ON MULTIGRADED REES ALGEBRAS OF IDEALS OF POSITIVE GRADE

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0. Introduction

Let A be a local ring with the maximal ideal m. Let $I \subset A$ be an ideal of $\operatorname{grade} I > 0$. The purpose of this paper is to study the Cohen-Macaulay and Gorenstein properties of the multigraded Rees algebras $R_A(I^{k_1}, \ldots, I^{k_r}) = A[I^{k_1}t_1, \ldots, I^{k_r}t_r]$, where $t_1, \ldots, t_r$ are indeterminates and $k_1, \ldots, k_r \in \mathbb{N}^*$. The geometric object associated to $R_A(I^{k_1}, \ldots, I^{k_r})$ is the multiprojective scheme $\operatorname{Proj} R_A(I^{k_1}, \ldots, I^{k_r})$ embedded into some projective space $\mathbb{P}^{l_1}_A \times_A \ldots \times_A \mathbb{P}^{l_r}_A$. The scheme $\operatorname{Proj} R_A(I^{k_1}, \ldots, I^{k_r})$ is isomorphic to $\operatorname{Proj} R_A(I^{k_1+\cdots+k_r})$, which is the blow-up of $\operatorname{Spec} A$ along the subscheme $V(I^{k_1+\cdots+k_r})$. From the homological point of view these multigraded Rees algebras must thus be closely connected to the Rees algebras of powers of ideals.

The Cohen-Macaulay and Gorenstein properties of $R_A(I^{k_1}, \ldots, I^{k_r})$ have previously been considered in [HHR] for equimultiple ideals I. The main results from [HHR] concerning the Gorensteiness of $R_A(I^{k_1}, \ldots, I^{k_r})$ have partly been generalized to arbitrary ideals in [R]. For an equimultiple ideal, we could calculate (by an idea of E. Hyry) the local cohomology and the specific canonical modules of several graded algebras by using a slightly generalized concept of the Segre product of two graded rings in the following sense: Goto and Watanabe had determined the local cohomology of the Segre product of two graded rings over a field. Their arguments could be extended in [HHR] to graded rings over Artinian rings; in particular, we could compute the local cohomology of the Segre products

$$(\bigoplus_{n\geq 0} I^n/I^{n+1}) \sharp (A/I)[t_1, \ldots, t_r]$$

and

$$(\bigoplus_{n\geq 0} I^n/I^{n+q}) \sharp (A/I^q)[t_1, \ldots, t_r] \quad (q > 1),$$

where $t_1, \ldots, t_r$ are indeterminates and I is an m-primary ideal. Then, for any equimultiple ideal, one could proceed by standard arguments.

In [R], by a completely different approach the Cohen-Macaulay type of a multigraded Rees ring $R_A(I^{k_1},\dots,I^{k_r})$ for an arbitrary ideal I has been determined. This computation together with the observation that the canonical module of $R_A(I)$ can be easily calculated in terms of the canonical module of $R_A(I^{k_1},\dots,I^{k_r})$ led to the more general results in [R].

In this paper, we also consider the multigraded Rees ring $R_A(I^{k_1},\dots,I^{k_r})$ for an arbitrary ideal I. In Theorem 2.4 we first give a necessary and sufficient condition for the Cohen-Macaulayness of $R_A(I^{k_1},\dots,I^{k_r})$ in terms of the local cohomology of the usual Rees algebra $R_A(I)$. It then turns out in the main Theorem 3.16 that if $R_A(I^{k_1},\dots,I^{k_r})$ is Cohen-Macaulay, $R_A(I^{k_1},\dots,I^{k_r})$ is Gorenstein if and only if the Rees algebra $R_A(I^{k_1+\dots+k_r})$ is Gorenstein. Consequently a suitable characterization of the Gorensteiness of $R_A(I^q)$, $q\in\mathbb{N}^*$, for an arbitrary ideal I of grade $I>0$ is desirable in this context. This is the second aim of this paper. The Gorenstein properties of ordinary Rees algebras $R_A(I^q)$, $q\in\mathbb{N}^*$ have already been studied to some extent in [HRZ], [HRS], [H] and [Z]. Assuming $R_A(I)$ to be Cohen-Macaulay, it was shown in [HRZ] that if A is a local Gorenstein ring and $\operatorname{ht} I>1$, $R_A(I^q)$ is Gorenstein if and only if the associated graded ring $gr_A(I)$ is Gorenstein with the a-invariant $-(q+1)$. This is a generalization of a result of Ikeda ([I]), which says that if $R_A(I)$ is Cohen-Macaulay and grade $I>1$, $R_A(I)$ is Gorenstein if and only if $\omega_A\cong A$ and $\omega_{gr_A(I)}\cong gr_A(I)(-2)$. The result of Ikeda has been extended to the case grade $I=1$ by Goto and Nishida in [GN]. They first prove that if $R_A(I)$ is Gorenstein, then $\omega_A\cong \operatorname{Hom}_A(I,A)=I^{-1}$ and there exists an exact sequence

$$0\longrightarrow gr_A(I)(-2)\longrightarrow \omega_{gr_A(I)}\longrightarrow \operatorname{Ext}^1_A(A/I,A)(-1)\longrightarrow 0.$$

They then show that these conditions are under certain assumptions sufficient for the Gorensteiness of $R_A(I)$. Meanwhile Trung, Viêt and Zarzuela have proved in [TVZ] that these conditions are also sufficient in general. Here we follow the line of thinking in [GN] in the following sense.

If $I\subset A$ is an ideal, let $I^*=\bigcup_{n\geq 1} I^{n+1}:I^n$ be the corresponding Ratliff-Rush ideal (see [Mc]). Let $R^*_A(I)$ and $gr^*_A(I)$ denote the Rees algebra and the associated graded ring of the filtration

$$A\supset I^*\supset I^{2*}\supset I^{3*}\supset\dots.$$

Set $I^{-n}=\operatorname{Hom}_A(I^n,A)$ for $n>0$. We shall show in Theorem 3.9 that if $R_A(I^q)$ is Gorenstein, there exists an exact sequence

$$0\longrightarrow R^*_A(I)(-q)\longrightarrow \omega_{R_A(I)}\longrightarrow \bigoplus_{n=1}^{q-1} I^{n-q}\longrightarrow 0$$

and we have $\omega_A \cong I^{-q}$. If, moreover, $\underline{H}_{\mathfrak{M}}^{\dim A}(R_A(I)) = 0$, where $\mathfrak{M}$ is the homogeneous maximal ideal of $R_A(I)$, there also exists an exact sequence

$$0 \longrightarrow gr_A^*(I)(-(q+1)) \longrightarrow \omega_{gr_A(I)} \longrightarrow \bigoplus_{n=1}^{q} I^{n-q-1}/I^{n-q} \longrightarrow 0.$$

We then prove in Theorem 3.12 under the assumption $\underline{H}_{\mathfrak{M}}^{\dim A}(R_A(I)) = 0$ that if $\operatorname{grade} I > 1$ the conditions $\omega_A \cong A$, $\omega_{gr_A(I)} \cong gr_A^*(I)(-(q+1))$ imply the Gorensteiness of $R_A(I^q)$. In the case $q = 1$, this reduces to the above mentioned theorem of Goto and Nishida.

In Corollary 3.14 we show that if A is a local Cohen-Macaulay ring and $R_A(I^q)$ is Gorenstein, $gr_A(I)$ is Gorenstein if and only if either I is principal or $\operatorname{ht} I > 1$ and $gr_A(I)$ is Cohen-Macaulay.

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1. Preliminaries

We begin by fixing some notation and by recalling certain basic facts about the local cohomology theory of multigraded rings and modules (for details see [HHR], [HIO], [GW1], [GW2]).

We use the following multi-index notation. The norm of a multi-index $\mathbf{n} \in \mathbf{Z}^r$ is $|\mathbf{n}| = n_1 + \ldots + n_r$. If $\mathbf{m}, \mathbf{n} \in \mathbf{Z}^r$ are multi-indexes, their product $\mathbf{mn} = (m_1n_1, \ldots, m_rn_r)$ and dot-product $\mathbf{m} \cdot \mathbf{n} = m_1n_1 + \ldots + m_rn_r$. If $m_i < n_i$ ($m_i \le n_i$) for every i, we set $\mathbf{m} < \mathbf{n}$ ($\mathbf{m} \le \mathbf{n}$). We denote $\mathbf{1} = (1, \ldots, 1)$.

In the following we call $\mathbf{Z}^r$-graded rings and modules r-graded or simply multigraded. Rings are always assumed to be Noetherian and $\mathbf{N}^r$-graded. Let $S = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} S_{\mathbf{n}}$ be an r-graded ring. Denote $S^+ = \bigoplus_{\mathbf{n}\neq\mathbf{0}} S_{\mathbf{n}}$, $S_i^+ = \bigoplus_{n_i>0} S_{\mathbf{n}}$ ($i = 1, \ldots, r$) and $S^{++} = \bigoplus_{\mathbf{n}>\mathbf{0}} S_{\mathbf{n}} = S_1^+ \cap \ldots \cap S_r^+$. If $s \in S_{\mathbf{n}}$, we say that s has *total degree* $|\mathbf{n}|$. From any graded ring R we can always form an r-graded ring $R^{r-gr} = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} R_{|\mathbf{n}|}$, which we call the *$r$-graded ring corresponding to R*.

From now on we assume that $S = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} S_{\mathbf{n}}$, where $S_{\mathbf{0}} = A$ is a local ring. If m is the maximal ideal of A, the ring S now has a unique homogeneous maximal ideal $\mathfrak{M} = m \oplus S^+$. We have the multigraded local cohomology modules $\underline{H}_{\mathfrak{M}}^i(M)$. Put $d = \dim S$. An r-graded S-module ω_S is called a *canonical module* of S if

$$\underline{\operatorname{Hom}}_S(\underline{H}_{\mathfrak{M}}^d(S), \underline{E}_S(k)) \cong \omega_S \otimes_A \widehat{A},$$

where $\underline{E}_S(k)$ is the injective envelope of k in the category of r-graded S-modules. If a canonical module exists, it is finitely generated and unique up to an isomorphism. Moreover, it always satisfies the condition (S_2) and $\dim S/P = \dim S$ for all $P \in \operatorname{Ass} \omega_S$.

