(n, G)$ of Proposition 4.1, in bijection with an orbit of the above action. Let "$\sim$" be the equivalence relation on $Z_\gamma \times Z_\gamma$ defined by $(g_1, h_1) \sim (g_2, h_2)$ iff $(g_1, h_1)[\alpha, \beta] = (g_2, h_2)[\alpha, \beta]$. Let

$$\phi\colon Z_\gamma \times Z_\gamma \to \pi_0(Z_\alpha)\backslash\pi_0(Z_\gamma)/\pi_0(Z_\beta)$$

be the map defined by $\phi(g, h) := [g^{-1}h]$. Part (ii) of Theorem B then follows from the following lemma.

Lemma 5.2. *$(g_1, h_1) \sim (g_2, h_2)$ iff $\phi(g_1, h_1) = \phi(g_2, h_2)$.*

Proof. Suppose that $(g_1, h_1) \sim (g_2, h_2)$. This means that there exist $s_-, s_+ \in Z_\gamma$, with $[s_-] = [s_+]$ in $\pi_0(Z_\gamma)$, such that the following equality

$$(g_1\alpha g_1^{-1}, h_1\beta h_1^{-1}) = (s_- g_2\alpha g_2^{-1} s_-^{-1}, s_+ h_2\beta h_2^{-1} s_+^{-1})$$

holds in $Z_\gamma \times Z_\gamma$. This implies that

$$g_1 = s_-\, g_2\, A \quad \text{and} \quad h_1 = s_+\, h_2\, B$$

with $A \in Z_\alpha$ and $B \in Z_\beta$ (the centralizers of the images of α and β). Therefore $g_1^{-1}h_1 = A^{-1}g_2^{-1}s_-^{-1}s_+\, h_2\, B$, which implies $\phi(g_1, h_1) = \phi(g_2, h_2)$.

To prove the converse, observe that

- $(g, h) \sim (C\, g, C\, h)$ for $C \in Z_\gamma$.
- $(g, h) \sim (gA, hB)$ for $A \in Z_\alpha$ and $B \in Z_\beta$.
- $(g, h) \sim (g, uh)$ for u in the identity component of Z_γ.

Suppose that $\phi(g_1, h_1) = \phi(g_2, h_2)$. This means that there are $A \in Z_\alpha$, $B \in Z_\beta$ and u in the identity component of Z_γ auch that $g_1^{-1}h_1 = A^{-1}g_2^{-1}uh_2B$ (u can be put in the middle since the identity component of Z_γ is a normal subgroup of Z_γ). One then has

$$(g_1, h_1) \sim (e, g_1^{-1}h_1) = (e, A^{-1}g_2^{-1}uh_2B) \sim (g_2A, u\, h_2B) \sim (g_2, h_2)\ .$$

□

Proof of Proposition C. Let ξ, α and β as in the statement of Proposition C. Let $\gamma := \alpha_{|SO(n-1)} = \beta_{|SO(n-1)}$. Then, $\xi \in \bar{J}^{-1}([\gamma])$ and, by 4.2, one has $\tilde{J}(\xi) = [\alpha, \beta]$ in $\mathcal{R}_\gamma(n, G)$. By 4.4, $\xi = \xi_{\alpha,\beta}$. Therefore, $C(\xi^\flat) = C(\xi^\flat_{\alpha,\beta}) = [\bar{\psi}(\alpha, \beta)]$, the last equality coming from Equation (3.9) of 3.7 and that $C(\xi^\flat)$ can be represented by the transition function [23, Th. 18.4]. □

Proof of Theorem A. Since $SO(1)$ is trivial, $Z_\gamma = G$ which is suppose to be connected. Therefore, Part (i) is a particular case of Part (i) of Theorem B. Let $c\colon I \to S^n$ parametrizing the meridian arc, as in (3.4). Let $\alpha, \beta\colon SO(2) \to G$ be two homomorphisms representing $J(\xi)$. One can find a and b so that α and β are the isotropy representations associated to a and b. As G is connected, the submanifold $P_0 := p^{-1}(c(I))$ of $E(\xi)$ is connected and there is a smooth lifting $\tilde{c}$ of c such that $\tilde{c}(-1) = a$ and $\tilde{c}(1) = b$. As $SO(1)$ is trivial, $\tilde{c}$ is isotropic. Part (ii) of Theorem A then follows from Proposition C. □

Proof of Proposition D. Recall that any element of $\pi_1(G, e)$ can be represented by a homomorphism (a geodesic in a maximal compact subgroup K of G, with a K-bi-invariant Riemannian metric, being a 1-parameter subgroup [11, Ch. IV, § 6]). Therefore, if η is a G-bundle over S^2, there exists a homomorphism $\alpha\colon S^1 \to G$ such that $C(\eta) = [\alpha]$. For $q \in \mathbf{N}$, let $\alpha_q\colon S^1 \to G$ given by $\alpha_q(z) := \alpha(z)^q$. If α is not trivial, the classes $[\alpha_q]$ are all distinct in $\mathcal{R}(2, G)$. Indeed, the set $\mathcal{R}(2, G)$ is in bijection with lattice points in a Weyl chamber of the Lie algebra of a maximal torus of G and the point representing α_q is q times those representing α.

Suppose first that η is not trivial. Hence, α is not trivial and $[\alpha_{q+1}, \alpha_q]$ are all different classes in $\mathcal{R}_\gamma(2, G)$ with $[\bar{\psi}(\alpha_{q+1}, \alpha_q])= C(\eta)$. The result then follows from Propositions C and 4.1.

When η is trivial, one takes any non-trivial homomorphism $\alpha\colon SO(2) \to G$. The classes $[\alpha_q, \alpha_q]$ in $\mathcal{R}_\gamma(2, G)$ represent infinitely many distinct $SO(2)$-equivariant G-bundles ξ_q with trivial $\xi_q^\flat$. □

Proof of Proposition E. If $n \geq 3$, the group $SO(n)$ is semi-simple and the set $\mathcal{R}(n, G)$ is finite. The latter follows from the following known results:

- a homomorphism is determined by its tangent map at the identity (as a homomorphism of Lie algebras).
- the Lie algebra of G contains only finitely many semi-simple Lie subalgebras, up to inner automorphism [21, Prop. 12.1].
- there are only finitely many homomorphisms between two semisimple Lie algebras.

Also, if G is compact, the group Z_γ is compact and then $\pi_0(Z_\gamma)$ is finite. Proposition E then follows from Theorem B. □

Remark 5.3. To remove the hypothesis "G compact" from Proposition E, it is enough to consider the case G connected. Indeed, $\mathcal{R}_\gamma(n, G)$ is a quotient of $\mathcal{R}_\gamma(n, G_e)$, where G_e is the connected component of e. One would then need the following kind of result: if H is a compact Lie subgroup of a connected Lie group G, then $\pi_0(Z(H))$ is finite. We do not know whether this true.

6. $SO(n+1)$-EQUIVARIANT BUNDLES

In this section, we describe the (n,G)-bundles which are $SO(n+1)$-equivariant G-bundles, for the natural action of $SO(n+1)$ on S^n. Let $\delta\colon SO(n) \xrightarrow{\approx} SO(n)$ be the conjugation by the diagonal $(n\times n)$-matrix $\mathrm{Diag}\,(1,\dots,1,-1)$ (or, equivalently, $\mathrm{Diag}\,(-1,\dots,-1,1)$). If $\alpha\colon SO(n)\to G$ is a smooth homomorphism, observe that $\mathrm{Res}\,[\alpha] = \mathrm{Res}\,[\alpha\circ\delta]$ in $\mathcal{R}(n-1,G)$.

