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LIFTING HOMOTOPY T-ALGEBRA MAPS TO STRICT MAPS

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ABSTRACT. The settings for homotopical algebra—categories such as simplicial groups, simplicial rings, A_∞ spaces, E_∞ ring spectra, etc.—are oftentimes equivalent to categories of algebras over some monad or triple T. In such cases, T is acting on a nice simplicial model category in such a way that the T descends to a monad on the homotopy category and defines a category of *homotopy* T-algebras. In this setting there is a forgetful functor from the homotopy category of T-algebras to the category of homotopy T-algebras.

Under suitable hypotheses we provide an obstruction theory, in the form of a Bousfield-Kan spectral sequence, for lifting a homotopy T-algebra map to a strict map of T-algebras. Once we have a map of T-algebras to serve as a basepoint, the spectral sequence computes the homotopy groups of the space of T-algebra maps and the edge homomorphism on π_0 is the aforementioned forgetful functor. We discuss a variety of settings in which the required hypotheses are satisfied, including monads arising from algebraic theories and from operads.

We provide examples in G-spaces, G-spectra, rational E_∞-algebras, and A_∞-algebras under an Eilenberg-MacLane commutative ring spectrum. We give explicit calculations showing that the forgetful functor from the homotopy category of E_∞ ring spectra to the category of H_∞ ring spectra is generally neither full nor faithful. We also apply a result of the second named author and Nick Kuhn to compute the homotopy type of the space $E_\infty(\Sigma^\infty_+ \operatorname{Coker} J, L_{K(2)}R)$.

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1. INTRODUCTION

In the work of Ando, Hopkins, Rezk, and Strickland on the Witten genus [AHS01, AHS04, AHR06] the authors first construct a lift of the Witten genus to a multiplicative map of cohomology theories, then to an H_∞ map (i.e., a map preserving power operations), and finally to an E_∞ map

$$MString \to tmf.$$

In each of these steps they are asking that the map commute with more structure and it is natural to ask if there are general techniques for lifting a map to one that commutes with this structure.

Their construction of an H_∞ map makes use of ideas from Ando's thesis [And92, And95], where he defines H_∞ maps from complex cobordism to Lubin-Tate spectra using a connection to isogenies of Lubin-Tate formal group laws. Thus the result arises from a computation: Since the H_∞ condition can be formulated in the stable homotopy category, a map is H_∞ if and only if an associated sequence of cohomological equations hold. The applicability of such techniques is one of the reasons that the category of H_∞ ring spectra is computationally more accessible. Although every E_∞ map forgets to an H_∞ map, constructing E_∞ maps is much more subtle and requires rather different techniques.

We develop an obstruction-theoretic spectral sequence to detect when an H_∞ map can be lifted to an E_∞ map and other problems of this type. As a consequence of our approach we can also see *how much* information is lost under the passage from E_∞ to H_∞ ring spectra. The first category can be described as the category of algebras over a monad/triple T in a category of spectra while the second is the category of such algebras in the homotopy category. Phrased in these terms, it is expected that a great deal is forgotten in the passage from E_∞ to H_∞ ring spectra. But to date, there have been no examples demonstrating this. Since our methods apply more generally to studying categories of algebras over a monad T (satisfying some hypotheses), we set up our machinery in the more abstract setting.

In Section 2 we provide a rapid review of the theory of monads and how they naturally encode algebraic structures. We emphasize the examples coming from algebraic theories and from operads since they make up the majority of our examples. In Section 3, we recall some conditions which guarantee the existence of a simplicial model structure on the category of algebras over a monad. These conditions are often satisfied and cover a broad range of standard examples. Although most of this material is quite standard and can be skipped by experts, we found it useful to recall the relevant background. Some crucial technical lemmas in Sections 3.2 and 3.3 will be used to establish the correct homotopical properties of the simplicial resolution from which we construct the spectral sequence.

This spectral sequence, which we call the T-algebra spectral sequence, arises as a special case of Bousfield's spectral sequence computing the homotopy of the totalization of a cosimplicial space [Bou89] under a number of assumptions. As shown in Section 4.3, these assumptions hold in many cases of interest such as categories of algebras over a cofibrant operad in spaces or in R-module spectra for cofibrant commutative ring spectrum R, G-spaces and G-spectra (provided G is sufficiently nice), and many algebraic categories such as simplicial groups and rings.

In Section 5 we apply this machinery to compute this spectral sequence in several cases of interest. The reader interested in applications is encouraged to skip directly to this section. We include some calculations analyzing the spaces of equivariant maps in G-spaces and G-spectra. In these examples we explicitly analyze the forgetful functor landing in G-objects in the homotopy category of spaces and spectra. We then work through a number of examples analyzing spaces of A_∞ and E_∞ maps in categories of Eilenberg-MacLane spectra. In particular we provide two examples arising via unstable rational homotopy theory demonstrating that the forgetful functor from the homotopy category of E_∞ ring spectra to H_∞ ring spectra is generally neither full nor faithful. To the authors' knowledge, these are the first such examples.

Finally, we include a result of Nick Kuhn and the second author about the infinite loop space Coker J to give a non-trivial situation in which H_∞ maps and homotopy classes of E_∞ maps coincide. Namely maps from $\Sigma^\infty_+ \operatorname{Coker} J$ to any $K(2)$-local E_∞ ring spectrum.

Theorem 1.1. Suppose T is a monad acting on a simplicial category $\mathscr{C}$ and X and Y are T-algebras such that:

(a) T is Quillen (Definition 3.2),
(b) T commutes with geometric realization,
(c) and X is resolvable with replacement $\widetilde{X}$ (Definition 3.22).

Let $U\colon \mathscr{C}_T \to \mathscr{C}$ denote the forgetful functor from the category of T-algebras to $\mathscr{C}$. Then there exists an obstruction-theoretic spectral sequence satisfying:

(1) $E_1^{0,0} = ho\mathscr{C}(UX, UY)$.
(2) $E_2^{0,0} = (ho\mathscr{C})_{hT}(UX, UY)$. That is, a homotopy class $[f]\colon UX \to UY$ survives to the E_2 page if and only if it is a map of hT-algebras[1] in the homotopy category.
(3) Provided a T-algebra map $\varepsilon\colon X \to Y$ to serve as a base point, the spectral sequence conditionally converges to the homotopy of the derived mapping space
$$\pi^s \pi_t \mathscr{C}^d(T^\bullet U\widetilde{X}, Y) \Longrightarrow \pi_* \mathscr{C}^d_T(X, Y).$$
(4) In this case the differentials $d_r[f]$ provide obstructions to lifting $[f]$ to a map of T-algebras.
(5) The edge homomorphisms
$$\begin{aligned}\pi_0 \mathscr{C}^d_T(X,Y) \twoheadrightarrow E_\infty^{0,0} & \\ &\hookrightarrow E_2^{0,0} = (ho\mathscr{C})_{hT}(UX,UY) \\ &\hookrightarrow E_1^{0,0} = ho\mathscr{C}(UX,UY)\end{aligned}$$
are the corresponding forgetful functors.
(6) The spectral sequence is contravariantly functorial in $X \in ho\mathscr{C}_T$ and covariantly functorial in $Y \in ho\mathscr{C}_T$ and T satisfying the hypotheses.