There is the theorem of *local duality*, which says that if A is complete, S is Cohen-Macaulay if and only if every finitely generated r-graded S-module M satisfies

$$\underline{\mathrm{Hom}}_S(\underline{H}^i_{\mathfrak{M}}(M), \underline{E}_S(k)) = \underline{\mathrm{Ext}}^{d-i}_S(M, \omega_S) \quad (i = 0, \ldots, d).$$

An important corollary of this theorem says that if S is Cohen-Macaulay and has a canonical module ω_S, then also every r-graded ring T defined over a local ring and admitting a finite ring homomorphism $S \longrightarrow T$ has a canonical module

$$\omega_T = \underline{\mathrm{Ext}}^e_S(T, \omega_S),$$

where $e = \dim S - \dim T$. The ring S is Gorenstein if and only if it is Cohen-Macaulay and $\omega_S \cong S(\mathbf{n})$ for some $\mathbf{n} \in \mathbf{Z}^r$.

Recall then the notion of the *a-invariant*. If R is a graded d-dimensional ring defined over a local ring and has the homogeneous maximal ideal $\mathfrak{N}$, the a-invariant of R is

$$a(R) = \max\{m \in \mathbf{N} | [\underline{H}^d_{\mathfrak{N}}(R)]_m \neq 0\}.$$

If R has a canonical module, also

$$a(R) = -\min\{m \in \mathbf{N} | [\omega_R]_m \neq 0\}.$$

The a-invariant of an r-graded ring S is $\mathbf{a}(S) = (a_1, \ldots, a_r)$, where

$$a_j = \max\{n_j | \mathbf{n} \in \mathbf{Z}^r \text{ and } [\underline{H}^d_{\mathfrak{M}}(S)]_{\mathbf{n}} \neq 0\}$$

In the case S has a canonical module, we also have

$$a_j = -\min\{n_j | \mathbf{n} \in \mathbf{Z}^r \text{ and } [\omega_S]_{\mathbf{n}} \neq 0\}.$$

If S is Gorenstein, $\omega_S \cong S(\mathbf{a}(S))$.

If $\mathbf{k} \in (\mathbf{N}^*)^r$, the *Veronesean subring* $S^{(\mathbf{k})}$ of S is $S^{(\mathbf{k})} = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} S_{\mathbf{kn}}$. If M is an r-graded S-module, the *Veronesean submodule* $M^{(\mathbf{k})}$ of M is the r-graded $S^{(\mathbf{k})}$-module $M^{(\mathbf{k})} = \bigoplus_{\mathbf{n} \in \mathbf{Z}^r} M_{\mathbf{kn}}$. If S can be generated by elements of total degree one over A, we have $(\underline{H}^i_{\mathfrak{M}}(M))^{(\mathbf{k})} = \underline{H}^i_{\mathfrak{M}^{(\mathbf{k})}}(M^{(\mathbf{k})})$. Moreover, if $\dim S = \dim S^{(\mathbf{k})}$ and S has a canonical module ω_S, so does $S^{(\mathbf{k})}$ and the canonical module of $S^{(\mathbf{k})}$ is $\omega_{S^{(\mathbf{k})}} = (\omega_S)^{(\mathbf{k})}$.

In many occasions it is useful to consider the ring S endowed with a different grading. Given a homomorphism $\varphi: \mathbf{Z}^r \to \mathbf{Z}^q$ satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$ we put

$$S^{\varphi} = \bigoplus_{\mathbf{m} \in \mathbf{N}^q} \Big(\bigoplus_{\varphi(\mathbf{n}) = \mathbf{m}} S_{\mathbf{n}} \Big).$$

For any r-graded S-module M there is the corresponding r-graded S^φ-module

$$M^\varphi = \bigoplus_{\mathbf{m}\in\mathbf{Z}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} M_\mathbf{n}\Big).$$

It is easy to see that for all r-graded S-modules N and every homomorphism $\varphi: \mathbf{Z}^r \to \mathbf{Z}^q$ satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$ $(\underline{\mathrm{Ext}}^i_S(M,N))^\varphi = \underline{\mathrm{Ext}}^i_{S^\varphi}(M^\varphi, N^\varphi)$. The following lemmas show that the local cohomology groups and the canonical module behave well under a change of grading (see [HHR, §1]).

1.1. Lemma. *Let S be an r-graded ring defined over a local ring and let $\mathfrak{M}$ be the homogeneous maximal ideal of S. Let M be an r-graded S-module. If $\varphi: \mathbf{Z}^r \to \mathbf{Z}^q$ is a homomorphism satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$, we have*

$$(\underline{H}^i_\mathfrak{M}(M))^\varphi = \underline{H}^i_{\mathfrak{M}^\varphi}(M^\varphi).$$

1.2. Remark. If φ is an isomorphism $\mathbf{Z}^r \to \mathbf{Z}^r$ such that $S^\varphi = S$ and $M^\varphi = M$, it especially follows that $(\underline{H}^i_\mathfrak{M}(M))^\varphi = \underline{H}^i_\mathfrak{M}(M)$.

1.3. Lemma. *Let S be an r-graded ring defined over a local ring. Suppose $\varphi: \mathbf{Z}^r \to \mathbf{Z}^q$ is a homomorphism satisfying $\varphi(\mathbf{N}^r) \subset \mathbf{N}^q$ and $\varphi^{-1}(\mathbf{0}) \cap \mathbf{N}^r = \mathbf{0}$. If S has a canonical module ω_S, so does S^φ and the canonical module of S^φ is*

$$\omega_{S^\varphi} = (\omega_S)^\varphi.$$

Let A be a ring and let $I_1, \ldots, I_r \subset A$ be ideals. Set $\mathbf{I} = (I_1, \ldots, I_r)$. The *multi-Rees ring* $R_A(\mathbf{I})$ is the r-graded ring

$$R_A(\mathbf{I}) = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} I_1^{n_1} \cdots I_r^{n_r}.$$

We often identify $R_A(\mathbf{I})$ with the subring $A[I_1t_1, \ldots, I_rt_r]$ of $A[t_1, \ldots, t_r]$. If $\mathrm{ht}\, I_j > 0$ $(j = 1, \ldots, r)$, we have $\dim R_A(\mathbf{I}) = \dim A + r$. In this paper we concentrate to the case where all the ideals $I_1, \ldots, I_r$ are powers of the same ideal $I \subset A$. We use the notation $\mathbf{I}_r$ for the r-tuple $(I, \ldots, I)$ and set $\mathbf{I}_r^\mathbf{k} = (I^{k_1}, \ldots, I^{k_r})$ for $\mathbf{k} \in (\mathbf{N}^*)^r$. The *associated r-graded ring* $gr_A(\mathbf{I}_r) = R_A(\mathbf{I}_r)/IR_A(\mathbf{I}_r)$. If A is local and $\mathrm{ht}\, I > 0$, we have $\dim gr_A(\mathbf{I}_r) = \dim A + r - 1$.

2. The Cohen-Macaulay property of the multi-Rees algebras

Let A be a local ring and $I \subset A$ an ideal of $\mathrm{ht}\, I > 0$. Let $\mathbf{k} \in (\mathbf{N}^*)^r$. We want to calculate the local cohomology of the multigraded Rees algebra $R_A(\mathbf{I}_r^\mathbf{k})$. Since $R_A(\mathbf{I}_r^\mathbf{k}) = (R_A(\mathbf{I}_r))^{(\mathbf{k})}$, we can first consider the ring $R_A(\mathbf{I}_r)$. We have

$$R_A(\mathbf{I}_r) = \bigoplus_{\mathbf{n}\in\mathbf{N}^r} [R_A(I)]_{|\mathbf{n}|}$$

so that $R_A(\mathbf{I}_r)$ is the r-graded ring $(R_A(I))^{r-gr}$ corresponding to $R_A(I)$. We therefore begin with the following general lemma:

2.1. Lemma. *Let R be a graded ring defined over a local ring and let $\mathfrak{M}$ be the homogeneous maximal ideal of R. Put $S = R^{r-gr}$ and $\mathfrak{N} = \mathfrak{M}^{r-gr}$. Let $j, l \in \{1, \ldots, r\}$ $(j \neq l)$. Then*

(1) $\underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_r)} = \underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,\overset{(j)}{n_j+n_l},\ldots,\overset{(l)}{0},\ldots,n_r)}$ *if* $n_j \geq 0$ *and* $n_l \geq 0$.

(2) $\underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_r)} = 0$ *if* $n_j < 0$ *and* $n_l \geq 0$.

(3) $\underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_r)} = \underline{H}^i_{\mathfrak{N}}(S_l^+)_{(n_1,\ldots,\overset{(j)}{n_j+n_l},\ldots,\overset{(l)}{0},\ldots,n_r)}$ *if* $n_j < 0$ *and* $n_l < 0$.

Moreover, $\underline{H}^i_{\mathfrak{N}}(S_l^+)_{(n_1,\ldots,n_r)} = 0$ *if* $n_j \geq 0$ *and* $n_l = 0$.

Proof. By symmetry we can assume that $j = r-1$ and $l = r$ (see Remark 1.2). Consider the exact sequence

$$0 \longrightarrow S_r^+ \longrightarrow S \longrightarrow S/S_r^+ \longrightarrow 0.$$

From this sequence we get for all $\mathbf{n} \in \mathbf{N}^r$ the exact sequence

$$[\underline{H}^{i-1}_{\mathfrak{N}}(S/S_r^+)]_{\mathbf{n}} \longrightarrow [\underline{H}^i_{\mathfrak{N}}(S_r^+)]_{\mathbf{n}} \longrightarrow [\underline{H}^i_{\mathfrak{N}}(S)]_{\mathbf{n}} \longrightarrow [\underline{H}^i_{\mathfrak{N}}(S/S_r^+)]_{\mathbf{n}}.$$

Since $[\underline{H}^i_{\mathfrak{N}}(S/S_r^+)]_{\mathbf{n}} = 0$ if $n_r \neq 0$, we obtain an isomorphism

$$[\underline{H}^i_{\mathfrak{N}}(S_r^+)]_{\mathbf{n}} \cong [\underline{H}^i_{\mathfrak{N}}(S)]_{\mathbf{n}}.$$

Similarly there is an isomorphism

$$[\underline{H}^i_{\mathfrak{N}}(S_{r-1}^+)]_{\mathbf{n}} \cong [\underline{H}^i_{\mathfrak{N}}(S)]_{\mathbf{n}}$$

for $n_{r-1} \neq 0$. Also note that the map $S_r^+ \longrightarrow S_{r-1}^+$, $s \mapsto t_{r-1}t_r^{-1}s$, $s \in S_r^+$ induces an isomorphism

$$[\underline{H}^i_{\mathfrak{N}}(S_r^+)]_{(n_1,\ldots,n_{r-1},n_r)} \cong [\underline{H}^i_{\mathfrak{N}}(S_{r-1}^+)]_{(n_1,\ldots,n_{r-1}+1,n_r-1)}.$$

