10-COMMUTATOR AND 13-COMMUTATOR

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ABSTRACT. Skew-symmetris sum of $N!$ compositions of N vector fields in all possible order are called N-commutator. We construct 10-commutator and 13-commutator on $Vect(3)$ and 10-commutator on a space of divergenceless vector fields $Vect_0(3)$. We show that 2-commutator, 10-commutator and 13-commutator form final list of N-commutators on $Vect(3)$ and under these polylinear operations $Vect(3)$ has a structure of sh-Lie algebra. We establish that the list of 2-and 10-commutators on $Vect_0(3)$ is also final. Constructions are based on calculations of powers of odd derivations.

Let $(A, \circ)$ be an algebra with vector space A and multiplication $\circ$. Let $\mathbf{C} < t_1, \dots, t_k >$ be a space of non-commutative non-associative polynomials. Any $f \in \mathbf{C} < t_1, \dots, t_k >$ induces a k-ary map

$$f : \underbrace{A \times \cdots \times A}_{k} \to A,$$

that correspond to any $a_1, \dots, a_k \in A$ element $f(a_1, \dots, a_k)$ calculated by multiplication $\circ$. If this map is trivial, i.e., $f(a_1, \dots, a_k) = 0$, for any $a_1, \dots, a_k \in A$ then $f = 0$ is said *identity* on $(A, \circ)$. If f is polylinear, then f induces a k-*ary multiplication* on A. For example, if $s_2 = t_1 t_2 - t_2 t_1 \in \mathbf{C} < t_1, t_2 >$, then

$$s_2(a, b) = a \circ b - b \circ a$$

is ordinary commutator.

Let

$$s_k = \sum_{\sigma \in Sym_k} sign\, \sigma\, t_{\sigma(1)}(\cdots(t_{\sigma(k-1)} t_{\sigma(k)})\cdots)$$

be standard skew-symmetric polynomial. Let $Diff_n$ be a space of differential operators with n variables. For simplicity assume that variables are from $\mathbf{C}[x_1, \dots, x_n]$. Let $Diff_n^{[d]}$ be a subspace of differential operators of order d:

$$Diff_n^{[d]} = \left\langle u\partial^\alpha | |\alpha| = \sum_{i=1}^n \alpha_i = d \right\rangle.$$

We can interpret differential operators of first order as vector fields and identify $Diff_n^{[1]}$ with a space of vector fields $Vect(n)$. Consider s_k as a k-ary operation on a space of differential operators $Diff_n$. So, $s_k(X_1,\dots,X_k)$ is a skew-symmetric sum of compositions of k operators $X_{\sigma(1)}\cdots X_{\sigma(k)}$ by all $k!$ permutations. In general, composition of k operators of orders $d_1,\dots,d_k$ is a differential operator of order $d_1+\cdots+d_k$. Therefore,

$$X_1\in Diff_n^{[d_1]},\dots,X_k\in Diff_n^{[d_k]}\Rightarrow s_k(X_1,\dots,X_k)\in Diff_n^{[d_1+\cdots+d_k]}.$$

In fact differential order of $s_k(X_1,\dots,X_k)$ is less than $d_1+\ldots+d_k$. For example, differential order of $s_k(X_1,\dots,X_k)$ is no more than n, if all $X_1,\dots,X_k$ are operators of order 1 (vector fields) on n-dimensional manifold for any k [2]. Moreover, for some k might happen that s_k will be well-defined operation on $Diff_n^{[1]}$:

$$X_1,\dots,X_k\in Diff_n^{[1]}\Rightarrow s_k(X_1,\dots,X_k)\in Diff_n^{[1]}.$$

In [1] is established that s_{n^2+2n-2} is well-defined on $Vect(n)=Diff_n^{[1]}$ and in [2] is proved that $s_{n^2+2n-1}=0$ is identity on $Vect(n)$. For example, $Vect(2)$ has 6-commutator and skew-symmetric identity of degree 7. Hamiltonian vector fields on 2-dimensional plane has 5-commutator and skew-symmetric identity of degree 6.

Question 1. $(n>1)$. Is it true that $N=n^2+2n-1$ is index of nilpotency for operator D, i.e., $D^{n^2+2n-1}=0$, but $D^{n^2+2n-2}\neq 0$?

We think that coefficient at $\prod_i \eta_i \prod_{(i,j)\neq(n,n)}\partial_i\eta_j\prod_{i\neq n}\partial_i^2\eta_i\partial_n$ of D^{n^2+2n-2} is non-zero. Computer calculations on Mathematica shows that this coefficient is equal $1,2,3600$ for $n=2,3,4$.

Question 2. $(n>3)$. Is it true that s_{n^2+2n-2} is a unique N-commutator well-defined on $Vect(n)$, for $N>2$? In other words, is it true that

$$D^N\in Der\,\mathcal{L}_n, n>3, \Rightarrow N=2 \text{ or } n^2+2n-2?$$

In our paper we prove that for $n=3$ answer to this question is negative. according our results $Vect(3)$ has 2-commutator, 10-commutator and 13-commutator and this list of N-commutators is complete. Notice that 13-commutator is connected with skew-symmetric identity of degree 14, but 10-commutator has no such connection with skew-symmetric identity of degree 11: s_{11} even is not well-defined operation on $Vect(3)$. Some quantitive parameters about D^{10} and D^{13}. D^{10} has three escort invariants. They have types $(2,7,1)$, $(3,5,2)$ and $(3,6,0,1)$. It has 489 terms of type $(2,7,1)$, 3093 terms of type $(3,5,2)$, 480 terms of type $(3,6,0,1)$ and all together 4062 terms. D^{13} has one escort invariant. It has type $(3,8,2)$ and has 261 terms.

In other words, $s_{10}(X_1,\dots,X_{10})$ can be presented as a sum of 4062 10×10-determinants of three types. Similarly, $s_{13}(X_1,\dots,X_{13})$ can be presented as a sum of 261 matrices of order 13×13.

To see that D^{10} is well-defined on $Vect_0(3)$, we change all terms of D^{10} like $\partial^\alpha\eta_3\partial_3$, $\alpha_3>0$, to $-\partial^{\alpha-\varepsilon_3+\varepsilon_1}\eta_1\partial_1-\partial^{\alpha-\varepsilon_3+\varepsilon_2}\eta_2\partial_2$. We obtain element with 864 terms, among them 82 has type $(2,7,1)$, 76 has type $(3,6,0,1)$ and 706 has type $(3,5,2)$.

It is easy to see that D^k is a sum of compositions of the form $D\star_1(D\star_2\cdots(D\star_{k-1}D)\cdots)$, where $(\star_1,\dots,\star_{k-1})$ is a sequence of two symbols $\circ$ or $\bullet$ such that there are no two consequative $\bullet$ and whole number of $\bullet$ is no more than $|I|$. In particular we see that the differential order of D^k is no more than $min(k+1/2,|I|)$. This estimate is not strong. One can see that, for $n=2,3$ differential orders of D^k are given as follows

$n=2$

k	1	2	3	4	5	6	7
$\partial deg\, D^k$	1	1	2	2	2	1	$-\infty$

$n=3$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\partial deg\, D^k$	1	1	2	2	3	3	3	3	3	1	2	2	1	$-\infty$

If $D\in Der_0\mathcal{L}_n$, i.e., $Div\,D=0$, then the growth of differential orders of D^k given as

$n=2$

k	1	2	3	4	5	6	7
$\partial deg\, D^k$	1	1	2	2	1	$-\infty$	$-\infty$

$n=3$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\partial deg\, D^k$	1	1	2	2	3	3	3	3	3	1	2	$-\infty$	$-\infty$	$-\infty$

We pay attention to a drammatical jumping of $\partial deg\, D^k$ in $(n,k)=(3,10)$. Here we see that D^{10} is a derivation or that the 10-commutator is defined correctly on $Vect(3)$. One can check that $Div\,D^{10}=0$ and hence 10-commutator is a well defined commutator on $Vect_0(3)$ also.

Theorem 0.1. *Let* $D=\sum_{i=1}^3\eta_i\partial_i\in Der\,\mathcal{L}_3$ *be odd derivation. Then*

$$D^{10}\in Der\,\mathcal{L}_3\otimes\mathbf{C}[x_1,x_2,x_3],$$
$$D^{13}\in Der\,\mathcal{L}_3\otimes\mathbf{C}[x_1,x_2,x_3],$$
$$D^{14}=0.$$

If $D^N \in Der\,\mathcal{L}_3 \otimes \mathbf{C}[x_1,x_2,x_3]$, *then* $N=2,10,13$.

Theorem 0.2. *Let* $D=\sum_{i=1}^3 \eta_i\partial_i \in Der\,\mathcal{L}_3$ *be odd derivation and* $Div\,D=\sum_{i=1}^3 \partial_i\eta_i = 0$. *Then*

$$D^{10} \in Der\,\mathcal{L}_3 \otimes \mathbf{C}[x_1,x_2,x_3],$$

$$Div\,D^{10}=0,$$

$$D^{11}=0.$$

If $D^N \in Der\,\mathcal{L}_3 \otimes \mathbf{C}[x_1,x_2,x_3]$, *and* $Div\,D=0$, *then* $N=2$ *or* 10.

1. sl_n-module structure on $U=\mathbf{C}[x_1,\dots,x_n]$ and $Diff_n(U)$

Endow U by a structure of module over Lie algebra $gl_n =< x_i\partial_j : i,j=1,\dots,n, i\neq j >$. Define an action of gl_n on generators of U by

$$x_i\partial_j(\partial^\alpha(u_s)) = -\delta_{i,s}\partial^\alpha(u_j) + \sum_{i=1}^n \alpha_j \partial^{\alpha-\epsilon_j+\epsilon_i}(u_s)$$

and prolong this action to U as an even derivation:

$$a(XY)=a(X)Y+Xa(Y),$$

for any $X,Y\in U$. Prolong the gl_n-module structure by natural way to $Diff_n(U)$. Notice that gl_n acts on $Diff_n(U)$ as a derivation

$$a(FG)=a(F)G+Fa(G),$$

and as gl_n-module subspaces $< u_i\partial_j : i,j=1,\dots,n > \subset Diff_n(U)$ and $< \partial_i(u_j) : i,j=1,\dots,n >$ are isomorphic to adjoint module.

Denote by $\pi_1,\dots,\pi_{n-1}$ fundamental weights of sl_n and by $R(\gamma)$ the irreducible sl_n-module with highest weight γ. Let

$$\mathcal{D}^{[s]} =< \partial^\alpha : \alpha\in \mathbf{Z}_+^n, |\alpha|=s >$$

and

$$U_s =< \partial^\alpha u_i : \alpha\in\mathbf{Z}_+^n, |\alpha|=s, i=1,\dots,n > .$$

Since ∂_i are even and u_i are odd elements, take place the following isomorphisms of sl_n-modules

$$\mathcal{D}^{[s]} \cong R(s\pi_1),$$

$$U_s \cong R(s\pi_1)\cong R(\pi_{n-1}).$$

In particular,

$$\mathcal{D} :=< \partial_i : i=1,\dots,n > \cong R(\pi_{n-1}),$$

$$\mathcal{D}^{[2]} :=< \partial_i\partial_j : i,j=1,\dots,n > \cong R(2\pi_1),$$

$$U_0 =< u_i : i=1,\dots,n > \cong R(\pi_1),$$

$$U_1 =< \partial_i(u_j) : i,j=1,\dots,n > \cong R(\pi_1)\otimes R(\pi_{n-1}) R(\pi_1+\pi_{n-1})\oplus R(0),$$

$U_2 =< \partial_i\partial_j(u_s) : i,j,s = 1,\dots,n >\cong R(2\pi_1)\otimes R(\pi_{n-1}) \cong R(2\pi_1+\pi_{n-1})\oplus R(\pi_1)$.
We use the following well-known isomorphisms without special mentioning:

$$\wedge^{n-1}R(\pi_1) \cong R(\pi_{n-1}),$$
$$\wedge^{n}R(\pi_1) \cong R(0),$$
$$\wedge^{n^2-1}R(\pi_1+\pi_{n-1}) \cong R(\pi_1+\pi_{n-1}),$$
$$\wedge^{n^2}R(\pi_1+\pi_{n-1}) \cong R(0).$$

Lemma 1.1. *$a(D^k) = 0$ for any $a \in gl_n$.*

Proof. If $k = 1$ then action of $a \in gl_n$ corresponds to adjoint derivation and D corresponds to Euler operator. Therefore,

$$a(D) = [a, \sum_{i=1}^{n} u_i\partial_i] = 0.$$

If our statement is true for $k-1$ then

$$a(D^k) = kD^{k-1}[a,D] = 0.$$

□

2. Escort invariants of N-commutators

Let $L = W_n$ be Witt algebra and $U = \mathbf{C}[x_1,\dots,x_n]$ be natural L-module. Then

- $L = \oplus_{i\geq -1}L_i$ is a graded Lie algebra,
$$L_s =< x^\alpha\partial_j : |\alpha| = s+1 >,$$
- $U = \oplus_{i\geq 0}U_i$ be associative commutative graded algebra with 1,
$$U_s =< x^\alpha : |\alpha| = s >,$$
- L acts on U as a derivation algebra, i.e.,
$$X(uv) = X(u)v + u(Xv),$$
for any $X \in L, u, v \in U$ and
- this action is graded:
$$L_iU_j \subseteq U_{i+j}, \quad i \geq -1, j \geq 0.$$

In particular, L_0 is a Lie algebra isomorphic to gl_n and all homogeneous components L_s and U_s have structures of gl_n-modules. Then as sl_n-modules,

$$L_{-1} =< \partial_i : i = 1,\dots,\partial_n >\cong R(\pi_{n-1}),$$
$$L_0 =< x_i\partial_j : i,j = 1,\dots,n >\cong R(\pi_1)\oplus R(\pi_{n-1})\oplus R(0),$$
$$L_1 =< x_ix_j\partial_s : i,j,s = 1,\dots,n >\cong R(2\pi_1+\pi_{n-1})\oplus R(\pi_1).$$

Let M be graded L-module. It is called (L,U)*-module* if it has additional structure of graded module over U such that

$$X(um)=X(u)m+uX(m),$$

for any $X\in L, u\in U, m\in M$. Call M (L,U)-module with *a base* N if $N=M^{L_{-1}}=<m\in M: X(m)=0,\forall X\in L_{-1}>$ and M is free U-module with base N. If $M_1,\ldots,M_k$ and M are (L,U)-modules with bases and N is a base of M, then a space of polylinear maps $C(M_1,\ldots,M_k;M)=<\psi: M_1\times\cdots\times M_k\to M>$ is (L,U)-module with base and this base as a vector space is isomorphic to $C(M_1,\ldots,M_k;M)$. In particular, to any L_{-1}-invariant polylinear map $\psi\in C(M_1,\ldots,M_k;M)$ one can correspond some polylinear map $esc(\psi)\in C(M_1,\ldots,M_k;N)$ called *escort* of ψ, by

$$esc(\psi)(m_1,\ldots,m_k)=pr(\psi(m_1,\ldots,m_k)),$$

where

$$pr: M\to N,$$

is a projection map to N, i.e., $pr(x^\alpha m)=\delta_{\alpha,0}m$. Inversly, for any $\phi\in C(M_1,\ldots,M_k;N)$ one can correspond some L_{-1}-invariant polylinear map $\psi=E\phi\in C(M_1,\ldots,M_k;M)$ by

$$E\phi(X_1,\ldots,X_k)=\sum_{a_1\in M_1,\ldots,a_k\in M_k}E_{a_1}(X_1)\cdots E_{a_k}(X_k)\phi(a_1,\ldots,a_k),$$

where a_i run basic elements of M_i of the form $x^\alpha n_i$, n_i run basic elements of a base of M_i. If $M_1=\cdots=M_k=L$ all are adjoint modules then

$$E_{x^\alpha\partial_i}(v\partial_j)=\delta_{i,j}\frac{\partial^\alpha(v)}{\alpha!}.$$

Details of such constructions see [3].