This result will be proven in Section 4.1. Note that we do not require any properness assumptions on our model category, since we avoid using E_2 model structures or Bousfield localizations.

Bousfield has shown that this spectral sequence can still be applied even without the existence of a base point—a useful generalization since a space of T-algebra maps may well be empty. In this case there is an obstruction theory (see Remark 4.4) for lifting a map in $\mathscr{C}$ to a map of T-algebras so that one can obtain a base point [Bou89, §5]. The farther one can lift this base point up the totalization tower, the greater the range in which one can define the spectral sequence and differentials.

When the relevant mapping spaces in $\mathscr{C}$ have the homotopy type of H-spaces, e.g., if $\mathscr{C} = \mathcal{S}pectra$, then one can choose these obstructions to land in the E_2 page of the spectral sequence. This is codified in Theorem 4.5. In practice, the E_2 page is significantly smaller than the E_1 page and these obstruction groups can be shown to vanish in some special cases (see Section 5).

Related work. The T-algebra spectral sequence arises by taking a functorial resolution of the source X. Namely we replace X by the two sided bar construction $B(F_T, T, UX)$ where U is the forgetful functor $\mathscr{C}_T \to \mathscr{C}$ and F_T is its left adjoint. For this approach, one wants general conditions under which the replacement is cofibrant, weakly equivalent to X, and equipped with a suitable

[1] Note that the notion of *homotopy* T-algebra is entirely distinct from the notion of a homotopy algebra in the context of algebras over operads. If T comes from an operad then our T-algebras are homotopy algebras.

filtration for obtaining a spectral sequence. A number of special cases of this theory are well known, and the arguments for spaces and spectra can be found in the literature. Although the two-sided bar construction has been a standard tool in homotopy theory for decades, we know of no reference in which its homotopical properties are developed with sufficient breadth for our purposes. The technical work in Sections 3.2 and 3.3 is concerned with filling this gap.

There are a couple of alternative methods for constructing maps of structured ring spectra. This work can be considered an extension of the obstruction theory for maps of A_∞ simplicial R-modules and A_∞ ring spectra that appears in Rezk's thesis [Rez96] and his presentation of the Hopkins-Miller theorem [Rez97]. Indeed the latter work was a significant source of inspiration for this project. Angeltveit [Ang08] has also constructed an obstruction theory, which appears to be part of a spectral sequence, for computing maps of A_∞ ring spectra.

The Goerss-Hopkins spectral sequence also computes the homotopy of the derived mapping space between two spectra which are algebras over a suitable operad, such as an E_∞ operad [GH04, GH05]. This spectral sequence uses an E_2 model structure which guarantees an algebraic description of the E_2 term and is generally distinct from the T-algebra spectral sequence. In particular, their edge homomorphism is generally a Hurewicz homomorphism which generally is distinct from the forgetful functor above. However, in future work, the second author will show that in special cases such as those considered in Section 5.3 the spectral sequences do agree.

Determining the E_2 term of the T-algebra spectral sequence is generally quite difficult. Indeed, the main results of [AHS04, And95, JN10] could be expressed as partial computations of $d_1\colon E_1^{0,0} \to E_1^{1,0}$. The difficulties here are generic; there are very few examples where one has enough knowledge of the power operations to compute the E_2 term explicitly.

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Conventions/Terminology. We will say that a category is bicomplete if it has all small colimits and limits. As we have no need to discuss 'large' colimits and limits we will henceforth omit this adjective.

Let Δ denote the category of finite non-empty linearly ordered sets with order preserving maps. We will write $s\mathscr{C} := \mathscr{C}^{\Delta^{\mathrm{op}}}$ for the category of simplicial objects in $\mathscr{C}$. Throughout we will regard $s\mathcal{S}et$, the category of simplicial sets, as equipped with the Quillen model structure, in which cofibrations are monomorphisms, fibrations are Kan fibrations, and weak equivalences are those maps which induce weak equivalences of topological spaces after geometric realization.

For our purposes, the category $\mathcal{T}op$ of topological spaces will be the cartesian closed category of compactly generated weak Hausdorff spaces. There is a model structure on $\mathcal{T}op$ such that geometric realization $s\mathcal{S}et \to \mathcal{T}op$ is a left Quillen functor and a Quillen equivalence.

We will make the convention that a simplicial category is tensored and cotensored over simplicial sets. This convention is standard when discussing simplicial model categories, but unusual in enriched category theory.

Definition 1.2. Let $G\colon \mathscr{C} \to \mathscr{D}$ be a functor, and let $D\colon \mathscr{I} \to \mathscr{C}$ be a diagram in $\mathscr{C}$.

- G *preserves* colimits of D if $GD \to GA$ is a colimit in $\mathscr{D}$ whenever $D \to A$ is a colimit in $\mathscr{C}$.
- G *reflects* colimits of D if $D \to A$ is a colimit in $\mathscr{C}$ whenever $GD \to GA$ is a colimit in $\mathscr{D}$.

- G *creates* colimits of D if D has a colimit whenever GD has a colimit and G preserves and reflects colimits of D.

We say that G preserves, reflects, or creates colimits if it does so for all diagrams. Similar terminology is used for preservation, reflection, or creation of limits.

2. Algebras over a monad

This section reviews monads and their categories of algebras, focusing on conditions which ensure that limits and colimits in the categories of algebras exist and can be computed with information from the underlying categories.

In Section 2.1 we begin with a familiar example, focusing on points which are key to the general theory. A wealth of additional examples can be found in the framework of algebraic theories which we recall in Section 2.2. In Section 2.3 we extend this discussion to the simplicially enriched context. Finally we recall some relevant facts about operads from [Rez97] in Section 2.4. In these last two sections the monads coming from simplicial algebraic theories and from operads will be introduced and form the lion's share of our examples.

2.1. **Monadicity and categories of algebras.** Given a set S we can take the free group FS on S whose underlying set consists of all finite reduced words whose letters are signed elements of S. Multiplication is then defined by composing words. We can also take a group G, forget its group structure, and regard it is as a set $X = UG$. These constructions are clearly functorial and participate in an adjunction

$$\mathcal{G}roup \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{S}et$$

where U is right adjoint to F. Let $T = UF$ denote the endofunctor of $\mathcal{S}et$ given by the composite of these two functors.