We then have for any $n_{r-1} \neq 0, n_r \neq -1$

$$\begin{aligned}\underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_{r-1},n_r)} &\cong \underline{H}^i_{\mathfrak{N}}(S_{r-1}^+)_{(n_1,\ldots,n_{r-1},n_r)} \\ &\cong \underline{H}^i_{\mathfrak{N}}(S_r^+)_{(n_1,\ldots,n_{r-1}-1,n_r+1)} \\ &\cong \underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_{r-1}-1,n_r+1)}.\end{aligned}$$

If we now replace n_{r-1} and n_r with $n_{r-1}+n_r$ and 0 respectively, the repeated use of this formula implies (1). Also (2) follows, since for $k \gg 0$ we get

$$\begin{aligned}\underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_{r-1},n_r)} &\cong \underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_{r-1}-1,n_r+1)} \\ &\cong \ldots \\ &\cong \underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\ldots,n_{r-1}-k,n_r+k)} \\ &= 0.\end{aligned}$$

To prove (3) observe that for $n_{r-1} \neq -1$

$$\begin{aligned} \underline{H}^i_{\mathfrak{N}}(S^+_r)_{(n_1,\dots,n_{r-1},n_r)} &\cong \underline{H}^i_{\mathfrak{N}}(S^+_{r-1})_{(n_1,\dots,n_{r-1}+1,n_r-1)} \\ &\cong \underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\dots,n_{r-1}+1,n_r-1)}. \end{aligned}$$

By replacing n_{r-1} and n_r again with $n_{r-1}+n_r$ and 0 respectively, we get (3). To prove the last claim note that for $n_{r-1} \neq -1, n_r \neq 1$

$$\begin{aligned} \underline{H}^i_{\mathfrak{N}}(S^+_r)_{(n_1,\dots,n_{r-1},n_r)} &\cong \underline{H}^i_{\mathfrak{N}}(S)_{(n_1,\dots,n_{r-1}+1,n_r-1)} \\ &\cong \underline{H}^i_{\mathfrak{N}}(S^+_r)_{(n_1,\dots,n_{r-1}+1,n_r-1)} \end{aligned}$$

so that for $k \gg 0$

$$\begin{aligned} \underline{H}^i_{\mathfrak{N}}(S^+_r)_{(n_1,\dots,n_{r-1},0)} &\cong \underline{H}^i_{\mathfrak{N}}(S^+_r)_{(n_1,\dots,n_{r-1}+1,-1)} \\ &\cong \dots \\ &\cong \underline{H}^i_{\mathfrak{N}}(S^+_r)_{(n_1,\dots,n_{r-1}+k,-k)} \\ &= 0. \end{aligned}$$

2.2. Theorem. *Let R be a graded ring defined over a local ring and let $\mathfrak{M}$ be the homogeneous maximal ideal of R. Put $S = R^{r-gr}$ and $\mathfrak{N} = \mathfrak{M}^{r-gr}$. Then*

$$[\underline{H}^i_{\mathfrak{N}}(S)]_{\mathbf{n}} = \begin{cases} [\underline{H}^i_{\mathfrak{M}}(R)]_{|\mathbf{n}|} & \text{if } \mathbf{n} \geq \mathbf{0} \\ [\underline{H}^{i+1-r}_{\mathfrak{M}}(R)]_{|\mathbf{n}|} & \text{if } \mathbf{n} < \mathbf{0} \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We use induction on r. The case $r=1$ being trivial assume $r>1$. Observe that $S^{(r-1)-gr} = S/S^+_r$. Let $\mathbf{n} \in \mathbb{N}^r$ and consider the exact sequence

$$\begin{aligned} &[\underline{H}^{i-1}_{\mathfrak{N}}(S)]_{\mathbf{n}} \longrightarrow \\ &\quad \underline{H}^{i-1}_{\mathfrak{N}}(S/S^+_r)]_{\mathbf{n}} \longrightarrow [\underline{H}^i_{\mathfrak{N}}(S^+_r)]_{\mathbf{n}} \longrightarrow [\underline{H}^i_{\mathfrak{N}}(S)]_{\mathbf{n}} \longrightarrow [\underline{H}^i_{\mathfrak{N}}(S/S^+_r)]_{\mathbf{n}} \\ &\qquad \longrightarrow [\underline{H}^{i+1}_{\mathfrak{N}}(S^+_r)]_{\mathbf{n}} \end{aligned}$$

coming from the exact sequence

$$0 \longrightarrow S^+_r \longrightarrow S \longrightarrow S/S^+_r \longrightarrow 0.$$

If $\mathbf{n} \geq \mathbf{0}$, Lemma 2.1 first implies that

$$[\underline{H}^i_{\mathfrak{N}}(S)]_{\mathbf{n}} \cong [\underline{H}^i_{\mathfrak{N}}(S)]_{(n_1,\dots,n_{r-1}+n_r,0)}.$$

Moreover, we get

$$[\underline{H}^i_{\mathfrak{N}}(S_r^+)]_{(n_1,\dots,n_{r-1}+n_r,0)} = [\underline{H}^{i+1}_{\mathfrak{N}}(S_r^+)]_{(n_1,\dots,n_{r-1}+n_r,0)} = 0$$

so that there is an isomorphism

$$[\underline{H}^i_{\mathfrak{N}}(S)]_{(n_1,\dots,n_{r-1}+n_r,0)} \cong [\underline{H}^i_{\mathfrak{N}}(S/S_r^+)]_{(n_1,\dots,n_{r-1}+n_r,0)}.$$

The claim now follows from the induction hypothesis. Suppose then that $\mathbf{n} < \mathbf{0}$. We get

$$[\underline{H}^i_{\mathfrak{N}}(S)]_{\mathbf{n}} \cong [\underline{H}^i_{\mathfrak{N}}(S_r^+)]_{(n_1,\dots,n_{r-1}+n_r,0)},$$

by Lemma 2.1, (3),

$$\cong \underline{H}^{i-1}_{\mathfrak{N}}(S/S_r^+)]_{(n_1,\dots,n_{r-1}+n_r,0)},$$

since in the exact sequence mentioned above

$$[\underline{H}^i_{\mathfrak{N}}(S)]_{(n_1,\dots,n_{r-1}+n_r,0)} = [\underline{H}^{i-1}_{\mathfrak{N}}(S)]_{(n_1,\dots,n_{r-1}+n_r,0)} = 0$$

by Lemma 2.1, (2). This implies the claim by the induction hypothesis. The case, where some $n_j < 0$ and some $n_l \geq 0$ follows immediately from Lemma 2.1, (2).

We also need the following lemma (see [HHR, Theorem 2.2 and its proof]).

2.3. Lemma. *Let A be a local ring and let $I_1,\dots,I_r \subset A$ be ideals such that* $\operatorname{ht} I_j > 0$ *($j = 1,\dots,r$). Then* $\mathbf{a}(R_A(I_1,\dots,I_r)) = -\mathbf{1}$.

We now give a necessary and sufficient condition for the Cohen-Macaulayness of the multi-Rees algebra $R_A(\mathbf{I}_r^{\mathbf{k}})$.

2.4. Theorem. *Let A be a local ring of dimension d and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *Let $\mathbf{k} \in (\mathbb{N}^*)^r$. Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(I)$. Then $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Cohen-Macaulay if and only if $[\underline{H}^i_{\mathfrak{M}}(R_A(I)]_{\mathbf{k}\cdot\mathbf{n}} = 0$ for $i < d+1$ and $\mathbf{n} < \mathbf{0}$ or $\mathbf{n} \geq \mathbf{0}$.*

Proof. Since $R_A(\mathbf{I}_r^{\mathbf{k}}) = (R_A(\mathbf{I}_r))^{(\mathbf{k})}$ and $R_A(\mathbf{I}_r) = (R_A(I))^{r-gr}$, it follows from Theorem 2.2 that

$$[\underline{H}^i_{\mathfrak{N}}(R_A(\mathbf{I}_r^{\mathbf{k}}))]_{\mathbf{n}} = [\underline{H}^i_{\mathfrak{N}}(R_A(\mathbf{I}_r))]_{\mathbf{kn}} = \begin{cases} [\underline{H}^i_{\mathfrak{M}}(R_A(I))]_{\mathbf{k}\cdot\mathbf{n}} & \text{if } \mathbf{n} \geq \mathbf{0} \\ [\underline{H}^{i+1-r}_{\mathfrak{M}}(R_A(I))]_{\mathbf{k}\cdot\mathbf{n}} & \text{if } \mathbf{n} < \mathbf{0} \\ 0 & \text{otherwise}. \end{cases}$$

If $\underline{H}^i_{\mathfrak{N}}(R_A(\mathbf{I}_r^{\mathbf{k}})) = 0$ for $i < d+r$, this immediately implies $[\underline{H}^i_{\mathfrak{M}}(R_A(I)]_{\mathbf{k}\cdot\mathbf{n}} = 0$ for $i < d+1$ and $\mathbf{n} < \mathbf{0}$ or $\mathbf{n} \geq \mathbf{0}$. Since $a(R_A(I)) = -1$ by Lemma 2.3, we have $[\underline{H}^i_{\mathfrak{M}}(R_A(I))]_{\mathbf{k}\cdot\mathbf{n}} = 0$ for $i \geq d+1$ and $\mathbf{n} \geq \mathbf{0}$. If $[\underline{H}^i_{\mathfrak{M}}(R_A(I)]_{\mathbf{k}\cdot\mathbf{n}} = 0$ for $i < d+1$ and $\mathbf{n} < \mathbf{0}$ or $\mathbf{n} \geq \mathbf{0}$, we thus get $\underline{H}^i_{\mathfrak{N}}(R_A(\mathbf{I}_r^{\mathbf{k}})) = 0$ for $i < d+r$ and the claim has so been proved.

Theorem 2.4 immediately implies the following two corollaries.

2.5. Corollary. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *Let* $\mathbf{k} \in (\mathbf{N}^*)^r$. *Denote* $q = \gcd(k_1, \ldots, k_r)$. *If $R_A(I^q)$ is Cohen-Macaulay, also $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Cohen-Macaulay.*

2.6. Corollary. *Let A be a local ring of dimension d and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *Let* $\mathbf{k} \in (\mathbf{N}^*)^r$. *If $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Cohen-Macaulay, also $R_A(I^{|\mathbf{k}|})$ is Cohen-Macaulay.*

Also recall the following result proved in [HHR] (see [HHR, Theorem 2.2]):

2.7. Theorem. *Let A be a local ring of dimension d and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be the homogeneous maximal ideals of $R_A(I)$ and $R_A(\mathbf{I}_r)$ respectively. The following conditions are equivalent.*

(1) $R_A(\mathbf{I}_r)$ *is Cohen-Macaulay.*

(2) $[\underline{H}^i_{\mathfrak{N}}(gr_A(\mathbf{I}_r)]_{\mathbf{n}} = 0$ *when* $i < d + r - 1$ *and* $\mathbf{n} \neq -\mathbf{1}$,
$\mathbf{a}(gr_A(\mathbf{I}_r)) < \mathbf{0}$.

