MIRROR SYMMETRY FORMULAE FOR THE ELLIPTIC GENUS OF COMPLETE INTERSECTIONS

VASSILY GORBOUNOV AND SERGE OCHANINE

Understanding the mathematics behind Quantum and in particular Conformal Field Theories has been a challenge for more then twenty five years. The usual ground for mathematical interpretations of Quantum Field Theory predictions has been the Topological Quantum Field Theory, which is a certain reduction of the genuine Quantum Field Theory. There have been significant mathematical advances in this area. Mirror symmetry is among the major motivations behind these advances. Vaguely stated, mirror symmetry is a duality between complex and symplectic geometry. As was originally discovered by physicists [6], among its specific manifestations is a striking connection between the"number of curves of a given genus" in a symplectic manifold and the periods of a holomorphic form of a different complex manifold. Ever since, the"explicit" construction of the mirror partner for a given manifold has become the central task for the mathematicians. There is a vast amount of work done in this direction. In this paper, we follow the line pioneered by Gepner [8] and developed by Vafa [25] who discovered that the Conformal Field Theory defined by a manifold, the so-called the sigma model, can be identical to the Conformal Field Theory of an a different type, the so-called Landau-Ginsburg model. This point of view was further developed by Witten [28]. As far as the application of this idea to mirror symmetry is concerned, the major reference for the purposes of this paper is the work of Hori and Vafa [15]. They showed that the mirror partner of a large class of manifolds turns out to be a Landau-Ginsburg model of some kind, or its orbifold. The"proof" that these models form a mirror pair would consist of picking an invariant which is known to be identical for mirror partners and by a calculation showing that it is indeed the same for given hypothetical mirror partners. In [15] such an invariant is given by the periods of a holomorphic form on a manifold and the so called BPS masses in the Landau-Ginsburg model. Of course the identities implied by mirror symmetry in Topological Quantum Field Theory are a reduction of stronger identities in the original Quantum Field Theory. The rigorous mathematical structure behind Quantum Field Theory is not known at the present time, so exploring the mathematical consequences of mirror symmetry at this higher level is difficult. Some time ago, Malikov, Schechtman and Vaintrob [19] introduced a construction of a mathematical approximation to the structure of Quantum Field Theory defined by a manifold. It is so called the Chiral de Rham complex. Mathematically, these ideas were further developed in [11], [10], [4], [5], [16]. The relevance of this construction to physics was explained only recently in [30], [17]. Despite being only an approximation to

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the genuine Quantum Field Theory, the Chiral de Rham complex carries features not available in the Topological Quantum Field Theory. One of them is the elliptic genus. Introduced in mathematics by in [21], it has been almost immediately connected to quantum physics [27],[26] and later shown to be identical for mirror partners, thus providing another test for a pair of manifolds to be mirror partners. Moreover, as explained in [29], there is a physics counterpart of the elliptic genus in a large class of Conformal Field Theories, in particular in Landau-Ginsburg models and their orbifolds, and this physics elliptic genus coincides with the one defined in topology for the sigma model. Taking on Vafas and Wittens ideas, a number of physicists came up with a new type of formulas for the elliptic genus of some classes of manifolds in terms of their mirror Landau-Ginsburg partner [18], [2], [3], [7] . For about ten years, no mathematical proof of these formulas was produced. The first paper where such a proof was given in the case of Calabi-Yau hypersurfaces was [9]. The scope of this paper was much broader and the result for the elliptic genus fell out as a simple consequence of a much deeper connection found between the Landau-Ginsburg model and the Chiral de Rham complex of a hypersurface.

The purpose of the present paper is to prove, by more or less elementary means, that the Landau-Ginsburg mirror partners found in [15] for complete intersection in products of projective spaces have the physics elliptic genus identical to the topological elliptic genus. It is interesting to note that our result provides an extra test for the constructions in [15], and as such refines the conditions for the existence of a mirror Landau-Ginsburg theory as a mirror partner for some complete intersections given in [15]. We are planning to return to the case of complete intersections in more general toric varieties considered in [15] in a future work.

Although inspired by physics, with the exception of the introductory section 0, this paper is mathematically self-contained. It starts with a brief introduction to Jacobi functions and the correspondent elliptic genera. We then proceed to derive the main formula (theorem 9) using the residue theorem for functions of several variables. The formula is derived for elliptic genera of any level which includes the case of Hirzebruch's level N genus [13], [14]. In section 10, the formula is specialized to the level 2 genus discussed in physics literature. A different proof of the level 2 case is given in section 11.

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0. Landau Ginsburg orbifolds and their elliptic genus

In this section we briefly state the results about the elliptic genus of LG theories and their orbifolds without going into details. In physics literature the elliptic genus is defined as a character of an action of an infinite dimensional Lie algebra, namely the so called $N = 2$ algebra. For the purpose of this note, Landau-Ginzburg field theories are described mathematically by a non-compact manifold and a function W on it, called *superpotential*, which has isolated singularities. These data are sufficient for defining and calculating explicitly the elliptic genus of some relevant for this note LG theories [29]. An important class of quantum field theories related to LG is defined by orbifolds of LG with respect to a finite group action. The elliptic genus of the LG orbifolds relevant for us was calculated in a number of papers [2], [3], [7], [18] (see also [9] for more mathematical approach). Such a genus is either a Jacobi form or a modular function depending on the data defining the LG model. In this note we consider LG defined by $(\mathbb{C}^N, W)$, where the superpotential W is a holomorphic and quasi-homogeneous function of $z_1, z_2, \ldots, z_N$ and its orbifolds with respect to the action of some finite group of symmetries. The conditions on the W are the following: It should be possible to assign some weights $k_i \in \mathbb{Z}$ to the variables z_i for $i \in \mathbb{N}$ and a degree of homogeneity $d \in \mathbb{N}$ to W so that

$$W(\lambda^{k_i} z_i) = \lambda^d W(z_i)$$

for all i and $\lambda \in \mathbb{C}$. Let $q_i = k_i/d$.