The unit of this adjunction is a natural transformation $e\colon \mathrm{Id} \to T$ given by identifying taking an element of a set with to its associated word of length one. Using the underlying group structure on X one can multiply the elements in a word to obtain a *structure map*

$$\mu_X\colon TX \to X.$$

Alternatively we could construct this map by applying U to the counit

$$\varepsilon\colon FU \to \mathrm{Id}$$

of this adjunction. In particular, we have such a map for anything in the image of T and obtain a natural transformation

$$\mu_T\colon T^2 \to T.$$

The (large) category of endofunctors of $\mathcal{S}et$ admits a monoidal structure under composition and we can see that (T, e, μ_T) is an associative monoid in this category, in other words, T is a *monad* on $\mathcal{S}et$.

In the case of $X = UG$ we see that the map μ_X is compatible with this structure in the sense that the two double composites of straight arrows in (2.1) are equal and each composite of a curved arrow followed by a straight arrow is the identity morphism.

$$\begin{tikzcd} TTX \arrow[r, shift left, "\mu_T"] \arrow[r, shift right, "T\mu_X"'] & TX \arrow[l, bend right, "e_{TX}"'] \arrow[r, "\mu_X"] & X \arrow[l, bend right, "e_X"'] \end{tikzcd} \tag{2.1}$$

An object $X \in \mathcal{S}et$ with a map $\mu_X\colon TX \to X$ satisfying these identities is called a T-algebra in $\mathcal{S}et$. We obtain a category $\mathcal{S}et_T$ of T-algebras in $\mathcal{S}et$ by restricting to those set maps which commute with the structure morphisms. To be explicit, the morphisms between two T-algebras (X,μ_X) and (Y,μ_Y) are those maps $f\colon X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\mu_X"'] & TY \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

or, alternatively,

$$\mathcal{S}et_T(X,Y) = \mathrm{eq}\left[\mathcal{S}et(X,Y) \underset{\mu_X^*}{\overset{(\mu_Y\circ Tf)^*}{\rightrightarrows}} \mathcal{S}et(TX,Y)\right]. \tag{2.2}$$

The category of T-algebras in $\mathcal{S}et$ admits an obvious forgetful functor to $\mathcal{S}et$ and we saw above that the forgetful functor $U\colon \mathcal{G}roup \to \mathcal{S}et$ factors through $\mathcal{S}et_T$. It is not difficult to see that the latter functor defines an equivalence of categories. Indeed, if G is a group then we can see that some of the maps in (2.1) can be realized by applying U to following diagram of groups:

$$\begin{tikzcd} FTG \arrow[r, shift left, "\mu_T"] \arrow[r, shift right, "F\mu_{UG}"'] & FG \arrow[l, bend right=60, "e"'] \arrow[r] & G. \end{tikzcd} \tag{2.3}$$

The map on the right exhibits G as the coequalizer of the two straight arrows on the left. Moreover, the map e exhibits this coequalizer as a *reflexive* coequalizer. In this sense we see that every group has a *functorial resolution* by free groups. The forgetful functor from $\mathcal{S}et_T$ to $\mathcal{S}et$ admits a left adjoint F_T which factors T as $T = UF_T$. Similarly, we see that every T-algebra admits a functorial resolution by free T-algebras. After forgetting down to $\mathcal{S}et$ these coequalizer diagrams become *split coequalizer diagrams* [Bor94b, Lemma 4.3.3], i.e., diagrams of the form (2.1). Split coequalizer diagrams have the useful property that they are preserved by *all* functors [Bor94c, Prop. 2.10.2].

Using these functorial resolutions and that a morphism of groups is an isomorphism if and only if it induces an isomorphism between the underlying sets we can see that the lifted functor $U\colon \mathcal{G}roup \to \mathcal{S}et_T$ is essentially surjective. By applying the functorial resolution again and (2.2) we obtain an equivalence of categories $\mathcal{G}roup \to \mathcal{S}et_T$.

These arguments are completely general:

Theorem 2.4. Barr-Beck/Monadicity Any functor $U\colon \mathscr{D} \to \mathscr{C}$, which admits a left adjoint F, lifts to a functor to the category of $T = UF$-algebras in $\mathscr{C}$. Moreover this functor is an equivalence of categories if and only if

(a) U reflects isomorphisms, i.e., a map f in $\mathscr{D}$ is an isomorphism if and only if Uf is.
(b) If U takes a pair of arrows of the form (2.3), where G is assumed to be a T-algebra, to a split coequalizer, then the pair of arrows in (2.3) admits a coequalizer which is preserved by U.

Proof. This version of the Barr-Beck theorem is a slight variation of [Bor94b, 4.4.4]. Here we assume the existence of a left adjoint and have a slightly weaker assumption in Item b, but Borceux's proof applies directly. □

Theorem 2.4 can be used to identify many categories as categories of algebras over a monad. Since we want $\mathscr{C}_T$ to have an ample supply of colimits and limits for constructing model structures we postpone introducing these examples for the moment so that we can record when such constructions exist.

Proposition 2.5. [Bor94b, 4.3.1, 4.3.2] Suppose T is a monad acting on $\mathscr{C}$, then

(1) The forgetful functor $U\colon \mathscr{C}_T \to \mathscr{C}$ creates all limits.
(2) The forgetful functor $U\colon \mathscr{C}_T \to \mathscr{C}$ creates all colimits which commute with T in $\mathscr{C}$.

Proposition 2.6. [EKMM97, Prop. II.7.4] Suppose $\mathscr{C}$ is cocomplete and T commutes with reflexive coequalizers, then $\mathscr{C}_T$ is cocomplete and the forgetful functor reflects all reflexive coequalizers.

Proof. Let $\alpha\colon \operatorname{colim} TX_i \to T\operatorname{colim} X_i$ be the map induced by applying T to the maps $X_i \to \operatorname{colim} X_i$. Since T commutes with reflexive coequalizers, every colimit in $\mathscr{C}_T$ can be calculated via the following formula in $\mathscr{C}$:

$$\operatorname{coeq}\left[\begin{tikzcd} T\operatorname{colim} TX_i \arrow[r, shift left, "T\operatorname{colim}\mu_{X_i}"] \arrow[r, shift right, "\mu\circ\alpha"'] & T\operatorname{colim} X_i \arrow[l, bend right=40, "Te_{X_i}"'] \end{tikzcd}\right] \tag{2.7}$$

□

Alternatively, if we suppose that $\mathscr{C}$ is bicomplete and T preserves κ-filtered colimits for some regular cardinal κ then $\mathscr{C}_T$ is bicomplete by [Bor94b, 4.3.6]. We often want T, or equivalently U, to preserve *both* filtered colimits and reflexive coequalizers (for some examples where this does not hold see [Bor94b, §4.6]). In such a case we can apply the following useful form of the Barr-Beck theorem provided we restrict to *locally presentable categories* [Bor94b, §5.2].