(3) $[\underline{H}^i_{\mathfrak{M}}(R_A(I)]_n = 0$ *when* $i < d+1$ *and* $n \notin \{-r+1, \ldots, -1\}$.

(4) $[\underline{H}^i_{\mathfrak{M}}(gr_A(I)]_n = 0$ *when* $i < d$ *and* $n \notin \{-r, \ldots, -1\}$,
$a(gr_A(I)) < 0$.

3. The Gorenstein property of the multi-Rees algebras

Let A be a local ring and $I \subset A$ an ideal of $\operatorname{grade} I > 0$. In this chapter we want to study the Gorenstein property of the multi-Rees algebra $R_A(\mathbf{I}_r^{\mathbf{k}})$, where $\mathbf{k} \in (\mathbf{N}^*)^r$. We begin with a series of lemmas.

3.1. Lemma. *Let A be a local ring and let $I_1, \ldots, I_r \subset A$ be ideals. Set $S = R_A(I_1, \ldots, I_r)$ and $J = I_1 \cdots I_r$. Then*

$$\underline{\operatorname{Hom}}_S(JS, S) = \bigoplus_{\mathbf{n} \in \mathbf{N}^r} \operatorname{Hom}_A(J, I_1^{n_1} \cdots I_r^{n_r}).$$

Proof. Suppose first that $f \in [\underline{\operatorname{Hom}}_S(JS, S)]_{\mathbf{n}}$. Then $f = (f_{\mathbf{m}})_{\mathbf{m} \in \mathbf{N}^r}$, where $f_{\mathbf{m}} \in \operatorname{Hom}_A(JS_{\mathbf{m}}, S_{\mathbf{m}+\mathbf{n}})$. If $a \in J$ and $b \in I_1^{m_1} \cdots I_r^{m_r}$, we have

$$f_{\mathbf{m}}((ab)t_1^{m_1} \cdots t_r^{m_r}) = (bt_1^{m_1} \cdots t_r^{m_r}) f_{\mathbf{0}}(a) = f_{\mathbf{0}}(ab) t_1^{m_1} \cdots t_r^{m_r} \in S_{\mathbf{m}+\mathbf{n}}.$$

It follows that $f_{\mathbf{m}}(ct_1^{m_1} \cdots t_r^{m_r}) = f_{\mathbf{0}}(c) t_1^{m_1} \cdots t_r^{m_r}$ for all $c \in I_1^{m_1+1} \cdots I_r^{m_r+1}$. This shows that f is uniquely determined by an element of $\operatorname{Hom}_A(JS_{\mathbf{0}}, S_{\mathbf{n}}) = \operatorname{Hom}_A(J, I_1^{n_1} \cdots I_r^{n_r})$ and the lemma follows.

3.2. Lemma. *Let A be a local ring and let $I_1, \ldots, I_r \subset A$ be ideals such that* $\operatorname{grade} I_j > 0$ $(j = 1, \ldots, r)$. *Set $S = R_A(I_1, \ldots, I_r)$ and $J = I_1 \cdots I_r$. If* $\operatorname{grade}(JS + S^{++}) > 1$, *we have* $[\underline{\operatorname{Ext}}^1_S(S/S^{++}, S)]_{\mathbf{n}} = 0$ *for* $\mathbf{n} \geq \mathbf{0}$.

Proof. Let $a_j \in I_j$ $(j = 1, \dots, r)$ be non-zero divisors. Denote $a = a_1 \cdots a_r$ and $s = at_1 \cdots t_r$. Multiplication by s then induces an exact sequence

$$0 \longrightarrow S(-\mathbf{1}) \longrightarrow S \longrightarrow S/sS \longrightarrow 0.$$

From this we get the exact sequence

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Hom}}_S(S/S^{++}, S(-\mathbf{1})) \longrightarrow \underline{\mathrm{Hom}}_S(S/S^{++}, S) \\ \longrightarrow \underline{\mathrm{Hom}}_S(S/S^{++}, S/sS) \longrightarrow \underline{\mathrm{Ext}}^1_S(S/S^{++}, S(-\mathbf{1})) \\ \longrightarrow \underline{\mathrm{Ext}}^1_S(S/S^{++}, S) \longrightarrow \dots \end{aligned}$$

Since $\underline{\mathrm{Hom}}_S(S/S^{++}, S) = 0$ and $\underline{\mathrm{Ext}}^1_S(S/S^{++}, S(-\mathbf{1})) \longrightarrow \underline{\mathrm{Ext}}^1_S(S/S^{++}, S)$ is a zero map, there is an isomorphism

$$\underline{\mathrm{Hom}}_S(S/S^{++}, S/sS) \cong \underline{\mathrm{Ext}}^1_S(S/S^{++}, S(-\mathbf{1})).$$

Because

$$\underline{\mathrm{Hom}}_S(S/S^{++}, S/sS) \cong sS : S^{++}/sS,$$

we get

$$[\underline{\mathrm{Ext}}^1_S(S/S^{++}, S)]_{\mathbf{n}} \cong [sS : S^{++}/sS]_{\mathbf{n}+\mathbf{1}}$$

for all $\mathbf{n}$.

We shall next show that for $\mathbf{n} \geq \mathbf{1}$ $[sS : S^{++}]_{\mathbf{n}} = [sS : (JS + S^{++})]_{\mathbf{n}}$. Let $ct_1^{n_1} \cdots t_r^{n_r} \in [sS : S^{++}]_{\mathbf{n}}$. Then

$$(Jt_1 \cdots t_r)(ct_1^{n_1} \cdots t_r^{n_r}) \subset (at_1 \cdots t_r)(I_1^{n_1} \cdots I_r^{n_r} t_1^{n_1} \cdots t_r^{n_r}).$$

This implies $Jc \subset aI_1^{n_1} \cdots I_r^{n_r} \subset aI_1^{n_1-1} \cdots I_r^{n_r-1}$ so that

$$J(ct_1^{n_1} \cdots t_r^{n_r}) \subset (at_1 \cdots t_r)(I_1^{n_1-1} \cdots I_r^{n_r-1} t_1^{n_1-1} \cdots t_r^{n_r-1}).$$

Hence $ct_1^{n_1} \cdots t_r^{n_r} \in [sS : (JS + S^{++})]_{\mathbf{n}}$ and the above claim has been proved.

If $\mathrm{grade}(JS + S^{++}) > 1$, we have $sS : (JS + S^{++}) = sS$ and get thus $[\underline{\mathrm{Ext}}^1_S(S/S^{++}, S)]_{\mathbf{n}} = 0$ for $\mathbf{n} \geq \mathbf{0}$ as wanted.

3.3. Lemma. *Let A be a local ring and let $I_1, \dots, I_r \subset A$ be ideals such that* $\mathrm{grade}\, I_j > 0$ $(j = 1, \dots, r)$. *Set $S = R_A(I_1, \dots, I_r)$ and $J = I_1 \cdots I_r$. If* $\mathrm{grade}(JS + S^{++}) > 1$, *the canonical homomorphism*

$$I_1^{n_1-1} \cdots I_r^{n_r-1} \longrightarrow \mathrm{Hom}_A(J, I_1^{n_1} \cdots I_r^{n_r})$$

is an isomorphism for all $\mathbf{n} \geq \mathbf{1}$.

Proof. By dualizing the exact sequence

$$0 \longrightarrow S^{++} \longrightarrow S \longrightarrow S/S^{++} \longrightarrow 0$$

by S one obtains the diagram

$$\begin{array}{ccccccccc} \underline{\mathrm{Hom}}_S(S/S^{++},S) & \longrightarrow & S & \longrightarrow & \underline{\mathrm{Hom}}_S(S^{++},S) & \longrightarrow & \underline{\mathrm{Ext}}^1_S(S/S^{++},S) & \longrightarrow & 0 \\ & & & & \downarrow \varrho & & & & \\ & & & & \underline{\mathrm{Hom}}_S(JS,S) & & & & , \end{array}$$

where ϱ is the degree $\mathbf{1}$ isomorphism induced by the isomorphism $JS \longrightarrow S^{++}$, $s \mapsto st_1 \cdots t_r$, $s \in JS$. Since $\operatorname{grade} J > 0$, we have $\underline{\mathrm{Hom}}_S(S/S^{++},S) = 0$. By Lemma 3.2 we know that $[\underline{\mathrm{Ext}}^1_S(S/S^{++},S)]_{\mathbf{n}} = 0$ for $\mathbf{n} \geq \mathbf{0}$. The diagram then implies that for $\mathbf{n} \geq \mathbf{1}$ there is an isomorphism $S_{\mathbf{n}-\mathbf{1}} \longrightarrow [\underline{\mathrm{Hom}}_S(JS,S)]_{\mathbf{n}}$. In this isomorphism $s \in S_{\mathbf{n}-\mathbf{1}}$ is mapped to the element $s' \mapsto (s't_1 \cdots t_r)s$, $s' \in JS$ of $[\underline{\mathrm{Hom}}_S(JS,S)]_{\mathbf{n}}$. Because

$$\underline{\mathrm{Hom}}_S(JS,S) = \bigoplus_{n \in \mathbb{N}^r} \mathrm{Hom}_A(J, I_1^{n_1} \cdots I_r^{n_r})$$

by Lemma 3.1, we get an isomorphism $I_1^{n_1-1} \cdots I_r^{n_r-1} \longrightarrow \mathrm{Hom}_A(J, I_1^{n_1} \cdots I_r^{n_r})$, which maps an element $a \in I_1^{n_1-1} \cdots I_r^{n_r-1}$ to the element $a' \mapsto aa'$, $a' \in J$ of $\mathrm{Hom}_A(J, I_1^{n_1} \cdots I_r^{n_r})$ as desired.

3.4. Lemma. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{grade} I > 0$. *Suppose that $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Cohen-Macaulay for some $\mathbf{k} \in (\mathbb{N}^*)^r$. Then*

(1) The canonical homomorphism $I^n : I^m \longrightarrow \mathrm{Hom}_A(I^m, I^n)$ is an isomorphism for $m \leq |\mathbf{k}| \leq n$.