Apply this theory for L-module of differential operators $M=Diff_n=<u\partial^\alpha: u\in U,\alpha\in\mathbf{Z}_+^n>$. Endow $M=Diff_n$ by grading:

$$M_s=<x^\alpha\partial^\beta:|\alpha|=s>.$$

$Diff_n$ has a structure of associative algebra, in particular, it is a Lie algebra under commutator. As a Lie algebra it has a subalgebra isomorphic to W_n, and hence it has a structure of adjoint module over W_n. Make $Diff_n$ U-module under action $u(v\partial^\beta)=uv\partial^\beta$. We see that $M^{L_{-1}}=<\partial^\alpha:\alpha\in\mathbf{Z}_+^n>$ and M is free U-module with base $M^{L_{-1}}$. Therefore, $Diff_n$ is (L,U)-module.

Define $s_k\in C^k(L,M)$ by

$$s_k(X_1,\ldots,X_k)=\sum_{\sigma\in Sym_k}sign\,\sigma\,X_{\sigma(1)}\cdots X_{\sigma(k)}.$$

We see that s_k is L_{-1}-invariant and graded:

$$\partial_i(s_k(X_1,\dots,X_k)) = \sum_{s=1}^{k} s_k(X_1,\dots,X_{s-1},[\partial_i,X_s],X_{s+1},\dots,X_k),$$

for any $\partial_i \in L_{-1}, X_1,\dots,X_k \in L,$ and

$$s_k(L_{i_1},\dots,L_{i_k}) \subseteq M_{i_1+\dots+i_k},$$

Fix some ordering on the set of basic elements of W_n. Let us take, for example, the following ordering: $x^\alpha\partial_i < X^\beta\partial_j$, if $i<j$ or $|\alpha|<|\beta|$ if $i=j$ or $\alpha<\beta$ in lexicographic order if $i=j$ and $|\alpha|=|beta|$. As we mentioned above any L_{-1}-invariant cochain $C^k(W_n, Diff_n)$ can be restored by its escort. In particular, s_k can be restored by its escort. Any escort is defined as a polylinear map on its support. Call a subspace of k-chains $a_1\wedge\dots\wedge a_k \in \wedge^k L$ generated by basic vectors $a_1,\dots,a_k$ such that

$$s_k(a_1,\dots,a_k) \in <\partial^\alpha : \alpha\in \mathbf{Z}_+^n>$$

as a *support* of s_k. Then $supp(s_k)$ has a structure of sl_n-module as a sl_n-submodule of $\wedge^k L$. We know that sl_n-module $Diff_n^{L_{-1}} =<\partial^\alpha : \alpha\in\mathbf{Z}_+^n>$ is isomorphic to a direct sum of sl_n-modules $R(p\pi_{n-1})$:

$$<\partial^\alpha : |\alpha|=p, \alpha\in\mathbf{Z}_+^n> \cong R(p\pi_{n-1})>.$$

Then $supp(s_k)$ is also a direct sum of sl_n-submodules $supp_p(s_k)$, where $supp_p(s_k)$ is a sl_n-submodule of $\wedge^k L$ generated by suppport k-chains $a_1\wedge\dots\wedge a_k$ such that

$$s_k(a_1,\dots,a_k) \in <\partial^\alpha : \alpha\in\mathbf{Z}_+^n, |\alpha|=p>.$$

So, we see that any standard skew-symmetric polynomial s_k induces a serie of sl_n-invariant maps

$$supp_p(s_k) \to R(p\pi_{n-1}).$$

Call such maps *escort invariants*. So, the calculation problem of k-commutators is equivalent to the problem of finding escort invariants.

Example. $esc(s_k)=0$ if $k\geq n^2+2n-1$ and s_{n^2+2n-2} has exactly one escort invariant $R(\pi_1)\otimes R(\pi_{n-1})\otimes\wedge^{n-1}R(2\pi_1+\pi_{n-1}) \to R(\pi_{n-1})$.

3. Differential polynomials super-agebra $\mathcal{L}_n$

Let $\mathbf{Z}$ be set of integers, $\mathbf{Z}_+$ a set of non-negative integers, $\mathbf{Z}^n$ a set of n-typles $\alpha=(\alpha_1,\dots,\alpha_n), \alpha_i\in\mathbf{Z}, i\in I,$ and $\mathbf{Z}_+^n = \{\alpha\in\mathbf{Z}^n|\alpha_i \geq$

$0, i \in I\}$. Let $\varepsilon_i \in \mathbf{Z}^n$ with i-th component 1 and other components are 0. Then $\alpha = \sum_{i=1}^n \alpha_i \varepsilon_i$, for any $\alpha \in \mathbf{Z}^n$. Set

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

Endow sets $\mathbf{Z}_+^n$ and $\mathbf{Z}_+^n \times \{1, \ldots, n\}$ by linear ordering: $\alpha < \beta$, if

$$|\alpha| < |\beta|$$

or

$$|\alpha| = |\beta|, \alpha_1 = \beta_1, \ldots, \alpha_{s-1} = \beta_{s-1}, \alpha_s > \beta_s,$$

for some $s = 1, \cdots, n$. Set $(\alpha, i) < (\beta, j)$, if $i < j$ or $i = j, \alpha < \beta$.

Let $\mathcal{L}_n$ be an super-commutative associative algebra over a field K generated by odd elements $e_{\alpha,i}$, where $\alpha \in \mathbf{Z}_+^n, i \in I$. Then

$$e_{\alpha,i} e_{\beta,j} = -e_{\beta,j} e_{\alpha,i},$$

$$e_{\alpha,i}(e_{\beta,j} e_{\gamma,s}) = (e_{\alpha,i} e_{\beta,j}) e_{\gamma,s},$$

for any $\alpha, \beta, \gamma \in \mathbf{Z}_+^n, i, j, s \in I$. Elements $e_{\alpha,i} e_{\beta,j} \cdots e_{\gamma,s}$ with $(\alpha, i) < (\beta, j) < \cdots < (\gamma, s)$ form base of $\mathcal{L}_n$. We fix this base and call such elements *base* elements of $\mathcal{L}_n$. Call number of indexes $i, j, \cdots, s$ of base element e as its *length* and denote $l(e)$.

Any base element of $\mathcal{L}_n$ can be presented as $e = e^{[-1]} e^{[0]} e^{[1]} \cdots e^{[r]}$, where $e^{[s]}$ is a product of ordered generators of a form $e_{\alpha,i}$ with $|\alpha| = s+1$. Call $e^{[s]}$ s*-component* of e and its length $l(e^{[s]})$, denote it $l_s(e)$, call as s*-Length* of e. Thus,

$$l(e) = \sum_{i \geq -1} l_i(e).$$

Let $\partial_i = \frac{\partial}{\partial_i}, i \in I$, are partial derivations of $U = K[x_1, \ldots, x_n]$. Prolong these maps to maps of $\mathcal{L}_n$ by

$$\partial_i e_{\beta,j} = e_{\alpha+\varepsilon_i, j}.$$

It is easy to see that ∂_i satisfies Leibniz rule

$$\partial_i(e_{\beta,j} e_{\gamma,s}) = (\partial_i e_{\beta,j}) e_{\gamma,s} + e_{\beta,j}(\partial_i e_{\gamma,s}),$$

for any $\beta, \gamma \in \mathbf{Z}_+^n$. So, we have constructed commuting even derivations $\partial_1, \ldots, \partial_n \in Der(\mathcal{L}_n \otimes U)$ and

$$e_{\alpha,i} = \partial^\alpha e_{0,i},$$

for any $\alpha \in \mathbf{Z}_+^n, i \in I$. Here $0 = (0, \ldots, 0) \in \mathbf{Z}_+^n$.

Space $\mathcal{L}_n$ has three kinds of gradings. The first one, $\mathbf{Z}^n$-grading is defined by

$$||e_{\alpha,i}|| = \alpha - \varepsilon_i$$

and for other base elements are prolonged by multiplicativity,

$$||e_{\alpha,i}e_{\beta,j}\cdots e_{\gamma,s}|| = \alpha-\varepsilon_i+\beta-\varepsilon_j+\cdots+\gamma-\varepsilon_s.$$

The second grading is induced by $\mathbf{Z}^n$-grading. It is $\mathbf{Z}$-grading defined on base element $e=e_{\alpha,i}e_{\beta,j}\cdots e_{\gamma,s}$ by

$$|e| = -l(e)+|\alpha|+|\beta|+\cdots+|\gamma|.$$

The third grading is defined by length. Let $l(\xi)=s$, if ξ is a nontrivial linear combination of homogeneous base elements of length s.

Call

$$wt(e)=|\alpha|+\cdots+|\beta|-l(e)$$

weight of e. A parity on $\mathcal{L}_n$ is defined by length. Let $\mathcal{L}_n^{[l]}$ be linear span of base elements u with $l(u)=l$. Let $\mathcal{L}_n^{[l,w]}$ be a linear span of base elements u with $l(e)=l, wt(e)=w$.

Example.

$$\mathcal{L}_n^{[1]}=\langle e_{\alpha,i}|\alpha\in\mathbf{Z}^n, i=1,\dots,n\rangle\,,$$
$$\mathcal{L}_n^{[n]}=\langle e_{0,1}\cdots e_{0,n}\rangle\,,$$
$$\mathcal{L}_n^{[1,-1]}=\langle e_{0,i}\rangle\,.$$

Proposition 3.1. *$\mathcal{L}_n$ is associative, super-commutative graded algebra:*

$$(uv)w=u(vw),$$
$$uv=(-1)^{q(u)q(v)}vu,$$
$$\mathcal{L}_n=\oplus_{l\geq 1, w\geq -n}\mathcal{L}_n^{[l,w]},$$
$$\mathcal{L}_n^{[l,w]}\mathcal{L}_n^{l_1,w_1]}\subseteq\mathcal{L}_n^{[l+l_1,w+w_1]}.$$

for any $u,v,w\in\mathcal{L}_n$.

Note that any base element $u\in\mathcal{L}_n$ can be presented in a form $u_{-1}u_0\cdots u_r$ where $u_s, s=-1,0,\dots,r$ are base elemenents and u_s are products of generators of weight s. We say that base element $u\in\mathcal{L}_n$ has *type* $(l_{-1},l_0,\dots,l_r)$, if u is a product of l_s generators of weight s, for $s=-1,0,\dots,r$.

Lemma 3.2. *Any base element $u\in\mathcal{L}_n$ satisfy the following conditions*

$$\sum_{i\geq -1} l_i(u)=l(u),$$
$$\sum_{i\geq -1} i\,l_i(u)=wt(u),$$
$$l_i(u)\leq n\binom{n+i}{i+1},\qquad i\geq -1.$$

Proof. First two relations are reformulations of grading property of $\mathcal{L}_n$ (proposition 3.1). As far as last two relations, they follow from the fact

$$|\{\alpha \in \mathbf{Z}_+^n | |\alpha| = i+1\}| = \binom{n+i}{i+1}.$$

Example. Let $u = \eta_1 \partial_1^2 \eta_2 \partial_1 \partial_2 \eta_2$. Then u is odd base element of type $(1,0,2)$ and $l(u) = 3, wt(u) = 1$.

Let $Diff_n$ be an algebra of differential operators on $\mathcal{L}_n$. It has a base consisting differential operators of a form $u\partial^\alpha$, where $\alpha \in \mathbf{Z}_+^n$ and u is a base element of $\mathcal{L}_n$. Endow $Diff_n$ by multiplication $\cdot$ given by

$$u\partial^\alpha \cdot v\partial^\beta = \sum_\gamma \binom{\alpha}{\gamma} u\partial^\gamma v \partial^{\alpha+\beta-\gamma}.$$

Here

$$\binom{\beta}{\gamma} = \prod_{i=1}^n \binom{\beta_i + \gamma_i}{\gamma_i}.$$

Multiplication $\cdot$ corresponds to composition of differential operators.

Endow $Diff_n$ also by two more multiplications $\circ$ and $\bullet$. They are given by the following rules

$$u\partial^\alpha \circ v\partial^\beta = \sum_{\gamma \neq 0} \binom{\alpha}{\gamma} u\partial^\gamma v \partial^{\alpha+\beta-\gamma},$$

$$u\partial^\alpha \bullet v\partial^\beta = uv\partial^{\alpha+\beta}.$$

We see that

$$X \cdot Y = X \circ Y + X \bullet Y,$$

for any $X, Y \in Diff_n$.