Proposition 2.8. Suppose $U\colon \mathscr{D} \to \mathscr{C}$ is a functor between two locally presentable categories such that

(a) U preserves limits,
(b) U reflects κ-filtered colimits for some regular cardinal κ,
(c) and U preserves and reflects reflexive coequalizers,

then U admits a left adjoint F, $\mathscr{D}$ is equivalent to the category of $T = UF$-algebras in $\mathscr{C}$, and T commutes with reflexive coequalizers and κ-filtered colimits.

Proof. Since $\mathscr{C}$ and $\mathscr{D}$ are locally presentable the first two conditions guarantee that U admits a left adjoint F [Bor94b, 5.5.7]. The third condition is stronger than the two conditions of the monadicity theorem, and thus we have $\mathscr{D} \simeq \mathscr{C}_T$. The last two conditions guarantee that $T = UF$ commutes with the stated colimits. □

The above theorems illustrate the importance of reflexive coequalizers and filtered colimits in $\mathscr{C}_T$. These are particular examples of *sifted* colimits, which are colimits indexed over $\mathscr{I}$ such that the diagonal map $\mathscr{I} \to \mathscr{I} \times \mathscr{I}$ is final. Sifted colimits can also be characterized as those colimits which commute with finite products in $\mathcal{S}et$. One of the main results of [ARV10, 2.1] is that if $\mathscr{C}$ is finitely cocomplete then T commutes with all (κ-)sifted colimits if and only if T commutes with all reflexive coequalizers and (κ-)filtered colimits.

2.2. **Algebraic theories.** Monads which commute with sifted colimits arise naturally in the study of algebraic theories in the sense of Lawvere [Law63]. Recall that an algebraic theory is a category $\mathscr{T}$ equipped with a product preserving functor $i\colon \mathcal{F}in\mathcal{S}et^{\mathrm{op}} \to \mathscr{T}$ which is essentially surjective. If we label finite sets by their cardinality, this condition is equivalent to saying that every object of $\mathscr{T}$ is isomorphic to $i(1)^n$ for some natural number n. If $\mathscr{C}$ is a category with finite products, a $\mathscr{T}$-model in $\mathscr{C}$ is a product preserving functor $A\colon \mathscr{T} \to \mathscr{C}$. The collection of $\mathscr{T}$-models in $\mathscr{C}$ forms a category $\mathscr{C}_{\mathscr{T}}$ where the morphisms are natural transformations.

We should think of $\mathscr{T}$ as encoding the operations on an object of $\mathscr{C}_{\mathscr{T}}$. For example, suppose k is a commutative ring and define a theory $\mathscr{T}$ as the subcategory of the opposite category of k-algebras whose nth object $i(n)$ is the free k-algebra $k\langle x_1, \cdots, x_n\rangle$.

Note that for each k-algebra A, we obtain a $\mathcal{T}$-model in $\mathcal{S}et$ by

$$i(n) \mapsto k\text{-}\mathcal{A}lg(k\langle x_1, \cdots, x_n\rangle, A) \cong A^n.$$

Conversely, if $A \in \mathcal{S}et_{\mathcal{T}}$ we can identify A with the set $A(i(1))$ equipped with the operations encoded by the functor A. For example, consider the maps in

$$\mathcal{T}(i(2), i(1)) \cong k\text{-}\mathcal{A}lg(k\langle x_1\rangle, k\langle x_1, x_2\rangle)$$

which send x_1 to $x_1 + x_2$ and $x_1 \cdot x_2$ respectively. These two maps define respective natural operations

$$(-)+(-)\colon A(i(1))^2 \to A(i(1))$$
$$(-)\cdot(-)\colon A(i(1))^2 \to A(i(1)).$$

The first map is commutative since $x_1 + x_2 = x_2 + x_1$, while the latter generally is not. By combining maps in $\mathcal{T}$ we can see that the latter operation will distribute over the former. All of these operations and their relations coming from $\mathcal{T}$ show that $A(i(1))$ is a k-algebra.

Example 2.9.

(1) If $\mathcal{T} = \mathcal{F}in\mathcal{S}et^{\mathrm{op}}$ and i is the identity functor then $\mathcal{C}_{\mathcal{T}}$ is the category of theories in $\mathcal{C}$.

(2) Let $\mathcal{T}_{Gp}$ be the category whose objects are indexed by natural numbers and whose morphisms are

$$\mathcal{T}_{Gp}(m,n) = \mathcal{G}roup(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free group on m elements. If we let i be the functor which takes a finite set X to the element of $\mathcal{T}$ labeled by $|X|$ then $\mathcal{S}et_{\mathcal{T}_{Gp}}$ is equivalent to the category of groups.

(3) Let G be a group and let $\mathcal{T}_G$ be the theory defined as in (2) but with

$$\mathcal{T}_G(m,n) = G\text{-}\mathcal{S}et(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free G-set on m elements, then $\mathcal{S}et_{\mathcal{T}_G}$ is equivalent to the category of G-sets.

(4) Let $\mathcal{T}_{Ab}$ be the theory defined as in (2) but with

$$\mathcal{T}_{Ab}(m,n) = \mathcal{A}b\mathcal{G}roup(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free abelian group on m elements, then $\mathcal{S}et_{\mathcal{T}_{Ab}}$ is equivalent to the category of abelian groups.

(5) Let k be a commutative ring and $\mathcal{T}_{Ass_k}$ be the theory defined as in (2) but with

$$\mathcal{T}_{Ass_k}(m,n) = \mathcal{A}ss\mathcal{A}lg_k(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free associative k-algebra on m elements, then $\mathcal{S}et_{\mathcal{T}_{Ass_k}}$ is equivalent to the category of associative k-algebras.

(6) Let $\mathcal{T}_{Comm}$ be the theory defined as in (2) but with

$$\mathcal{T}_{Comm}(m,n) = \mathcal{C}omm\mathcal{A}lg(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free commutative ring on m elements, then $\mathcal{S}et_{\mathcal{T}_{Comm}}$ is equivalent to the category of commutative rings.

(7) Let k be a commutative ring and $\mathcal{T}_{Lie_k}$ be the theory defined as in (2) but with

$$\mathcal{T}_{Lie_k}(m,n) = \mathcal{L}ie_k(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free Lie algebra over k on m elements, then $\mathcal{S}et_{\mathcal{T}_{Lie_k}}$ is equivalent to the category of Lie algebras over k.