Suppose the potential is invariant under a finite abelian group of symmetries G. Denote by R_i the function on G satisfying $g(z_i) = \exp(2\pi i R_i(g)) z_i$. The invariance of W means of course that for all i and $g \in G$

$$W(g(z_i)) = W(z_i).$$

The elliptic genus $Ell(q, y)$ of such an LG orbifold defined by the data (W, G) following [2], [3], is given as follows:

$$Ell(q,y) = \frac{1}{|G|} \sum_{g_1, g_2 \in G} \prod_{i=1}^{N} y^{-R_i(g_1)} \frac{\Theta_1((1-q_i)z + R_i(g_1) + R_i(g_2)\tau|\tau)}{\Theta_1(q_i z + R_i(g_1) + R_i(g_2)\tau|\tau)},$$

where $y = \exp(2\pi i z)$ and $q = \exp(2\pi i \tau)$.

Example. Consider the following LG data taken from [1]: In the space $\mathbb{C}^9$ introduce coordinates X_a $(a = 0, 1, 2, 3)$, Y_b $(b = 0, 1, 2)$, Z_c $(c = 0, 1)$, and define the potential by the formula

$$W = \sum_{r=0}^{1} (X_r + X_r Y_r^2 + Y_r Z_r^2) + X_2^3 + X_2 Y_2^2 + X_3^3.$$

In this case $d = 3$ and $q_i = 1/3$. It is easy to check that W is invariant under the following action of $\mathbb{Z}/12$ with generator ω:

$$\omega(X_a, Y_b, Z_c) = (\omega^4 X_a, \omega^{-2} Y_b, \omega Z_c).$$

Calculating the appropriate number $R_i(\omega)$ which define such an action we obtain:

$$R_a(\omega) = 1/3, \quad R_b(\omega) = -1/6, \quad R_c(\omega) = 1/12.$$

Therefore the above formula for the elliptic genus of the LG orbifold defined by the data $(W, \mathbb{Z}/12)$ becomes:

$$Ell(q,y) =$$
$$\frac{1}{12}\sum_{i,j=0}^{12}\left(y^{-\frac{j}{3}}\frac{\Theta_1(\frac{2z}{3}-\frac{i+j\tau}{3}|\tau)}{\Theta_1(\frac{z}{3}+\frac{i+j\tau}{3}|\tau)}\right)^4\left(y^{\frac{j}{6}}\frac{\Theta_1(\frac{2z}{3}+\frac{i+j\tau}{6}|\tau)}{\Theta_1(\frac{z}{3}-\frac{i+j\tau}{6}|\tau)}\right)^3\left(y^{-\frac{j}{12}}\frac{\Theta_1(\frac{2z}{3}-\frac{i+j\tau}{12}|\tau)}{\Theta_1(\frac{z}{3}+\frac{i+j\tau}{12}|\tau)}\right)^2.$$

It is easy to check that this formula agrees with our formula for the elliptic genus of the complete intersection in $\mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^1$ given by equations:

$$\sum_{a=0}^{3} X_a^3 = 0, \quad \sum_{b=0}^{2} X_b Y_b^2 = 0, \quad \sum_{c=0}^{1} Y_c Z_c^2 = 0$$

(see section 10).

In sections 1–7, k will be an algebraically closed field of characteristic $p \geq 0$, G a finite group of order n, and $p \nmid n$.

1. Regular Representations

Consider the k-vector space $V = \mathrm{Map}(G,k)$ of all k-valued functions on G. Clearly, $\dim_k V = n$. The group G acts on V by the formula

$$(g\cdot f)(h) = f(hg), \qquad (g,h \in G, f \in V).$$

With this action, V is called the *regular representation of* G.

We now consider the case when G is abelian and write the group operation in G additively. Let $\widehat{G} = \mathrm{Hom}(G, k^*)$ be the character group of G. The following is well known (cf. [23], 2.4):

Theorem 1. (1) *$\widehat{G} \subset V$ is a basis of V.*

(2) *The one-dimensional subspace V_χ of V generated by $\chi \in \widehat{G}$ consists of those $f \in V$ which satisfy*

$$f(u+g) \equiv \chi(g)f(u) \qquad \textit{for all} \quad g \in G.$$

(3) *The G-modules V_χ are pairwise non-isomorphic.*

(4) *Every irreducible G-module is of degree 1 and is isomorphic to one of the V_χ.*

2. Generalized Jacobi Functions

Let now E be an elliptic curve over k, $k(E)$ be the field of rational functions on E, and $G \subset E$ be any finite subgroup of order n. As above, we assume that $p \nmid n$.

Let Δ be the divisor

$$\Delta = \sum_{g \in G} (g),$$

and let $\mathcal{L}(G) \subset k(E)$ be the associated vector space:

$$\mathcal{L}(G) = \{ f \in k(E) \mid \operatorname{div}(f) \succcurlyeq -\Delta \} .$$

By the Riemann-Roch theorem, $\dim_k \mathcal{L}(G) = n$. The group G acts on E by translations and leaves Δ unchanged. Therefore $\mathcal{L}(G)$ is invariant under the induced G-action on $k(E)$. Thus $\mathcal{L}(G)$ is naturally an n-dimensional representation of G.

Theorem 2. *As a G-representation, $\mathcal{L}(G)$ is isomorphic to the regular representation of G.*

Proof. Let $u_0 \in E$ be any point such that $nu_0 \neq 0$. Define a G-linear map

$$\phi : \mathcal{L}(G) \longrightarrow V = \operatorname{Map}(G, k)$$

by

$$\phi(f)(g) = f(u_0 + g), \qquad (f \in \mathcal{L}(G), g \in G).$$

This is well-defined, since $u_0 \notin G$ and therefore $u_0 + g$ is not a pole of f. Since $\mathcal{L}(G)$ and V have the same dimension n, we only need to prove that ϕ is injective.