For a base element $X = u\partial^\alpha \in Diff_n$ define *length* $l(X)$, *weight* $wt(X)$, *parity* $q(X)$ and *differential order* $\partial deg(X)$ by

$$l(X) = l(u),$$

$$wt(X) = wt(u) + |\alpha|,$$

$$q(X) = l(u),$$

$$\partial deg(X) = |\alpha|.$$

Let

$$Diff_n^{[d]} = \langle X | \partial deg(X) = d \rangle .$$

$$Diff_n^{[l,w]} = \langle X | l(X) = l, wt(X) = w \rangle ,$$

$$Diff_n^{[l,w,d]} = \langle X | l(X) = l, wt(X) = w, \partial deg(X) = d \rangle .$$

Denote a space of differential operators of first order $Diff_n^{[1]}$ by W_n.

For a differential operator $X = \sum_{\alpha\in \mathbf{Z}^n_+} v_\alpha \partial^\alpha \in Diff_n$, define its differential order $deg(X)$ as maximal $|\alpha|$, such that $v_\alpha \neq 0$.

Proposition 3.3. *Space of differential operators under different multiplications have the following properties.*

The algebra $(Diff_n, \cdot)$ is associative super-algebra:

$$X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z,$$

for any $X, Y, Z \in Diff_n$. This agebra is graded,

$$Diff_n = \oplus_{l>0, w\geq -n} Diff_n^{[l,w]},$$
$$Diff_n^{[l,w]} \cdot Diff_n^{[l_1,w_1]} \subseteq Diff_n^{[l+l_1,w+w_1]}.$$

The algebra $(W_n, \circ)$ is super-left-symmetric:

$$(X, Y, Z) = (-1)^{q(X)q(Y)}(Y, X, Z),$$

for any differential operators of first order X, Y, Z, where $(X, Y, Z) = X \circ (Y \circ Z) - (X \circ Y) \circ Z$ is associator. Moreover, super-left-symmetric rule is true for any $X, Y \in Diff_n^{[1]}, Z \in Diff_n$. This algebra is graded,

$$W_n = \oplus_{l>0, w\geq -n} W_n^{[l,w]},$$
$$W_n^{[l,w]} \circ W_n^{[l_1,w_1]} \subseteq W_n^{[l+l_1,w+w_1]}.$$

The algebra $(Diff_n, \bullet)$ is associative super-commutative:

$$X \bullet Y = (-1)^{q(X)q(Y)} Y \bullet X,$$
$$X \bullet (Y \bullet Z) = (X \bullet Y) \bullet Z,$$

for any $X, Y, Z \in Diff_n$. This algebra is graded under length, weight and differential order,

$$Diff_n = \oplus_{l>0, w\geq -n, d\geq 0} Diff_n^{[l,w,d]},$$
$$Diff_n^{[l,w,d]} \bullet Diff_n^{[l_1,w_1,d_1]} \subseteq Diff_n^{[l+l_1,w+w_1,d+d_1]}.$$

Any differential operator of first order under multiplication $\circ$ acts on $(Diff_n, \bullet)$ as a derivation:

$$X \circ (Y \bullet Z) = (X \circ Y) \bullet Z + (-1)^{q(X)q(Y)} Y \bullet (X \circ Z),$$

for any $X \in W_n, Y, Z \in Diff_n$.

Proof. Notice that natural action of W_n on $\mathcal{L}_n$ coincides with left-symmetric product:

$$X(\eta) = X \circ \eta,$$

for any $X \in W_n, \eta \in \mathcal{L}_n$. Therefore, we have the following connection between composition and left-symmetric multiplications:

$$(X \cdot Y)(\eta) \neq (X \circ Y)(\eta))$$

but

$$(X \cdot Y)(\eta) = X \circ Y(\eta),$$

for any $X, Y \in Diff_n, \eta \in \mathcal{L}_n$. Moreover,composition of differential operators of first order can be expressed in terms of left-symmetric multiplication,

$$(X \cdot Y)(\eta) = X \circ (Y \circ \eta),$$

for any $X, Y \in W_n, \eta \in \mathcal{L}_n$. Thus,

$$(X \circ Y + X \bullet Y)(\eta) = X \circ (Y \circ \eta),$$

and

$$X \circ (Y \circ \eta) - (X \circ Y)(\eta) = (X \bullet Y)(\eta).$$

Since $X \circ Y \in W_n$, this means that

$$X \circ (Y \circ \eta) - (X \circ Y) \circ \eta = (X \bullet Y)(\eta). \tag{1}$$

for any $X, Y \in W_n, \ \eta \in \mathcal{L}_n$. By these facts we see that

$$([X, Y] \cdot Z)(\eta) = (X \cdot Y - (-1)^{q(X)q(Y)} Y \cdot X)(Z(\eta))$$

$$= (X \circ Y + X \bullet Y - (-1)^{q(X)q(Y)} Y \circ X - (-1)^{q(X)q(Y)} Y \bullet X) \circ (Z(\eta))$$

$$= (X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ (Z(\eta)).$$

On the other hand

$$([X, Y] \cdot Z)(\eta) = (X \cdot (Y \cdot Z) - (-1)^{q(X)q(Y)} Y \cdot (X \cdot Z))(\eta)$$

$$= X \circ (Y \cdot Z)(\eta) - (-1)^{q(X)q(Y)} Y \circ (X \cdot Z)(\eta)$$

$$= X \circ (Y \circ Z(\eta)) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z(\eta)).$$

Hence,

$$(X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ (Z(\eta)) = X \circ (Y \circ Z(\eta)) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z(\eta)).$$

In other words,

$$(X \circ Y - (-1)^{q(X)q(Y)} Y \circ X) \circ Z = X \circ (Y \circ Z) - (-1)^{q(X)q(Y)} Y \circ (X \circ Z),$$

for any $X, Y \in W_n, Z \in Diff_n$.

Other statements of our proposition are evident. $\square$

For a base element $X = u\partial^\alpha \in Diff_n$ say that it has *type* $(l_{-1}, l_0, l_1, \ldots, l_r; d)$ if u has type $(l_{-1}, l_0, \ldots, l_r)$ and $|\alpha| = d$.

Example. Let $X = \eta_1\eta_3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1\partial_2\partial_3\eta_3\partial_1\partial_2$. Then X is base element of $Diff_3$ of type $(2, 3, 0, 1; 1)$, weight 2 and differential order 2.

Lemma 3.4. *Any base element $X \in Diff_n$ satisfies the following conditions:*

$$\sum_{i\geq -1} l_i(X) = l(X),$$

$$\sum_{i\geq -1} i\, l_i(X) + deg(X) = wt(X),$$

$$l_i(X) \leq n\binom{n+i}{i+1}, \qquad i \geq -1.$$

Proof. Follows from proposition 3.3 and Lemma 3.2. □

Let $Diff_n^{(l_{-1},l_0,\dots,l_r;d)}$ be a subspace of $Diff_n$ generated by base elements of type $(l_{-1}, l_0, \dots, l_r; d)$. Let

$$\tau_{(l_{-1},l_0,\dots,l_r;d)} : Diff_n \to Diff_n^{(l_{-1},l_0,\dots,l_r;d)},$$

$$\tau_d : Diff_n \to Diff_n^{[d]}$$

be projection maps.

Polynomial space $U = K[x_1, \dots, x_n]$ has natural gradings:

$$||x^\alpha|| = \alpha, \qquad |x^\alpha| = |\alpha|.$$

It has standard base $\{x^\alpha = \prod_{i=1}^n x_i^{\alpha_i} | \alpha \in \mathbf{Z}^n\}$. These gradings on $\mathcal{L}_n$ and U induce gradings on $\mathcal{L}_n \otimes U$.

In previous section we define parity q on $\mathcal{L}_n \otimes U$. Below we set

$$\eta_i = e_{0,i}.$$

So, instead of $e_{\alpha,i}$ we can write $\partial^\alpha \eta_i$. Then for $\eta = \partial^{\alpha_1}\eta_{i_1} \cdots \partial^{\alpha_k}\eta_{i_k}$ we have

$$l(\eta) = k.$$

We identitfy $\mathcal{L}_n$ with $\mathcal{L}_n \otimes 1$ and consider $\mathcal{L}_n$ as a subalgebra of $\mathcal{L}_n \otimes U$.

4. Differential operators of first order on $\mathcal{L}_n$

$W_n = Diff_n^{[1]}$ has two algebraic structures. The first one, a structure of super-Lie algebra, is well-known. Let

$$[D_1, D_2] = D_1 D_2 - (-1)^{q(D_1)q(D_2)} D_2 D_1.$$

be super-commutator. Then

$$[D_1, D_2] = -(-1)^{q(D_1)q(D_2)}[D_2, D_1],$$

$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{q(D_1)q(D_2)}[D_2, [D_1, D_3]].$$

Notice that

$$q(\xi\partial_i) = q(\xi),$$

for any $\xi \in \mathcal{L}_n$. Recall that for any $D \in W_n$, corresponding adjopint operator

$$ad\, D : W_n \Rightarrow W_n$$

is a derivation of W_n. Therefore, W_n can be interpretered as a derivation super-Lie algebra of $\mathcal{L}_n$.

The second structure of algebra on W_n can be done by left-symmetric multiplication. It is less known. Define a product $\circ$ by

$$(\xi\partial_i) \circ (\eta\partial_j) = \xi\partial_i(\eta)\partial_j.$$

Then for ant $D_1, D_2, D_3 \in W_n$,

$$(D_1, D_2, D_3) = (-)^{q(D_1)q(D_2)}(D_2, D_1, D_3)$$

(left-symmetric identity). Here

$$(D_1, D_2, D_3) = D_1 \circ (D_2 \circ D_3) - (D_1 \circ D_2) \circ D_3$$

is associator.

Remark. Let $Diff_n^{(k)}$ be subspace of $Diff_n$ of order no more than k. Well known that

$$Diff_0^{(0)} = U \subset Diff_n^{(1)} \subset Diff_n^{(2)} \subset \cdots$$

is an increasing filtration on $Diff_n$,

$$Diff_n^{(k)} \cdot Diff_n^{(s)} \subseteq Diff_n^{(k+s)}, \qquad k, s \geq 0.$$

So, $Diff_n^{(k)}$ has a structure of algebra under composition operation, if $k = 0$, $Diff_n^{(1)}$ has an algebraic structure under commutator. One cas ask about algebraic structures on $Diff_n^{(k)}$ for $k > 0$. In other words, is it possible to find some $N = N(n, k)$, such that

$$X_1, \ldots, X_N \in Diff_n^{(k)} \Rightarrow s_N(X_1, \ldots, X_N) \in Diff_n^{(k)}.$$

One can prove the following

Theorem 4.1. *Let $n > 1$. Then $s_{(n+1)^2} = 0$ is identity on $Diff_n^{(1)}$ and s_{n^2+2n}, s_{n^2+2n-1} are well-defined operations on $Diff_n^{(1)}$. Moreover, $s_{n^2+2n}(X_1, \ldots, X_{n^2+2n}) \in Diff_n^{(0)}$, for all $X_1, \ldots, X_{n^2+2n} \in Diff_n^{(1)}$.*

5. Calculation of D^n

Let $\eta_1, \ldots, \eta_n$ are odd elements and

$$D = \sum_{i=1}^{n} \eta_i\partial_i,$$

$$F = D \circ D = \sum_{i,j=1}^{n} \eta_i\partial_i\eta_j\partial_j..$$

Notice that

- $D \in W_n^{[1,0]}$
- F is even element of W_n
- $l_{-1}(F) = 1, l_0(F) = 1, l_s(F) = 0, s > 0.$

Therefore,

$$D^k \in Diff_n^{[k,0]}.$$

Define left-symmetric power $D^{\circ k}$ by

$$\begin{array}{ll} D^{\circ k} = & D \circ D^{\circ(k-1)}, \text{ if } k > 1, \\ D^{\circ 1} = & D \qquad . \end{array}$$

Similarly one defines bullet power $D^{\bullet k}$ and associative power $D^{\cdot k}$. Since multiplications $\cdot$ and $\bullet$ are associative, in last cases $D^{\bullet k}$ and $D^{\cdot k}$ have usual properties of powers

$$D^{\bullet k} \bullet D^{\bullet s} = D^{\bullet(k+s)},$$

$$D^{\cdot k} \bullet D^{\cdot s} = D^{\cdot(k+s)},$$

These facts are not true for left-symmetric powers. For example,

$$D \circ (D \circ D^{\circ 2}) = (D \circ D) \circ D^{\circ 2},$$

but

$$D \circ D^{\circ 2} \circ D) \neq (D \circ D^{\circ 2}) \circ D.$$

Lemma 5.1. $D^{\cdot 2} = F.$

Proof.

$$D^{\cdot 2} = D \cdot D = \sum_{i,j=1}^{n} \eta_i \partial_i \eta_j \partial_j + \sum_{i,j=1}^{n} \eta_i \eta_j \partial_i \partial_j.$$

Since $\eta_i \eta_j = -\eta_j \eta_i$ and $\partial_i \partial_j = \partial_j \partial_i,$ we have

$$\sum_{i,j=1}^{n} \eta_i \eta_j \partial_i \partial_j = 0.$$

Thus,

$$D^2 = \sum_{i,j=1}^{n} \eta_i \partial_i \eta_j \partial_j = D \circ D = F.$$

Lemma 5.2. $D^{\circ(2n)} = F^{\circ n}$ *for any* $n = 1, 2, 3, \cdots$

Proof. We use induction on n.
If $n = 1,$ then nothing is to prove.
Suppose that

$$D^{\circ(2(n-1))} = F^{\circ(n-1)}$$

for some $n > 1.$ Then by definition

$$D^{\circ(2n)} = D \circ (D \circ D^{\circ 2(n-1)})$$

Since D is odd, by left-symmetric property of $(W_n, \circ)$ (proposition 3.3)

$$(D, D, G) =$$

for any $G \in W_n$. Thus,

$$D \circ (D \circ G) = (D \circ D) \circ G.$$

Therefore,

$$D^{\circ(2n)} = (D \circ D) \circ D^{\circ(2(n-1))}.$$

By inductive suggestion,

$$D^{\circ(2n)} = F \circ F^{\circ(n-1)} = F^{\circ n}.$$

Lemma 5.3. $F \circ F^{\bullet k} = kF^{\bullet(k-1)} \bullet F^{\circ 2}$

Proof. Since $F \in W_n$ is even derivation, any any left-symmetric multiplication operator acts on $(Diff_n, \bullet)$ as a super-derivation (proposition 3.3) we have

$$F \circ (F^{\bullet} F) = (F \circ F) \bullet F + F \bullet (F \circ F).$$

By commutativity of bullet-multiplication this means that

$$F \circ F^{\bullet 2} = 2F \bullet F^{\circ 2}.$$

Easy induction on k based on a such arguments shows that our lemma is true in general case.

