A note on the graded ring of a polarized Calabi-Yau 3-fold

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MPI / 93-16

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Introduction.

Recently, Ein and Lazarsfeld ([E-L, 2,3]) obtained the best possible effective estimate on the base point freeness of adjoint linear series of a non-singular polarized 3-fold. But it still remains a problem to find a reasonable effective estimate on the very ampleness of them (cf. [E-L, 1,2]).

In this paper, inspired by lectures given by Ein and Reid at Utah University in November 1992, we shall prove the following theorem concerning with a polarized Calabi-Yau 3-fold. This is an improvement of our previous result [O, (3.1)].

Main Theorem. *Let (X, L) be a polarized Calabi-Yau 3-fold, i.e., X is a non-singular projective complex 3-fold with $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$, and L is an ample line bundle on X. Assume that $|mL|$ is free and $\Phi_{|mL|}$ is birational for every $m \geq f$, where f is a positive integer. Put $R_n = H^0(\mathcal{O}_X(nL))$. Then, for every $n \geq 2f$, we have:*

(1) $R_f.R_n = R_{n+f}$,
(2) *nL is simply generated, i.e., the graded $\mathbb{C}$-algebra $\oplus_{k\geq 0} R_{kn}$ is generated by R_n. In particular, nL is very ample for every $n \geq 2f$.*

Let $(X, L) = ((3) \cap (4) \subset \mathbf{P}(1,1,1,1,1,2), \mathcal{O}_X(1))$ be a general complete intersection of hypersurfaces of degree 3 and 4 in the weighted projective space $\mathbf{P}(1,1,1,1,1,2)$. Then, as is observed in [O, (3.7)], (X, L) is a polarized Calabi-Yau 3-fold which satisfies that $|mL|$ is free and $\Phi_{|mL|}$ is birational for $m \geq 1$, but L itself is not very ample. So our estimate is best possible for at least $f = 1$.

On the other hand, we know that for a polarized Calabi-Yau 3-fold (X, L), $|mL|$ is free for every $m \geq 4$ by [E-L, 2,3], and $\Phi_{|mL|}$ is birational for every $m \geq 5$ by [O, (1.1)]. Moreover, $\Phi_{|4L|}$ is birational except for a few cases (for detail, see [O, (1.1)]). Thus we can derive a next effective estimate on the very ampleness (more strongly, on the simply generatedness) from our main theorem:

Corollary. *Let (X, L) be a polarized Calabi-Yau 3-fold. Then,*

(1) *mL is simply generated for every $m \geq 10$,*
(2) *mL is simply generated for every $m \geq 8$ if $h^0(\mathcal{O}_X(L)) \geq 2$.*

A pair $(X, L) = ((10) \subset \mathbf{P}(1,1,1,2,5), \mathcal{O}_X(1))$ shows that both of the estimates on the base point freeness and on the birationality quoted above are best possible for polarized Calabi-Yau 3-folds. But the author does not know whether the estimate in the corollary is best possible or not.

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We shall prove our main theorem by making use of the following theorem concerning with curves mainly due to Fujita, Green, and Reid.

Theorem 1 ([F1, Theorem A7], [G, Theorem (4.e.1)], [R, Lemma 2.5]). *Let C be a non-singular projective curve, and L_1 and L_2 be line bundles on C such that*

(1) *$|L_1|$ is free and $\Phi_{|L_1|}$ is birational, and*
(2) *either $h^0(K_C + L_1 - L_2) \leq h^0(L_1) - 2$ and $h^0(L_2) \neq 0$, or $L_2 = K_C$ and $g(C) \geq 1$.*

Then, the following natural multiplication map is surjective:

$$H^0(L_1) \otimes H^0(L_2) \longrightarrow H^0(L_1 + L_2).$$

The author would like to express his thanks to Professor Dr.'s L. Ein and R. Lazarsfeld for their encouragement, to Professor Dr. H. Clemens for informing him the nice workshop at Utah, and to Professor Dr. F. Hirzebruch for offering him an opportunity to visit Max-Planck-Institut für Mathematik. This article was written up during his stay in the institute.

Set up of the proof.

Let (X, L) be a polarized Calabi-Yau 3-fold and f be an integer in the main theorem. Let $s, t \in H^0(\mathcal{O}_X(fL))$ be general elements. Put $S := div(s)$ and $C := div(t|_S)$. Then, C is a smooth curve with $K_C = 2fL_C$, where L_C is the restriction of L to C. Note that $\Phi_{|fL|}$ and $\Phi_{|K_C|}$ define birational morphisms on C. In particular, $C \not\simeq \mathbb{P}^1$. We need the following lemmas in order to prove our main theorem.

Lemma 2 ([O, (0.2), (0.3)]).