Theorem 6.1. *Let ξ be a (n,G)-bundle. If ξ comes from an $SO(n+1)$-equivariant G-bundle then $J(\xi)$ is of the form $([\alpha],[\alpha\circ\delta])$.*

For any $[\alpha]\in\mathcal{R}(n-1,G)$ there is a unique $\xi\in\mathcal{R}(n,G)$ which comes from a $SO(n+1)$-equivariant G-bundle and such that $J(\xi) = ([\alpha],[\alpha\circ\delta])$.

Proof. For $\theta\in[0,\pi]$, let $R_\theta\in SO(n+1)$ be the rotation of angle θ in the plane of the last 2 coordinates. Let $R := R_\pi$, the diagonal matrix with entries $(1,\dots,1,-1,-1)$.

Let ξ be an $SO(n+1)$-equivariant bundle. Choose $a,b\in E(\xi)$, with $p(a) = (0,\dots,-1)$ and let $b := R\cdot a$. For $A\in SO(n)$, one has $R^{-1}AR = \delta(A)$ and

$$\begin{aligned} b\,\beta(A) &= A\cdot b = A\cdot(R\cdot a) = R\cdot(R^{-1}AR)\cdot a \\ &= R\cdot a\,\alpha(\delta(A)) = b\,\alpha(\delta(A)), \end{aligned} \tag{6.2}$$

whence $\beta = \alpha\circ\delta$, which proves Part (i).

Let $\alpha\colon SO(n)\to G$ be a smooth homomorphism and set $\gamma := \alpha_{|SO(n-1)}$. Suppose that ξ is an $SO(n+1)$-equivariant G-bundle with $a\in E(\xi)$ such that $p(a) = (0,\dots,-1)$ and $A\cdot a = a\,\alpha(A)$ for $A\in SO(n)$. Then $\xi\in\bar{J}^{-1}([\gamma])$. let $c\colon I\to E(\xi)$ be the curve $c(t) := R_{\theta(t)}\cdot a$, where $\theta(t) := \frac{\pi}{2}(t-1)$. Using that $R_\theta^{-1}BR_\theta = B$ for $B\in SO(n-1)$, on checks, as in equation (6.2), that c is a γ-isotropic lifting. By (4.2) and equation (6.2), one has $\tilde{J}_\gamma(\xi) = ([\alpha],[\alpha\circ\delta])$ in $\mathcal{R}_\gamma(n,G)$. By Proposition 4.1, this proves that ξ is determined by α, which proves the uniqueness statment of Part (ii).

It remains to construct, for a smooth homomorphism $\alpha\colon SO(n)\to G$, an $SO(n+1)$-equivariant G-bundle with $J(\xi) = ([\alpha],[\alpha\circ\delta])$. Consider the map $p\colon SO(n+1)\to S^n$ sending a matrix to its last column. This makes an $SO(n+1)$-equivariant $SO(n)$-bundle (the principal $SO(n)$-bundle associated to the tangent bundle of S^n). Let ξ be the G-bundle obtained by the Borel construction, using the homomorphism $\alpha\circ\delta\colon SO(n)\to G$

$$E(\xi) := SO(n+1)\times G\big/\{(BA,g) = (B,\alpha(\delta(A))g)\}. \tag{6.3}$$

This is an $SO(n+1)$-equivariant G-bundle. Choosing $a := (R,e)$ and $b := (I,e)$, one sees that

$$A\cdot a = (AR,e) = (R\delta(A),e) = (R,\alpha(A)e) = a\,\alpha(A)$$

and

$$A\cdot b = (A,e) = (I,\alpha(\delta(A))e) = b\,\alpha(\delta(A)).$$

Therefore, ξ is an $SO(n+1)$-equivariant G-bundle with $J(\xi) = ([\alpha],[\alpha\circ\delta])$. □

Remark 6.4. Theorem 6.1 and its proof show that any $SO(n+1)$-equivariant G bundle is derived from the tangent bundle to S^n by the Borel construction (formula (6.3)). This can be compared with [10, § 6].

7. Examples and applications

Notation: if X is a set, we denote by ΔX the diagonal in $X \times X$.

7.1. *$n = 2$ and $G = U(m)$.* A homomorphism $\alpha\colon S^1 \to U(m)$ has, up to conjugacy, a unique diagonal form $\alpha(z) = \mathrm{Diag}\,(z^{p_1}, \dots, z^{p_m})$, with $p_1 \geq \cdots \geq p_m$. The same holds for β. By Theorem A, $\mathcal{E}(2, U(m))$ is then in bijection with the set of pairs (p, q) of m-tuples of non-decreasing integers. In $\pi_1(U(m)) = \mathbf{Z}$, one has $[\alpha] = \sum_{i=1}^m p_i$ and $[\beta] = \sum_{i=1}^m q_i$, so, by Proposition C:

$$C(\xi^\flat) = \sum_{i=1}^{m} (p_i - q_i).$$

If one wishes instead to characterize $\xi^\flat$ by its first Chern number $c(\xi^\flat) \in H^2(S^2) = \mathbf{Z}$, then $c(\xi^\flat) = -C(\xi^\flat)$ [19, p. 445].

For instance, if τ is the unit tangent bundle over S^2 with the natural action, then $\alpha(z) = z$, $\beta(z) = z^{-1}$, so $C(\tau) = -2$ and $c(\tau) = 2 = \chi(S^2)$.

Note that, if ξ comes from a $SO(3)$-bundle, then, by Theorem 6.1, one has $q_i = -p_i$ (like for τ above). In particular $c(\xi^\flat)$ must be even (see also [10, (6.3)]).

7.2. *$n = 2$ and $G = O(2)$.* For $q \in \mathbf{Z}$, let $\alpha_q\colon SO(2) \to O(2)$ be the homomorphism $A \to A^q$. The set $\mathcal{R}(2, O(2))$ is in bijection with $\mathbf{N}$ given by $\alpha_q \mapsto |q|$. The same recipe produces a bijection $\pi_1(O(2), e)/\pi_0(O(2)) \cong \mathbf{N}$. Let $\xi_{p,q} := \xi_{\alpha_p, \alpha_q}$ be the $(2, O(2))$-bundles constructed in (3.7). By Proposition 4.1 and (4.3), each $\mathcal{E}(2, O(2))$ is represented by some $\xi_{p,q}$, with the only relation $\xi_{p,q} = \xi_{-p,-q}$. One has $J(\xi_{p,q}) = (|p|, |q|)$ and $C(\xi_{p,q}) = |p - q|$. Therefore, $J^{-1}(r, s)$ contains 1 element if $rs = 0$ and 2 otherwise.

7.3. *$n = 2$ and $G = SO(m)$ with $m \geq 3$.* A maximal torus of $SO(m)$ is formed by matrices containing 2-blocks concentrated around the diagonal, so isomorphic to $SO(2)^k$ where $k = [m/2]$. As in 7.1, by Theorem A, $\mathcal{E}(2, SO(m))$ is then in bijection with the set of pairs (p, q) of k-tuples of non-decreasing integers. The bundle $\xi^\flat$ is determined by its second Stiefel-Whitney number $w(\xi^\flat) \in \mathbf{Z}_2$ which is then given by

$$w(\xi^\flat) = \sum_{i=1}^{[\frac{m}{2}]} (p_i - q_i) \pmod{2}.$$

Again, ξ comes from a $SO(3)$-equivariant bundle iff $q_i = -p_i$ and then $\xi^\flat$ is trivial.

7.4. *$n = 2k + 1 \geq 3$ and G is a compact classical group other than $SO(2m)$.* The important thing is that $SO(n-1)$ contains a maximal torus of $SO(n)$. Therefore, by [3, Ch. 6, Corollary 2.8], for any embedding $\psi\colon G \hookrightarrow U(m)$, the representations $\psi\circ\alpha$ and $\psi\circ\beta$ are conjugate in in $U(M)$. For G a compact classical group other than $SO(2m)$, this implies that α and β are conjugate in G [20, Pro. 8, p.56]. Therefore, the image of J is the diagonal $\Delta\mathcal{R}(n,G)$. We do not know whether this true for $G=SO(2m)$ (see [20, Rm. p.57] for a possible source of counter-examples).