(8) Let $\mathcal{T}_{C^\infty}$ be the theory defined as in (2) but with $\mathcal{T}_{C^\infty}(m,n) = C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, the set of smooth maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, then $\mathcal{S}et_{\mathcal{T}_{C^\infty}}$ is equivalent to the category of C^∞-rings [Dub81, MR91].

The list in Example 2.9 is far from comprehensive and is limited only by the authors' imagination and the readers' patience.

If $\mathscr{T}$ is a theory, we obtain $\mathscr{T}$-models $\mathscr{T}\{m\}$ in $\mathcal{S}et$ by setting $\mathscr{T}\{m\}(-) = \mathscr{T}(m,-)$, which we can think of as the free objects on a set of m-elements. This construction lifts to a (covariant!) functor $\mathscr{T}\{-\}\colon \mathcal{F}in\mathcal{S}et \to \mathcal{S}et_{\mathscr{T}}$. Since $\mathcal{S}et$ is the closure of $\mathcal{F}in\mathcal{S}et$ under sifted colimits, and sifted colimits commute with products in $\mathcal{S}et$ we see that we can canonically prolong this to a functor from $\mathcal{S}et$. This functor admits a forgetful right adjoint given by evaluating at $i(1)$.

Just as in Section 2.1, we can compose these adjoints to obtain a monad $T = U\mathscr{T}\{-\}$ on $\mathcal{S}et$. Since $\mathcal{S}et_{\mathscr{T}}$ is locally presentable and $U\mathscr{T}\{-\}$ preserves sifted colimits we can apply Proposition 2.8 and see that $\mathcal{S}et_{\mathscr{T}}$ is equivalent to the category of $T = U\mathscr{T}\{-\}$ algebras in $\mathcal{S}et$. More explicitly, we have

$$TX = \int^{n\in \mathcal{F}in\mathcal{S}et} X^n \times \mathscr{T}(n,1). \tag{2.10}$$

Remark 2.11. We can apply the discussion of limits and colimits in Section 2.1 to analyze these constructions in $\mathcal{S}et_{\mathscr{T}}$. For example, since $\mathcal{S}et$ is complete, $\mathcal{S}et_T$ is complete and the forgetful functor creates all limits. By applying (2.10) we see that since sifted colimits commute with products in sets and all colimits commute with coends, T commutes with sifted colimits. Hence the forgetful functor reflects all sifted colimits. The remaining colimits can be constructed via (2.7).

2.3. **Simplicial categories of T-algebras.** The theory of T-algebra limits and colimits from Section 2.2 admits a straightforward extension to the enriched context. For general background on enriched categories and functors between them the reader is encouraged to consult [Bor94a, §6.2] or [Kel05].

Since we are interested in studying the *space* of maps between two T-algebras we give this extension in the case that $\mathscr{C}$ is a simplicial category. To obtain categorical information analogous to the previous section we will replace all of our categories with simplicial categories, all of our functors with simplicial functors, and all of our natural transformations with simplicial natural transformations. Note that one can regard any topological category as a simplicial category via the symmetric monoidal functor Sing.

Recall that we require a simplicially enriched category $\mathscr{C}$ to have a tensor bifunctor

$$\otimes\colon \mathscr{C} \times s\mathcal{S}et \to \mathscr{C}.$$

This is related to the simplicial mapping functor $\mathscr{C}(-,-)$ and the simplicial cotensor $(-)^-$ via the following adjunction isomorphisms

$$s\mathcal{S}et(K, \mathscr{C}(C,D)) \cong \mathscr{C}(C\otimes K, D) \cong \mathscr{C}(C, D^K).$$

Proposition 2.12. Suppose that

(a) $\mathscr{C}$ is a bicomplete simplicial category.
(b) T is a simplicial monad acting on $\mathscr{C}$.
(c) T commutes with either
  (i) reflexive coequalizers or
  (ii) filtered colimits.

Then $\mathscr{C}_T$ is a bicomplete simplicial category such that

(1) The forgetful functor $\mathscr{C}_T \to \mathscr{C}$ creates limits and cotensors
(2) The simplicial tensor is constructed as follows:

$$X \otimes_T V = \operatorname{coeq}\left[\begin{tikzcd} F_T(TX\otimes V) \arrow[r, shift left, "F_T(\mu\otimes V)"] \arrow[r, shift right, "\mu\circ\alpha"'] & F_T(X\otimes V) \arrow[l, bend right=40, "F_T(e\otimes V)"'] \end{tikzcd}\right] \tag{2.13}$$

Here $\alpha\colon F_T(TX \otimes V) \to F_T(X \otimes V)$ is adjoint to the assembly map $TX \otimes V \to T(X \otimes V)$.

Proof. First we check that $\mathscr{C}_T$ is bicomplete: By Proposition 2.5 $\mathscr{C}_T$ is complete. Under Hypothesis c.i we can apply Proposition 2.6 to see that $\mathscr{C}_T$ is cocomplete. When Hypothesis c.ii holds, cocompleteness follows from [Bor94b, Prop. 4.3.6].

The hom spaces of $\mathscr{C}_T$ are defined by taking the equalizer, in $s\mathcal{S}et$, of the diagram in (2.2). The fact that U creates cotensors appears in [EKMM97, §VII Prop. 2.10] In order for the adjunctions to hold the tensor must be defined by (2.13) and we see that cotensors are created via U. □

Note that if T commutes with reflexive coequalizers we can compute the simplicial tensor in $\mathscr{C}$.

Algebraic theories are extended similarly to the simplicial context: Regarding the category of finite sets as a simplicial category with discrete mapping objects, a simplicial algebraic theory is just a product preserving functor $\mathcal{F}in\mathcal{S}et^{\mathrm{op}} \to \mathcal{T}$ to a simplicial category $\mathcal{T}$ which is essentially surjective as an ordinary functor. Similarly, a T-model in a simplicial category $\mathscr{C}$ with finite products is just a product preserving simplicial functor $\mathcal{T} \to \mathscr{C}$.

Example 2.14. Each of the examples listed in Example 2.9 naturally defines a simplicial theory. The $\mathcal{T}$-models in simplicial sets are respectively equivalent to the categories of simplicial groups, simplicial abelian groups, etc.

Since all limits and colimits in $s\mathcal{S}et$ are computed level-wise, and sifted colimits commute with products in $\mathcal{S}et$, we see that sifted colimits commute with products in $s\mathcal{S}et$. It follows that our discussion of limits and colimits in Remark 2.11 extends to this context:

Proposition 2.15. Let T be a monad associated to a simplicial algebraic theory. Then the category $s\mathcal{S}et_T$ of simplicial T-algebras satisfies the conditions of Proposition 2.12 and hence is a bicomplete simplicial category with tensor defined by (2.13).