Suppose $f \neq 0$ and $\phi(f) = 0$. Then $f(u_0 + g) = 0$ for all $g \in G$. Thus f has $\geq n$ zeroes. On the other hand, since $f \in \mathcal{L}(G)$, it has $\leq n$ poles. Since $\deg \operatorname{div}(f) = 0$, we conclude that f has a simple pole at each $g \in G$ and a simple zero at each $u_0 + g$. I.e.

$$\operatorname{div}(f) = -\sum_{g \in G} (g) + \sum_{g \in G} (u_0 + g).$$

Thus the image of $\operatorname{div}(f)$ under Abel's map

$$\operatorname{Div}(E) \longrightarrow E$$

is $nu_0 \neq 0$. The contradiction shows that $f = 0$. □

Corollary 1. *For each $\chi \in \widehat{G}$, there is a non-zero function $f_\chi \in \mathcal{L}(G)$ satisfying*

$$f_\chi(u + g) \equiv \chi(g) f_\chi(u)$$

for all $g \in G$. This function is determined uniquely up to a non-zero multiplicative constant.

Definition 1. *We call f_χ a (generalized) Jacobi function belonging to $\chi \in \widehat{G}$.*

Remark 1. The classical Jacobi functions correspond to the case where G is a cyclic group of order $2(\operatorname{char} k \neq 2)$.

3. The Divisor of a Jacobi Function

We now describe explicitly the divisor of a Jacobi function.

Theorem 3. *A non-zero function $f \in \mathcal{L}(G)$ is a Jacobi function if and only if* $\operatorname{div}(f)$ *is invariant under translations by elements of G.*

Proof. If f is a Jacobi function, the formula

$$f_\chi(u+g) \equiv \chi(g) f_\chi(u),$$

shows that the functions

$$u \longmapsto f(u) \quad \text{and} \quad u \longmapsto f(u+g)$$

have the same divisor. Thus $\operatorname{div}(f)$ is invariant by G.

Conversely, let f be a non-constant function in $\mathcal{L}(G)$, and suppose $\operatorname{div}(f)$ is invariant by translation. Then there is a constant $\chi(g) \in k^*$ such that

$$f_\chi(u+g) \equiv \chi(g) f_\chi(u),$$

for all $u \in E$. Since

$$\chi(g_1+g_2) f(u) = f(u+g_1+g_2) = \chi(g_2) f(u+g_1) = \chi(g_1)\chi(g_2) f(u),$$

we have $\chi(g_1+g_2) = \chi(g_1)\chi(g_2)$. Thus $\chi \in \widehat{G}$, and f is a Jacobi function belonging to χ. □

It is now easy to describe the divisors of the Jacobi functions. Let E_n be the group of n-division points on E. Since n is the order of G, we have $G \subset E_n$. For every coset $\gamma \in E_n/G$, let

$$\Gamma_\gamma = \sum_{r \in \gamma} (r) \in \operatorname{Div}(E).$$

Theorem 4. *A non-constant function $f \in \mathcal{L}(G)$ is a Jacobi function if and only if*

$$\operatorname{div}(f) = -\Delta + \Gamma_\gamma$$

for some non-zero $\gamma \in E_n/G$.

Proof. Using theorem 3, we only need to describe the non-zero principal divisors D satisfying $D \succcurlyeq -\Delta$ and invariant under G. Since the polar part of D is non-zero and invariant under G, it must be equal to $-\Delta$. Since $D+\Delta$ is invariant, and is of degree n, it is necessarily of the form

$$\sum_{g \in G} (r+g)$$

for some $r \notin G$, i.e. of the form Γ_γ. □

4. Modulus and Conjugation

Let f be a Jacobi function with character χ and divisor $\operatorname{div}(f) = -\Delta + \Gamma_\gamma$, $\gamma \in E_n/G$. For each $r \in \gamma$, the involution $u \mapsto r - u$ takes $\operatorname{div}(f)$ to $-\operatorname{div}(f)$. Therefore, $f(u)f(r-u)$ is a non-zero constant that we call the *modulus* of f and designate $c(r)$. Thus $c : \gamma \longrightarrow k^\times$. It is easy to check that

$$c(r+g) = \chi(g)c(r)$$

for all $g \in G$.

Similarly, the involution $u \mapsto -u$ takes $-\Delta + \Gamma_\gamma$ to $-\Delta + \Gamma_\gamma$. This $u \mapsto f(-u)$ is also a Jacobi function with character χ^{-1}. In particular, when $n = 2$, E_n/G has order 2, and $\chi^{-1} = \chi$. It follows that $f(-u) = af(u)$ where $a^2 = 1$. Thus f is either even or odd. Since f has a pole of order 1 at $O \in E$, it has to be odd.

5. Jacobi Functions on the Tate Curve

Consider a local field k complete with respect to a discrete valuation v, and let $q \in k^\times$ be any element satisfying $v(q) < 1$. It is well known (cf. [24], Appendix C, §14) that $E = k^\times/q^{\mathbb{Z}}$ can be identified with the Tate curve

$$y^2 + xy = x^3 + a_4 x + a_6,$$

where

$$a_4 = \sum_{m \geq 1} (-5m^3) \frac{q^m}{1 - q^m}$$

and

$$a_6 = \sum_{m \geq 1} \left(-\frac{5m^3 + 7m^5}{12} \right) \frac{q^m}{1 - q^m}.$$

Let $G \subset E$ be a finite subgroup of order n. As usual, we assume that $\operatorname{char} k \nmid n$. Let $\gamma \in E_n/G$ be any non-trivial coset. We will construct an explicit Jacobi function for G, whose zeroes are the points of γ.