7.5. *$n=2k+1\geq 3$ and $G=SO(n)$.* The set $\mathcal{R}(n,SO(n))$ has just two elements, represented by the trivial homomorphism and the identity id of $SO(n)$. If $\iota\colon SO(n-1)\subset SO(n)$ denotes the inclusion, then $Z_{[\iota]}$ contains 2 elements, represented by the identity matrix and the diagonal matrix D with entries $(-1,\dots,-1,1)$. Let δ be the inner automorphism of $SO(n)$ given by the conjugation by D. By Theorem B, the set $\mathcal{E}(n,SO(n))$ for $n=2k+1\geq 3$ then contains 3 elements:

(i) the trivial bundle $S^n\times SO(n)$ with the action $A\cdot(z,B)=(A\cdot z,B)$. The isotropy representations are both trivial.
(ii) the trivial bundle $S^n\times SO(n)$ with the action $A\cdot(z,B)=(A\cdot z,AB)$. It is characterized by $\bar J(\xi)=[\iota]$ and $J_\iota(\xi)=([\mathrm{id}],[\mathrm{id}])$. This does not come from an $SO(n+1)$-equivariant bundle.
(iii) the principal $SO(n)$-bundle $\mathcal{T}S^n$ associated with the tangent bundle of S^n. It is characterized by $\bar J(\mathcal{T}S^n)=[\iota]$ and $J_\iota(\mathcal{T}S^n)=([\mathrm{id}],[\delta])$. This comes from an $SO(n+1)$-equivariant bundle.

Observe that $([\mathrm{id}],[\delta])=([\delta],[\mathrm{id}])$ in $\mathcal{R}_\iota(n,SO(n))$. By Proposition C, this implies that $C(\mathcal{T}S^n)=-C(\mathcal{T}S^n)$. This proves again the classical fact that $C(\mathcal{T}S^{2k+1})\in\pi_{2k}(SO(2k+1))$ is of order 2 [4, Cor. IV.1.11].

7.6. *$n=2k\geq 6$ and $G=SO(n)$.* The set $\mathcal{R}(n,SO(n))$ contains 3 elements, represented by the trivial homomorphism, the identity id of $SO(n)$ and the conjugation δ by the diagonal matrix with entries $(-1,\dots,-1,1)$. The non-trivial homomorphisms restrict to the inclusion $\iota\colon SO(n-1)\subset SO(n)$. The group Z_ι is trivial. Therefore, J is injective and the set $\mathcal{E}(n,SO(n))$ for $n=2k$ then contains 5 elements:

(i) the trivial bundles $S^n\times SO(n)$ with the actions $A\cdot(z,B)=(A\cdot z,B)$, $A\cdot(z,B)=(A\cdot z,AB)$ and $A\cdot(z,B)=(A\cdot z,\delta(A)B)$. Their images by J is the diagonal $\Delta\mathcal{R}(n,SO(n))$.
(ii) the principal $SO(n)$-bundle $\mathcal{T}S^n$ associated with the tangent bundle of S^n. One has $J(\mathcal{T}S^n)=([\mathrm{id}],[\delta])$.
(iii) the (n,G)-bundle $-\mathcal{T}S^n$ with $J(-\mathcal{T}S^n)=([\delta],[\mathrm{id}])$. Its underlying $SO(n)$-principal bundle is stably trival with Euler number -2.

The trivial bundle with action $A\cdot(z,B)=(A\cdot z,B)$, as well as the bundles in (ii) and (iii) are the ones coming from $SO(n+1)$-equivariant bundles.

7.7. *$n=4$ and $G=SO(3)$.* The groups $SO(4)$ and $SO(3)$ are built up out of the unit quaternions S^3 by $SO(4)\cong(S^3\times S^3)/\{(1,1),(-1,-1)\}$ and $SO(3)\cong S^3/\{\pm1\}$. Recall that these isomorphisms are constructed as follows: the orthogonal transformation $A_{p,q}\in SO(4)$ associated to $(p,q)\in S^3\times S^3$ is

$A_{p,q}(x) := px\bar{q}$, where $x \in \mathbf{H}$ is a quaternion and $\mathbf{H}$ is made isomorpohic to $\mathbf{R}^4$ by choosing $(i, j, k, 1)$ as a basis. The correspondence $p \to A_{p,p}$ then induces the inclusion $\iota\colon SO(3) \subset SO(4)$. As for the automorphism δ of $SO(4)$ of § 6, the conjugation by $D := \mathrm{Diag}\,(-1, \dots, -1, 1)$, as $Dx = \bar{x}$, one checks easily that $\delta(A_{p,q}) = A_{q,p}$.

The non-equivariant isomorphism class of a $SO(3)$-principal bundle η is characterized by $C(\eta) \in \pi_3(SO(3)) = \pi_3(S^3) = \mathbf{Z}$. It is also determined by its first Pontrjagin number $p(\eta) \in 4\,\mathbf{Z}$, with the relation $p(\eta) = 4\,C(\eta)$.

The set $\mathcal{R}(4, SO(3))$ contains 3 elements represented by the trivial homomorphism and those induced by the projections $S^3 \times S^3 \to S^3$ given by $\sigma_1(p, q) := p$, $\sigma_2(p, q) := q$. The last two restrict over $SO(3)$ to the identity id of $SO(3)$. The group Z_ι being trivial, the map J is injective. This shows that the set $\mathcal{E}(4, SO(3))$ contains 5 elements:

(i) the trivial bundles $S^4 \times SO(3)$ with the actions $A \cdot (z, B) = (A \cdot z, B)$ and $A \cdot (z, B) = (A \cdot z, \sigma_i(A)B)$ for $i = 1, 2$. Their images by J is the diagonal $\Delta\mathcal{R}(4, SO(3))$.

(ii) the principal $SO(3)$-bundle $\mathcal{H}\colon \mathbf{R}P^7 \to S^4$ coming from the quaternionic Hopf bundle $S^7 \to S^4$; the $SO(4)$-action comes from the $SU(2) \times SU(2)$-action on S^7 given by $(p, q) \cdot (z_1, z_2) = (pz_1, qz_2)$. One has $J(\mathcal{H}) = ([\sigma_1], [\sigma_2])$ and $p(\mathcal{H}^\flat) = 4$.

(iii) the (n, G)-bundle $-\mathcal{H}$ with $J(-\mathcal{H}) = ([\sigma_2], [\sigma_1])$ and $p(\mathcal{H}^\flat) = -4$.

The trivial bundle with action $A \cdot (z, B) = (A \cdot z, B)$, as well as the bundles in (ii) and (iii) are the ones coming from $SO(5)$-equivariant bundles.

7.8. $n = 4$ *and* $G = SO(4)$. Taking the notations of 7.7, the set $\mathcal{R}(4, SO(4))$ contains 5 elements represented by:

- the trivial homomorphism.
- those induced by $\sigma_1(p, q) := (p, p)$ and $\sigma_2(p, q) := (q, q)$.
- the identity id of $SO(4)$.
- the homomorphism $\delta(p, q) := (q, p)$.

The non-trivial homomorphisms all restrict to to ι over $SO(3)$. The group Z_ι is trivial and then J is injective. The non-equivariant isomorphism class of a $SO(4)$-principal bundle η is characterized by $C(\eta) \in \pi_3(SO(4)) = \pi_3(S^3 \times S^3) = \mathbf{Z} \oplus \mathbf{Z}$. More usually, one takes the pair of integers $(p(\eta), e(\eta))$ formed by the first Pontrjagin number ($\in 2\mathbf{Z}$) and the Euler number of η. The linear map which sends $C(\eta)$ to $(p(\eta), e(\eta))$ has matrix $\left(\begin{smallmatrix} 2 & 2 \\ 1 & -1 \end{smallmatrix}\right)$ (as can be checked on the examples below).