(1) *$h^0(\mathcal{O}_X(mL)) = \frac{1}{6}(m^3 - m)L^3 + mh^0(\mathcal{O}_X(L)) \geq 1$ and $h^i(\mathcal{O}_X(mL)) = 0$ for every $m \geq 1$ and $i \geq 1$.*
(2) *$h^i(\mathcal{O}_X(mL)) = 0$ for every $m \in \mathbf{Z}$ and $i = 1, 2$.*
(3) *$h^1(\mathcal{O}_S(mL_S)) = 0$ and the following natural restriction maps are surjective for every $m \in \mathbf{Z}$:*

$$r_S : H^0(\mathcal{O}_X(mL)) \longrightarrow H^0(\mathcal{O}_S(mL_S)),$$

$$r_C : H^0(\mathcal{O}_S(mL_S)) \longrightarrow H^0(\mathcal{O}_C(mL_C)).$$

Lemma 3. *The following natural multiplication map is surjective for every $r \geq 0$:*

$$H^0((2f + r)L_C) \otimes H^0(fL_C) \longrightarrow H^0((3f + r)L_C).$$

Proof. If $r = 0$, the assertion directly follows from Theorem 1 because $2fL_C = K_C$. In what follows, we assume that $r > 0$. We shall apply Theorem 1 to the pair $L_1 := fL_C$ and $L_2 := (2f + r)L_C$. In order to do this, since $h^0((2f + r)L_C) = h^0(K_C + rL_C) \neq 0$, it is enough to check the next inequality ($\sharp$):

$$(\sharp)\qquad h^0(K_C + L_1 - L_2) \le h^0(L_1) - 2.$$

Since $K_C + L_1 - L_2 = (f-r)L_C$, we can calculate $h^0(K_C + L_1 - L_2)$ from the exact sequences $0 \longrightarrow \mathcal{O}_X(-rL) \longrightarrow \mathcal{O}_X((f-r)L) \longrightarrow \mathcal{O}_S((f-r)L_S) \longrightarrow 0$, and $0 \longrightarrow \mathcal{O}_S(-rL_S) \longrightarrow \mathcal{O}_S((f-r)L_S) \longrightarrow \mathcal{O}_C((f-r)L_C) \longrightarrow 0$ as follows:

$$\begin{aligned} &h^0(K_C + L_1 - L_2) \\ &= h^0(\mathcal{O}_X((f-r)L)) = \begin{cases} \frac{(f-r)^3-(f-r)}{6}L^3 + (f-r)h^0(\mathcal{O}_X(L)) & f > r \\ 1 & f = r \\ 0 & f < r\,. \end{cases} \end{aligned}$$

On the other hand, from the exact sequences
$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(fL) \longrightarrow \mathcal{O}_S(fL_S) \longrightarrow 0$, and $0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(fL_S) \longrightarrow \mathcal{O}_C(fL_C) \longrightarrow 0$, we have:

$$h^0(L_1) = h^0(\mathcal{O}_X(fL)) - 2 = \frac{f^3 - f}{6}L^3 + fh^0(\mathcal{O}_X(L)) - 2.$$

If $r \ge f$, then $h^0(K_C + L_1 - L_2) \le 1$. But, since $\Phi_{|L_1|}$ is birational and since $S \not\simeq \mathbb{P}^1$, we have $h^0(L_1) \ge 3$. Thus ($\sharp$) holds if $r \ge f$. We shall treat the case when $0 < r < f$. Note that $f \ge 2$. In this case, the difference between $h^0(L_1) - 2$ and $h^0(K_C + L_1 - L_2)$ is calculated as follows:

$$\begin{aligned} &h^0(L_1) - 2 - h^0(K_C + L_1 - L_2) \\ &= \frac{f^3 - f}{6}L^3 + fH^0(\mathcal{O}_X(L)) - 2 - 2 - \frac{(f-r)^3 - (f-r)}{6}L^3 - (f-r)h^0(\mathcal{O}_X(L)) \\ &= \frac{3f(f-r) + (r^3 - r)}{6}L^3 + rh^0(\mathcal{O}_X(L)) - 4 \\ &\ge \frac{f(f-1)}{2}L^3 + h^0(\mathcal{O}_X(L)) - 4\,. \end{aligned}$$

If $f \ge 3$, we have $\frac{f(f-1)}{2}L^3 + h^0(\mathcal{O}_X(L)) - 4 \ge \frac{1}{2}.3.2.1 + 1 - 4 \ge 0$ and ($\sharp$) holds. Assume that $f = 2$. Then we have $r = 1$ and $h^0(L_1) - 2 - h^0(K_C + L_1 - L_2) = L^3 + h^0(\mathcal{O}_X(L)) - 4$. We shall show that $L^3 + h^0(\mathcal{O}_X(L)) - 4 \ge 0$, under the assumption that $2L$ is free and $\Phi_{|2L|}$ is birational, by arguing contradiction. Assume that $L^3 + h^0(\mathcal{O}_X(L)) \le 3$. Then, since $L^3 \ge 1$ and since $h^0(\mathcal{O}_X(L)) \ge 1$, the pair $(L^3, h^0(\mathcal{O}_X(L)))$ is one of the following 3 candidates: $(1,1), (2,1), (1,2)$. If $(L^3, h^0(\mathcal{O}_X(L))) = (1,2)$, then (X, L) is isomorphic to $(X = (6) \cap (6) \subset \mathbb{P}(1,1,2,2,3,3), \mathcal{O}_X(1))$ by [F2] or [O, (5.1)]. But, in this case, as is easily seen by writing down the equation of X, we have $deg\Phi_{|2L|} = 4$, which contradicts our assumption that $\Phi_{|2L|}$ is birational. If $(L^3, h^0(\mathcal{O}_X(L))) = (1,1)$ or $(2,1)$, then we have $h^0(\mathcal{O}_X(2L)) = L^3 + 2h^0(\mathcal{O}_X(L)) = 3$ or 4. Thus $\Phi_{|2L|}$ is a map from