If $\alpha \in k^\times$, we will write $\bar{\alpha}$ for its image in $E = k^\times/q^{\mathbb{Z}}$. Choose $g_1 = 1, g_2, \ldots, g_n \in k^\times$ so that

$$G = \{\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n\}.$$

Then choose $r_1, r_2, \ldots, r_n \in k^\times$ so that

$$\gamma = \{\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n\}$$

and

$$\prod_i r_i = \prod_i g_i.$$

This can be done as follows. First, choose $r \in k^\times$ so that $\bar{r} \in E$ represents γ. Then clearly,

$$\gamma = \{\bar{r}\bar{g}_1, \bar{r}\bar{g}_2, \dots, \bar{r}\bar{g}_n\}.$$

Since $\bar{r}^n = 1$ in E, we have $r^n = q^s$ for some $s \in \mathbb{Z}$. Let

$$r_1 = q^{-s}r, \qquad r_i = rg_i \quad (i \geq 2).$$

Then

$$\prod_i r_i = q^{-s}r^n \left(\prod_i g_i\right) = \prod_i g_i.$$

Now consider the basic Theta function

$$\begin{aligned}\Theta(u) &= (1-u^{-1})\prod_{k\geq 1}(1-q^k u)(1-q^k u^{-1}) \\ &= \prod_{k\geq 1}(1-q^k u)\prod_{k\leq 0}(1-q^{-k}u^{-1}).\end{aligned}$$

This is an "analytic" function on $k^\times$ that has simple zeroes at the points of $q^{\mathbb{Z}}$ (cf. [22]) and satisfies

$$\Theta(q^{-1}u) = -u\Theta(u).$$

Along with $\Theta(u)$ we will consider its translates $\Theta_\alpha(u)$ ($\alpha \in k^\times$) defined by

$$\Theta_\alpha(u) = \Theta(\alpha^{-1}u).$$

The function $\Theta_\alpha(u)$ has simple zeroes at the points of $\alpha q^{\mathbb{Z}}$ and satisfies

$$\Theta_\alpha(q^{-1}u) = -\alpha^{-1}u\Theta_\alpha(u).$$

Consider

$$f(u) = \prod_{i=1}^{n} \frac{\Theta_{r_i}(u)}{\Theta_{g_i}(u)}.$$

Theorem 5. *The function f is q-periodic and defines a Jacobi function for G having γ as the coset of zeroes.*

Proof. It is enough to prove the first statement since this would imply that f defines a function on E with simple poles at the points of G and simple zeroes at the points of γ. We have

$$f(q^{-1}u) = \prod_{i=1}^{n} \frac{\Theta_{r_i}(q^{-1}u)}{\Theta_{g_i}(q^{-1}u)} = \prod_{i=1}^{n} \frac{(-r_i u)\Theta_{r_i}(u)}{(-g_i u)\Theta_{g_i}(u)} = f(u).$$

□

6. A Special Case

We now assume that q is not a root of 1, and choose an arbitrary n-th root q_0 of q. For $n \geq 1$ and $a \in \mathbb{Z}$, define

$$\vartheta_a^n(u) = \prod_{\ell \geq 1,\, \ell \equiv a(n)} (1 - q^\ell u) \prod_{\ell \leq 0,\, \ell \equiv a(n)} (1 - q^{-\ell} u^{-1}).$$

With the notations of the previous section, we have $\Theta(u) = \vartheta_0^1(u)$.

Theorem 6. *The function*

$$f(u) = \frac{\vartheta_{-1}^n(u)}{u\, \vartheta_0^n(u)}$$

defines a Jacobi function on $E = k^\times / q^{\mathbb{Z}}$ for the group $G \subset E$ consisting of the images of the n-th roots of 1, and that has simple zeroes at the images of the n-th roots of q.

Proof. Let $\varepsilon_1 = 1, \varepsilon_2, \ldots, \varepsilon_n$ be the n-th roots of 1 in k. Thus

$$G = \{\bar{\varepsilon}_1, \bar{\varepsilon}_2, \ldots, \bar{\varepsilon}_n\}, \quad \gamma = \{\bar{\varepsilon}_1 \bar{q}_0, \bar{\varepsilon}_2 \bar{q}_0, \ldots, \bar{\varepsilon}_n \bar{q}_0\}.$$

Notice that

$$\prod (\varepsilon_i q_0) = \left(\prod \varepsilon_i\right) q.$$

Thus we can take

$$r_1 = q^{-1} q_0, \qquad r_i = \varepsilon_i q_0 \quad (i \geq 2).$$

According to the previous theorem, the function

$$\prod_{i=1}^n \frac{\Theta_{r_i}(u)}{\Theta_{\varepsilon_i}(u)}$$

has all the required properties. It remains to identify the numerator and denominator explicitly. We will be using the identities

$$\prod (1 - \varepsilon_i t) = \prod (1 - \varepsilon_i^{-1} t) = 1 - t^n,$$

which are easily proven by noticing that the three polynomials in t have same degree n, same roots, and same constant term 1.

We have:

$$\begin{aligned}
\prod_i \Theta_{\varepsilon_i}(u) &= \prod_i (1 - \varepsilon_i u^{-1}) \prod_{k \geq 1} \prod_i (1 - q^k \varepsilon_i^{-1} u) \prod_{k \geq 1} \prod_i (1 - q^k \varepsilon_i u^{-1}) \\
&= (1 - u^{-n}) \prod_{k \geq 1} (1 - q^{nk} u^n) \prod_{k \geq 1} (1 - q^{nk} u^{-n}) \\
&= \vartheta_0^n(u).
\end{aligned}$$

Similarly,

$$\begin{aligned}\prod_i \Theta_{\varepsilon_i q_0}(u) &= \prod_i (1-\varepsilon_i q_0 u^{-1}) \prod_{k\geq 1}\prod_i (1-q^k\varepsilon_i^{-1}q_0^{-1}u)\prod_{k\geq 1}\prod_i (1-q^k\varepsilon_i q_0 u^{-1})\\ &= (1-qu^{-n})\prod_{k\geq 1}(1-q^{nk-1}u^n)\prod_{k\geq 1}(1-q^{nk+1}u^{-n})\\ &= \vartheta_{-1}^n(u).\end{aligned}$$

Finally,

$$\prod_i \Theta_{r_i}(u) = \prod_i \Theta_{\varepsilon_i q_0}(u)\frac{\Theta_{q^{-1}q_0}(u)}{\Theta_{q_0}(u)}$$

and

$$\Theta_{q_0}(u) = \Theta(q_0^{-1}u) = \Theta(q^{-1}(qq_0^{-1}u)) = -qq_0^{-1}u\,\Theta_{q^{-1}q_0}(u),$$

i.e.