(2) $I^{\mathbf{k}\cdot\mathbf{n}} : I^{|\mathbf{k}|} = I^{\mathbf{k}\cdot(\mathbf{n}-\mathbf{1})}$ for $\mathbf{n} \geq \mathbf{1}$.

Proof. We apply Lemma 3.3 with $I_j = I^{k_j}$ $(j = 1, \ldots, r)$. Since $S = R_A(\mathbf{I}_r^{\mathbf{k}})$ is Cohen-Macaulay, we have $\mathrm{ht}(JS + S_j^+) > 1$ $(j = 1, \ldots, r)$ and so also $\operatorname{grade}(JS + S^{++}) = \mathrm{ht}(JS + S^{++}) > 1$. It follows that the canonical homomorphism $I^{\mathbf{k}\cdot(\mathbf{n}-\mathbf{1})} \longrightarrow \mathrm{Hom}_A(I^{|\mathbf{k}|}, I^{\mathbf{k}\cdot\mathbf{n}})$ is an isomorphism for all $\mathbf{n} \geq \mathbf{1}$. We then observe that (2) is a consequence of (1). To prove (1) note first that in the case $\mathbf{n} = \mathbf{1}$ the above isomorphism gives an isomorphism $A \longrightarrow \mathrm{Hom}_A(I^{|\mathbf{k}|}, I^{|\mathbf{k}|})$. Consider the exact sequence

$$0 \longrightarrow I^{|\mathbf{k}|} \longrightarrow I^m \longrightarrow I^m/I^{|\mathbf{k}|} \longrightarrow 0.$$

Dualizing by I^n gives the sequence

$$\mathrm{Hom}_A(I^m/I^{|\mathbf{k}|}, I^n) \longrightarrow \mathrm{Hom}_A(I^m, I^n) \longrightarrow \mathrm{Hom}(I^{|\mathbf{k}|}, I^n).$$

Since $\operatorname{grade} I > 0$, we have $\operatorname{Hom}_A(I^m/I^{|\mathbf{k}|}, I^n) = 0$. Then

$$\operatorname{Hom}_A(I^m, I^n) \subset \operatorname{Hom}_A(I^{|\mathbf{k}|}, I^n) \subset \operatorname{Hom}_A(I^{|\mathbf{k}|}, I^{|\mathbf{k}|}) = A.$$

Hence we can make the identification

$$\operatorname{Hom}_A(I^m, I^n) = \{a \in A | aI^m \subset I^n\} = I^n : I^m$$

and the lemma has so been proved.

Let A be a ring and $I \subset A$ an ideal. Consider the so-called Ratliff-Rush ideal

$$I^* = \bigcup_{p>0} I^{p+1} : I^p$$

(see [Mc, Chapter 11]). Note that $I^{n+p} : I^p \subset I^{n+p+1} : I^{p+1}$ for all $n, p \geq 0$. When $\operatorname{grade} I > 0$, it is well-known that

$$I^{n*} = \bigcup_{p>0} I^{n+p} : I^p$$

for all $n \in \mathbf{N}$ (see [Mc, Proposition (11.1), (v)]).

3.5. Lemma. *Let A be a local ring and $I \subset A$ an ideal of $\operatorname{grade} I > 0$. Suppose that $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Cohen-Macaulay for some $\mathbf{k} \in (\mathbf{N}^*)^r$. If $\mathbf{n} \in \mathbf{N}^r$, let $\mathbf{s}(\mathbf{n}) = (s_1(n_1), \ldots, s_r(n_r)) \in \mathbf{N}^r$, where $n_j + s_j(n_j) \in k_j\mathbf{N}$, $0 \leq s_j(n_j) < k_j$ $(j = 1, \ldots, r)$. Then $I^{|\mathbf{n}|+p} : I^p = (I^{|\mathbf{n}|})^*$ for $p \geq |\mathbf{s}(\mathbf{n})|$.*

Proof. It is enough to prove that $I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|} : I^{|\mathbf{s}(\mathbf{n})|} \supset (I^{|\mathbf{n}|})^*$. We show that

$$I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|+q|k|} : I^{|\mathbf{s}(\mathbf{n})|+q|k|} \subset I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|} : I^{|\mathbf{s}(\mathbf{n})|}$$

for all $q > 0$. If $a \in I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|+q|\mathbf{k}|} : I^{|\mathbf{s}(\mathbf{n})|+q|k|}$, we obtain

$$aI^{|\mathbf{s}(\mathbf{n})|} \subset I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|+q|\mathbf{k}|} : I^{q|\mathbf{k}|}.$$

Now $\mathbf{n} + \mathbf{s}(\mathbf{n}) = \mathbf{km}$ for some $\mathbf{m} \in \mathbf{N}^r$. Since $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Cohen-Macaulay, it follows from Lemma 3.4 that

$$I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|+q|\mathbf{k}|} : I^{q|\mathbf{k}|} = I^{\mathbf{k}\cdot(\mathbf{m}+q\mathbf{1})} : I^{q|\mathbf{k}|} = I^{\mathbf{k}\cdot\mathbf{m}} = I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|}.$$

Therefore $a \in I^{|\mathbf{n}|+|\mathbf{s}(\mathbf{n})|} : I^{|\mathbf{s}(\mathbf{n})|}$ and the claim has been proved.

We are now ready to consider the Gorensteiness of $R_A(I^q)$ $(q \in \mathbf{N}^*)$.

3.6. Proposition. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *Set $S = R_A(I)$. Let $q \in \mathbf{N}^*$. If $n \in \mathbf{N}$, let $r(n), s(n) \in \mathbf{N}$ be the numbers determined by the conditions $n + s(n) = r(n)q$, $0 \leq s(n) < q$. Denote $T = R_A(I^q)$. If T is Cohen-Macaulay and has a canonical module, also S has a canonical module and*

$$[\omega_S]_n = [\underline{\operatorname{Hom}}_T(I^{s(n)}T, \omega_T)]_{r(n)}$$

for all $n \geq 1$. If $a \in I^m$ and $\varphi \in [\omega_S]_n$,

$$((at^m)\varphi)(bt^k) = \varphi(abt^{k+r(n+m)-r(n)})$$

for $b \in I^{s(m+n)+kq}$, $k \in \mathbf{N}$. Moreover,

$$[\underline{\operatorname{Hom}}(IS, \omega_S)]_n = [\underline{\operatorname{Hom}}_T(I^{s(n)+1}T, \omega_T)]_{r(n)}.$$

Proof. Put $U = A[I^q t^q]$ and denote $U^p = (It)^p U$ for $0 \leq p < q$. First observe that U is a subring of S and S is a finite U-module. In fact, as a U-module

$$S = \bigoplus_{p=0}^{q-1} U^p.$$

If $\varphi: \mathbf{Z} \longrightarrow \mathbf{Z}$ is the map $n \mapsto qn$, $n \in \mathbf{Z}$, we have $U = T^\varphi$ and $U^p = (I^p T)^\varphi(-p)$. Since T and U are isomorphic as rings, also U is Cohen-Macaulay and so

$$\omega_S = \underline{\operatorname{Hom}}_U(S, \omega_U) = \bigoplus_{p=0}^{q-1} \underline{\operatorname{Hom}}_U(U^p, \omega_U).$$

By Lemma 1.3 we know that $\omega_U = (\omega_T)^\varphi$. Then

$$\begin{aligned}[\underline{\operatorname{Hom}}_U(U^p, \omega_U)]_n &= [\underline{\operatorname{Hom}}_{T^\varphi}((I^p T)^\varphi, (\omega_T)^\varphi)]_{n+p} \\ &= [(\underline{\operatorname{Hom}}_T(I^p T, \omega_T))^\varphi]_{n+p}.\end{aligned}$$

It follows that $[\underline{\operatorname{Hom}}_U(U^p, \omega_U)]_n = 0$ if $p \neq s(n)$. So

$$[\omega_S]_n = [\underline{\operatorname{Hom}}_U(U^{s(n)}, \omega_U)]_n = [\underline{\operatorname{Hom}}_T(I^{s(n)}T, \omega_T)]_{r(n)}.$$

If $a \in I^m$, $b \in I^{s(n+m)+kq}$ and $\varphi \in [\omega_S]_n$, the claim concerning $((at^m)\varphi)(bt^k)$ follows easily from the observation that we can write

$$m + s(m+n) + kq = s(n) + kq + (r(n+m) - r(n))q.$$

Since

$$IS = \bigoplus_{p=0}^{q-1} IU^p$$

and

$$\underline{\operatorname{Hom}}_S(IS, \omega_S) = \underline{\operatorname{Hom}}_S(IS, \underline{\operatorname{Hom}}_U(S, \omega_U)) = \underline{\operatorname{Hom}}_U(IS, \omega_U),$$

it follows in a similar way that

$$[\underline{\operatorname{Hom}}(IS, \omega_S)]_n = [\underline{\operatorname{Hom}}_U(IU^{s(n)}, \omega_U)]_n = [\underline{\operatorname{Hom}}_T(I^{s(n)+1}T, \omega_T)]_{r(n)}.$$

In the proof of the following lemma we use the ideas presented in the proof of [Z, Proposition (1.1)].

3.7. Lemma. *Let A be a complete local ring of dimension d and $I \subset A$ an ideal of $\operatorname{ht} I > 0$. Set $S = R_A(I)$ and $G = gr_A(I)$. Let $\mathfrak{M}$ be the homogeneous maximal ideal of S. There exists a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_S & \longrightarrow & \underline{\operatorname{Hom}}_S(S^+,\omega_S) & \longrightarrow & \omega_A & \longrightarrow & \underline{\operatorname{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_S(k)) \\
 & & \uparrow \tau & & \downarrow \varrho & & & & \\
0 & \longrightarrow & \omega_S & \longrightarrow & \underline{\operatorname{Hom}}_S(IS,\omega_S) & \longrightarrow & \omega_G & \longrightarrow & \underline{\operatorname{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_S(k)),
\end{array}$$

where τ is a homomorphism of degree -1 and ϱ is the degree 1 isomorphism induced by the isomorphism $IS \longrightarrow S^+$, $s \mapsto st$, $s \in IS$. Moreover, if $n > 1$, the homomorphisms $\tau_n\colon [\omega_S]_n \longrightarrow [\omega_S]_{n-1}$ are injective and for every $\alpha \in [\omega_S]_n$ $\tau(\alpha)$ is uniquely determined by the property that $(ct)\tau(\alpha) = c\alpha$ for all $c \in I$.