X to $\mathbb{P}^2$ or $\mathbb{P}^3$, which again contradicts our assumption that $\Phi_{|2L|}$ is birational. Q.E.D. of Lemma 3.

Proof of the Main Theorem (1).

By lemma 3 and lemma 2 (3), we can get the following 3 exact sequences:

$$H^0((2f+r)L_S)\otimes H^0(fL_S)\xrightarrow{r_C} H^0((2f+r)L_C)\otimes H^0(fL_C)\longrightarrow 0\,,$$

$$0\longrightarrow H^0((2f+r)L_S)\xrightarrow{t} H^0((3f+r)L_S)\xrightarrow{r_C} H^0((3f+r)L_C)\longrightarrow 0\,,$$

$$H^0((2f+r)L_C)\otimes H^0(fL_C)\xrightarrow{m_C} H^0((3f+r)L_C)\longrightarrow 0\,,$$

where m_C is the natural multiplication map. Then, by an easy diagram chasing, we see that the next natural multiplication map is surjective for $r\geq 0$:

$$m_S: H^0((2f+r)L_S)\otimes H^0(fL_S)\longrightarrow H^0((3f+r)L_S)\,.$$

In fact, for $x\in H^0((3f+r)L_S)$, put $y=r_C(x)$. Then, there exists an element $z\in H^0((2f+r)L_C)\otimes H^0(fL_C)$ such that $y=m_C(z)$. Take $w\in H^0((2f+r)L_S)\otimes H^0(fL_S)$ such that $z=r_C(w)$. Since $r_C(x-m_S(z))=r_C(x)-m_Cr_C(w)=0$, there is an element $v\in H^0((2f+r)L_S)$ such that $x-m_S(z)=v.t$. Thus $x=m_S(z+v\otimes t)$ and m_S is surjective. Now, by the surjection m_S and by lemma 2 (3), we can see, in the same way as before, that the following natural multiplication map is surjective for every $r\geq 0$:

$$m_X: H^0(\mathcal{O}_X(fL))\otimes H^0(\mathcal{O}_X((2f+r)L))\longrightarrow H^0(\mathcal{O}_X((3f+r)L))\,.$$

Hence $R_f.R_n=R_{n+f}$ for every $n\geq 2f$. Q.E.D. of the Main Theorem (1).

Proof of the Main Theorem (2).

Let n be an integer such that $n\geq 2f$. By dividing n by f, we can write n as $n=qf+r$ where q and r are integers which satisfy that $q\geq 2$ and $0\leq r\leq f-1$. In order to prove (2), it is enough to show that $R_{kn}=R_{(k-1)n}.R_n$ for every $k\geq 2$. Since $k\geq 2$ and $q\geq 2$, we have:

$$kn-3(f+r)=(kq-4)f+f+(k-3)r\geq f-r\geq 0\,.$$

Thus, we can applying the main theorem (1) to the pair $(kn, f+r)$, and get:

$$R_{kn}=R_{kn-(f+r)}.R_{(f+r)}\,.$$

Since $k\geq 2$ and $q\geq 2$, we have:

$$\{kn-(f+r)\}-(q-2)f-3f=\{(k-1)q-2\}f+(k-1)r\geq 0\,.$$

Thus, by applying the main theorem (1) repeatedly to the pairs

$(kn-(f+r),f),(kn-(f+r)-f,f),...,(kn-(f+r)-(q-2)f,f)$, we have:

$$\begin{aligned}
&R_{kn-(f+r)}\\
&= R_{kn-(f+r)-f}.R_f\\
&= R_{kn-(f+r)-f-f}.R_f.R_f\\
&= R_{kn-(f+r)-(q-2)f}.R_f^{q-2}\\
&= R_{kn-(f+r)-(q-2)f-f}.R_f.R_f^{q-2}\\
&= R_{(k-1)n}.R_f^{q-1}\,.
\end{aligned}$$

Thus $R_{kn} = R_{(k-1)n}.R_f^{q-1}.R_{f+r}$. But, since $R_f^{q-1}.R_{f+r} \subset R_{f(q-1)+f+r} = R_{fq+r} = R_n$, we have $R_{kn} \subset R_{(k-1)n}.R_n$. Thus we get the desired equality $R_{kn} = R_{(k-1)n}.R_n$. Q.E.D. of the Main Theorem (2).

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