$$\frac{\vartheta_{-1}^n(u)}{u\,\vartheta_0^n(u)}$$

is a constant multiple of

$$\prod_{i=1}^n \frac{\Theta_{r_i}(u)}{\Theta_{\varepsilon_i}(u)}$$

and therefore has all the required properties. □

Remark 2. In the case $n=2$, the previous theorem gives

$$\begin{aligned}f(u) &= \frac{1}{u(1-u^{-2})}\prod_{k\geq 1}\frac{(1-q^{2k-1}u^2)(1-q^{2k-1}u^{-2})}{(1-q^{2k}u^2)(1-q^{2k}u^{-2})}\\ &= \frac{1}{u-u^{-1}}\prod_{k\geq 1}\frac{(1-q^{2k-1}u^2)(1-q^{2k-1}u^{-2})}{(1-q^{2k}u^2)(1-q^{2k}u^{-2})}.\end{aligned}$$

The formal substitution $u = e^{z/2}$ leads to

$$zf(e^{z/2}) = \frac{z/2}{\sinh(z/2)}\prod_{k\geq 1}\frac{(1-q^{2k-1}e^z)(1-q^{2k-1}e^{-z})}{(1-q^{2k}e^z)(1-q^{2k}e^{-z})},$$

which is a familiar expression for the generating function of the level 2 elliptic genus (see below).

7. Modulus and Normalization on the Tate curve

Continuing with the situation of the preceding section, we are first going to compute the modulus for the Jacobi function defined by

$$f(u) = \frac{\vartheta_{-1}^n(u)}{u\,\vartheta_0^n(u)},$$

i.e. compute the constant value of $f(u)f(ru^{-1})$ for $r \in \gamma$. We start with $r = q_0$. The following formulas can be easily obtained from the definition of ϑ_a^n:

$$\vartheta_0^n(u^{-1}) = -u^n \vartheta_0^n(u), \qquad \vartheta_a^n(u^{-1}) = -u^n \vartheta_{-a}^n(u) \quad (a \not\equiv 0(n))$$

and

$$\vartheta_0^n(q_0 u) = -q^{-1} u^{-n} \vartheta_1^n(u), \qquad \vartheta_{-1}^n(q_0 u) = \vartheta_0^n(u).$$

It follows that

$$\vartheta_0^n(q_0 u^{-1}) = -q^{-1} u^n \vartheta_{-1}^n(u), \qquad \vartheta_{-1}^n(q_0 u^{-1}) = -u^n \vartheta_0^n(u).$$

and therefore

$$f(q_0 u^{-1}) = \frac{qu\vartheta_0^n(u)}{q_0 \vartheta_{-1}^n(u)} = q_0^{n-1} f(u)^{-1}.$$

Thus

$$c(q_0) = q_0^{n-1}.$$

Notice now that the formula for f does not depend on the choice of q_0. Therefore we have the following

Theorem 7. *For the Jacobi function defined by*

$$f(u) = \frac{\vartheta_{-1}^n(u)}{u\, \vartheta_0^n(u)},$$

and any n-th root r of q in $k^\times$, we have

$$c(r) = r^{n-1}.$$

We now normalize f by requiring that

$$\mathrm{Res}_{u=1}\left(f(u)\frac{du}{u}\right) = \frac{1}{n}.$$

This normalization is formally equivalent to the requirement that $zf(e^{z/n}) = 1 + o(z)$. Since

$$\begin{aligned}
\frac{(u-1)\vartheta_{-1}^n(u)}{u\,\vartheta_0^n(u)} &= \frac{u-1}{u(1-u^{-n})} \cdot \frac{\vartheta_{-1}^n(u)}{\prod_{k\geq 1}(1-q^{nk}u^n)(1-q^{nk}u^{-n})} \\
&= \frac{u^{n-1}}{1+u+u^2+\cdots+u^{n-1}} \cdot \frac{\vartheta_{-1}^n(u)}{\prod_{k\geq 1}(1-q^{nk}u^n)(1-q^{nk}u^{-n})},
\end{aligned}$$

we see that the normalized Jacobi function is $f(u)N$, where

$$N = \frac{\prod_{k\geq 1}(1-q^{nk})^2}{\vartheta_{-1}^n(1)}.$$

The modulus of the normalized function is

$$c(r) = r^{n-1}N^2.$$

and satisfies

$$c(r)^n = q^{n-1}N^{2n}.$$

In particular, if $n = 2$, we have

$$N = \prod_{k \geq 1} \left(\frac{1 - q^{2k}}{1 - q^{2k-1}} \right)^2,$$

and

$$c(r)^2 = q \prod_{k \geq 1} \left(\frac{1 - q^{2k}}{1 - q^{2k-1}} \right)^8,$$

which is the familiar expression for the modular form ε for one of the three level two elliptic genera.

8. Elliptic Genus

From now on, $k = \mathbb{C}$, $E = \mathbb{C}/L$ for some lattice L, $G = L_0/L$ and $n = [L_0 : L]$. Let f be a Jacobi function with character χ. Then f is an elliptic function with period lattice L. We *normalize* f so that it has residue $\operatorname{Res}_{z=0}(f) = 1$. Then the Taylor expansion of $zf(z)$ is a formal power series with constant term 1 and defines, via the Hirzebruch formalism, a multiplicative genus

$$\varphi : \Omega_*^U \longrightarrow \mathbb{C},$$

which we refer to as the *level n elliptic genus* defined by f. The case $n = 2$ is best known. In this case, $zf(z)$ is an even series in z and φ factors through a genus

$$\varphi : \Omega_*^{SO} \longrightarrow \mathbb{C}.$$

9. Complete Intersections

In this section, we will compute the elliptic genus of complete intersections satisfying a non-degeneracy condition as a summation over some division points of $E \times E \times \cdots \times E$.