2.4. **Monads from operads.** A *symmetric sequence* in $s\mathcal{S}et$ is a sequence

$$C = \{C(n)\}_{n\geq 0}$$

of spaces such that $C[n]$ has a right action by Σ_n. A map of symmetric sequences is a levelwise equivariant map.

An *operad* is a symmetric sequence such that for each partition of n with k parts, $i_1+\cdots+i_k = n$, there is a structure map

$$\begin{aligned} C(k) \times C(i_1) \times \cdots \times C(i_k) &\longrightarrow C(n). \\ (f, g_1, \ldots, g_n) &\longmapsto f\{g_1, \ldots, g_k\} \end{aligned}$$

These structure maps satisfy the following axioms [May72]:

(a) For $g \in C(n)$, $1\{g\} = g$.
(b) For $f \in C(k)$, $f\{1, \ldots, 1\} = f$.
(c) For $f \in C(k)$, $g_j \in C(i_j)$, and $h_{j,l} \in C(m_{j,l})$,

$$f\{g\{h_{1,1}, \ldots, h_{1,i_1}\}, \ldots, f\{g\{h_{k,1}, \ldots, h_{k,i_k}\}\} = (f\{g_1, \ldots, g_k\})\{h_{1,1}, \ldots, h_{k,i_k}\}$$

(d) For f and g as above and $\sigma \in \Sigma_k$,

$$(f\sigma)\{g_1, \ldots, g_k\} = f\{g_{\sigma(1)}, \ldots, g_{\sigma(k)}\}$$

(e) For f and g as above and $\sigma_j \in \Sigma_{i_j}$,

$$f\{g_1\sigma_1, \ldots, g_k\sigma_k\} = (f\{g_1, \ldots, g_k\}) \circ (\sigma_1 \times \cdots \times \sigma_k).$$

Given our emphasis on monads, it will be helpful to have an alternative definition which we now summarize—additional details are found in [Rez97, §11]. For the remainder of this section we assume that $\mathscr{C}$ is simplicial symmetric monoidal category with tensor $\otimes$ such that:

(a) $\otimes$ distributes over coproducts in $\mathscr{C}$ and
(b) there is a symmetric monoidal functor $F : s\mathcal{S}et \to \mathscr{C}$, such that the tensor of a space K and an object X of $\mathscr{C}$ is defined by $FK \otimes X$.

Now given a symmetric sequence C, we have an associated functor $T_C \colon \mathscr{C} \to \mathscr{C}$ defined on objects by

$$T_C(X) = \coprod_{n\geq 0} C(n) \otimes_{\Sigma_n} X^{\otimes n}.$$

A map of symmetric sequences yields a natural transformation of functors, and this construction yields a functor from symmetric sequences to endofunctors of $\mathscr{C}$.

There is an external product, which we again denote by $\otimes$:

$$(C \otimes D)(n) = \coprod_{i+j=n} C(i) \times D(j) \times_{\Sigma_i \times \Sigma_j} \Sigma_n$$

Since the symmetric monoidal structure on $\mathscr{C}$ distributes over coproducts we see:

$$T_C \times T_D \cong T_{C\otimes D}.$$

Now we define the circle product by:

$$(C \circ D)(n) = C(n) \times_{\Sigma_n} D^{\otimes n}.$$

We can now check that the construction $C \mapsto F_C$ defines a monoidal functor from symmetric sequences under the circle product to endofunctors under composition. Now we obtain our alternative definition: an operad $\mathscr{O}$ is a symmetric sequence which is a monoid for the circle product; the associated endofunctor is then a monad.

Proposition 2.16. Suppose that $\mathscr{C}$ is a symmetric monoidal bicomplete simplicial category $\mathscr{C}$ such that:

(a) There is a symmetric monoidal functor

$$i \colon s\mathcal{S}et \to \mathscr{C}$$

defining the simplicial tensor.

(b) The monoidal product in $\mathscr{C}$ commutes with either
   (i) reflexive coequalizers or
   (ii) filtered colimits.

Then for any operad $\mathscr{O}$ of simplicial sets, the category of $\mathscr{O}$-algebras in $\mathscr{C}$ is a bicomplete simplicial category.

Proof. Let $T_{\mathscr{O}}$ be the monad associated to the operad $\mathscr{O}$. Suppose that I indexes either a reflexive coequalizer or filtered colimit diagram (depending on which part of hypothesis b holds). Since colimits commute with coproducts, we see that $T_{\mathscr{O}}$ commutes with colimits over I:

$$\begin{aligned} T_{\mathscr{O}} \operatorname*{colim}_I X_i &= \coprod_{n\geq 0} \mathscr{O}(n) \otimes (\operatorname*{colim}_I X_i)^{\otimes n} \\ &\cong \coprod_{n\geq 0} \mathscr{O}(n) \otimes \operatorname*{colim}_I (X_i^{\otimes n}) \\ &\cong \operatorname*{colim}_I \coprod_{n\geq 0} \mathscr{O}(n) \otimes (X_i^{\otimes n}) \\ &= \operatorname*{colim}_I T_{\mathscr{O}} X_i. \end{aligned}$$

Hence by Proposition 2.12, $T_{\mathscr{O}}$ has a bicomplete category of algebras.

Similarly, we obtain natural maps $\eta_{K,X}$ via the following composite

$$\begin{aligned} K\otimes T_{\mathscr{O}}X &= K\otimes \coprod_{n\geq 0}\mathscr{O}(n)\otimes X^{\otimes n} \\ &\cong \coprod_{n\geq 0} K\otimes(\mathscr{O}(n)\otimes X^{\otimes n}) \longrightarrow \coprod_{n\geq 0} K^n\otimes(\mathscr{O}(n)\otimes X^{\otimes n}) \\ &\cong \coprod_{n\geq 0}(\mathscr{O}(n)\otimes(K\otimes X)^{\otimes n}) \\ &= T_{\mathscr{O}}(K\otimes X). \end{aligned}$$

Pullback along these maps defines a morphism of simplicial mapping spaces given on k-simplices by

$$\mathscr{C}(\Delta^k\otimes X,Y)\xrightarrow{T_{\mathscr{O}}}\mathscr{C}(T_{\mathscr{O}}(\Delta^k\otimes X),T_{\mathscr{O}}Y)\xrightarrow{\eta^*_{\Delta^k,X}}\mathscr{C}(\Delta^k\otimes T_{\mathscr{O}}X,T_{\mathscr{O}}Y).$$

This map clearly preserves simplicial units. Compatibility with composition is verified by the commutative diagram in Fig. 2.17 where we let $d\colon \Delta^k\to\Delta^k\times\Delta^k$ denote the diagonal map.