One sees that the set $\mathcal{E}(4, SO(3))$ then contains 17 elements:

(i) the trivial bundles $S^4 \times SO(3)$ with the 5 actions using the above representations. Their images by J are just the diagonal elements $\Delta\mathcal{R}(4, SO(4))$.

(ii) the principal $SO(4)$-bundle $\widehat{\mathcal{H}}$ whose total space is $E(\widehat{\mathcal{H}}) := \mathbf{R}P^7 \times_{SO(3)} SO(4)$. One has $J(\widehat{\mathcal{H}}) = ([\sigma_1], [\sigma_2])$ and $C(\widehat{\mathcal{H}}^\flat) = (1, 1)$ therefore its characteristic classes are $(p(\mathcal{H}^\flat), e(\mathcal{H}^\flat)) = (4, 0)$. Also its "inverse" $-\widehat{\mathcal{H}}$, with $J(-\widehat{\mathcal{H}}) = ([\sigma_2], [\sigma_1])$ and $(p(\mathcal{H}^\flat), e(\mathcal{H}^\flat)) = (-4, 0)$.

(iii) the principal $SO(4)$-bundle $\mathcal{T}S^4$ associated with the tangent bundle of S^4. One has $J(\mathcal{T}S^4) = ([\mathrm{id}], [\delta])$. Therefore, $C(\mathcal{T}S^4) = (1, -1)$ and $(p(\mathcal{T}^\flat), e(\mathcal{T}^\flat)) = (0, 2)$. Again, one can consider its inverse.

(iv) the (n, G)-bundles ξ_i $(i = 1, 2)$ with $J(\xi_i) = ([\mathrm{id}], [\sigma_i])$ and their inverses $-\xi_i$. They satisfy $C(\xi_1) = (0, -1)$ and $C(\xi_2) = (1, 0)$, or, equivalently:

$$(p(\xi_1^\flat), e(\xi_1^\flat)) = (-2, 1) \quad \text{and} \quad (p(\xi_2^\flat), e(\xi_2^\flat)) = (2, 1).$$

(v) the (n, G)-bundle $\xi_{i,\delta}$ $(i = 1, 2)$ with $J(\xi_i) = ([\delta], [\sigma_i])$ and their inverses. They satisfy $C(\xi_{1,\delta}) = (-1, 0)$ and $C(\xi_{1,\delta}) = (0, 1)$, or, equivalently:

$$(p(\xi_{1,\delta}^\flat), e(\xi_{1,\delta}^\flat)) = (-2, -1) \quad \text{and} \quad (p(\xi_{2,\delta}^\flat), e(\xi_{2,\delta}^\flat)) = (2, -1).$$

Only the trivial bundle with action $A \cdot (z, B) = (A \cdot z, B)$ and the bundles in (ii) and (iii) come from $SO(5)$-equivariant bundles.

7.9. $G = U(m)$. In order to have non-trivial $(n, U(m))$-bundles, one must have $\dim U(m) > \dim SO(n)$. We check that we are then in the stable range, where, by Bott periodicity,

$$\pi_{n-1}(U(m)) \approx \pi_{n-1}(U(m+k)) \approx \begin{cases} 0 & \text{if } n \text{ is odd} \\ \mathbf{Z} & \text{if } n \text{ is even.} \end{cases}$$

Problem: *which integers occur as $C(\xi^\flat)$ for a $(n, U(m))$-bundle ξ ?*

8. A more general setting

The orthogonal action of $SO(n)$ on S^n is an example of the *special* Π-manifolds defined by Jänich [13, 1.2]. Other examples include the "cohomogeneity one" actions studied by E. Straume [25] (see [7] for a recent application and other references). In this section we give the classification of equivariant (Π, G)-bundles over special Π-manifolds. We will assume in this section that Π and G are both *compact* Lie groups.

Let X be a smooth, connected, closed n-dimensional manifold with a smooth Π-action. Choose a Π-invariant Riemannian metric on X, and then each tangent space T_xX contains a Π_x invariant subspace V_x perpendicular to the orbit $\Pi \cdot x$. Then X is called a *special* Π-manifold if for each $x \in X$ the representation of Π_x on the normal space V_x is the direct sum of a trivial representation and a transitive representation.

It follows that the orbit space $M = X/\Pi$ admits a natural structure as a topological manifold with boundary [13, 1.3], with dimension equal to $n - \dim(\Pi/H)$

where H is the principal isotropy type. Under the "functional" smooth structure [2, VI.6], the orbit space M is a smooth manifold with boundary. The pair $(X, \pi\colon X \to M)$ is called a special Π-manifold over M.

Special Π manifolds over M were classified by Jänich [13, 3.2], and independently by W.-C. Hsiang and W.-Y. Hsiang [12] (see also [2, V.5, VI.6]).

Let $\partial M_A = \{B_\alpha\}_{\alpha \in A}$ denote the set of boundary components of M. An *admissible* isotropy group system (H, U_A) over M consists of a closed subgroup H of Π and a set $U_A = \{U_\alpha\}_{\alpha \in A}$ of closed subgroups in Π containing H, with the property that for each $\alpha \in A$ there exists a transitive representation in which H appears as the isotropy group of a non-zero vector. Let $\Gamma = N(H)/H$ and $\Omega_\alpha = N(U_\alpha) \cap N(H)/H$ for each $\alpha \in A$. The idea of the classification is to re-construct X from the unique principal Γ-bundle P over M such that $P_{|M_0} = \{x \in X \mid \Pi_x = H\}$, and a reduction of the structural group of $P|B_\alpha$ to Ω_α over each of the boundary components B_α.

Let $B_\alpha \times [0,1]$ be a collar neighbourhood of some component B_α in M, and let $Y_\alpha = \pi^{-1}(B_\alpha \times [0,1])$ denote its pre-image in X. The key fact is the following identification of Y_α as a Π-space.

Theorem 8.1. (Tube Theorem [2, V.4.2]) *Let $Y = Y_\alpha$, $B = B_\alpha$ and $\Omega = \Omega_\alpha$. There exists a right Ω-principal bundle $Q = Q_\alpha$ over B, and a Π-equivariant diffeomorphism*

$$M_\psi \times_\Omega Q \xrightarrow{\approx} Y$$

commuting with the projection to $[0,1]$, where M_ψ denotes the mapping cylinder of the canonical projection $\psi\colon \Pi/H \to \Pi/U$.

Let $X_0 = X - \bigcup_\alpha \pi^{-1}(B_\alpha \times [0, 1/2))$ and $M_0 = X_0/\Pi$. The Tube Theorem shows that X is Π-diffeomorphic to the union

$$X = X_0 \cup \bigcup_\alpha Y_\alpha = \Pi/H \times_\Gamma P \cup \bigcup_\alpha M_{\psi_\alpha} \times_{\Omega_\alpha} Q_\alpha$$

with the identification on the overlaps $B_\alpha \times [1/2, 1]$ induced by a reduction of structural groups $P|B_\alpha \cong \Gamma \times_{\Omega_\alpha} Q_\alpha$.

Two special Π-manifolds (X_1, π_1) and (X_2, π_2) over M are called *isomorphic* when there exists a Π-equivariant diffeomorphism $f\colon X_1 \to X_2$ so that the induced diffeomorphism $\bar{f}\colon M \to M$ is the identity. By the the smooth isotopy lifting theorem of G. Schwarz [22, Corollary 2.4] this is equivalent to Jänich's original definition where f was assumed to be only isotopic to the identity, by an isotopy fixing ∂M pointwise. Let $\mathcal{S}[H, U_A]$ denote the set of isomorphism classes of special Π-manifolds over M, with isotropy group system fine equivalent to (H, U_A) (cf. [13, §2]).