Let $M = (m_{ij})$ be a $l \times t$ matrix over $\mathbb{Z}$, and let P be the product

$$P = \mathbb{C}\mathrm{P}^{N_1 - 1} \times \mathbb{C}\mathrm{P}^{N_2 - 1} \times \cdots \times \mathbb{C}\mathrm{P}^{N_t - 1}.$$

For $1 \leq j \leq t$, let η_j be the pull-back over P of the canonical line bundle of $\mathbb{C}\mathrm{P}^{N_j - 1}$. Then for $1 \leq i \leq l$, let ξ_i be the line bundle

$$\xi_i = \eta_1^{m_{i1}} \otimes \eta_2^{m_{i2}} \otimes \cdots \otimes \eta_t^{m_{it}}$$

over P. We will write H_i for the stably almost complex manifold (hypersurface) dual to ξ_i and let $X(M)$ be the transverse intersection

$$X(M) = H_1 \cap H_2 \cap \cdots \cap H_l.$$

Writing $Mz = (\mu_1 z, \mu_2 z, \ldots, \mu_l z)$, we define l linear forms $\mu_i : \mathbb{C}^t \longrightarrow \mathbb{C}$, i.e.

$$\mu_i z = m_{i1} z_1 + m_{i2} z_2 + \cdots + m_{it} z_t.$$

Let φ be the elliptic genus of level n defined by a Jacobi function for $G \subset E$ with character χ. The standard computation using Hirzebruch's formalism leads to the following

Theorem 8. *$\varphi(X(M))$ is the coefficient of $z_1^{-1} z_2^{-1} \cdots z_t^{-1}$ in the Laurent expansion at $z = (0, 0, \ldots, 0)$ of*

$$F(z) = F(z_1, z_2, \ldots, z_t) = \frac{f(z_1)^{N_1} f(z_2)^{N_2} \cdots f(z_t)^{N_t}}{f(\mu_1 z) f(\mu_2 z) \cdots f(\mu_l z)}.$$

□

In accordance with the notation $[m]$ for the multiplication-by-m map on E, we will write

$$[M] : E^t \longrightarrow E^s, \qquad [\mu_i] : E^t \longrightarrow E, \qquad [\mu] : E^t \longrightarrow E$$

for the maps induced by left multiplication by M, by μ_i, and by $\mu = \sum \mu_i$.

For the rest of this section we will assume that $l = t$ and that $\det M \neq 0$. Let $G^t = G \times \cdots \times G \subset E^t$ and $H \subset E^t$ be the inverse image of G^t under $[M]$. Clearly, H is a subgroup of E^t containing G^t (since the entries of M are integers), and it is easy to check that $[H : G^t] = (\det M)^2$.

For each j, $1 \leq j \leq t$, choose a zero $r_j \in E$ of f so that

$$\operatorname{div}(f) = -\sum_g (g) + \sum_g (r_j + g),$$

and let $r = (r_1, r_2, \ldots, r_t) \in E^t$. Also, choose a $s = (s_1, s_2, \ldots, s_t) \in E^t$ satisfying $[M]s = r$ (this uses $\det M \neq 0$).

Theorem 9. *If for every j, $\sum_i m_{ij} \equiv N_j \mod \exp(G)$, we have*

$$\varphi(X(M)) = \frac{(-1)^{t+1}}{(\det M) c(r)} \sum_{h \in H/G^t} \chi(-[\mu]h) f(s+h)^N,$$

where we use the abbreviations:

$$c(r) = c(r_1) c(r_2) \cdots c(r_t)$$

and

$$f(s+h)^N = f(s_1 + h_1)^{N_1} f(s_2 + h_2)^{N_2} \cdots f(s_t + h_t)^{N_t},$$

and where the sum runs over representatives of the cosets in H/G^t.

Remark 3. The condition $\sum_i m_{ij} \equiv N_j \mod \exp(G)$ is equivalent to $c_1(X(M)) \equiv 0 \mod \exp(G)$.

The proof of this theorem is based on the following, slightly modified version of the global residue theorem for functions of several complex variables as described in chapter 5 of [12] (pp. 655–656).

Theorem 10. *Let V be a compact complex manifold of dimension t, and let*

$$D_1, D_2, \ldots, D_\nu$$

($t \leq \nu$) be effective divisors having the property that the intersection of every t of them, $D_{n_1} \cap D_{n_2} \cap \cdots \cap D_{n_t}$, is a finite set of points, whereas the intersection of every $t+1$ of them is empty. Let $D = D_1 + D_2 + \cdots + D_\nu$, and let ω be a meromorphic t-form on V with polar divisor D. Then for every $P \in D_{n_1} \cap D_{n_2} \cap \cdots \cap D_{n_t}$ the residue $\operatorname{Res}_P \omega$ *is defined and*

$$\sum_P \operatorname{Res}_P \omega = 0.$$

The residue $\operatorname{Res}_P \omega$ for $P \in D_{n_1} \cap D_{n_2} \cap \cdots \cap D_{n_t}$ can be defined as follows: choose local coordinates $(z_1, z_2, \ldots, z_t)$ near P and write ω near P as

$$\omega = \frac{\psi(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_t}{\phi_1(z)\phi_2(z)\cdots\phi_t(z)},$$

where $\psi, \phi_1, \phi_2, \ldots \phi_t$ are pairwise relatively prime holomorphic functions. Then

$$\operatorname{Res}_P \omega = \frac{1}{(2\pi i)^t} \int_\Gamma \omega,$$

where Γ is the suitably oriented real cycle

$$\Gamma = \{z \mid \phi_1(z) = \varepsilon, \phi_2(z) = \varepsilon, \ldots, \phi_t(z) = \varepsilon\},$$

for a small ε.