$$\begin{tikzcd}
\mathscr{C}(\Delta^k\otimes Y,Z)\times\mathscr{C}(\Delta^k\otimes X,Y) \arrow[dr, "T_{\mathscr{O}}"] \arrow[dd, "1\times(\Delta^k\otimes 1)"'] & & \mathscr{C}(\Delta^k\otimes T_{\mathscr{O}}(Y),T_{\mathscr{O}}Z)\times\mathscr{C}(\Delta^k\otimes T_{\mathscr{O}}(X),T_{\mathscr{O}}Y) \arrow[dd, "1\times(\Delta^k\otimes 1)"] \\
& \mathscr{C}(T_{\mathscr{O}}(\Delta^k\otimes Y),T_{\mathscr{O}}Z)\times\mathscr{C}(T_{\mathscr{O}}(\Delta^k\otimes X),T_{\mathscr{O}}Y) \arrow[ur, "\eta^*\times\eta^*"] \arrow[dd, "1\times T_{\mathscr{O}}(\Delta^k\otimes 1)"'] & \\
\mathscr{C}(\Delta^k\otimes Y,Z)\times\mathscr{C}((\Delta^k\times\Delta^k)\otimes X,\Delta^k\otimes Y) \arrow[dr, "T_{\mathscr{O}}"] \arrow[dd, "\circ"'] & & \mathscr{C}(\Delta^k\otimes T_{\mathscr{O}}Y,T_{\mathscr{O}}Z)\times\mathscr{C}((\Delta^k\times\Delta^k)\otimes T_{\mathscr{O}}X,\Delta^k\otimes T_{\mathscr{O}}Y) \arrow[dd, "\circ"] \\
& \mathscr{C}(T_{\mathscr{O}}(\Delta^k\otimes Y),T_{\mathscr{O}}Z)\times\mathscr{C}(T_{\mathscr{O}}((\Delta^k\times\Delta^k)\otimes X),T_{\mathscr{O}}(\Delta^k\otimes Y)) \arrow[dd, "\circ"'] & \\
\mathscr{C}((\Delta^k\times\Delta^k)\otimes X,Z) \arrow[dr, "T_{\mathscr{O}}"] \arrow[dd, "d^*"'] & & \mathscr{C}((\Delta^k\times\Delta^k)\otimes(T_{\mathscr{O}}X),T_{\mathscr{O}}Z) \arrow[dd, "d^*"] \\
& \mathscr{C}(T_{\mathscr{O}}((\Delta^k\times\Delta^k)\otimes X),T_{\mathscr{O}}Z) \arrow[ur, "\eta^*"] \arrow[dd, "d^*"'] & \\
\mathscr{C}(\Delta^k\otimes X,Z) \arrow[dr, "T_{\mathscr{O}}"] & & \mathscr{C}(\Delta^k\otimes T_{\mathscr{O}}(X),T_{\mathscr{O}}Z) \\
& \mathscr{C}(T_{\mathscr{O}}(\Delta^k\otimes X),T_{\mathscr{O}}Z) \arrow[ur, "\eta^*"] &
\end{tikzcd}$$

FIGURE 2.17. Diagram to verify compatibility of $T_{\mathscr{O}}$ with composition.

To verify that the structure maps for $T_{\mathscr{O}}$ are simplicial natural transformations, one checks that the following diagrams commute where e and μ_T are the unit and multiplication map for our monad $T_{\mathscr{O}}$ and $f \in \mathscr{C}(\Delta^k \otimes X, Y)$:

$$\begin{tikzcd}
& \mathscr{C}(T_{\mathscr{O}}(\Delta^k \otimes X), T_{\mathscr{O}}Y) \arrow[d, "\eta^*"] \\
\mathscr{C}(\Delta^k \otimes X, Y) \arrow[ur, "T_{\mathscr{O}}"] \arrow[dr, "e"'] & \mathscr{C}(\Delta^k \otimes (T_{\mathscr{O}}X), T_{\mathscr{O}}Y) \arrow[d, "e^*"] \\
& \mathscr{C}(\Delta^k \otimes X, T_{\mathscr{O}}Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^k \otimes T_{\mathscr{O}}^2 X \arrow[r, "\eta"] \arrow[d, "\mu_T"] & T_{\mathscr{O}}(\Delta^k \otimes T_{\mathscr{O}}X) \arrow[r, "\eta"] & T_{\mathscr{O}}^2(\Delta^k \otimes X) \arrow[r, "T_{\mathscr{O}}^2 f"] \arrow[d, "\mu_T"] & T_{\mathscr{O}}^2 Y \arrow[d, "\mu_T"] \\
\Delta^k \otimes T_{\mathscr{O}}X \arrow[rr, "\eta"] & & T_{\mathscr{O}}(\Delta^k \otimes X) \arrow[r, "T_{\mathscr{O}} f"] & T_{\mathscr{O}}Y
\end{tikzcd}$$

□

The hypotheses concerning colimits for this proposition hold whenever the symmetric monoidal structure comes from a *closed* symmetric monoidal structure and hence distributes over colimits. For example, simplicial sets, simplicial abelian groups, and simplicial R-modules all satisfy the conditions of Proposition 2.16 with their cartesian symmetric monoidal structure. The categories of pointed compactly generated spaces or pointed simplicial sets, each equipped with the smash product, satisfy these conditions. Any of the closed symmetric monoidal categories of spectra satisfy the hypotheses.

3. Model structures

In Section 3.1 we recall conditions that guarantee that the category of T-algebras has a suitable homotopy theory. After establishing the existence of a model structure, we construct functorial simplicial resolutions of algebras in Section 3.2 which serve as input into our spectral sequences.

Here, we choose to work in the context of simplicial model categories. A disadvantage of this approach is that some of our assumptions—most notably the existence of colimits and the standard issues concerning cofibrancy and fibrancy—should not be strictly necessary (see for example [Lur12, §6.2]).

An advantage of this approach is that the theory is well-developed, well-understood, and relatively straightforward to apply to many categories of interest. A great deal of the material for this section can be found in the appendices of [Lur09].

3.1. **Model structure on $\mathscr{C}_T$.**

Now we would like to identify our simplicial structure on $\mathscr{C}_T$ as part of a simplicial model structure [Qui67]. These model categories satisfy the following equivalent form of Quillen's corner axiom.

Axiom. SM7 Given any cofibration $f: X \to Y$ in $s\mathcal{S}et$ and fibration $g: K \to L$ in $\mathscr{M}$, the induced morphism

$$A^L \longrightarrow A^K \times_{B^K} B^L \tag{3.1}$$

is a fibration which is a weak equivalence if either f or g is.