Lemma 5.4. $D^4 = F^{\circ 2} + F^{\bullet 2}$

Proof. By Lemma 5.1 and by associativity of $\circ$,

$$D^4 = D^2 \cdot D^2 = F \cdot F = F \circ F + F \bullet F.$$

Lemma 5.5. $D^6 = F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}$.

Proof. By Lemma 5.1 and Lemma 5.4,

$$D^6 = D^2 \cdot D^4 = D^2 \circ D^4 + D^2 \bullet D^4$$

$$= F \circ (F^{\circ 2} + F^{\bullet 2}) + F \bullet (F^{\circ 2} + F^{\bullet 2})$$

$$F^{\circ 3} + F \circ F^{\bullet 2} + F \bullet F^{\circ 2} + F^{\bullet 3}.$$

Thus by Lemma 5.3,

$$D^6 = F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3}.$$

Lemma 5.6. $D^8 = F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}$.

Proof. By Lemma 5.3

$$F \circ F^{\bullet 3} = 3F^{\circ 2} \bullet F^{\bullet 2}.$$

Therefore, by Lemma 5.1, Lemma 5.5

$$D^8 = D^2 \cdot D^6 = D^2 \circ D^6 + D^2 \bullet D^6$$

$$= F \circ (F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3})$$
$$+F \bullet (F^{\circ 3} + 3F \bullet F^{\circ 2} + F^{\bullet 3})$$

$$= F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 3F \bullet F^{\circ 3} + 3F^{\circ 2} \bullet F^{\bullet 2}$$
$$+F \bullet F^{\circ 3} + 3F^{\bullet 2} \bullet F^{\circ 2} + F^{\bullet 4}$$

$$= F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4}.$$

Lemma 5.7.

$$D^{10} = F^{\circ 5}$$
$$+5(F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3}))$$
$$+5(2F^{\circ 3} \bullet F^{\bullet 2} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2})$$
$$+4F^{\circ 2} \bullet F^{\bullet 3} + 6F^{\circ 2} \bullet F^{\bullet 3} + F^{\bullet 5}$$

Proof. By Lemma 5.6 and Lemma 5.3

$$D^{10} = D^2 \circ D^8 = D^2 \circ D^8 + D^2 \bullet D^8$$

$$= F \circ (F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4})$$
$$+F \bullet (F^{\circ 4} + 3F^{\circ 2} \bullet F^{\circ 2} + 4F \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 2} + F^{\bullet 4})$$

$$= F^{\circ 5} + 3F \circ (F^{\circ 2} \bullet F^{\circ 2}) + 4F \circ (F \bullet F^{\circ 3}) + 6F \circ (F^{\circ 2} \bullet F^{\bullet 2}) + F \circ (F^{\bullet 4})$$
$$+F \bullet F^{\circ 4} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2} + 4F^{\bullet 2} \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 3} + F^{\bullet 5})$$

$$= F^{\circ 5} + 6F^{\circ 2} \bullet F^{\circ 3} + 4F^{\circ 2} \bullet F^{\circ 3} + 4F \bullet F^{\circ 4}$$
$$+6F^{\circ 3} \bullet F^{\bullet 2} + 12F^{\circ 2} \bullet F^{\circ 2} \bullet F + 4F^{\circ 2} \bullet F^{\bullet 3}$$
$$+F \bullet F^{\circ 4} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2} + 4F^{\bullet 2} \bullet F^{\circ 3} + 6F^{\circ 2} \bullet F^{\bullet 3} + F^{\bullet 5})$$

$$= F^{\circ 5}$$
$$+10F^{\circ 2} \bullet F^{\circ 3} + 5F \bullet F^{\circ 4}$$
$$+10F^{\circ 3} \bullet F^{\bullet 2} + 15F \bullet F^{\circ 2} \bullet F^{\circ 2}$$
$$+4F^{\circ 2} \bullet F^{\bullet 3} + 6F^{\circ 2} \bullet F^{\bullet 3}$$
$$+F^{\bullet 5}.$$

By Lemma 5.3,

$$F\circ(F^{\circ 3}\bullet F)=F^{\circ 4}\bullet F+F^{\circ 3}\bullet F^{\circ 2}.$$

Thus

$$2F^{\circ 2}\bullet F^{\circ 3}+F\bullet F^{\circ 4}=$$
$$F^{\circ 2}\bullet F^{\circ 3}+F\circ(F\bullet F^{\circ 3}).$$

Our lemma is proved.

Lemma 5.8. *For any* $G\in Diff_n$,

$$F\circ(\prod_{r=1}^{n}\eta_r\, G)=0.$$

Proof. We have

$$\eta_i\partial_i(\eta_j)\partial_j(\eta_1\cdots\eta_n)$$
$$=\sum_{s=1}^{n}\xi_s,$$

where

$$\xi_s=\eta_i\partial_i\eta_j\eta_1\cdots\eta_{s-1}\partial_j(\eta_s)\eta_{s+1}\cdots\eta_n.$$

If $s\neq i$, then

$$\xi_s=\pm\eta_i\eta_i\xi_{i,s},$$

where

$$\xi_{i,s}=\partial_i\eta_j\partial_j\eta_s\prod_{r\neq i,s}\eta_r.$$

Since $\eta_i\eta_i=0$, this means that

$$\xi_s=0,$$

if $s\neq i$. If $s=i$, then

$$\xi_s=\pm\eta_i\partial_i\eta_j\partial_j\eta_i\prod_{r\neq i}\eta_r=\partial_i\eta_j\partial_j\eta_i(\prod_r\eta_r).$$

We have

$$\sum_{i,j=1}^{n}\partial_i\eta_j\partial_j\eta_i=\theta_1+\theta_2+\theta_3,$$

where

$$\theta_1=\sum_{i<j}\partial_i\eta_j\partial_j\eta_i,$$
$$\theta_2=\sum_{i}\partial_i\eta_i\partial_i\eta_i,$$
$$\theta_3=\sum_{i>j}\partial_i\eta_j\partial_j\eta_i,$$

Since elements $\partial_i\eta_j$ and $\partial_j\eta_i$ are odd,

$$\theta_1+\theta_3=0,\qquad \theta_2=0.$$

Thus,

$$F \circ (\prod_{r=1}^{n} \eta_r G) = (\sum_{i,j=1}^{n} \partial_i \eta_j \partial_j \eta_i) \prod_r \eta_r G = 0.$$

Let

$$Diff_n^{[s]} =< u\partial^\alpha | u \in \mathcal{L}_n, \alpha \in \Gamma_n, |\alpha| = s >$$

be a space of differential operators of order s and

$$\tau_s : Diff_n \to Diff_n^{[s]}$$

be projection map.

Lemma 5.9. *If* $n = 3$, $D = \sum_{i=1}^n u_i \partial_i$, *and* u_i *are odd, then*

$$\tau_1 D = F^{\circ 5},$$

$$\tau_2 D^{10} = 5(F^{\circ 2} \bullet F^{\circ 3} + F \circ (F \bullet F^{\circ 3})),$$

$$\tau_3 D^{10} = 5(2F^{\circ 3} \bullet F^{\bullet 2} + 3F \bullet F^{\circ 2} \bullet F^{\circ 2}),$$

$$\tau_s D^{10} = 0, \qquad s > 3.$$

Proof. Follows from Lemma 5.7 and from the fact that $F^{\bullet s} = 0$, if $s > n$.

Conclusion. To find D^{10} we need to calculate $F^{\circ s}$, for $s = 1, 2, 3$ and $F^{\bullet 2}$.

6. Second bullet-power of F

The following calculations are not difficult.

$$F^{\bullet 2}_{\eta_1\eta_2;\partial_1^2} = -2\eta_1\eta_2\partial_1\eta_1\partial_2\eta_1,$$

$$F^{\bullet 2}_{\eta_1\eta_3;\partial_1^2} = -\eta_1\eta_3 2\partial_1\eta_1\partial_3\eta_1\partial_1^2,$$

$$F^{\bullet 2}_{\eta_2\eta_3;\partial_1^2} = -2\eta_2\eta_3\partial_2\eta_1\partial_3\eta_1\partial_1^2,$$

7. Second left-symmetric power of F

It is not hard to obtain the following results.

$$F^{\circ 2}_{\eta_1;\partial_1} =$$
$$\eta_1(-2\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2 - 2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3 + \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2$$
$$-\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3 + \partial_3\eta_1\partial_1\eta_2\partial_2\eta_3 + \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3)\partial_1,$$

$$F^{\circ 2}_{\eta_2;\partial_1} =$$
$$\eta_2(-\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2 - \partial_1\eta_1\partial_3\eta_1\partial_2\eta_3 - \partial_2\eta_1\partial_3\eta_1\partial_1\eta_3$$
$$-\partial_2\eta_1\partial_3\eta_2\partial_2\eta_3 + \partial_3\eta_1\partial_2\eta_2\partial_2\eta_3 + \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3)\partial_1,$$

$$F^{\circ 2}_{\eta_3;\partial_1} =$$
$$\eta_3(-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2 - \partial_1\eta_1\partial_3\eta_1\partial_3\eta_3 - \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2$$
$$+\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2 - \partial_2\eta_1\partial_3\eta_2\partial_3\eta_3 + \partial_3\eta_1\partial_3\eta_2\partial_2\eta_3)\partial_1,$$

$$F^{\circ 2}_{\eta_1\eta_2;\partial_1} =$$
$$\eta_1\eta_2(-\partial_1\eta_1\partial_1\partial_2\eta_1 - \partial_1\eta_2\partial_2^2\eta_1 - \partial_1\eta_3\partial_2\partial_3\eta_1$$
$$+\partial_2\eta_1\partial_1^2\eta_1 + \partial_2\eta_2\partial_1\partial_2\eta_1 + \partial_2\eta_3\partial_1\partial_3\eta_1)\partial_1,$$

$$F^2(\eta_1\eta_3;\partial_1) =$$
$$\eta_1\eta_3(-\partial_1\eta_1\partial_1\partial_3\eta_1 - \partial_1\eta_2\partial_2\partial_3\eta_1 - \partial_1\eta_3\partial_3^2\eta_1$$
$$+\partial_3\eta_1\partial_1^2\eta_1 + \partial_3\eta_2\partial_1\partial_2\eta_1 + \partial_3\eta_3\partial_1\partial_3\eta_1)\partial_1,$$

$$F^{\circ 2}_{\eta_2\eta_3;\partial_1} =$$
$$\eta_2\eta_3(-\partial_2\eta_1\partial_1\partial_3\eta_1 - \partial_2\eta_2\partial_2\partial_3\eta_1 - \partial_2\eta_3\partial_3^2\eta_1$$
$$+\partial_3\eta_1\partial_1\partial_2\eta_1 + \partial_3\eta_2\partial_2^2\eta_1 + \partial_3\eta_3\partial_2\partial_3\eta_1)\partial_1.$$

8. Third left-symmetric power of F

In this section we give results of some calculations concerning $F^{\circ 3} = F\circ(F\circ F)$

$$F^{\circ 3}_{\eta_1;\partial_1} =$$
$$\eta_1(2\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3+2\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3-6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3$$
$$-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3-2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3-2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3$$
$$+2\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3-2\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3+\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3$$
$$-\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3+2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_3\eta_3-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3$$
$$+\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3-\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3-2\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3)\partial_1$$

(all together 15 terms)

$$F^{\circ 3}_{\eta_2;\partial_1} =$$
$$\eta_2(2\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3$$
$$-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3-\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3$$
$$+\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3 + \partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3)\partial_1.$$

(all together 8 terms)

$$F^{\circ 3}_{\eta_3;\partial_1} =$$
$$\eta_3(\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3 - 2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_3 - 2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3$$
$$+\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3 - 2\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3 - \partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3$$
$$-\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3 + 2\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3)\partial_1.$$

(all together 8 terms)

$$F^{\circ 3}_{\eta_1\eta_2;\partial_1} =$$
$$\eta_1\eta_2(3\partial_1\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2^2\eta_1 + 3\partial_1\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 + 4\partial_1\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1$$
$$-\partial_1\eta_1\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 - 2\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_1\partial_2\eta_1 + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2^2\eta_2$$
$$-5\partial_1\eta_1\partial_2\eta_1\partial_1\eta_3\partial_1\partial_3\eta_1 + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_3\partial_2\partial_3\eta_2 - 2\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1\partial_2\eta_2$$
$$+\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1^2\eta_1 - 3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_1\partial_3\eta_2 + 3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1^2\eta_3$$
$$+\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\partial_2\eta_3 - 4\partial_1\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 - 2\partial_1\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1$$
$$-2\partial_1\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 + \partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2^2\eta_3 + 3\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_2\eta_1$$
$$+\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_2\partial_3\eta_3 - 3\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\partial_2\eta_3 + \partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_2\eta_2$$
$$-3\partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_3\eta_3 + \partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1^2\eta_1 + \partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1\partial_2\eta_3$$
$$+\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1 + 2\partial_1\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 - 3\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1$$
$$-3\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 + 3\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_1 - 4\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1$$
$$+2\partial_2\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 - 2\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\partial_2\eta_1 + \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2^2\eta_2$$
$$+2\partial_2\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2 - \partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_3 + 2\partial_2\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_2$$
$$+3\partial_2\eta_1\partial_1\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 + \partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_1\partial_3\eta_1 - \partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_2$$
$$-\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1\partial_3\eta_2 + \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_3 + 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_3$$
$$+2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_3 - 4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1^2\eta_1 - \partial_2\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1^2\eta_2$$
$$-\partial_2\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1^2\eta_3 - 2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1\partial_2\eta_1 + \partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3$$
$$-\partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_3 - \partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1^2\eta_1 + 2\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1$$
$$+2\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 + \partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 - \partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1$$
$$-2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 - 2\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2^2\eta_1 + 2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2^2\eta_3$$
$$-2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 + 2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3 - \partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2^2\eta_2$$
$$-\partial_3\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2^2\eta_3 - 4\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1 - \partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_2$$
$$+2\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_3 - \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 - \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3$$
$$+3\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_1\partial_2\eta_1 - 2\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3 + \partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1\partial_2\eta_2$$
$$+\partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1^2\eta_1 + \partial_3\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_2\eta_3 + \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_3$$
$$+\partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1^2\eta_1 - 3\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_1 - \partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_1$$
$$+\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1)\partial_1.$$