Proof. Set $T = R_A(I^q)$ and $U = A[I^q t^q]$. Since U is isomorphic with T as ring, also U is Cohen-Macaulay. Moreover, U is a subring of S over which S is finitely generated. By dualizing the exact sequences

$$0 \longrightarrow S^+ \longrightarrow S \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow IS \longrightarrow S \longrightarrow G \longrightarrow 0$$

by ω_U we obtain the exact sequences of S-modules

$$\underline{\operatorname{Hom}}_U(A,\omega_U) \longrightarrow \underline{\operatorname{Hom}}_U(S,\omega_U) \longrightarrow \underline{\operatorname{Hom}}_U(S^+,\omega_U) \longrightarrow \underline{\operatorname{Ext}}^1_U(A,\omega_U) \longrightarrow \underline{\operatorname{Ext}}^1_U(S,\omega_U),$$

$$\underline{\operatorname{Hom}}_U(G,\omega_U) \longrightarrow \underline{\operatorname{Hom}}_U(S,\omega_U) \longrightarrow \underline{\operatorname{Hom}}_U(IS,\omega_U) \longrightarrow \underline{\operatorname{Ext}}^1_U(G,\omega_U) \longrightarrow \underline{\operatorname{Ext}}^1_U(S,\omega_U).$$

Now $\dim G = \dim A = d$, but $\dim U = \dim S = d+1$. Let $\mathfrak{M}$ be the maximal ideal of S. By local duality we get

$$\underline{\operatorname{Hom}}_U(A,\omega_U) = \underline{\operatorname{Hom}}_U(\underline{H}^{d+1}_{\mathfrak{M}}(A),\underline{E}_U(k)) = 0$$

and

$$\underline{\operatorname{Hom}}_U(G,\omega_U) = \underline{\operatorname{Hom}}_U(\underline{H}^{d+1}_{\mathfrak{M}}(G),\underline{E}_U(k)) = 0.$$

Since $\underline{E}_S(k) = \underline{\operatorname{Hom}}_U(S,\underline{E}_U(k))$, it follows that

$$\underline{\operatorname{Ext}}^1_U(S,\omega_U) = \underline{\operatorname{Hom}}_U(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_U(k)) = \underline{\operatorname{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_S(k)).$$

We also have $\omega_S = \underline{\mathrm{Hom}}_U(S,\omega_U)$, $\omega_A = \mathrm{Ext}^1_U(A,\omega_U)$, $\omega_G = \mathrm{Ext}^1_U(G,\omega_U)$ and

$$\underline{\mathrm{Hom}}_S(S^+,\omega_S) = \underline{\mathrm{Hom}}_S(S^+,\underline{\mathrm{Hom}}_U(S,\omega_U)) = \underline{\mathrm{Hom}}_U(S^+,\omega_U),$$

$$\underline{\mathrm{Hom}}_S(IS,\omega_S) = \underline{\mathrm{Hom}}_S(IS,\underline{\mathrm{Hom}}_U(S,\omega_U)) = \underline{\mathrm{Hom}}_U(IS,\omega_U).$$

We thus get the exact sequences

$$0 \longrightarrow \omega_S \longrightarrow \underline{\mathrm{Hom}}_S(S^+,\omega_S) \longrightarrow \omega_A \longrightarrow \underline{\mathrm{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_S(k))$$

and

$$0 \longrightarrow \omega_S \longrightarrow \underline{\mathrm{Hom}}_S(IS,\omega_S) \longrightarrow \omega_G \longrightarrow \underline{\mathrm{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_S(k)).$$

Since $[\omega_A]_n = 0$ if $n \geq 1$, the map $[\omega_S]_n \longrightarrow [\underline{\mathrm{Hom}}_S(S^+,\omega_S)]_n$ is an isomorphism for $n \geq 1$. Because $a(S) = -1$, $[\omega_S]_n = 0$ for $n \leq 0$. By means of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_S & \longrightarrow & \underline{\mathrm{Hom}}_S(S^+,\omega_S) & \longrightarrow & \omega_A & \longrightarrow & \underline{\mathrm{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_S(k)) \\ & & & & \big\downarrow \varrho & & & & \\ 0 & \longrightarrow & \omega_S & \longrightarrow & \underline{\mathrm{Hom}}_S(IS,\omega_S) & \longrightarrow & \omega_G & \longrightarrow & \underline{\mathrm{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S),\underline{E}_S(k)), \end{array}$$

we can now define a degree -1 homomorphism $\tau : \omega_S \longrightarrow \omega_S$ such that the diagram commutes. It follows easily from the definition of τ that $(ct)\tau(\alpha) = \tau((ct)\alpha) = c\alpha$ for all $c \in I$. To see that this property uniquely determines $\tau(\alpha)$ assume that $\beta \in \omega_S$ and $ct\beta = c\alpha$ for all $c \in I$. Then $ct(\tau(\alpha) - \beta) = 0$ for all $c \in I$, which implies that $S^+ \subset \mathrm{Ann}(\tau(\alpha) - \beta)$. If $\tau(\alpha) \neq \beta$, there would now exist $P \in \mathrm{Ass}\,\omega_S$ such that $S^+ \subset P$. We would then have $\dim S/P < \dim S$, which is impossible, since $\dim S/P = \dim S$ for all $P \in \mathrm{Ass}\,\omega_S$. So $\tau(\alpha) = \beta$ and the lemma has been proved.

Let A be a ring and $I \subset A$ an ideal of grade $I > 0$. Consider the Ratliff-Rush ideals I^{n*} $(n \in \mathbb{N})$. By [Mc, Proposition (11.1), (vi)] $I^{n*}I^{m*} \subset I^{n+m*}$ for all $m, n \in \mathbb{N}$. The ideals I^{n*} $(n \in \mathbb{N})$ so define a filtration

$$A \supset I^* \supset I^{2*} \supset I^{3*} \supset \ldots.$$

Let

$$R^*_A(I) = \bigoplus_{n\in\mathbb{N}} I^{n*} \quad \text{and} \quad gr^*_A(I) = \bigoplus_{n\in\mathbb{N}} I^{n*}/I^{n+1*}$$

denote the corresponding Rees-algebra and the associated graded ring respectively. By [Mc, Theorem (12.3)] we have $I^{n*} = I^n$ for all $n \in \mathbb{N}$ if and only if $\mathrm{grade}(gr_A(I))^+ > 0$. We thus get $R^*_A(I) = R_A(I)$ and $gr^*_A(I) = gr_A(I)$ if and only if $\mathrm{grade}(gr_A(I))^+ > 0$. By [V, p. 157] this is the case, for example, when $R_A(I)$ has the property (S_2). It is also useful to note the following simple lemma.

3.8. Lemma. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{grade} I > 0$. *Then $gr_A^*(I) \cong gr_A(I)$ if and only if* $\operatorname{grade}(gr_A(I))^+ > 0$.

Proof. Suppose $gr_A^*(I) \cong gr_A(I)$. There exists then for every $n \in \mathbb{N}$ an isomorphism $I^n/I^{n+1} \longrightarrow I^{n*}/I^{n+1*}$. From the isomorphism $A/I \longrightarrow A/I^*$ we get $I = I^*$. It follows by induction on n that $I^n = I^{n*}$ for every $n \in \mathbb{N}$. As mentioned above this is now equivalent to $\operatorname{grade}(gr_A(I))^+ > 0$.

We denote $I^{-n} = \operatorname{Hom}_A(I^n, A)$ for $n > 0$. For all $n \in \mathbb{N}$ there is the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(I^n/I^{n+1}, A) \longrightarrow \operatorname{Hom}_A(I^n, A) \longrightarrow \operatorname{Hom}_A(I^{n+1}, A)$$

coming from the sequence

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow I^n/I^{n+1} \longrightarrow 0.$$

Since $\operatorname{grade} I > 0$, we have $\operatorname{Hom}_A(I^n/I^{n+1}, A) = 0$ so that there is a monomorphism

$$0 \longrightarrow I^{-n} \longrightarrow I^{-(n+1)}.$$

By means of this monomorphism we shall consider I^{-n} as an A-submodule of $I^{-(n+1)}$.

3.9. Theorem. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{grade} I > 0$. *Set $S = R_A(I)$ and $G = gr_A(I)$. Suppose that $R_A(I^q)$ is Gorenstein for some $q \in \mathbb{N}^*$. Then*

(1) There exists an exact sequence of graded S-modules

$$0 \longrightarrow R_A^*(I)(-q) \longrightarrow \omega_S \longrightarrow \bigoplus_{n=1}^{q-1} I^{n-q} \longrightarrow 0.$$

(2) $\omega_A \cong I^{-q}$

(3) Set $d = \dim A$ and let $\mathfrak{M}$ be the homogeneous maximal ideal of S. If $\underline{H}_{\mathfrak{M}}^d(S) = 0$, we also have an exact sequence of graded S-modules

$$0 \longrightarrow gr_A^*(I)(-(q+1)) \longrightarrow \omega_G \longrightarrow \bigoplus_{n=1}^{q} I^{n-q-1}/I^{n-q} \longrightarrow 0.$$

Proof. We may assume that A is complete. Lemma 3.7 implies the existence of the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \omega_S & \longrightarrow & \underline{\operatorname{Hom}}_S(S^+, \omega_S) & \longrightarrow & \omega_A & \longrightarrow & \underline{\operatorname{Hom}}_S(\underline{H}_{\mathfrak{M}}^d(S), \underline{E}_S(k)) \\ & & \big\downarrow \varrho & & & & \\ 0 \longrightarrow \omega_S & \longrightarrow & \underline{\operatorname{Hom}}_S(IS, \omega_S) & \longrightarrow & \omega_G & \longrightarrow & \underline{\operatorname{Hom}}_S(\underline{H}_{\mathfrak{M}}^d(S), \underline{E}_S(k)), \end{array}$$

where ϱ is an isomorphism of degree 1. Put $T = R_A(I^q)$. Since T is Gorenstein, $\omega_T = T(-1)$. By Lemma 3.1 it follows from Proposition 3.6 that

$$\begin{aligned}\omega_S &= \bigoplus_{n\geq 1} [\underline{\mathrm{Hom}}_T(I^{s(n)}T, \omega_T)]_{r(n)} \\ &= \bigoplus_{n\geq 1} [\underline{\mathrm{Hom}}_T(I^{s(n)}T, T)]_{r(n)-1} \\ &= \bigoplus_{n\geq 1} \mathrm{Hom}_A(I^{s(n)}, I^{s(n)+n-q}).\end{aligned}$$

Similarly

$$\underline{\mathrm{Hom}}_S(IS, \omega_S) = \bigoplus_{n\geq 1} \mathrm{Hom}_A(I^{s(n)+1}, I^{s(n)+n-q}).$$