Theorem 8.2. ([13, 3.2]) *Let (H, U_A) be an admissible isotropy group system over M, where M is a smooth, compact connected manifold with boundary. Then*

$$\mathcal{S}[H, U_A] \cong [M, \partial M_A; B\Gamma, B\Omega_A] \ .$$

In order to classify equivariant (Π, G)-bundles (E, p) over a special Π-manifold X, called *special* (Π, G)*-bundles* for short, we will generalize the results of Lashof [16] to describe the bundles over Π-spaces X_0 and Y_α with one orbit, and then follow Jänich's method to glue the pieces together.

First some general definitions: if $\rho\colon H \to G$ is a (smooth) homomorphism, $[\rho]$ denotes the set of homomorphisms $\rho'\colon H \to G$ such that $\rho'(h) = g\rho(h)g^{-1}$ for some $g \in G$ and all $h \in H$. We will say that *the fibre over x belongs to* $[\rho]$ if for each $z \in p^{-1}(x)$ there exists $\rho' \in [\rho]$ such that $hz = z \cdot \rho'(h)$ for all $h \in H$. Then let

$$X^{[\rho]} = \{x \in X^H \mid \text{ the fibre over } x \text{ belongs to } [\rho]\}$$

Let $X_0^{[\rho]} = X_0 \cap X^{[\rho]}$ and notice that $X_0^{[\rho]} \subset P = \{x \in X \mid \Pi_x = H\}$. By [16, Lemma 1.2], the space

$$E^\rho = \{z \in E \mid hz = z \cdot \rho(h),\ \forall h \in H\}$$

is an Z_ρ-bundle over $X^{[\rho]}$, where Z_ρ is the centralizer of ρ in G. The group $\widehat{G} = \Pi \times G$ has a left action on the total space E given by the formula $(\gamma, g)\cdot z = \gamma z \cdot g^{-1}$ for any $(\gamma, g) \in \widehat{G}$ and any $z \in E$. Let us set

$$H\langle\rho\rangle := \{(h, \rho(h)) \mid h \in H\} \subset \Pi \times G$$

and

$$\Gamma\langle\rho\rangle := N(H\langle\rho\rangle)/H\langle\rho\rangle\ .$$

Then E^ρ is just the fixed set of $H\langle\rho\rangle$ in E under this action.

Two special (Π, G)-bundles (E_1, p_1) and (E_2, p_2) over M are called *equivalent* when there exists a Π-equivariant G-bundle isomorphism $\phi\colon E_1 \to E_2$, covering a Π-equivariant diffeomorphism $f\colon X_1 \to X_2$, so that the induced diffeomorphism $\bar{f}\colon M \to M$ is the identity.

We now give the classification of the part of (E, p) lying over M_0. After the remarks above, we see that it follows directly from the slice theorem [2, II.5.8].

Theorem 8.3. ([16, 1.9]) *Let $\rho\colon H \to G$ be a smooth homomorphism. The equivalence classes of (Π, G)-equivariant bundles (E_0, p_0) over M_0 with all fibres belonging to $[\rho]$ are in bijection with the homotopy classes of maps $[M_0, B\Gamma\langle\rho\rangle]$.*

We next define an *admissible* (Π, G)-isotropy group system over M to be a set $(H, U_A; \rho, \rho_A)$, where H and U_A are as above, $\rho\colon H \to G$ is a homomorphism, and $\rho_A = \{\rho_\alpha\}_{\alpha\in A}$ is a set of homomorphisms $\rho_\alpha\colon U_\alpha \to G$ such that $\rho_\alpha | H = \rho$. We define

$$\Omega_\alpha\langle\rho_\alpha\rangle = N(U_\alpha\langle\rho_\alpha\rangle) \cap N(H\langle\rho\rangle)/H\langle\rho\rangle$$

and

$$\Omega_A\langle\rho_A\rangle = \{\Omega_\alpha\langle\rho_\alpha\rangle\}_{\alpha\in A}\ .$$

A special (Π, G)-bundles (E, p) *realizes* an admissible (Π, G)-isotropy group system $(H, U_A; \rho, \rho_A)$ over M if

(i) there exists points $\{y_\alpha \in Y_\alpha\}$ such that $\Pi_{y_\alpha} = U_\alpha$,
(ii) for each y_α, the normal space V_{y_α} to the orbit $\Pi \cdot y_\alpha$ has a point $z_\alpha \in V_{y_\alpha}$ with $\Pi_{z_\alpha} = H$,
(iii) the images $c_\alpha(t)$ of rays in V_{y_α} joining z_α to y_α, have isotropic liftings $\tilde{c}_\alpha(t)$ to E such that $\tilde{c}_\alpha(t) \subset E^\rho$ for $0 \leq t < 1$ and $\tilde{c}_\alpha(1) \subset E^{\rho_\alpha}$.

It is not difficult to check (following [13, §2]) that every special (Π, G)-bundle (E, p) over M realizes some admissible (Π, G)-isotropy group system $(H, U_A; \rho, \rho_A)$. This isotropy group system is unique up to a natural notion of equivalence, extending the "fine-orbit structure" of Jänich.

We say that two isotropy group systems $(H, U_A; \rho, \rho_A)$ and $(H', U'_A; \rho', \rho'_A)$ are *fine-equivalent* if the following conditions hold:

(i) there exists an element $\gamma \in \Pi \times G$ such that $H'\langle\rho'\rangle = \gamma H\langle\rho\rangle\gamma^{-1}$, and
(ii) there exist $n_\alpha \in NH\langle\rho\rangle$ such that $U'_\alpha\langle\rho'_\alpha\rangle = (\gamma n_\alpha) U_\alpha\langle\rho_\alpha\rangle(\gamma n_\alpha)^{-1}$.

Let $\mathcal{S}[H, U_A; \rho, \rho_A]$ denote the set of equivalence classes of special (Π, G)-bundles over M realizing the given (Π, G)-isotropy group system, up to fine equivalence.

Theorem 8.4. *Let $(H, U_A; \rho, \rho_A)$ be an admissible (Π, G)-isotropy group system over M, where M is a smooth, compact connected manifold with boundary. Then*

$$\mathcal{S}[H, U_A; \rho, \rho_A] \cong \left[M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle\right] .$$

Proof. Suppose that we are given an admissible (Π, G)-isotropy group system over M. Let (E, p) be a special (Π, G)-bundle over M realizing the given (Π, G)-isotropy group system. By restricting the bundle to M_0, we get a map $\omega_0 \colon M_0 \to B\Gamma\langle\rho\rangle$ classifying the principal $\Gamma\langle\rho\rangle$-bundle $P\langle\rho\rangle$ (which completely determines (E_0, p_0)) by Theorem 8.3. We can apply the Tube Theorem 8.1 to the $\widehat{G} = \Pi \times G$ action on $p^{-1}(Y_\alpha) \subset E$, since $z \in E^\rho$ means that $\widehat{G}_z = H\langle\rho\rangle$ and similarly for $z \in E^{\rho_\alpha}$. This identifies the restriction of our bundle to the part over $B_\alpha \times [0,1]$ as $M_{\varphi_\alpha} \times_{\Omega_\alpha\langle\rho_\alpha\rangle} Q\langle\rho_\alpha\rangle$, where

$$\varphi_\alpha \colon \widehat{G}/H\langle\rho\rangle \to \widehat{G}/U_\alpha\langle\rho_\alpha\rangle$$

is the $\widehat{G}$-equivariant projection, and $Q\langle\rho_\alpha\rangle$ is a principal right $\Omega_\alpha\langle\rho_\alpha\rangle$-bundle over B_α. The classifying map ω_0 for $P\langle\rho\rangle$ therefore extends to a map

$$\omega \colon (M, \partial M_A) \to (B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle)$$

where the notation means that each boundary component B_α is mapped into the α-component of $B\Omega_A\langle\rho_A\rangle$. The restriction of ω to B_α classifies $Q\langle\rho_\alpha\rangle$. This shows that (E, p) is determined up to equivalence by ω.