(all together 76 terms)

$$
\begin{gathered}
F^{\circ 3}_{\eta_1\eta_3;\partial_1} = \\
\eta_1\eta_3(-\partial_1\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\partial_3\eta_1 - \partial_1\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3^2\eta_1 + 4\partial_1\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_1 \\
+4\partial_1\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1 + 3\partial_1\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_1 + 3\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_1\partial_3\eta_1 \\
+\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\partial_3\eta_2 + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_3\partial_3^2\eta_2 + \partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1\partial_3\eta_2 \\
-3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1^2\eta_2 - 3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\partial_2\eta_2 + \partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\partial_3\eta_3 \\
+\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1^2\eta_1 - 3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_3\partial_1\partial_3\eta_2 + 2\partial_1\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_1 \\
-5\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_1 + \partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\partial_3\eta_3 - 2\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_1 \\
+\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_3^2\eta_3 + \partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_3\eta_2 - 3\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\partial_2\eta_3 \\
-2\partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1\partial_3\eta_3 + \partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1^2\eta_1 - 3\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 \\
-2\partial_1\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 + 2\partial_1\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1 - \partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3^2\eta_1 \\
+4\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_1 + 2\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1 + \partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
-3\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 + 2\partial_1\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_1 + 2\partial_2\eta_1\partial_1\eta_2\partial_1\eta_3\partial_3^2\eta_1 \\
-\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\partial_3\eta_1 - \partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\partial_3\eta_2 - 4\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\partial_2\eta_1 \\
-\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\partial_3\eta_3 + 2\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_2 - 3\partial_2\eta_1\partial_1\eta_2\partial_3\eta_3\partial_1\partial_3\eta_1 \\
+2\partial_2\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_2 + 2\partial_2\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_2 + \partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3^2\eta_2 \\
-\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_2 - \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1^2\eta_1 - \partial_2\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_3\eta_2 \\
-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_2 + 4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1^2\eta_1 - 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_2 \\
+\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1^2\eta_2 + \partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1^2\eta_3 + 2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1\partial_3\eta_1 \\
-2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_2 + \partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3^2\eta_3 - \partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_3\eta_3 \\
-\partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1^2\eta_1 + 3\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 - \partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 \\
-\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1 - 2\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1 + 3\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\partial_2\eta_1 \\
+2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1 - \partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2 + 2\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_3 \\
-\partial_3\eta_1\partial_1\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1 + \partial_3\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3 - \partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_2 \\
-2\partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 + \partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_3 - \partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_2 \\
-2\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3 + \partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_2 + \partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1^2\eta_1 \\
+\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_3 + 3\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 - 3\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 \\
+2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1)\partial_1.
\end{gathered}
$$

(all together 76 terms)

$$
\begin{gathered}
F^{\circ 3}_{\eta_2\eta_3;\partial_1} = \\
\eta_2\eta_3(\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\partial_3\eta_1 + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_3\partial_3^2\eta_1 + \partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_1\partial_3\eta_1
\end{gathered}
$$

$$
\begin{aligned}
&+\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_2\partial_3\eta_2+\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_3^2\eta_2-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\partial_2\eta_2\\
&-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\partial_3\eta_3-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1^2\eta_1-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\partial_2\eta_1\\
&-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_2^2\eta_2-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_3\partial_2\partial_3\eta_2+2\partial_1\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_1\\
&+2\partial_1\eta_1\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1+2\partial_1\eta_1\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1-\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2^2\eta_1\\
&-\partial_1\eta_1\partial_3\eta_1\partial_1\eta_3\partial_2\partial_3\eta_1+\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_2\partial_3\eta_3+\partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_3\eta_1\\
&+\partial_1\eta_1\partial_3\eta_1\partial_2\eta_3\partial_3^2\eta_3-\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2^2\eta_3-\partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1\partial_2\eta_1\\
&-\partial_1\eta_1\partial_3\eta_1\partial_3\eta_3\partial_2\partial_3\eta_3-2\partial_1\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1-\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\partial_3\eta_1\\
&+2\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2^2\eta_1+\partial_2\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2\partial_3\eta_1+\partial_2\eta_1\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_1\\
&+\partial_2\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3^2\eta_1+\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3^2\eta_2-3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_2\eta_1\\
&-\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\partial_3\eta_3+\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_2-2\partial_2\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_3\eta_1\\
&+\partial_2\eta_1\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_2+2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_2-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\partial_2\eta_1\\
&-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_1\partial_3\eta_1-\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\partial_2\eta_2+2\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1^2\eta_1\\
&-\partial_2\eta_1\partial_3\eta_1\partial_2\eta_3\partial_1\partial_3\eta_2-\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\partial_2\eta_3-\partial_2\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1\partial_3\eta_3\\
&+2\partial_2\eta_1\partial_3\eta_1\partial_3\eta_3\partial_1^2\eta_1-2\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1+\partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1\\
&-2\partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2+\partial_2\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3^2\eta_3+\partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1\\
&-\partial_2\eta_1\partial_3\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3+\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1-\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1\\
&+\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_1-\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2^2\eta_1-2\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1\\
&+\partial_3\eta_1\partial_1\eta_2\partial_3\eta_3\partial_2^2\eta_1-2\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1+\partial_3\eta_1\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1\\
&+\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1+\partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1-\partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2\\
&+2\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_3-2\partial_3\eta_1\partial_2\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1+\partial_3\eta_1\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3\\
&-3\partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1-\partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2+\partial_3\eta_1\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3\\
&+\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1-2\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3+\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_2\\
&+\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_3-\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1)\partial_1.
\end{aligned}
$$

(all together 71 terms)

9. Quadratic differential part of D^{10}

For $G \in Diff_n$ denote by $G_{\eta_{i_1}\cdots\eta_{i_k};\partial^\alpha}$ projection to subspace of $Diff_n$ generated by differential operators of the form $\eta_{i_1}\cdots\eta_{i_k}\prod_{s,\beta\neq 0}\partial^\beta u_s\partial^\alpha$. For example, if

$$G = -5\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\partial_3^2\eta_3\partial_1 + \eta_1\eta_3\partial_2\eta_1\partial_1\eta_3\partial_1\partial_2^3\eta_3\partial_3^2$$
$$-\eta_1\eta_3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_1\partial_2\partial_3\eta_3\partial_2 + 9\eta_1\eta_3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_1\partial_2\partial_3\eta_3\partial_2$$
$$-7\eta_1\eta_3\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1^2\partial_2\partial_3^2\eta_3\partial_3^2,$$

then

$$G_{\eta_1\eta_3;\partial_3^2} = \eta_1\eta_3\partial_2\eta_1\partial_1\eta_3\partial_1\partial_2^3\eta_3\partial_3^2 - 7\eta_1\eta_3\partial_1\eta_1\partial_3\eta_2\partial_1\eta_3\partial_1^2\partial_2\partial_3^2\eta_3\partial_3^2.$$

Lemma 9.1. $F^{\circ 2}\bullet F^{\circ 3} + F\circ(F\bullet F^{\circ 3}) = 0.$

Proof. Let

$$Q = F^{\circ 2}\bullet F^{\circ 3} + F\circ(F\bullet F^{\circ 3}).$$

It is enough to prove that ∂_1^2-part of Q is equal to 0. Then by symmetry ∂_2^2-, ∂_3^2-parts of Q should be 0, and $\partial_1\partial_2$-, $\partial_1\partial_3$-, $\partial_2\partial_3$-parts of G also will vanish.

Let us show how to calculate $\eta_1\eta_2\eta_3\partial_1^2$-part of Q.

Notice that $\eta_1\eta_2\eta_3\partial_1^2$-part of $F^{\circ 2}\bullet F^{\circ 3}$, denote it by G_1, is equal to

$$G_1 =$$
$$F^{\circ 2}_{\eta_1;\partial_1}\bullet F^{\circ 3}_{\eta_2\eta_3;\partial_1} + F^{\circ 2}_{\eta_2;\partial_1}\bullet F^{\circ 3}_{\eta_1\eta_3;\partial_1} + F^{\circ 2}_{\eta_3;\partial_1}\bullet F^{\circ 3}_{\eta_1\eta_2;\partial_1} +$$
$$F^{\circ 2}_{\eta_1\eta_2;\partial_1}\bullet F^{\circ 3}_{\eta_3;\partial_1} + F^{\circ 2}_{\eta_1\eta_3;\partial_1}\bullet F^{\circ 3}_{\eta_2;\partial_1} + F^{\circ 2}_{\eta_2\eta_3;\partial_1}\bullet F^{\circ 3}_{\eta_1;\partial_1}.$$

Using results of sections 7, 8 we obtain that

$$F^{\circ 2}_{\eta_1;\partial_1}\bullet F^{\circ 3}_{\eta_2\eta_3;\partial_1} + F^{\circ 2}_{\eta_2;\partial_1}\bullet F^{\circ 3}_{\eta_1\eta_3;\partial_1} + F^{\circ 2}_{\eta_3;\partial_1}\bullet F^{\circ 3}_{\eta_1\eta_2;\partial_1}$$

$$= \eta_1\eta_2\eta_3(-5\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 + 14\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1$$
$$-\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_1 - 8\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1$$
$$+9\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 + 4\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1$$
$$+3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 - 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1$$
$$+8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 + 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1$$
$$-6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2 - \partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3^2\eta_3$$
$$+10\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_3 - 2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1$$
$$+4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3 - \partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_2^2\eta_2$$
$$-4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 - 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2$$
$$+2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3 - 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1$$
$$+2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 - 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3$$

$$\begin{gathered}
+7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_2 + 5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_3 \\
-10\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_2 + 5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_2 \\
-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 + 4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2 \\
-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_3 + 5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_2 \\
+4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_1\partial_2\eta_1 + 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3 \\
-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_2 - 7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_2 \\
-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_3 - 5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_3 \\
+2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1 - 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_2 \\
+7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_3 - 6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3 \\
-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_2 + 2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_2 \\
+7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_3 + 13\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
+4\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 - 13\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_1 \\
+9\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2^2\eta_1 + \partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
-8\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_1 - 5\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_1 \\
-3\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 - 14\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 \\
+\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 + \partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 - 4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 \\
+6\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 - 8\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 \\
+\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2 + \partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3 \\
-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1 + 13\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 \\
-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3 - 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_2 \\
+2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_1 - 4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_1 \\
-5\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 + \partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3 \\
+\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2^2\eta_2 + 5\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1 \\
-\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_2 - \partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_3 \\
+8\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 - \partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3 \\
-\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_2 + 13\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 \\
-2\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2 - 2\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3 \\
-\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2^2\eta_1 - \partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1)\partial_1^2.
\end{gathered}$$

Similarly,

$$F^{\circ 2}_{\eta_1\eta_2;\partial_1} \bullet F^{\circ 3}_{\eta_3;\partial_1} + F^{\circ 2}_{\eta_1\eta_3;\partial_1} \bullet F^{\circ 3}_{\eta_2;\partial_1} + F^{\circ 2}_{\eta_2\eta_3;\partial_1} \bullet F^{\circ 3}_{\eta_1;\partial_1} =$$

$$\begin{gathered}
\eta_1\eta_2\eta_3(-4\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1 + \partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_1 \\
+3\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 - 4\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1 \\
+\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 - 3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1
\end{gathered}$$

$$+8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1-8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1$$
$$-8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3\partial_2\partial_3\eta_1+4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1$$
$$+\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1+6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1$$
$$+8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1-5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1$$
$$+\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1-6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_1\partial_2\eta_1$$
$$+3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1^2\eta_1-5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1$$
$$+4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1-3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1^2\eta_1$$
$$-3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1+\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1$$
$$+3\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_1-4\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2^2\eta_1$$
$$-\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1+3\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_1$$
$$+3\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1+4\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1$$
$$-\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1-\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1$$
$$+2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1+4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1$$
$$-6\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1+4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1$$
$$+2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_1-5\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1$$
$$-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_1+4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_1$$
$$+\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1-\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1$$
$$-4\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1-5\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1$$
$$+\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2^2\eta_1+\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1)\partial_1^2.$$

Thus,

$$G_1=$$
$$\eta_1\eta_2\eta_3(-5\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1+10\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_1$$
$$-5\partial_1\eta_1\partial_2\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1+5\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_1$$
$$+5\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1+4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_1\partial_3\eta_1$$
$$-6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_2\partial_3\eta_2-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3^2\eta_3$$
$$+10\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2^2\eta_3-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_1\partial_2\eta_1$$
$$+4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_2\partial_3\eta_3-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3\partial_2^2\eta_2$$
$$+2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1-8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2$$
$$+2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3-3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1$$
$$-8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3+7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_2$$
$$+5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_3\eta_3\partial_2^2\eta_3-10\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_2$$
$$+5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_2-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1$$
$$+4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2-\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3\partial_3^2\eta_3$$
$$+5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_2-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_1\partial_2\eta_1$$
$$+8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\partial_3\eta_3-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2^2\eta_2$$

$$
\begin{gathered}
-7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_2 - 2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_3\eta_3 \\
+3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1^2\eta_1 - 5\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_3\eta_3\partial_1\partial_2\eta_3 \\
-3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1 - 8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_2 \\
+7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_3 + 4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 \\
-6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3 - \partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_2 \\
+2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_2 + 7\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_3 \\
-3\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1^2\eta_1 + 10\partial_1\eta_1\partial_2\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_1 \\
+5\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 - 10\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_1 \\
+5\partial_1\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2^2\eta_1 - 5\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_1 \\
-5\partial_1\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_1 - 10\partial_1\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_1 \\
-4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 + \partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2 \\
+\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3 + 8\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_1\partial_2\eta_1 \\
-2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2\partial_3\eta_3 - 2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3\partial_2^2\eta_2 \\
-4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 + \partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3 \\
+\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\partial_2^2\eta_2 + 4\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_1\partial_3\eta_1 \\
-\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_2\partial_3\eta_2 - \partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3^2\eta_3 \\
+4\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_1\partial_2\eta_1 - \partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2\partial_3\eta_3 \\
-\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\partial_2^2\eta_2 + 8\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_1\partial_3\eta_1 \\
-2\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_2\partial_3\eta_2 - 2\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3\partial_3^2\eta_3)\partial_1^2.
\end{gathered}
$$