It can be easily verified that if $F(z)$ is a meromorphic function near $O = (0, 0, \ldots, 0) \in \mathbb{C}^t$, then $\operatorname{Res}_O(F(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_t)$ is the coefficient of $z_1^{-1} z_2^{-1} \cdots z_t^{-1}$ in the Laurent expansion of $F(z)$ at O. We will also use the following formula, generalizing a fact well-known for functions of one variable:

Lemma 1. *If ω is written near P as*

$$\omega = \frac{\psi(z)\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_t}{\phi_1(z)\phi_2(z)\cdots\phi_t(z)},$$

and $\det(\partial\phi_i/\partial z_j)(P) \neq 0$, *then*

$$\operatorname{Res}_P \omega = \frac{\psi(P)}{\det(\partial\phi_i/\partial z_j)(P)}.$$

We now turn to the proof of theorem 10. In view of theorem 9, and the above remarks, we need to compute $\operatorname{Res}_O \omega$, where

$$\omega = \frac{f(z_1)^{N_1} f(z_2)^{N_2} \cdots f(z_t)^{N_t}\, dz_1 \wedge dz_2 \wedge \cdots \wedge dz_t}{f(\mu_1 z) f(\mu_2 z) \cdots f(\mu_l z)}.$$

We let $V = E^t$ and consider ω on V. According to theorem 11, we have

$$\sum_P \operatorname{Res}_P \omega = 0,$$

where P runs through the points $P \in G^t$, corresponding to the poles of $f(z)$ (first kind), and through the points $P \in s + H$ which are the simultaneous zeroes of $f([\mu_1]z), f([\mu_2]z), \ldots, f([\mu_l]z)$ (second kind).

Notice that the condition

$$\sum_i m_{ij} \equiv N_j \mod \exp(G)$$

guarantees that $F(z+g) = F(z)$ for all $g \in G^t$. It follows that the contribution of the points of the first kind to the sum of residues is $n^t \operatorname{Res}_O \omega$.

Turning to the points of the second kind, say $P = s + h$, we first notice that differentiating

$$f(r_i - z) = \frac{c(r_i)}{f(z)}$$

with respect to z and taking the limit as $z \to 0$ gives

$$f'(r_i) = -c(r_i).$$

Therefore

$$\frac{\partial f([\mu_i]z)}{\partial z_j}(s+h) = f'([\mu_i](s+h))m_{ij} = f'(r_i + [\mu_i]h))m_{ij} = -\chi([\mu_i]h)c(r_i)m_{ij}.$$

Thus

$$\det(\partial f([\mu_i]z)/\partial z_j)(s+h) = (-1)^t c(r)\chi([\mu]h),$$

and

$$\operatorname{Res}_{s+h} \omega = \frac{(-1)^t \chi(-[\mu]h) f(s+h)^N}{c(r)}.$$

Also, notice that this residue remains unchanged when h is replaced by $h + g$ with $g \in G^t$ (because of the condition $\sum_j m_{ij} \equiv N_j \mod \exp(G)$).

Theorem 10 is now an immediate consequence of the residue theorem 11.

10. The Level 2 Case

We now specialize the above formula for the level 2 elliptic genus. Let $\tau \in \mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$, $L_0 = \mathbb{Z} \oplus \mathbb{Z}\tau$, $L = \mathbb{Z} \oplus \mathbb{Z}(2\tau)$, $G = \{0, \tau\}$, and let $\chi : G \longrightarrow \{\pm 1\}$ be defined by $\chi(\tau) = -1$. The divisor of the corresponding Jacobi function f is

$$\operatorname{div}(f) = -(0) - (\tau) + (1/2) + (1/2 + \tau).$$

By comparing the divisors and the $1/z^2$ terms in the Taylor expansions at 0, we easily conclude that

$$f(z)^2 = \wp(z|\tau) - \wp(1/2 \,|\, \tau) = \wp(z|\tau) - e_1.$$

We will choose $r = 1/2$. Since f satisfies the differential equation

$$(f')^2 = f^4 - 2\delta f^2 + \varepsilon$$

(cf. [20]), we see that $c(1/2)^2 = f'(1/2)^2 = \varepsilon$, and, with an appropriate choice of the square root, $c(1/2) = \sqrt{\varepsilon}$.

Consider first the case of a hypersurface $X(m) \subset \mathbb{C}\mathrm{P}^{N-1}$ of degree m. Since we can take $s = 1/2m$, we have:

Theorem 11. *If N and m have same parity, then*

$$\varphi(X(m)) = \frac{1}{m\sqrt{\varepsilon}} \sum_{0\le a,b<m} (-1)^a f^N\left(\frac{1}{2m} + \frac{b}{m} + \frac{a\tau}{m}\right).$$

This is the Eguchi–Jinzenji formula derived in [7]. Notice that only the case where both N and m are even is of interest, since, for dimension reasons, the genus vanishes when N is odd.

Turning now to complete intersections, consider the case of $X(M) \subset \mathbb{C}\mathrm{P}^3 \times \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^1$, where

$$M = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

If we take $r = (1/2, 1/2, 1/2)$, then one can take $s = (1/6, 1/6, 1/6)$, since

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1/6 \\ 1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

Then, noticing that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 6 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 12 \end{pmatrix},$$

we easily conclude that

$$\frac{1}{12}\begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/6 \\ 1/12 \end{pmatrix} = v$$

generates a cyclic subgroup of order 12 of H/G^3, so that one can take

$$\{(a + b\tau)v \mid 0 \le a, b < 12\}$$

as representatives of the cosets of H/G^3. Also, notice that $[\mu]v = 1$, so

$$\chi(-[\mu](a + b\tau)v) = (-1)^b.$$

We obtain the following

Theorem 12. *For M as above,*

$$\varphi(X(M)) = \frac{1}{12\,\varepsilon^{3/2}} \sum_{0\le a,b<12} (-1)^b f\left(\frac{1}{6} + \frac{a+b\tau}{3}\right)^4 f\left(\frac{1}{6} - \frac{a+b\tau}{6}\right)^3 f\left(\frac{1}{6} + \frac{a+b\tau}{12}\right)^2.$$

The general case can be treated in exactly the same way, since one can always find invertible over $\mathbb{Z}$ matrices A, B such that AMB is diagonal.