Conversely, if we are given a map ω as above we can reconstruct a special (Π, G)-equivariant bundle over M realizing the isotropy group system, up to fine equivalence. It can be checked that this bundle is unique up to equivalence. □

Remark 8.5. Note that a special $\Pi \times G$-manifold over M is a special (Π, G)-bundle over M precisely when the subgroup $1 \times G$ acts freely on the total space.

The isotropy group system for the bundle is just the collection of isotropy groups for the $\Pi \times G$-action. This observation shows that Theorem 8.4 follows from Jänich's results.

Corollary 8.6. *Let* Π *and* G *be compact Lie groups. The set of special* (Π, G)*-equivariant bundles over* M *is finite, provided that* $\dim M \leq 1$ *and the isotropy groups are semi-simple.*

Proof. This is proved using [21], as for the finiteness of $\mathcal{E}(n, G)$. □

To conclude this section, we discuss the connection between these results and Theorem B. Let X be a special Π-manifold over M, and let $\mathcal{E}(X, \mathcal{F})$ denote the set of bundle isomorphism classes of principal (Π, G)-bundles over X with isotropy group system fine equivalent to $\mathcal{F} := (H, U_A; \rho, \rho_A)$. Here a bundle isomorphism is a Π-equivariant G-bundle isomorphism $\phi\colon E_1 \to E_2$ covering the *identity* on X.

Lemma 8.7. *There is a commutative diagram:*

$$\begin{tikzcd}
\mathcal{E}(X,\mathcal{F}) \arrow[r] \arrow[d, "v"] & \mathcal{S}[H, U_A; \rho, \rho_A] \arrow[r, "\lambda"] \arrow[d, "\approx"] & \mathcal{S}[H, U_A] \arrow[d, "\approx"] \\
\big[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\big] \arrow[r, "u"] & \big[M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle\big] \arrow[r] & \big[M, \partial M_A; B\Gamma, B\Omega_A\big]
\end{tikzcd}$$

where the horizontal composites have images represented by the element $[X]$.

Proof. The exact sequences $1 \to Z_\rho \to \Gamma\langle\rho\rangle \to \Gamma$ and $1 \to Z_{\rho_\alpha} \to \Omega_\alpha\langle\rho\rangle \to \Omega_\alpha$ of groups induce a map $u\colon \big[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\big] \to \big[M, \partial M_A; B\Gamma\langle\rho\rangle, B\Omega_A\langle\rho_A\rangle\big]$. By Theorem 8.4 and Theorem 8.2, we also have a *surjective* map

$$v\colon \mathcal{E}(X, \mathcal{F}) \to \big[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\big]$$

induced by u and our construction of equivariant bundles. □

Theorem 8.8. *Let* X *be a special* Π*-manifold over* M*, and* $\mathcal{F} = (H, U_A; \rho, \rho_A)$ *an admissible isotropy group system. Then*

$$v\colon \mathcal{E}(X, \mathcal{F}) \cong \big[M, \partial M_A; BZ_\rho, BZ_{\rho_A}\big]\ .$$

Proof. The map v is given in the diagram above, and we have already observed that it is surjective. Suppose that $\xi_1, \xi_2 \in \mathcal{E}(X, \mathcal{F})$ with $v(\xi_1) = v(\xi_2)$. Then we have a continuous map

$$(M \times I, \partial(M \times I)) \to (BZ_\rho, BZ_{\rho_A})$$

realizing the homotopy between the classifying maps for ξ_1 and ξ_2. By the surjectivity of v for (Π, G)-bundles over $M \times I$, we get a bundle (E, p) over $X \times I$ which restricts to ξ_1 and ξ_2 at the ends $X \times \partial I$. Since $E \cong E_0 \times I$, we get $\xi_1 \cong \xi_2$. □

Let $\mathrm{Aut}(X)$ be the group of Π-equivariant isotopy classes of Π-equivariant diffeomorphisms of X over the identity of M. The group $\mathrm{Aut}(X)$ preserving acts on $\mathcal{E}(X,\mathcal{F})$ by pulling back: $f\cdot\xi := (f^{-1})^*\xi$. This action is well defined by the equivariant Covering Homotopy Theorem of Palais [2, II.7.3]. If

$$\lambda\colon \mathcal{S}[H,U_A;\rho,\rho_A] \to \mathcal{S}[H,U_A]$$

denote the natural forgetful map, then there is an induced map

$$\psi\colon \mathcal{E}(X,\mathcal{F}) \to \lambda^{-1}(X)$$

given by applying our stronger equivalence relation on bundles (which allows ϕ to cover a self-diffeomorphism of X).

Proposition 8.9. *The map ψ induces a bijection between $\lambda^{-1}(X)$ and the quotient of $\mathcal{E}(X,\mathcal{F})$ by the action of* $\mathrm{Aut}(X)$*.*

Proof. The map from one set of bundles to the other is defined by regarding a (Π,G)-bundle over X as an element of $\lambda^{-1}(X)$, and this is well-defined since the equivalent relation in $\mathcal{S}[H,U_A;\rho,\rho_A]$ is stronger. Moreover, two bundles with isotropy group system $\mathcal{F}$ over X are equivalent if and only if they are in the same orbit of the action of $\mathrm{Aut}(X)$, hence our correspondence is injective. On the other hand, if (E',p') is a bundle with base space X' in $\lambda^{-1}(X)$, then there exists a Π-equivariant diffeomorphism $h\colon X\to X'$ covering the identity on M. Then $E := h^*(E')$ is an equivalent element in $\lambda^{-1}(X)$, and is a bundle over X, so our correspondence is surjective. ◻

These results and Theorem 8.4 can sometimes be used for explicit classification of equivariant (Π,G)-bundles over special Π-manifolds. Notice that Bredon in [2, V.7] together with [2, Theorem V.6.4] has determined $\mathrm{Aut}(X)$ in many cases of interest.

We shall now specialize to special Π-manifolds over $I := [-1,1]$, and extend Theorem B to this setting. Examples include cohomogeneity 1 actions on spheres, classified in [25, Thm. C, Table II]. The two components $\{\pm 1\}$ of ∂I are denoted $\{\pm\}$, and the notation $\mathcal{F} = (H,U_\pm;\rho,\rho_\pm)$ will be used for the admissible (Π,G)-isotropy group systems, as well as Γ, $\Omega_\pm$, etc. The classification of special Π-manifolds over I takes the following form.

Theorem 8.10. *Let $(H,U_\pm)$ be an admissible isotropy group system over I. Then $\mathcal{S}[H,U_\pm]$ is in bijection with the double cosets $\pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+)$.*

Proof. This follows directly from Theorem 8.2, since

$$[I,\partial I;B\Gamma,B\Omega_\pm] \cong \pi_1(B\Omega_-)\backslash\pi_1(B\Gamma)/\pi_1(B\Omega_+) \cong \pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+). \quad ◻$$

Remark 8.11. The bijection of Theorem 8.10 can be seen in a more constructive way. Given a special Π-manifold X over I, we can choose an *H-meridian* $c\colon I\to X$, i.e. a smooth section of $X\to I$ so that $\Pi_{c(t)} = H$ for t in the interior of I; this can be obtained from a smooth section of the (trivial) principal Γ-bundle $P\to I$.

Let $U_\pm := \Pi_{c(\pm 1)}$. We say that $(H, U_\pm)$ is an *H-meridian isotropy group system* for X. Choosing another smooth section of P gives isotropy groups conjugate to $U_\pm$ by elements in the same connected component of $N(H)$.