Now calculate $\eta_1\eta_2\eta_3\partial_1^2$-part of $F \circ (F \bullet F^{\circ 3})$. Set $G = F \bullet F^{\circ 3}$. It is easy to see that

$$G_{\eta_1\eta_2;\partial_1^2} = F_{\eta_1;\partial_1} \bullet F^{\circ 3}_{\eta_2;\partial_1} + F_{\eta_2;\partial_1} \bullet F^{\circ 3}_{\eta_1;\partial_1},$$

$$G_{\eta_1\eta_3;\partial_1^2} = F_{\eta_1;\partial_1} \bullet F^{\circ 3}_{\eta_3;\partial_1} + F_{\eta_3;\partial_1} \bullet F^{\circ 3}_{\eta_1;\partial_1},$$

$$G_{\eta_2\eta_3;\partial_1^2} = F_{\eta_2;\partial_1} \bullet F^{\circ 3}_{\eta_3;\partial_1} + F_{\eta_3;\partial_1} \bullet F^{\circ 3}_{\eta_2;\partial_1}.$$

By results of section 8,

$$
\begin{aligned}
G_{\eta_1\eta_2;\partial_1^2} &= \eta_1\eta_2 H_{\eta_1\eta_2}\partial_1^2, \\
G_{\eta_1\eta_3;\partial_1^2} &= \eta_1\eta_3 H_{\eta_1\eta_3}\partial_1^2, \\
G_{\eta_2\eta_3;\partial_1^2} &= \eta_2\eta_3 H_{\eta_2\eta_3}\partial_1^2,
\end{aligned}
$$

where,

$$
\begin{gathered}
H_{\eta_1\eta_2} = \\
4\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1 - 2\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_3 \\
-\partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_3 + 3\partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 \\
+\partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_3 + \partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2
\end{gathered}
$$

$$+2\partial_1\eta_3\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3,$$

$$H_{\eta_1\eta_3} =$$
$$-\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_3 + 3\partial_1\eta_1\partial_1\eta_2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2$$
$$-2\partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_2 + 4\partial_1\eta_1\partial_1\eta_3\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3$$
$$-2\partial_1\eta_2\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 - \partial_1\eta_2\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3$$
$$-\partial_1\eta_3\partial_2\eta_1\partial_2\eta_2\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3,$$

$$H_{\eta_2\eta_3} =$$
$$-3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_2\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2 - 3\partial_1\eta_1\partial_2\eta_1\partial_2\eta_3\partial_3\eta_1\partial_3\eta_2\partial_3\eta_3.$$

Thus, $\eta_1\eta_2\eta_3\partial_1^2$-part of $F\circ G$ by Lemma 5.8 is equal to $\eta_1\eta_2\eta_3\partial_1^2$-part of

$$F\circ(G_{\eta_1\eta_2;\partial_1^2} + G_{\eta_1\eta_3;\partial_1^2} + G_{\eta_2\eta_3;\partial_1^2}).$$

So, $\eta_1\eta_2\eta_3\partial_1^2$-part of $F\circ G$ is equal to

$$\eta_1\eta_2\eta_3(\partial_1(D)\circ H_{\eta_2\eta_3;\partial_1^2} + \partial_2(D)\circ H_{\eta_1\eta_3;\partial_1^2} + \partial_3(D)\circ H_{\eta_1\eta_2;\partial_1^2})\partial_1^2$$

Calculations show that this expression is equal to $-G_1$. So, we obtain that $\eta_1\eta_2\eta_3\partial_1^2$-part of $F^{\circ 2}\bullet F^{\circ 3} + F\circ(F\bullet F^{\circ 3})$ is equal 0.

Similar calculations show that sums of $\eta_i\eta_j\partial_1^2$-parts of $F^{\circ 2}\bullet F^{\circ 3}$ and $F\circ(F\bullet F^{\circ 3})$ are also vanish, if $i\neq j$. So, we have established that

$$F^{\circ 2}\bullet F^{\circ 3} + F\circ(F\bullet F^{\circ 3}) = 0.$$

10. Qubic differential part of D^{10}

In this section we use denotions and results of calculations of section 9.

Lemma 10.1. $F^{\circ 3}\bullet F^{\bullet 2} = 0$

Proof. Recall that $G = F\bullet F^{\circ 3}$. By associativity and super-commutativity of bullet-multiplication (proposition **??**) we have

$$F^{\circ 3}\bullet(F\bullet F) = F\bullet(F\bullet F^{\circ 3}).$$

So, $\eta_1\eta_2\eta_3\partial_1^2$-part of $F^{\circ 3}\bullet F^{\bullet 2}$ is equal to

$$F\bullet(\eta_1\eta_2 H_{\eta_1\eta_2} + \eta_1\eta_3 H_{\eta_1\eta_3} + \eta_2\eta_3 H_{\eta_2\eta_3})\partial_1^2$$
$$= \eta_1\eta_2\eta_3(\partial_1\eta_1\partial_1\bullet H_{\eta_2\eta_3}\partial_1^2 - \partial_2\eta_1\partial_1\bullet H_{\eta_1\eta_3}\partial_1^2 + \partial_3\eta_1\partial_1\bullet H_{\eta_1\eta_2}\partial_1^2).$$

By results of section 9 it is easy to obtain that

$$\partial_1\eta_1 H_{\eta_2\eta_3} = 0,$$
$$\partial_2\eta_1 H_{\eta_1\eta_3} = 0,$$
$$\partial_3\eta_1 H_{\eta_1\eta_2} = 0.$$

So, $\eta_1\eta_2\eta_3\partial_1^3$-part of $F^{\circ 3}\bullet F^{\bullet 2}$ is 0. Since number of bullets is two, $\eta_i\eta_j\partial_1^3$-parts and $\eta_i\partial_1^3$-parts of $F^{\circ 3}\bullet F^{\bullet 2}$ are also 0. So, ∂_1^3-part of $F^{\circ 3}\bullet F^{\bullet 2}$ vanishes. By symmetry ∂^α-part of $F^{\circ 3}\bullet F^{\bullet 2}$ vanishes also for any $\alpha\in\Gamma_3$, such that $|\alpha|=3$. Lemma is proved.

Lemma 10.2. $F\bullet F^{\circ 2}\bullet F^{\circ 2}=0$.

Proof. Let $R=F^{\circ 2}\bullet F^{\circ 2}$. We use calculations on $F^{\circ 2}$ (section 7) to obtain

$$\begin{gathered}R_{\eta_1\eta_2;\partial_1^2}\\ =\eta_1\eta_2(8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3-4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_3\partial_3\eta_3\\ +2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_1\eta_3\partial_3\eta_3+6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\\ -2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_2\eta_3\partial_3\eta_3+2\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3+\\ 4\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_2\eta_3\partial_3\eta_3)\partial_1^2,\end{gathered}$$

$$\begin{gathered}R_{\eta_1\eta_3;\partial_1^2}\\ =\eta_1\eta_3(-2\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_3-6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\\ -4\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3+8\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3\\ -4\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3+2\partial_2\eta_1\partial_3\eta_1\partial_1\eta_2\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3\\ -2\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_1\eta_3\partial_3\eta_3)\partial_1^2,\end{gathered}$$

$$\begin{gathered}R_{\eta_2\eta_3;\partial_1^2}\\ =\eta_2\eta_3(-6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_2\eta_2\partial_3\eta_2\partial_2\eta_3+6\partial_1\eta_1\partial_2\eta_1\partial_3\eta_1\partial_3\eta_2\partial_2\eta_3\partial_3\eta_3)\partial_1^2.\end{gathered}$$

We have

$$\begin{aligned}\eta_1\partial_1\eta_1\partial_1\bullet R_{\eta_2\eta_3;\partial_1^2}&=0,\\ \eta_2\partial_2\eta_1\partial_1\bullet R_{\eta_1\eta_3;\partial_1^2}&=0,\\ \eta_3\partial_3\eta_1\partial_1\bullet R_{\eta_1\eta_2;\partial_1^2}&=0.\end{aligned}$$

Therefore $\eta_1\eta_2\eta_3\partial_1^3$-part of $F\bullet F^{\circ 2}\bullet F^{\circ 2}$ is equal to

$$\begin{gathered}\eta_1\partial_1\eta_1\partial_1\bullet R_{\eta_2\eta_3;\partial_1^2}+\eta_2\partial_2\eta_1\partial_1\bullet R_{\eta_1\eta_3;\partial_1^2}+\eta_3\partial_3\eta_1\partial_1\bullet R_{\eta_1\eta_2;\partial_1^2}\\ =0.\end{gathered}$$

By symmetry, $\eta_1\eta_2\eta_3\partial_1^\alpha$-part of $F\bullet F^{\circ 2}\bullet F^{\circ 2}$ are also 0 for any $\alpha\in\Gamma_3$, such that $|\alpha|=3$. As we mentioned above η_i- and $\eta_i\eta_j$-parts of elements obtained by two bullets are equal to 0. Lemma is proved.

By Lemma 5.9

$$\tau_3(F^5)=5(2F^{\circ 3}\bullet F^{\bullet 2}+3F\bullet F^{\circ 2}\bullet F^{\circ 2}).$$

Therefore, we come to the following

Conclusion. $\tau_3(D^{10})=0$.

Proof of Theorem 0.1

By Theorem 0.1 of [2] if $\eta_{i_1}\cdots\eta_{i_r}\partial^\alpha$-part of D^{10} is nonzero, then

$$|\alpha|\le 3.$$

So,

$$D^{10}=\tau_1(D^{10}).$$

11. N-commutators and super-derivations

In this section we explain how escort invariants appear in calculating powers of odd derivations.

Suppose now $I=\{1,\dots,n\}$ and $D=\sum_{i=1}^n u_i\partial_i\in Der\,\mathcal{L}$ odd super-derivation. For $\alpha\in\mathbf{Z}_+^n$ set

$$x^{(\alpha)}=\frac{x^\alpha}{\alpha!}.$$

Denote by $Supp(s_k)$ a set of k-typles $\{(\alpha^{(1)},i_1),\cdots,(\alpha^{(k)},i_k)\}$ with $\alpha^{(1)},\dots,\alpha^{(k)}\in\mathbf{Z}_+^n, i_1,\dots,i_k\in I$, such that $\sum_{p=1}^k\alpha^{(p)}-\epsilon_{i_p}$ has a form $-\beta$ for some $0\ne\beta\in\mathbf{Z}_+^n$.

Theorem 11.1.

$$k!D^k=$$

$$\sum\partial^\alpha(u_{i_1})\partial^\beta(u_{i_2})\cdots\partial^\gamma(u_{i_k})esc(s_k)(x^{(\alpha)}\partial_{i_1},x^{(\beta)}\partial_{i_2},\dots,x^{(\gamma)}\partial_{i_k}),$$

where summation is by $\{(\alpha,i_1),(\beta,i_2),\dots,(\gamma,i_k)\}\in Supp(s_k)$.

If we use order on basic elements $x^\alpha\partial_i$ we can omit the coefficient $k!$:

$$D^k=\sum_{(\alpha^{(1)},i_1)<\cdots<(\alpha^{(k)},i_k)}\partial^{\alpha^{(1)}}(u_{i_1})\cdots\partial^{\alpha^{(k)}}(u_{i_k})esc(s_k)(x^{(\alpha^{(1)})}\partial_{i_1},\dots,x^{(\alpha^{(k)})}\partial_{i_k}).$$

Proof. Recall that $U=\mathbf{C}[x_1,\dots,x_n]$ and ∂_i are partial derivations of U. Let Gr_k be a Grassman algebra with exterior generators $\eta_1,\dots,\eta_k$, i.e., it is associative super-commutative algebra of dimension 2^k. For $U=\mathbf{C}[x_1,\dots,x_n]$ take its Grassman envelope

$$\mathcal{U}=U\otimes Gr_k.$$

Prolong derivation $\partial_i\in Der\,U$ to a derivation of $\mathcal{U}$ by

$$\partial_i(v\otimes\omega)=\partial_i(v)\otimes\omega.$$

We obtain commuting system of even derivations $\mathcal{D}=\{\partial_1,\dots,\partial_n\}$ of $\mathcal{U}$. For any $f_1,\dots,f_n\in\mathcal{U}$ and $\alpha\in\mathbf{Z}_+^n$ elements $\partial^\alpha u_j$ are odd. So, we obtain $\mathcal{D}$-differential super-algebra $\mathcal{U}$ and we can consider its algebra of super-derivations $\mathcal{L} =< f\partial_i : f \in \mathcal{U} >$ and its algebra of super-differential operators

$$\mathcal{D}iff =< f\partial^\alpha : \alpha \in \mathbf{Z}_+^n, f \in \mathcal{U} > .$$

We can endow $\mathcal{D}iff$ by composition operation, by left-symmetric multiplication and by bullet multiplication. In particular, we can consider $\mathcal{L}$ as a left-symmetric algebra and as a super-Lie algebra. Thus,

$$\mathcal{L} \cong W_n \otimes Gr_k$$

is isomorphic to a current algebra with coefficients not in Laurent polynomials as usual, but in exterior algebra.

We see that for any $f_1, \ldots , f_n$ we can consider a homomorphism

$$\mathcal{L}_n \to \mathcal{U}, u_i \mapsto f_i, i = 1, \ldots , n.$$

and this homomorphism can be extended to a homomorphism of left-symmetric or Lie algebras

$$Der\,\mathcal{L}_n \to \mathcal{L}$$

and to a homomorphism of associative (left-symmetric) algebras

$$Diff \to \mathcal{D}iff$$

We can use this homomorphism in calculating F^k for $F = \sum_{i=1}^n f_i\partial_i \in \mathcal{L}$. In other words, in the formula for D^k we can make substitutions $u_i \mapsto f_i$ and calculate obtained expressions in $\mathcal{U}$.

Use this method for calculating coefficients $\lambda_{\{(\alpha,i_1),(\beta,i_2),\ldots,(\gamma,i_k);\mu\}}$, where

$$D^k = \sum \lambda_{\{(\alpha,i_1),(\beta,i_2),\ldots,(\gamma,i_k);\mu\}}\partial^\alpha(u_{i_1})\partial^\beta(u_{i_2})\cdots\partial^\gamma(u_{i_k})\partial^\mu.$$

Since numbers of u_i-indexes and ∂_i-indexes are equal, summation here is done by $\alpha, \beta, \ldots , \gamma, \mu \in \mathbf{Z}_+^n$ and $i_1, \ldots , i_k \in \{1, \ldots , n\}$ such that $\alpha+\beta+\cdots+\gamma+\mu = \sum_{s=1}^k \epsilon_{i_s}$. In other words, summation here is done by $\{(\alpha, i_1), (\beta, i_2), \ldots , (\gamma, i_k)\} \in Supp_{|\mu|}(s_k)$.