If $1 \leq n \leq q$, $s(n) = q - n$ and so

$$\mathrm{Hom}_A(I^{s(n)}, I^{s(n)+n-q}) = \mathrm{Hom}_A(I^{q-n}, A) = I^{n-q},$$

$$\mathrm{Hom}_A(I^{s(n)+1}, I^{s(n)+n-q}) = \mathrm{Hom}_A(I^{q-n+1}, A) = I^{n-q-1}.$$

If $n > q$, $s(n) + n - q \geq q$ and it follows from Lemmas 3.4 and 3.5 that

$$\mathrm{Hom}_A(I^{s(n)}, I^{s(n)+n-q}) = I^{s(n)+n-q} : I^{s(n)} = (I^{n-q})^*,$$

$$\mathrm{Hom}_A(I^{s(n)+1}, I^{s(n)+n-q}) = I^{s(n)+n-q} : I^{s(n)+1} = (I^{n-q-1})^*.$$

The claim (1) is now immediate. Since T is Cohen-Macaulay, we have $[\underline{H}^d_{\mathfrak{M}}(S)]_0 = [\underline{H}^d_{\mathfrak{M}}(T)]_0 = 0$. Then

$$[\underline{\mathrm{Hom}}_S(\underline{H}^d_{\mathfrak{M}}(S), \underline{E}_S(k))]_0 = \mathrm{Hom}_A([\underline{H}^d_{\mathfrak{M}}(S)]_0, E_A(k)) = 0.$$

Since also $[\omega_S]_0 = 0$, the diagram implies $\omega_A \cong [\underline{\mathrm{Hom}}_S(IS, \omega_S)]_1 = \mathrm{Hom}_A(I^q, A)$ so that (2) has been proved. To prove (3) observe that in degrees $1 \leq n \leq q$ one can identify the second row of the diagram with the sequence

$$0 \longrightarrow I^{n-q} \longrightarrow I^{n-q-1} \longrightarrow I^{n-q-1}/I^{n-q} \longrightarrow 0.$$

In the case $n \leq q$, we thus get that

$$[\omega_G]_n = I^{n-q-1}/I^{n-q}.$$

If $n > q$, we have

$$[\omega_G]_n = [\underline{\mathrm{Hom}}_S(IS, S)]_n / [\omega_S]_n = (I^{n-q})^* / (I^{n-q-1})^*$$

and the claim has so been proved.

Suppose that $R_A(I^q)$ is Gorenstein. In [HRZ] the a-invariant of $gr_A(I)$ was computed in the case $\operatorname{grade} I > 1$ if $R_A(I)$ was Cohen-Macaulay. Now using Theorem 3.9 we can generalize this result as follows.

3.10. Corollary. *Let A be a local ring of dimension d and $I \subset A$ an ideal of $\operatorname{grade} I > 0$. Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(I)$ and assume that $\underline{H}^d_{\mathfrak{M}}(R_A(I)) = 0$. Suppose that $R_A(I^q)$ is Gorenstein for some $q \in \mathbf{N}^*$. If $\operatorname{grade} I = 1$, we have $a(gr_A(I)) > -(q+1)$, and if $\operatorname{grade} I > 1$, $a(gr_A(I)) = -(q+1)$.*

Proof. Set $G = gr_A(I)$. Assume first that $\operatorname{grade} I = 1$. Then $I^{-1} \neq A$. According to Theorem 3.9 we then have $[\omega_G]_q \neq 0$ so that $a(G) > -(q+1)$. In the case $\operatorname{grade} I > 1$, we obtain $[\omega_G]_n = I^{n-q}/I^{n-q-1} = 0$ for $1 \leq n \leq q$, but $[\omega_G]_{q+1} = A/I^* \neq 0$, which means that $a(G) = -(q+1)$.

We want to show next that if $\operatorname{grade} I > 1$, the conditions mentioned in Theorem 3.9 are also sufficient for the Gorensteiness of $R_A(I^q)$.

3.11. Lemma. *Let A be a complete local ring of dimension d and $I \subset A$ an ideal of $\operatorname{grade} I > 0$. Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(I)$ and assume that $\underline{H}^d_{\mathfrak{M}}(R_A(I)) = 0$. Let $q \in \mathbf{N}^*$. Suppose that $R_A(I^q)$ is Cohen-Macaulay. Set $S = R_A(I)$, $G = gr_A(I)$. If $[\omega_S]_q \cong A$ and if there exists an isomorphism*

$$gr^*_A(I)(-(q+1)) \longrightarrow \bigoplus_{n \geq q+1} [\omega_G]_n,$$

then we have an isomorphism

$$R^*_A(I)(-q) \longrightarrow \bigoplus_{n \geq q} [\omega_S]_n$$

of graded S-modules.

Proof. Let $\tau: \omega_S \longrightarrow \omega_S(-1)$ be the homomorphism of Lemma 3.7. We first define homomorphisms $\psi_n: (I^{n-q})^* \longrightarrow [\omega_S]_n$ $(n \geq q)$ so that the diagram

$$\begin{array}{ccc} [\omega_S]_n & \xrightarrow{\tau} & [\omega_S]_{n-1} \\ \uparrow \psi_n & & \uparrow \psi_{n-1} \\ (I^{n-q})^* & \longrightarrow & (I^{n-q-1})^* \end{array}$$

commutes. By assumption we can find an isomorphism $\psi_q: A \longrightarrow [\omega_S]_q$. According to Lemma 3.6 we can identify $[\omega_S]_n$ with

$$[\underline{\operatorname{Hom}}_T(I^{s(n)}T, \omega_T)]_{r(n)},$$

where $s(n), r(n) \in \mathbb{N}$ and $n + s(n) = r(n)q$, $0 \le s(n) < q$. Set $\xi = \psi_q(1) \in [\omega_S]_q$. Let $a \in (I^{n-q})^*$. To define $\psi_n(a)$ we need a homomorphism $I^{s(n)}T \longrightarrow \omega_T$ of degree $r(n)$. By Lemma 3.5 we have $(I^{n-q})^* = I^{s(n)+n-q} : I^{s(n)}$. Since $aI^{s(n)} \subset I^{s(n)+n-q}$ and ξ is a homomorphism $T \longrightarrow \omega_T$ of degree 1 , we can define $\psi_n(a)$ by setting

$$\psi_n(a)(bt^k) = \xi(abt^{k+r(n)-1}) \quad (b \in I^{s(n)+qk}, k \in \mathbb{N}).$$

The definition now implies easily that $\psi_n(ca) = ct\psi_{n-1}(a)$ for $a \in (I^{n-q-1})^*, c \in I$. By Lemma 3.7 this means that $\psi_{n-1}(a) = \tau(\psi_n(a))$. So the diagram commutes. Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\omega_S]_n & \longrightarrow & [\omega_S]_{n-1} & \longrightarrow & [\omega_G]_n & \longrightarrow & 0 \\ & & \uparrow \psi_n & & \uparrow \psi_{n-1} & & & & \\ 0 & \longrightarrow & (I^{n-q})^* & \longrightarrow & (I^{n-q-1})^* & \longrightarrow & (I^{n-q-1})^*/(I^{n-q})^* & \longrightarrow & 0 \end{array}$$

By the induction hypothesis ψ_{n-1} is an isomorphism. It follows that the induced homomorphism $(I^{n-q-1})^*/(I^{n-q})^* \longrightarrow [\omega_G]_n$ is an epimorphism. Since there by assumption exists an isomorphism $(I^{n-q-1})^*/(I^{n-q})^* \longrightarrow [\omega_G]_n$, this epimorphism must be an isomorphism. By the five-lemma ψ_n is an isomorphism. Since $\psi_{n+m}(ca) = ct^m\psi_n(a)$ for all $c \in I^m$, $a \in (I^{n-q})^*$ and $n \ge q$, the isomorphisms ψ_n now induce a S-linear isomorphism

$$\bigoplus_{n \ge q} (I^{n-q})^* \longrightarrow \bigoplus_{n \ge q} [\omega_S]_n$$

and the lemma has so been proved.

3.12 Theorem. *Let A be a local ring of dimension d and $I \subset A$ an ideal of* grade $I > 1$. *Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(I)$. Suppose $\underline{H}^d_{\mathfrak{M}}(R_A(I)) = 0$. Set $G = gr_A(I)$. Let $q \in \mathbb{N}^*$. Then $R_A(I^q)$ is Gorenstein if and only if the following conditions are satisfied*

(1) $R_A(I^q)$ is Cohen-Macaulay

(2) $\omega_A \cong A$

(3) $\omega_G \cong gr^*_A(I)(-(q+1))$.

If grade $G^+ > 0$, *condition (3) is equivalent with the condition*

(3') $\omega_G \cong gr_A(I)(-(q+1))$.

Proof. We may assume that A is complete. Put $S = R_A(I)$, $T = R_A(I^q)$. Since grade $I > 1$, we have $I^{n-q} = I^{n-q-1}$ for $1 \le n \le q$. According to Theorem 3.9 the Gorensteineness of T implies the conditions (1)–(3). Suppose then that the conditions (1)–(3) hold. Then $[\omega_G]_n = 0$ for $1 \le n \le q$ so that by the diagram of

Lemma 3.7 we get $[\omega_S]_q \cong [\omega_S]_{q-1} \cong \ldots \cong [\omega_S]_1 \cong \omega_A \cong A$. By Lemma 3.11 this implies that

$$\bigoplus_{n\geq q} [\omega_S]_n = \bigoplus_{n\geq q} (I^{n-q})^*.$$

Now $\omega_T = (\omega_S)^{(q)}$. Since T is Cohen-Macaulay, $(I^{q(n-1)})^* = I^{q(n-1)}$ (see p. 15). We get $\omega_T = T(-1)$ so that T is Gorenstein as wanted. The equivalence of (3) and (3') follows from the fact that grade $G^+ > 0$ implies $G = gr_A^*(I)$ (see p. 15).