As in (3.7), the special Π-manifold X can be reconstructed as a quotient of $I \times \Pi$:

$$X = (I \times \Pi/H) \,/ \{(\pm 1, g) \sim (\pm 1, gu_\pm),\ \forall\, u_\pm \in U_\pm\}$$

Therefore, X is determined by the subgroups $U_\pm$ (compare [7, Prop. 1.6]). Moreover, any set $(H, U'_\pm)$, where the $U'_\pm$ are conjugate to $U_\pm$, occurs as an H-meridian isotropy group system for some special Π-manifold X' over I (proved as in (4.3)). In this way, $\mathcal{S}[H, U_\pm]$ is a quotient of $\pi_0(N(H)) \times \pi_0(N(H))$. The diagonal group acts trivially, and by carefully examining the relevant equivalence relation one sees that $\mathcal{S}[H, U_\pm]$ is in bijection with $\pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+)$.

Let X be a special Π-manifold over I. We will now compute the set $\mathcal{E}(X, G)$ of isomorphism classes of Π-equivariant principal G-bundles over X. Fix an H-meridian isotropy group system $(H, U_\pm)$ for X induced by a smooth H-meridian $c\colon I \to X$. If $\xi = (E, p)$ is a (Π, G)-bundle over X, then we can choose an isotropic lift $\tilde{c}\colon I \to E$, and obtain an admissible isotropy group system $\mathcal{F} = (H, U_\pm; \rho, \rho_\pm)$ for the bundle.

Since the composition of an isotropic lift with a bundle isomorphism is again isotropic, the conjugacy classes $[\rho_\pm] \in \mathcal{R}(U_\pm, G)$ depend only on $[\xi] \in \mathcal{E}(X, G)$. This defines a map

$$J\colon \mathcal{E}(X, G) \to \mathcal{R}(U_-, G) \times \mathcal{R}(U_+, G)\ .$$

We write $\mathcal{R}(U_-, G) \times_H \mathcal{R}(U_+, G)$ for the set of pairs $([\rho_-], [\rho_+]) \in \mathcal{R}(U_-, G) \times \mathcal{R}(U_+, G)$ such that $\operatorname{Res}[\rho_-] = \operatorname{Res}[\rho_+]$ in $\mathcal{R}(H, G)$.

Theorem B generalizes to special manifolds over I as follows:

Theorem 8.12. *Let X be a special Π-manifold over I realizing the isotropy group system $(H, U_\pm)$. The set $\mathcal{E}(X, G)$ of isomorphism classes of (Π, G)-bundles over X is determined by the following properties:*

(i) *the image of J is $\mathcal{R}(U_-, G) \times_H \mathcal{R}(U_+, G)$.*
(ii) *Let $\rho_\pm\colon U_\pm \to G$ be two smooth homomorphisms such that $\rho_- |_H = \rho_+ |_H =: \rho$. Then there is a bijection between $J^{-1}([\rho_-], [\rho_+])$ and the set of double cosets $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$.*

Proof. Since $\rho_\pm$ comes from an admissible system, the image of J is contained in $\mathcal{R}(U_-, G) \times_H \mathcal{R}(U_+, G)$. On the other hand, let $\rho_\pm\colon U_\pm \to G$ be two smooth homomorphisms such that $\rho_- |_H = \rho_+ |_H =: \rho$. The special $(\Pi \times G)$-manifold constructed as in Remark 8.11, with isotropy group system $(H\langle\rho\rangle, U_\pm\langle\rho_\pm\rangle)$ (associated to an $H\langle\rho\rangle$-meridian) is a (Π, G)-bundle ξ over X (the special Π-manifold with H-meridian isotropy group system $(H, U_\pm)$). One has $J(\xi) = ([\rho_-], [\rho_+])$, which proves Part (i).

If $(E,p) \in J^{-1}([\rho_-],[\rho_+])$ then its isotropy group system is fine equivalent to $\mathcal{F} = (H, U_\pm; \rho, \rho_\pm)$. In the notation introduced earlier, we have

$$\mathcal{E}(X,\mathcal{F}) = J^{-1}([\rho_-],[\rho_+])$$

and the result now follows from Theorem 8.8. □

We also have a more explicit version of Proposition 8.9. Note that $J(f \cdot \xi) = J(\xi)$, for $f \in \mathrm{Aut}(X)$, so we must investigate the action of $\mathrm{Aut}(X)$ on a pre-image $\mathcal{E}(X,\mathcal{F}) \cong J^{-1}([\rho_-],[\rho_+])$. The group $\mathrm{Aut}(X)$ has a homotopy description: choose a base point $(\bullet) \in \Omega_-\backslash\Gamma/\Omega_+$ which corresponds to the class of X in $\pi_0(\Omega_-)\backslash\pi_0(\Gamma)/\pi_0(\Omega_+)$.

Proposition 8.13. (Bredon [2, V.7.3, VI.6.4]) *There is a group anti-isomorphism*

$$\mathrm{Aut}(X) \cong [I, \partial I; \Gamma, \Omega_\pm]_\bullet$$

Proof. The pointed maps $(I,\partial I) \to (\Gamma, \Omega_\pm)$ send ∂I into the component of $\Omega_-\backslash\Gamma/\Omega_+$ containing the base point $(\bullet)$. Let $f \in \mathrm{Aut}(X)$. Using our H-meridian c, one defines a smooth path $d\colon I \to \Pi/H$ by the formula

$$f(c(t)) = d(t) \cdot c(t).$$

The map f being Π-equivariant, one has, for all $h \in H$

$$hd(t) \cdot c(t) = h \cdot f(c(t)) = f(hc(t)) = f(c(t)) = d(t)c(t)$$

This implies, for all $t \in I$, that $d(t) \in N(H)/H = \Gamma$. For $t = \pm 1$, one gets in addition that $d(\pm 1) \in \Omega_\pm$. We check that this defines an anti-homomorphism from $\mathrm{Aut}(X)$ to the group $[I, \partial_- I, \partial_+ I; \Gamma, \Omega_-, \Omega_+]$. Now, if $d(t)$ represents a class in the latter, the formula

$$f_d(\alpha c(t)) := \alpha d(t) \cdot c(t) \quad , \quad \alpha \in \Pi, t \in I$$

defines an element f_d of $\mathrm{Aut}(X)$ and constitutes an inverse to the above anti-homomorphism. □

Suppose that the homomorphisms $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective. Then we have a fibre bundle

$$Z_{\rho_-}\backslash Z_\rho/Z_{\rho_+} \to \Omega_-\langle\rho\rangle\backslash\Gamma\langle\rho\rangle/\Omega_+\langle\rho\rangle \to \Omega_-\backslash\Gamma/\Omega_+. \tag{8.14}$$

Therefore, $\pi_1(\Omega_-\backslash\Gamma/\Omega_+)$ acts on $\pi_0(Z_{\rho_-}\backslash Z_\rho/Z_{\rho_+}) = \pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$.

Theorem 8.15. *Suppose that the homomorphisms $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective. Let $\rho_\pm\colon U_\pm \to G$ be two smooth homomorphisms such that $\rho_- |_H = \rho_+ |_H =: \rho$. Then the quotient of $J^{-1}([\rho_-],[\rho_+])$ by the action of* $\mathrm{Aut}(X)$ *is in bijection with with the quotient of $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$ by the action of $\pi_1(\Omega_-\backslash\Gamma/\Omega_+)$.*

Proof. By Theorem 8.8, there is a surjective map $\psi\colon J^{-1}([\rho_-],[\rho_+]) \to \lambda^{-1}([X])$. Since $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective, we have the fibration (8.14). From the homotopy exact sequence of this fibration, we see that $\lambda^{-1}([X])$ is in bijection with the quotient of $\pi_0\big(Z_{\rho_-}\backslash Z_\rho/Z_{\rho_+}\big)$ by the action of $\pi_1(\Omega_-\backslash\Gamma/\Omega_+)$. But this is just the action of $\mathrm{Aut}(X)$ by Proposition 8.13. $\square$

Example 8.16. Consider the standard action of $\Pi = SO(n)$ on S^n, with $H = SO(n-1)$. As $U_\pm = \Pi$, the homomorphisms $\Gamma\langle\rho\rangle \to \Gamma$ and $\Omega_\pm\langle\rho\rangle \to \Omega_\pm$ are surjective.