Take

$$F = X_1 \otimes \eta_1 + \cdots + X_k \otimes \eta_k \in \mathcal{L},$$

where $X_i \in W_n, i = 1, \ldots , k$ are even elements. It is evident that

$$F^k = s_k(X_1, \ldots , X_k) \otimes (\eta_1 \cdots \eta_k).$$

On the other hand, if $X_1 = x^{\alpha^{(1)}}\partial_{i_1}, X_2 = x^{\alpha^{(2)}}\partial_{i_2}, \ldots , X_k = x^{\alpha^{(k)}}\partial_{i_k}$, then F can be presented in the form

$$F = \sum_{i=1}^k f_i\partial_i \in \mathcal{L},$$

where

$$f_i = \sum_{s:i_s=i} x^{\alpha^{(s)}} \otimes \eta_s \in \mathcal{U}$$

summation by s such that $i_s = i$. So, substitutions

$$u_i \mapsto \sum_{s:i_s=i} x^{\alpha^{(s)}} \otimes \eta_s \in \mathcal{U}$$

in D^k and making calculations in $\mathcal{U}$ gives us on the one side

$$\lambda_{\{\alpha^{(1)},i_1),\dots,(\alpha^{(k)},i_k);\mu\}} \alpha^{(1)}! \cdots \alpha^{(k)}! \partial^\mu \otimes \eta_1 \cdots \eta_k + Y,$$

where

$$Y \in < x^\alpha \partial^\beta \otimes Gr_k : |\alpha| > 0, \alpha, \beta \in \mathbf{Z}_+^n >,$$

and one the other side

$$s_k(x^{\alpha^{(1)}} \partial_{i_1}, \dots, x^{\alpha^{(k)}} \partial_{i_k}) \otimes \eta_1 \cdots \eta_k.$$

Take projections $\mathcal{D}iff \to < 1 > \otimes \eta_1 \dots \eta_k$ from the both parts. We have

$$\lambda_{\{\alpha^{(1)},i_1),\dots,(\alpha^{(k)},i_k);\mu\}} \alpha^{(1)}! \cdots \alpha^{(k)}! \partial^\mu = esc(s_k)(x^{\alpha^{(1)}} \partial_{i_1}, \dots, x^{\alpha^{(k)}} \partial_{i_k}).$$

Thus,

$$esc(s_k)(x^{(\alpha^{(1)})} \partial_{i_1}, \dots, x^{(\alpha^{(k)})} \partial_{i_k}) = \lambda_{\{\alpha^{(1)},i_1),\dots,(\alpha^{(k)},i_k);\mu\}} \partial^\mu$$

that we need to prove. $\square$

Recall that k-commutator s_k is called well defined on W_n if

$$\forall X_1, \dots, X_k \in W_n \Rightarrow s_k(X_1, \dots, X_k) \in W_n$$

Denote by s_k° a map $\wedge^k W_n \to W_n$ given by

$$s_k^\circ(X_1, \dots, X_k) = \sum_{\sigma \in Sym_k} sign\, \sigma\, X_{\sigma(1)} \circ (X_{\sigma(2)} \circ (\cdots (X_{\sigma(k-1)} \circ X_{\sigma(k)}))),$$

where W_n is considered as a left-symmetric algebra under multiplication $f\partial_i \circ g\partial_j = f\partial_i(g)\partial_j$.

Corollary 11.2. *The following conditions are equivalent:*

- $D^k \in Der\, \mathcal{L}$
- $D^k = D^{\circ k}$
- s_k *is well defined operation on* W_n.
- $s_k = s_k^\circ$.

Theorem 11.1 has two-fold applications. We use it in constructing D^k by s_k and vice versa one can use D^k in calculating k-commutators.

Example 1. Let $k=1$. Then $s_1(x^\alpha\partial_i)=x^\alpha\partial_i$, therefore $supp(s_1)=<\partial_i : i=1,\dots,n>$ and $Supp(s_1)=\{(0,1),\dots,(0,n)\}$. Hence

$$D^1=\sum_{i=1}^n \partial^0(u_i)s_1(\partial_i)=\sum_{i=1}^n u_i\partial_i,$$

that we know well.

Example 2. Let us calculate $Y=s_{11}(X_1,\dots,X_{11})$ for $X_i=\partial_i, 1\le i\le 3,\ X_4=x_1\partial_1-x_3\partial_3,\ X_5=x_2\partial_1,\ X_6=x_3\partial_1,\ X_7=x_1\partial_2,\ X_8=x_2\partial_2-x_3\partial_3,\ X_9=x_1\partial_3,\ X_{10}=x_2\partial_3,\ X_{11}=x_3^2\partial_1$. Let Gr_{11} be Grassman algebra generated by 11 odd elements $\eta_1,\dots,\eta_{11}$. Take $\mathcal{U}=\mathbf{C}[x_1,x_2,x_3]\otimes Gr_{11}$. Recall that x_i and ∂_i are even variables. Consider a homomorphism of super-differential polynomials algebra $\mathcal{L}_3$ to $\mathcal{U}$ given by

$$u_1\mapsto \eta_1+x_1\eta_4+x_2\eta_5+x_3\eta_6+x_3^2\eta_{11},$$

$$u_2\mapsto \eta_2+x_1\eta_7+x_2\eta_8,$$

$$u_3\mapsto \eta_3-x_3\eta_4-x_3\eta_8+x_1\eta_9+x_2\eta_{10}.$$

In other words make in the formulas for $\tau_1(D^{11})$ and $\tau_2(D^{11})$ corresponding substitutions. Make all calculations in $\mathcal{L}_3$ using the formula $\partial_i(v\otimes\eta)=\partial_i(v)\otimes\eta,\ v\in\mathbf{C}[x_1,x_2,x_3], \eta\in Gr_{11}$. One obtains that the linear part of Y is equal to 0 and the quadratic part of Y is equal to $80\partial_1^2$. So, $Y=80\partial_1^2$.

The following results about N-commutators on $Vect(2)$ and $Vect_0(2)$ was established in [1]. 6-commutator on W_2 is well defined and it has one escort invariant

$$escort_{231}: L_0\otimes L_1\to R(\pi_1)\cong L_{-1},$$

$$escort_{231}(a,X)=d(a\circ Div\,X)+Div\,a\;d\,Div\,X-3d\,Div(a\circ X).$$

$s_6=0$ is identity on S_1 and 5-commutator is well defined on S_1 and it has one escort invariant

$$escort_{221}: R(2\pi_1)\otimes R(3\pi_1)\to R(\pi_1)\cong L_{-1},$$

$$escort_{221}(a,X)=-3d\,Div(a\circ X).$$

Below we do similar things for $Vect(3)$ and $Vect_0(3)$. Since calculations are too tedious and similar to calculations given above we formulate final results and give some examples.

12. 13-COMMUTATOR ON $Vect(3)$

Theorem 12.1. *If $n=3$, then 13-commutator on W_3*

$$X_1,\dots,X_{13}\in W_3\Rightarrow s_{13}(X_1,\dots,X_{13})\in W_3.$$

It has one escort invariant

$$escort_{382}=escort(s_{13}):L_0\otimes\wedge^2L_1\to\wedge^2R(\pi_1)\cong L_{-1}$$

defined by

$$escort_{382}(a,X,Y)=$$

$$-d(a\circ Div\,X)\wedge d(Div\,Y)+d(a\circ Div\,Y)\wedge d(Div\,X)$$
$$-2(Div\,a)d(Div\,X)\wedge d(Div\,Y)$$
$$+4\,d(Div\,X\circ a)\wedge d(Div\,Y)+4\,d(Div\,X)\wedge d(Div\,Y\circ a)$$
$$+8(da\overset{\circ}{\wedge}dX)\circ Div\,Y-8(da\overset{\circ}{\wedge}dY)\circ Div\,X.$$

Corollary 12.2. *$escort_{382}$ induces a homomorphism $\bar L_0\otimes\bar L_1\otimes R(2\pi_1)\to\wedge^3R(\pi_1)$ by*

$$(a\partial_i,v\partial_j,w)\mapsto da\wedge d\partial_i(v)\wedge d\partial_j(w).$$

Corollary 12.3. *$s_{13}=0$ is identity on $Vect_0(3)$. Moreover $s_{12}=0$ is identity on $Vect_0(3)$.*

Let

$$G_{ij}(a)=\partial_i(a)\partial_j-\partial_j(a)\partial_i,$$
$$\tilde u=u(x_1\partial_1+x_2\partial_2+x_3\partial_3).$$

Take place isomorphisms of sl_n-modules

$$L_1=\bar L_1+\tilde L_1,$$

where

$$\bar L_1=<X:Div(X)=0>\cong R(2\pi_1+\pi_{n-1})\cong R(2\pi_1+\pi_{n-1}),$$
$$\tilde L_1\cong R(\pi_1).$$

Then $\bar L_1$ is generated by elements of the form $G_{ij}(x^\alpha)$, where $i<j$ $\alpha\in\mathbf{Z}^n_+,|\alpha|=3$ and $\tilde L_1$ is has a basis $\{\tilde x_i:i=1,\dots,n\|$.

Let us give construction of escort invariant in terms of $\bar L_i$ and $\tilde L_i$. We see that $escort(s_{13})(a,X,Y)=0$, if $X,Y\in\tilde L_1$ or $X,Y\in\bar L_1$ or $a=\tilde 1=x_1\partial_1+x_2\partial_2+x_3\partial_3$. Below we use the following notation

$$a^{(i,j,k)}=\partial_1^i\partial_2^j\partial_3^s(a).$$

Non-zero components of $escort(s_{13})(a,X,Y)$ can be given by:

$$escort(s_{13})(G_{12}(a),G_{12}(b),\tilde x_1)=$$

$$(-32a^{(1,0,1)}b^{(0,3,0)}+32a^{(1,1,0)}b^{(0,2,1)}-32a^{(0,2,0)}b^{(1,1,1)}+32a^{(0,1,1)}b^{(1,2,0)})\partial_1$$
$$+(32a^{(1,0,1)}b^{(1,2,0)}-16a^{(2,0,0)}b^{(0,2,1)}+16a^{(0,2,0)}b^{(2,0,1)}-32a^{(0,1,1)}b^{(2,1,0)})\partial_2$$
$$+(-32a^{(1,1,0)}b^{(1,2,0)}+16a^{(2,0,0)}b^{(0,3,0)}+16a^{(0,2,0)}b^{(2,1,0)})\partial_3,$$

$$escort(s_{13})(G_{12}(a),G_{12}(b),\tilde{x}_2)=$$

$$(32a^{(1,0,1)}b^{(1,2,0)}-16a^{(2,0,0)}b^{(0,2,1)}+16a^{(0,2,0)}b^{(2,0,1)}-32a^{(0,1,1)}b^{(2,1,0)})\partial_1$$
$$+(32a^{(2,0,0)}b^{(1,1,1)}-32a^{(1,1,0)}b^{(2,0,1)}-32a^{(1,0,1)}b^{(2,1,0)}+32a^{(0,1,1)}b^{(3,0,0)})\partial_y$$
$$+(-16a^{(2,0,0)}b^{(1,2,0)}+32a^{(1,1,0)}b^{(2,1,0)}-16a^{(0,2,0)}b^{(3,0,0)})\partial_3$$

$$escort(s_{13})(G_{12}(a),G_{12}(b),\tilde{x}_3)=$$

$$(-32a^{(1,1,0)}b^{(1,2,0)}+16a^{(2,0,0)}b^{(0,3,0)}+16a^{(0,2,0)}b^{(2,1,0)})\partial_1$$
$$+(-16a^{(2,0,0)}b^{(1,2,0)}+32a^{(1,1,0)}b^{(2,1,0)}-16a^{(0,2,0)}b^{(3,0,0)})\partial_2$$

$$escort(s_{13})(G_{12}(a),G_{23}(b),\tilde{x}_1)=$$

$$(-16a^{(1,0,1)}b^{(0,2,1)}-16a^{(0,2,0)}b^{(1,0,2)}+16a^{(1,1,0)}b^{(0,1,2)}+16a^{(0,1,1)}b^{(1,1,1)})\partial_2$$
$$+(+16a^{(0,2,0)}b^{(1,1,1)}-16a^{(0,1,1)}b^{(1,2,0)}-16a^{(1,1,0)}b^{(0,2,1)}+16a^{(1,0,1)}b^{(0,3,0)})\partial_3$$

$$escort(s_{13})(G_{12}(a),G_{23}(b),\tilde{x}_2)=$$

$$(-16a^{(1,0,1)}b^{(0,2,1)}-16a^{(0,2,0)}b^{(1,0,2)}+16a^{(1,1,0)}b^{(0,1,2)}+16a^{(0,1,1)}b^{(1,1,1)})\partial_1$$
$$+(32a^{(1,1,0)}b^{(1,0,2)}+32a^{(1,0,1)}b^{(1,1,1)}-32a^{(2,0,0)}b^{(0,1,2)}-32a^{(0,1,1)}b^{(2,0,1)})\partial_2$$
$$+(-48a^{(1,1,0)}b^{(1,1,1)}-16a^{(1,0,1)}b^{(1,2,0)}+32a^{(2,0,0)}b^{(0,2,1)}$$
$$+16a^{(0,2,0)}b^{(2,0,1)}+16a^{(0,1,1)}b^{(2,1,0)})\partial_3$$

$$escort(s_{13})(G_{12}(a),G_{23}(b),\tilde{x}_3)=$$

$$(16a^{(1,0,1)}b^{(0,3,0)}-16a^{(1,1,0)}b^{(0,2,1)}+16a^{(0,2,0)}b^{(1,1,1)}-16a^{(0,1,1)}b^{(1,2,0)})\partial_1$$
$$+(-48a^{(1,1,0)}b^{(1,1,1)}-16a^{(1,0,1)}b^{(1,2,0)}+32a^{(2,0,0)}b^{(0,2,1)}$$
$$+16a^{(0,2,0)}b^{(2,0,1)}+16a^{(0,1,1)}b^{(2,1,0)})\partial_2$$
$$+(-32a^{(2,0,0)}b^{(0,3,0)}+64a^{(1,1,0)}b^{(1,2,0)}-32a^{(0,2,0)}b^{(2,1,0)})\partial_3.$$

If $\{i,j,s,k\} \subseteq \{1,2,3\}$, then any two pairs $(i,j),(s,k)$ has at least one common element. Therefore by symmetry one can easily write other formulas for $escort(s_{13})(G_{ij}(a), G_{sk}(b), \tilde{x}_r)$.