3.13. Example. ([HRS]) Let k be a field. Consider the ring

$$A = k[[X_1, \ldots, X_{11}]]/(X_1^2) =: k[[x_1, \ldots, x_{11}]],$$

where $k[[X_1, \ldots, X_{11}]]$ is the formal power series ring over k. The ring A is a hypersurface ring of multiplicity 2 and dimension 10. Let I denote the ideal generated by all monomials of degree 4 in $x_2, \ldots, x_{11}$ different from $x_2^2x_3^2$. Let m be the maximal ideal of A. We now have $I^2 = m^8$. Because $R_A(m)$ is Cohen-Macaulay and $a(G(m)) = -9$, we know by Theorem 3.12 that $R(I^2) = R(m^8)$ is Gorenstein. We shall show that in this case $gr_A(I)$ is not even quasi-Gorenstein. Set $S = R_A(I)$ and $G = gr_A(I)$. One now easily sees that there exists a short exact sequence

$$0 \longrightarrow S \longrightarrow R_A(m^4) \longrightarrow kx_2^2x_3^2(-1) \longrightarrow 0.$$

Let $\mathfrak{M}$ be the homogeneous maximal ideal of S. The corresponding cohomology sequence now implies $\underline{H}^i_{\mathfrak{M}}(S) = 0$ for $i \neq 1, 11$, but $\underline{H}^1_{\mathfrak{M}}(S) \cong k(-1)$. It then follows from the cohomolygy sequences corresponding to the short exact sequences

$$0 \longrightarrow S^+ \longrightarrow S \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow S^+(1) \longrightarrow S \longrightarrow G \longrightarrow 0$$

that $[\underline{H}^0_{\mathfrak{M}}(G)]_0 = [\underline{H}^1_{\mathfrak{M}}(S^+)]_1 = [\underline{H}^1_{\mathfrak{M}}(S)]_1 \neq 0$. Thus grade $G^+ = \operatorname{depth} G = 0$. Because $\omega_G \cong gr_A^*(I)(-3)$ by Theorem 3.12, it follows from Lemma 3.8 that $\omega_G \ncong G(-3)$.

3.14. Corollary. *Let A be a local Cohen-Macaulay ring and $I \subset A$ an ideal of* $\operatorname{ht} I > 0$. *Suppose that $R_A(I^q)$ is Gorenstein for some $q \in \mathbb{N}^*$. Then $gr_A(I)$ is Gorenstein if and only if either I is principal or* $\operatorname{ht} I > 1$ *and $gr_A(I)$ is Cohen-Macaulay.*

Proof. Denote $G = gr_A(I)$ and $a = a(G)$. Observe first that the assumptions in any case imply by [HRZ, Corollary (2.7)] that $R_A(I)$ is Cohen-Macaulay. Assume now that G is Gorenstein and $\operatorname{ht} I = 1$. Then $\omega_G = G(a)$. By Corollary 3.10 we have $a > -(q+1)$. Because G is Cohen-Macaulay, $gr_A^*(I) = G$. By Theorem 3.9 we then get $[\omega_G]_{q+1} = G_0$. So $I^{q+1+a}/I^{q+2+a} \cong A/I$, which implies that I^{q+1+a} is principal. Since $\operatorname{ht} I > 0$, it follows from [S, Proposition 1] that

I is principal. If $\operatorname{ht} I > 1$ and G is Cohen-Macaulay, [HRS, Theorem (2.3)] (or Theorem 3.12) implies that $\omega_G = G(-(q+1))$ so that G is Gorenstein.

We shall now consider the Gorensteiness of $R_A(\mathbf{I}_r^{\mathbf{k}})$ ($\mathbf{k} \in (\mathbf{N}^*)^r$). We have the following general result about the canonical module of an r-graded ring corresponding a graded ring.

3.15. Proposition *Let R be a graded ring defined over a local ring. Set $S = R^{r-gr}$. Suppose that $\dim S = \dim R + r - 1$. If R has a canonical module, then so does S and we have*

$$\omega_S = \bigoplus_{\mathbf{n}>\mathbf{0}} [\omega_R]_{|\mathbf{n}|}.$$

Proof. Set $d = \dim R$, so that $\dim R^{r-gr} = d + r - 1$. Denote $A = S_0$ and let $\mathfrak{M}$ be the homogeneous maximal ideal of R. Set $\mathfrak{N} = \mathfrak{M}^{r-gr}$. It follows from Theorem 2.2 that $[\underline{H}_{\mathfrak{N}}^{d+r-1}(S)]_{\mathbf{n}} = [\underline{H}_{\mathfrak{M}}^{d}(R)]_{|\mathbf{n}|}$ if $\mathbf{n} < \mathbf{0}$ and 0 otherwise. Then

$$\begin{aligned}
\omega_S &= \underline{\operatorname{Hom}}_S(\underline{H}_{\mathfrak{N}}^{d+r-1}(S), \underline{E}_S(k)) \\
&= \underline{\operatorname{Hom}}_A(\underline{H}_{\mathfrak{N}}^{d+r-1}(S), E_A(k)) \\
&= \bigoplus_{\mathbf{n}\in\mathbf{Z}^r} \operatorname{Hom}_A([\underline{H}_{\mathfrak{M}^{r-gr}}^{d+r-1}(S)]_{-\mathbf{n}}, E_A(k)) \\
&= \bigoplus_{\mathbf{n}>\mathbf{0}} \operatorname{Hom}_A([\underline{H}_{\mathfrak{M}}^{d}(R)]_{-|\mathbf{n}|}, E_A(k)) \\
&= \bigoplus_{\mathbf{n}>\mathbf{0}} [\underline{\operatorname{Hom}}_A(\underline{H}_{\mathfrak{M}}^{d}(R), E_A(k))]_{|\mathbf{n}|} \\
&= \bigoplus_{\mathbf{n}>\mathbf{0}} [\underline{\operatorname{Hom}}_R(\underline{H}_{\mathfrak{M}}^{d}(R), \underline{E}_R(k))]_{|\mathbf{n}|} \\
&= \bigoplus_{\mathbf{n}>\mathbf{0}} [\omega_R]_{|\mathbf{n}|}.
\end{aligned}$$

3.16. Theorem. *Let A be a local ring and $I \subset A$ an ideal of* $\operatorname{grade} I > 0$. *Let $\mathbf{k} \in (\mathbf{N}^*)^r$. Then $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein if and only if it is Cohen-Macaulay and $R_A(I^{|\mathbf{k}|})$ is Gorenstein.*

Proof. Denote $R = R_A(I)$, $S = R_A(\mathbf{I}_r)$ and $S' = R_A(\mathbf{I}_r^{\mathbf{k}})$, $R' = R_A(I^{|\mathbf{k}|})$.

Assume first that S' is Gorenstein. We then know by Lemma 2.3 that $\omega_{S'} = S'(-\mathbf{1})$. By Proposition 3.15 we have

$$\omega_S = \bigoplus_{\mathbf{n}>\mathbf{0}} [\omega_R]_{|\mathbf{n}|}.$$

Since $\omega_{S'} = (\omega_S)^{(\mathbf{k})}$, this implies

$$S'(-\mathbf{1}) = \bigoplus_{\mathbf{n}>\mathbf{0}} I^{\mathbf{k}\cdot(\mathbf{n}-\mathbf{1})} = \bigoplus_{\mathbf{n}>\mathbf{0}} [\omega_R]_{\mathbf{k}\cdot\mathbf{n}}.$$

Because $\omega_{R'} = (\omega_R)^{(|\mathbf{k}|)}$, it then follows that

$$\omega_{R'} = \bigoplus_{n>0} [\omega_R]_{|\mathbf{k}|n} = \bigoplus_{n>0} [\omega_R]_{\mathbf{k}\cdot(n\mathbf{1})} = \bigoplus_{n>0} I^{|\mathbf{k}|(n-1)} = R'(-1).$$

Since R' is Cohen-Macaulay by Corollary 2.5, we get that R' is Gorenstein.

Assume then that S' is Cohen-Macaulay and R' is Gorenstein. It follows from Theorem 3.9 that

$$\bigoplus_{n\geq|\mathbf{k}|} [\omega_R]_n = \bigoplus_{n\geq|\mathbf{k}|} (I^{n-|\mathbf{k}|})^*.$$

According to Proposition 3.15 we have

$$\omega_S = \bigoplus_{\mathbf{n}>\mathbf{0}} [\omega_R]_{|\mathbf{n}|}.$$

Then

$$\omega_{S'} = (\omega_S)^{(\mathbf{k})} = \bigoplus_{\mathbf{n}>\mathbf{0}} [\omega_R]_{\mathbf{k}\cdot\mathbf{n}} = \bigoplus_{\mathbf{n}>\mathbf{0}} (I^{\mathbf{k}\cdot(\mathbf{n}-\mathbf{1})})^*.$$

By Lemma 3.5 we know that $(I^{\mathbf{k}\cdot\mathbf{n}})^* = I^{\mathbf{k}\cdot\mathbf{n}}$ for all $\mathbf{n} \in \mathbf{N}^r$. So

$$\omega_{S'} = \bigoplus_{\mathbf{n}>\mathbf{0}} I^{\mathbf{k}\cdot(\mathbf{n}-\mathbf{1})} = S'(-\mathbf{1})$$

and the theorem has been proved.

3.17. Example. Consider the class of Cohen-Macaulay almost complete intersection ideals I of $\operatorname{ht} I > 1$ in a local Gorenstein ring A. For these ideals we know by [HRZ, Proposition (2.5)] that $gr_A(I)$ is Gorenstein and $a(gr_A(I)) = -\operatorname{ht} I < -1$. Since $R_A(I)$ is then Cohen-Macaulay, it follows from [HRZ, Theorem (3.5)] that $R_A(I^{\operatorname{ht} I-1})$ is Gorenstein. Therefore the multigraded Rees ring $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein if $|\mathbf{k}| = \operatorname{ht} I - 1$.

Take in particular $A = k[[X_1, \ldots, X_6]]$, where $k[[X_1, \ldots, X_6]]$ is the formal power series ring over a field k, and

$$I = (X_1^2 - X_2X_4, X_2^2 - X_3X_5, X_3^2 - X_1X_6, X_1X_2X_3 - X_4X_5X_6)$$

([HK]). Then $\operatorname{ht} I = 3$, $a(gr_A(I)) = -3$ so that $R(I^2)$ and $R(I, I)$ are Gorenstein.

By combining Theorem 3.16 with Theorem 3.12 we immediately get the following.

3.18. Corollary. *Let A be a local ring of dimension d and $I \subset A$ an ideal of* grade $I > 1$. *Let $\mathfrak{M}$ be the homogeneous maximal ideal of $R_A(I)$. Suppose $\underline{H}^d_{\mathfrak{M}}(R_A(I)) = 0$. Set $G = gr_A(I)$. Let $\mathbf{k} \in (\mathbb{N}^*)^r$. Then $R_A(\mathbf{I}_r^{\mathbf{k}})$ is Gorenstein if and only if the following conditions are satisfied*

(1) $R_A(\mathbf{I}_r^{\mathbf{k}})$ *is Cohen-Macaulay*

(2) $\omega_A \cong A$

(3) $\omega_G \cong gr^*_A(I)(-(|\mathbf{k}|+1))$.

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