11. A different approach for the level 2 case

Considering, as before, a hypersurface $X(m) \subset \mathbb{C}\mathrm{P}^{N-1}$ of degree m, and keeping the assumptions of the previous section, we first note that

$$\mathrm{div}(f) = -\sum_g (g) + \sum_g (r+g),$$

so that

$$\mathrm{div}(f(mz)) = -\sum_h (h) + \sum_h (s+h),$$

and thus the polar divisor of $1/f(mz)$ is

$$\mathrm{div}_\infty\left(\frac{1}{f(mz)}\right) = -\sum_h (s+h).$$

To calculate the residue of $1/f(mz)$ at $s+h$, let $t \mapsto \lambda(t)$ be a small loop going around 0 in the positive direction and for $h \in H$, let $\mu(t) = \lambda(t) + s + h$. Then, using the substitution $u = z + s + h$, we have:

$$\begin{aligned} \mathrm{Res}_{\,z=s+h}\left(\frac{1}{f(mz)}\right) &= \frac{1}{2\pi i}\oint_\mu \frac{dz}{f(mz)} \\ &= \frac{1}{2\pi i}\oint_\lambda \frac{du}{f(mu+r+mh)} \\ &= \frac{\chi(-mh)}{2\pi i}\oint_\lambda \frac{du}{f(mu+r)} \\ &= \frac{\chi(-mh)}{2\pi i c(r)}\oint_\lambda f(-mu)du \\ &= -\frac{\chi(-mh)}{mc(r)}. \end{aligned}$$

.

Theorem 13. *We have*

$$\frac{1}{f(mz)} = -\frac{1}{mc(1/2)}\sum_{0\le a,b<m}\frac{(-1)^a f'(z)}{f(z)-f(\omega_{a,b})},$$

where

$$\omega_{a,b} = \frac{1}{2m} + \frac{b}{m} + \frac{a\tau}{m}.$$

Proof. Specializing the above calculations to the level 2 case, we see that the polar divisor of $1/f(mz)$ is given by

$$\operatorname{div}_\infty\left(\frac{1}{f(mz)}\right) = -\sum_{0\le a<2m,\,0\le b<m}(\omega_{a,b}),$$

and that

$$\operatorname{Res}_{\omega_{a,b}}\left(\frac{1}{f(mz)}\right) = -\frac{(-1)^a}{mc(1/2)}.$$

Consider the function

$$h_{a,b}(z) = \frac{f'(z)}{f(z)-f(\omega_{a,b})},$$

i.e. the logarithmic derivative of $f(z)-f(\omega_{a,b})$. Let σ be the involution $\sigma(z)=\tau-z$. Then, since f is odd,

$$f(\tau-z) = -f(-z) = f(z),$$

i.e. $f\circ\sigma = f$. Also,

$$\sigma(\omega_{a,b}) = \tau - \frac{1}{2m} - \frac{b}{m} - \frac{a\tau}{m} = \frac{1}{2m} + \frac{-b-1}{m} + \frac{(m-a)\tau}{m} = \omega_{m-a,-b-1},$$

where $m-a$ and $-b-1$ should be interpreted modulo $2m$ and m respectively. As a consequence, we have $h_{a,b} = h_{m-a,-b-1}$.

The function $f(z)-f(\omega_{a,b})$ has two simple poles 0 and τ. Thus, it has two zeroes (counted with multiplicities), one of which is $\omega_{a,b}$. Since m is even, $b\not\equiv -b-1 \mod m$. Thus $\omega_{m-a,-b-1}$ is the other zero of $f(z)-f(\omega_{a,b})$ and both zeroes are simple. Thus

$$\operatorname{div}_\infty(h_{a,b}) = -(0)-(\tau)-(\omega_{a,b})-(\omega_{m-a,-b-1}),$$

with residue -1 at 0 and τ, and residue 1 at $\omega_{a,b}$ and $\omega_{m-a,-b-1}$. Let now

$$F(z) = -\frac{1}{mc(1/2)}\sum_{0\le a<m,\,0\le b<m}\frac{(-1)^a f'(z)}{f(z)-f(\omega_{a,b})} = -\frac{1}{2mc(1/2)}\sum_{0\le a<2m,\,0\le b<m}(-1)^a h_{a,b}(z).$$

The possible poles of F are 0, τ, and $\omega_{a,b}$ with $0\le a<2m$, $0\le b<m$. Let's compute the residues of F at these points. First, we have

$$\operatorname{Res}_0(F) = \operatorname{Res}_\tau(F) = -\frac{1}{2mc(1/2)}\sum_{0\le a<2m,\,0\le b<m}(-1)^a(-1) = 0,$$

since m is even. Then,

$$\operatorname{Res}_{\omega_{a,b}}(F) = -\frac{1}{2mc(1/2)}((-1)^a+(-1)^{m-a}) = -\frac{(-1)^a}{mc(1/2)}.$$

Thus, $1/f(mz)$ and $F(z)$ have the same polar part. It follows that

$$\frac{1}{f(mz)} = F(z)+C,$$

for some constant C. Replacing z with $\sigma(z)=\tau-z$, we have

$$\frac{1}{f(m\sigma(z))} = \frac{1}{f(m\tau-mz)} = \frac{1}{f(-mz)} = -\frac{1}{f(m\sigma(z))},$$

since $m\tau$ is a period of f for m even. Similarly, since $f(z)$ is invariant under σ and $f'(\sigma(z)) = -f'(z)$, we have $h_{a,b}(\sigma(z)) = -h_{a,b}(z)$ and therefore $F(\sigma(z)) = -F(z)$. It follows that $C = 0$.

As a corollary, we obtain a new proof of theorem 11. Indeed, according to theorem 8,

$$\varphi(X(m)) = \frac{1}{2\pi i}\oint_\lambda \frac{f(z)^N\,dz}{f(mz)},$$

where λ is a small circle around 0 traversed counter-clockwise. Thus

$$\varphi(X(m)) = -\frac{1}{2\pi imc(1/2)}\oint_\lambda \sum_{0\le a,b<m} \frac{(-1)^a f(z)^N f'(z)\,dz}{f(z) - f(\omega_{a,b})}.$$

Changing the variable to $u = f(z)$, we get

$$\varphi(X(m)) = -\frac{1}{2\pi imc(1/2)}\oint_\Lambda \sum_{0\le a,b<m} \frac{(-1)^a u^N, du}{u - f(\omega_{a,b})},$$

where Λ is a large circle around 0 traversed *clockwise*. The Cauchy integral formula gives

$$\varphi(X(m)) = \frac{1}{mc(1/2)}\sum_{0\le a,b<m} (-1)^a f(\omega_{a,b})^N,$$

which is exactly the formula in theorem 11.

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Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027