When $n = 2$, H is the trivial group. If G is connected, then $Z_\rho = G$ is connected and the map J is a bijection, as seen in Theorem A. Also, the group of Π-automorphism of S^2 is equal to Π, so $\mathrm{Aut}(S^2)$ is trivial.

If $n \geq 3$, the group $\Gamma = \Omega_\pm$ has 2 elements, the non-trivial one represented by the diagonal matrix $D := \mathrm{Diag}\,(1,\dots 1,-1,-1)$. The space $\Omega_-\backslash\Gamma/\Omega_+$ being reduced to a point, it follows from Theorem 8.15 that the group $\mathrm{Aut}(S^n)$ acts trivially on $\mathcal{E}(S^n, G)$. This has the following consequence:

Proposition 8.17. *Let $\rho_\pm\colon SO(n) \to G$ be two representations into a compact Lie group G such that $\rho_{\pm|SO(n-1)} = \rho$. Then, the element $\rho_-(D)\rho_+(D)^{-1}$ belongs to Z_ρ and represent the trivial element in $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$.*

Proof. As $\rho_{\pm|H} = \rho$, one has, for all $h \in H$,

$$\rho_+(D)^{-1}\rho(h)\rho_+(D) = \rho_-(D)^{-1}\rho(h)\rho_-(D).$$

Therefore, $\rho_-(D)\rho_+(D)^{-1} \in Z_\rho$.

Recall that our H-meridian for S^n is $c(t) = (0,\dots,\cos(\pi t/2),\sin(\pi t/2))$. By Proposition 4.1 and its proof there exists a (Π, G)-bundle ξ over S^n, with a ρ-isotropic lifting $\tilde{c}\colon I \to E(\xi)$ of c such that the isotropy representations associated to $\tilde{c}(\pm 1)$ are $\rho_\pm$. The curve $\tilde{c}$ is the horizontal lifting of c for a Π-invariant connection. There is no problem to extend the definitions of $c(t)$ and $\tilde{c}(t)$ for $t \in \mathbf{R}$. This defines $\mu \in G$ by

$$\tilde{c}(3) = \tilde{c}(-1)\cdot\mu \quad\text{and}\quad \tilde{c}(5) = \tilde{c}(1)\cdot\mu.$$

For $t \in [-1, 1]$, one has $D\cdot\tilde{c}(t) = \tilde{c}(2-t)\rho_+(D)$ (since this is true for $t = 1$ and both side are horizontal). For $t = -1$, this gives

$$\tilde{c}(-1)\mu\rho_+(D) = \tilde{c}(3)\rho_+(D) = D\cdot\tilde{c}(-1) = \tilde{c}(-1)\rho_-(D)$$

Therefore, $\mu = \rho_-(D)\rho_+(D)^{-1}$.

Now, let $\xi_1 := \delta^*\xi$; one has

$$E(\xi_1) = \{(x,u) \in S^n \times E(\xi) \mid \delta(x) = \pi(u)\}$$

and the Π-action on $E(\xi_1)$ comes from the diagonal action. A ρ-isotropic lifting $\tilde{c}_1\colon I \to E(\xi_1)$ of c is then given by

$$\tilde{c}_1(t) := (c(t), \tilde{c}(t-2)).$$

By Proposition 4.1, one has $\xi = \xi_1$ in $\mathcal{E}(S^n, G)$ iff $\tilde{J}_\rho(\xi) = \tilde{J}_\rho(\xi_1)$ in $\mathcal{R}_\rho(n, G)$. By 4.2, using the curves $\tilde{c}$ and $\tilde{c}_1$, one has in $\mathcal{R}_\rho(n, G)$

$$\tilde{J}_\rho(\xi) = [\rho_-, \rho_+] \text{ and } \tilde{J}_\rho(\xi_1) = [\mu^{-1}\rho_-\mu, \rho_+].$$

This proves that μ represents the unit element in $\pi_0(Z_{\rho_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\rho_+})$. □

Example 8.18. Consider the action of $SO(3)$ on the traceless (3×3)-symmetric matrixes by conjugation. Restricting this action to the unit sphere (for the invariant scalar product $\mathrm{tr}\,(AB)$) makes S^4 a special $SO(3)$-manifold over I (see [7, §3]) for more details on this classical cohomogeneity one $SO(3)$-action on S^4). Here $H = S(O(1)\times O(1)\times O(1))$, $U_- = S(O(1)\times O(2))$ and $U_+ = S(O(2)\times O(1))$. Observe that any homomorphism $\rho_\pm \colon U_\pm \to G$ is trivial unless its restriction to H is injective.

Take $G = SO(3)$ and use the standard inclusions of $U_\pm$ into $SO(3)$. Any non-trivial homomorphism ρ_- is conjugate to $\tilde{\rho}_- \colon (\varepsilon, r_\theta) \mapsto (\varepsilon, r_\theta^p)$ and any non-trivial ρ_+ is conjugate to $\tilde{\rho}_+ \colon (r_\theta, \varepsilon) \mapsto (r_\theta^q, \varepsilon)$, where $p, q \in \mathbf{N}_{\mathrm{odd}}$, the set of positive odd integers. Both $\tilde{\rho}_\pm$ restrict on H to the identification ρ of H with the diagonal matrices of $SO(3)$. One has $Z_\rho = H$, $Z_{\tilde{\rho}_-} = \{\mathrm{diag}\,(1,1,1), \mathrm{diag}\,(1,-1,-1)\}$ and $Z_{\tilde{\rho}_+} = \{\mathrm{diag}\,(1,1,1), \mathrm{diag}\,(-1,-1,1)\}$. Therefore, the set of double cosets $\pi_0(Z_{\tilde{\rho}_-})\backslash\pi_0(Z_\rho)/\pi_0(Z_{\tilde{\rho}_+})$ is trivial and $\mathcal{E}([S^4 \to I], SO(3))$ is in bijection, via the map J of theorem 8.12, with $\{0,0\} \cup \mathbf{N}_{\mathrm{odd}} \times \mathbf{N}_{\mathrm{odd}}$.

In [7, § 3], $(SO(3), SO(3))$-bundles $P_{p,q}$ over S^4 are constructed for $p, q \in \mathbf{Z}$ with $p \equiv 3(\mathrm{mod}\,4)$ (those come from $(S^3 \times S^3)$-bundles). These bundles satisfy $J(P_{p,q}) = (|p|, |q|)$, so, up to isomorphism, the sign of p and q does not matter.

Example 8.19. This is the complex analogue of Example 8.18. One considers the action of $SU(3)$ on the traceless (3×3)-Hermitian matrixes by conjugation and restrict it to the unit sphere. One thus gets a special $SU(3)$-manifold over I diffeomorphic to S^7. The isotropy groups are $H = S(U(1) \times U(1) \times U(1))$, $U_- = S(U(1) \times U(2))$ and $U_+ = S(U(2) \times U(1))$. As in Example 1, any homomorphism $\rho_\pm \colon U_\pm \to G$ is trivial unless its restriction to H is injective.

For $G = SU(3)$, one checks that any non trivial homomorphism of $U_\pm$ to G is conjugate to the standard inclusion. As $Z(H) = H$ in $SU(3)$, the map J is injective and $\mathcal{E}(S^7, SU(3))$ consists of two elements.

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