Let us show how to use theorem 12.1 in calculation of 13-commutator on $Vect(3)$.

Example 1. Take

$$X_1 = \partial_1, X_2 = \partial_2, X_3 = \partial_3, X_4 = x_1\partial_1, X_5 = x_2\partial_1, X_6 = x_3\partial_1, X_7 = x_1\partial_2,$$

$$X_8 = x_2\partial_2, X_9 = x_3\partial_2, X_{10} = x_1\partial_3, X_{11} = x_2\partial_3, X_{12} = x_1x_2\partial_3, X_{13} = x_3^2\partial_3.$$

We see that the number of elements of grade -1 is 3 and the number elements of grade 0 is 8. In 0-part here appear all base elements of gl_3 except $a = x_3\partial_3$. Elements of the grade 1 are two: $X = X_{12}$ and $Y = X_{13}$. So, to calculate 13-commutator of 13 elements $X_1,\dots,X_{13}$, we denote it $s_{13}(X_1,\dots,X_{13})$, we need to calculate $escort(s_{13})(a,X,Y)$. We have

$$Div\, X = \partial_3(x_1x_2) = 0,$$

therefore,

$$-d(a\circ Div\, X)\wedge d(Div\, Y) + d(a\circ Div\, Y)\wedge d(Div\, X) - 2(Div\, a)d(Div\, X)\wedge d(Div\, Y) = 0.$$

Further,

$$Div\, a = \partial_3(x_3) = 1, \quad Div\, X\circ a = Div(x_1x_2\partial_3(x_3)\partial_3) = 0,$$

and

$$+4\, d(Div\, X\circ a)\wedge d(Div\, Y) + 4\, d(Div\, X)\wedge d(Div\, Y\circ a) = 0.$$

Finally, $Div\, Y = 2x_3$ and

$$8\{\sum_{i,j=1}^{3} \partial_i(Div\, Y)d(a(x_j))\wedge d(\partial_j(X(x_i))) - \partial_i(Div\, X)d(a(x_j))\wedge d(\partial_j(Y(x_i)))\} =$$

$$8\sum_{i,j=1}^{3}\partial_i(2x_3)d(a(x_j))\wedge d(\partial_j(x_1x_2\partial_3(x_i))) =$$

$$16\sum_{j=1}^{3} d(a(x_j))\wedge d(\partial_j(x_1x_2)) = 0.$$

Therefore,

$$escort(s_{13})(x_3\partial_3, x_1x_2\partial_3, x_3^2\partial_3) = 0.$$

Example 2. Now change in example 1 X_{12} to

$$X_{12} = x_1x_3\partial_3,$$

other elements are as before. Two vector fields, nameley $X = X_{12}$ and $Y = X_{13}$ have non-constant divergences:

$$Div\, X = x_1, Div\, Y = 2x_3.$$

So, we can expect that $s_{11}(X_1, \ldots, X_{13})$ might be non-trivial vector field. We have

$$-d(a \circ Div\, X) \wedge d(Div\, Y) + d(a \circ Div\, Y) \wedge d(Div\, X) =$$
$$2dx_3 \wedge dx_1,$$

$$-2(Div\, a)d(Div\, X) \wedge d(Div\, Y) = -4dx_1 \wedge dx_3,$$

$$4\, d(Div\, X \circ a) \wedge d(Div\, Y) + 4\, d(Div\, X) \wedge d(Div\, Y \circ a) =$$
$$8\, dx_1 \wedge dx_3 + 8dx_1 \wedge dx_3,$$

$$8\, \{\sum_{i,j=1}^{3} \partial_i(Div\, Y)d(a(x_j)) \wedge d(\partial_j(X(x_i))) - \partial_i(Div\, X)d(a(x_j)) \wedge d(\partial_j(Y(x_i)))\} =$$
$$8\, \{2dx_3 \wedge d(\partial_3(x_1x_3)) =$$
$$-16\, dx_1 \wedge dx_3.$$

Thus,

$$escort(s_{13})(a, X, Y) = -6dx_1 \wedge dx_3.$$

Since the isomorphism $\wedge^2 R(\pi_1) \cong R(\pi_2)$ is established by

$$dx_1 \wedge dx_2 \mapsto \partial_3, dx_1 \wedge dx_3 \mapsto -\partial_2, dx_2 \wedge dx_3 \mapsto \partial_1$$

this means that

$$s_{13}(X_1, \ldots, X_{13}) = esc(s_{13})(X_1, \ldots, X_{13}) = 6\partial_2.$$

Example 3. Let now

$$X_{12} = x_1x_3\partial_3, X_{13} = x_2x_3\partial_3.$$

other elements as above. Then

$$-d(a \circ Div\, X) \wedge d(Div\, Y) + d(a \circ Div\, Y) \wedge d(Div\, X) = 0,$$

$$-2(Div\, a)d(Div\, X) \wedge d(Div\, Y) = -2dx_1 \wedge dx_2,$$

$$4\, d(Div\, X \circ a) \wedge d(Div\, Y) + 4\, d(Div\, X) \wedge d(Div\, Y \circ a) =$$
$$4\, dx_1 \wedge dx_2 + 4\, dx_1 \wedge dx_2 = 8\, dx_1 \wedge dx_2,$$

$$\sum_{i,j=1}^{3} \partial_i(Div\, Y)d(a(x_j))\wedge d(\partial_j(X(x_i)))-\partial_i(Div\, X)d(a(x_j))\wedge d(\partial_j(Y(x_i)))=0.$$

Thus,

$$escort(s_{13})(a,X,Y)=6dx_1\wedge dx_2,$$

and

$$s_{13}(X_1,\dots,X_{13})=escort(s_{13})(X_1,\dots,X_{13})=6\partial_3.$$

Example 4. Let now all $Y_i=X_i$ as before if $i<12$ and

$$Y_{12}=x_1x_3\partial_3, Y_{13}=x_2x_3^2\partial_3.$$

Then

$$s_{13}(Y_1,\dots,Y_{13})=E_{x_3^2\partial_3}(Y_{13})esc(s_{13})(X_1,\dots,X_{11},x_1x_3\partial_3,x_3^2\partial_3)+$$

$$E_{x_2x_3\partial_3}(Y_{13})esc(s_{13})(X_1,\dots,X_{11},x_1x_3\partial_3,x_2x_3\partial_3)=$$

(by results of example 2 and 3)

$$6x_2\partial_2+12x_3\partial_3.$$

13. 10-commutator on $Vect(3)$ and $Vect_0(3)$

Recall some denotions:

$$U_k=\langle x^\alpha||\alpha|=k\rangle\,,$$

the multiplication $\circ$ is left-symmetric and $\overset{\circ}{\wedge}$ means wedge-product corresponding to left-symmetric multiplication:

$$L_1\times U_2\to\wedge^2U_1,\qquad (u\partial_i,v)\mapsto du\partial_i\overset{\circ}{\wedge}dv:=du\wedge d\partial_i(v).$$

Below expressions like $Div\, a\circ X$ will mean $Div\,(a\circ X)$.

Theorem 13.1. 10*-commutator is well defined on* $L=Vect(3)$ *:*

$$X_1,\dots,X_{10}\in L\Rightarrow s_{10}(X_1,\dots,X_{10})\in L.$$

It has three escort invariants

$$escort_{271}:R(\pi_1)\otimes\wedge^2L_0\otimes L_2\to\wedge^2R(\pi_1)\cong L_{-1},$$

$$escort_{3601}:\wedge^3L_0\otimes L_2\to\wedge R(\pi_1)\cong L_{-1},$$

and

$$escort_{352}:\wedge^4L_0\otimes\wedge^2L_1\to\wedge^2R(\pi_1)\cong L_{-1}.$$

They can be given by

$$escort_{271}(u,a,b,X)=$$

$$du\wedge(11d(Div\ ([a,b]\circ X))+21d(a\circ Div(b\circ X)-b\circ Div(a\circ X))-44d([a,b]\circ Div\ X))$$
$$-32(d(a\circ u)\wedge d(Div(b\circ X))-d(b\circ u)\wedge d(Div(a\circ X)))$$
$$-50(d(a\circ u)\wedge d(b\circ Div\,X)-d(b\circ u)\wedge d(a\circ Div\,X))$$
$$+Div\,a\ du\wedge d(2b\circ Div\,X+9Div(b\circ X))-Div\,b\ du\wedge d(2a\circ Div\,X+9Div(a\circ X))$$
$$+8(Div\,a\ db\overset{\circ}{\wedge}d(u\,Div\,X)-Div\,b\ da\overset{\circ}{\wedge}d(u\,Div\,X))$$
$$+12(da\overset{\circ}{\wedge}d((b\circ X)\circ u)-db\overset{\circ}{\wedge}d((a\circ X)\circ u))$$
$$-28d[a,b]\overset{\circ}{\wedge}d(X\circ u)$$
$$+16(da\overset{\circ}{\wedge}d(X\circ(b\circ u))-db\overset{\circ}{\wedge}d(X\circ(a\circ u))),$$

$$escort_{3601}(a,b,c,X)=$$
$$-6(da\overset{\circ}{\wedge}d(b\circ(Div\ (c\circ X)))+db\overset{\circ}{\wedge}d(c\circ(Div\ (a\circ X)))+dc\overset{\circ}{\wedge}d(a\circ(Div\ (b\circ X))))$$
$$+6(da\overset{\circ}{\wedge}d(c\circ(Div\ (b\circ X)))+db\overset{\circ}{\wedge}d(a\circ(Div\ (c\circ X)))+dc\overset{\circ}{\wedge}d(b\circ(Div\ (a\circ X))))$$
$$-5(da\overset{\circ}{\wedge}d(Div\ ([b,c]\circ X))+db\overset{\circ}{\wedge}d(Div\ ([c,a]\circ X))+dc\overset{\circ}{\wedge}d(Div\ ([a,b]\circ X)))$$
$$-d[a,b]\overset{\circ}{\wedge}d(Div\ (c\circ X))-d[b,c]\overset{\circ}{\wedge}d(Div\ (a\circ X))-d[c,a]\overset{\circ}{\wedge}d(Div\ (b\circ X))$$
$$-12(d[a,b]\overset{\circ}{\wedge}d(c\circ Div\,X)+d[b,c]\overset{\circ}{\wedge}d(a\circ Div\,X)d[c,a]\overset{\circ}{\wedge}d(b\circ Div\,X))$$
$$+27d(a\circ[b,c]+b\circ[c,a]+c\circ[a,b])\overset{\circ}{\wedge}d(Div\,X)$$
$$+18Div\,a\ (db\overset{\circ}{\wedge}d(c\circ Div\,X)-dc\overset{\circ}{\wedge}d(b\circ Div\,X))$$
$$+18Div\,b\ (dc\overset{\circ}{\wedge}d(a\circ Div\,X)-db\overset{\circ}{\wedge}d(a\circ Div\,X))$$
$$+18Div\,c\ (da\overset{\circ}{\wedge}d(b\circ Div\,X)-db\overset{\circ}{\wedge}d(a\circ Div\,X))$$
$$-14Div\,a\ (db\overset{\circ}{\wedge}d(Div\ (c\circ X))-dc\overset{\circ}{\wedge}d(Div\ (b\circ X)))$$
$$-14Div\,b\ (dc\overset{\circ}{\wedge}d(Div\ (a\circ X))-da\overset{\circ}{\wedge}d(Div\ (c\circ X)))$$
$$-14Div\,c\ (da\overset{\circ}{\wedge}d(Div\ (b\circ X))-db\overset{\circ}{\wedge}d(Div\ (a\circ X)))$$

We are not able to write escort invariant of type $(3,5,2)$ in a compact form.

Corollary 13.2. *10-commutator on $L = Vect_0(3)$ is well defined and escort invariants are given by*

$$escort_{271} : R(\pi_1) \otimes \wedge^2 R(\pi_1+\pi_2) \otimes R(2\pi_1+\pi_2) \to \wedge^2 R(\pi_1),$$

$$escort_{271}(u,a,b,X) =$$

$$\begin{gathered} du \wedge (11d(Div\ ([a,b]\circ X)) + 21d(a\circ Div(b\circ X) - b\circ Div(a\circ X))) \\ -32(d(a\circ u)\wedge d(Div(b\circ X)) - d(b\circ u)\wedge d(Div(a\circ X))) \\ +12(da \overset{\circ}{\wedge} d((b\circ X)\circ u) - db \overset{\circ}{\wedge} d((a\circ X)\circ u)) \\ -28d[a,b] \overset{\circ}{\wedge} d(X\circ u) \\ +16(da \overset{\circ}{\wedge} d(X\circ(b\circ u)) - db \overset{\circ}{\wedge} d(X\circ(a\circ u))), \end{gathered}$$

$$escort_{3601} : \wedge^3 R(\pi_1+\pi_2) \otimes R(3\pi_1+\pi_{n-1}) \to \wedge^2 R(\pi_1)$$

$$escort_{3601}(a,b,c,X) =$$

$$\begin{gathered} -6(da \overset{\circ}{\wedge} d(b\circ(Div\ c\circ X)) + db \overset{\circ}{\wedge} d(c\circ(Div\ (a\circ X))) + dc \overset{\circ}{\wedge} d(a\circ(Div\ (b\circ X)))) \\ +6(da \overset{\circ}{\wedge} d(c\circ(Div\ (b\circ X))) + db \overset{\circ}{\wedge} d(a\circ(Div\ (c\circ X))) + dc \overset{\circ}{\wedge} d(b\circ(Div\ (a\circ X)))) \\ -5(da \overset{\circ}{\wedge} d(Div\ ([b,c]\circ X)) + db \overset{\circ}{\wedge} d(Div\ ([c,a]\circ X)) + dc \overset{\circ}{\wedge} d(Div\ ([a,b]\circ X))) \end{gathered}$$

$$escort_{352} : \wedge^4 R(\pi_1+\pi_2) \otimes \wedge^2 R(2\pi_1+\pi_2) \to \wedge^2 R(\pi_1) \cong L_{-1}$$

is too big to be presented here.

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