Definition 3.2. A monad T acting on a category $\mathscr{C}$ is *Quillen* if

(a) $\mathscr{C}$ is a simplicial model category.
(b) T is a simplicial monad acting on $\mathscr{C}$.
(c) $\mathscr{C}_T$ has a simplicial model structure such that the forgetful functor $U\colon \mathscr{C}_T \to \mathscr{C}$ is a simplicial right Quillen functor.
(d) A map f of T-algebras is a weak equivalence if and only if Uf is a weak equivalence.

A convenient way to show that T is Quillen is to assume we have a simplicial model structure on $\mathscr{C}$ and induce a model structure on $\mathscr{C}_T$ via F_T. We can do this if $\mathscr{C}$ is cofibrantly generated and T satisfies some mild hypotheses. In this case $\mathscr{C}$ has sets of generating cofibrations I and acyclic cofibrations J which are used to detect fibrations and acyclic fibrations. These sets of maps satisfy smallness hypotheses which are used to apply Quillen's small object argument and prove the lifting axioms. In the interest of being self-contained we recall some of the relevant definitions and results from [Hov99, §2.1].

Definition 3.3. Let $f\colon A \to B$ and $g\colon C \to D$ be maps in $\mathscr{C}$. We say that f has the *left lifting property* (LLP) with respect to g or equivalently that g has the *right lifting property* (RLP) with respect to f if for any maps $A \to C$ and $B \to D$ making the square below commute, there is a lift of f indicated by the dashed arrow commuting with the other maps in the diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

Definition 3.4. Let I be a class of maps in $\mathscr{C}$.

- A map is *I-injective* if it has the RLP with respect to every map in I. The class of I-injective maps is denoted I-inj.
- A map is *I-projective* if it has the LLP with respect to every map in I. The class of I-projective maps is denoted I-proj.
- A map is an *I-cofibration* if it has the LLP with respect to every map in I-inj. The class of I-cofibrations is (I-inj)-proj and denoted I-cof.
- A map is an *I-fibration* if it has the RLP with respect to every map in I-proj. The class of I-fibrations is (I-proj)-inj and denoted I-fib.

Recall that for a set of maps I in a category $\mathscr{C}$ a *relative I-cell complex* is a transfinite composition of pushouts of elements of I. We say that an object is an *I-cell complex* if the map from the initial object to it is a relative I-cell complex. If I is the set of generating cofibrations then we just say that the object is a cell complex.

Definition 3.5. Let I be a collection of morphisms in a cocomplete category $\mathscr{C}$ and κ a cardinal. We say that $A \in \mathscr{C}$ is *κ-small relative to I* if, for all κ-filtered ordinals λ and all λ-sequences X_i such that each map $X_\beta \to X_{\beta+1} \in I$ for $\beta+1 < \lambda$, the map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathscr{C}(A, X_\beta) \to \mathscr{C}(A, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism.

We say that A is *small relative* to I if it is κ-small relative I for some κ.

Definition 3.6. A model category $\mathscr{C}$ is *cofibrantly generated* provided there are sets of maps I (called the *generating cofibrations*) and J (called the *generating trivial cofibrations*) such that

- The domains of the maps of I are small relative to I-cell.
- The domains of the maps of J are small relative to J-cell.
- The class of fibrations is J-inj.

- The class of trivial fibrations is I-inj.

With this terminology available, we can now state conditions under which the maps in I and J generate a model structure.

Theorem 3.7. [Hov99, 2.1.19] Suppose $\mathscr{C}$ is a bicomplete category with a subcategory $\mathscr{W}$ and distinguished sets I and J of maps, then there is a cofibrantly generated model structure on $\mathscr{C}$ with I (resp. J) the generating (trivial) cofibrations, and $\mathscr{W}$ the weak equivalences if and only if the following conditions are satisfied:

(a) The subcategory $\mathscr{W}$ satisfies the two out of three property and is closed under retracts.
(b) The domains of I are small relative to I-cell.
(c) The domains of J are small relative to J-cell.
(d) I-inj $\subset (\mathscr{W} \cap J$-inj).
(e) Either $(\mathscr{W} \cap I$-cof$) \subset J$-cof, or $(\mathscr{W} \cap J$-inj$) \subset I$-inj.
(f) J-cell $\subset (\mathscr{W} \cap I$-cof).

Suppose that $\mathscr{C}$ is a model category and a functor

$$U : \mathscr{D} \to \mathscr{C}$$

admits a left adjoint. Then we say that U *right induces* a model structure on $\mathscr{D}$ if $\mathscr{D}$ admits a model structure such that a map f is a fibration (resp. weak equivalence) if and only if Uf is a fibration (resp. weak equivalence).

Theorem 3.8. [Sch07, App. A] Suppose that $\mathscr{C}$ is a cofibrantly generated simplicial model category with generating (acyclic) cofibrations I (resp. J) and $T = UF_T$ is a monad on $\mathscr{C}$ satisfying Proposition 2.12.

If the domains of $F_T I$ (resp. $F_T J$) are small relative to $F_T I$-cells (resp. $F_T J$-cells) and applying U to any $F_T J$-cell complex yields a weak equivalence in $\mathscr{C}$ then U right induces a cofibrantly generated simplicial model category structure on $\mathscr{C}_T$.

Proof. For the model category structure we apply Theorem 3.7. By Proposition 2.12 we know that $\mathscr{C}_T$ is a bicomplete category. Since we want U to right induce the model structure we set the weak equivalences to be those maps in $\mathscr{C}_T$ which project to weak equivalences in $\mathscr{C}$. We set $F_T I$ and $F_T J$ to be the generating cofibrations and acyclic cofibrations.

Property a is satisfied in $\mathscr{C}_T$ because it is satisfied in $\mathscr{C}$. While properties b and c are satisfied by assumption. By applying the adjunction we see the classes $F_T I$-inj and $F_T J$-inj are precisely those maps in $\mathscr{C}_T$ that map to I-inj and J-inj respectively under U. Consequently, Property d and the second condition in property e hold in $\mathscr{C}_T$ because they hold in $\mathscr{C}$.

To verify property f we note that by assumption $F_T J$-cell$\subset \mathscr{W}$. To see that every $F_T J$-cell is in $F_T I$-cof we must show that each such map has the LLP with respect to $F_T I$-inj. By property d we know these maps are $F_T J$-inj so each map in $F_T J$ has the LLP with respect to these maps, which we can use to inductively construct lifts in cellular pushout diagrams of the following form, where f is in $F_T I$-inj:

$$\begin{tikzcd}
\coprod F_T A_i \arrow[r] \arrow[d] & X_\alpha \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\coprod F_T B_i \arrow[r] \arrow[urr, dashed] & X_{\alpha+1} \arrow[r] \arrow[ur, dashed] & Z
\end{tikzcd}$$

This and transfinite induction can then be used to prove we get a lift in the general cellular diagram:

$$\begin{tikzcd} X_{-1} \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{colim}_{\beta<\lambda} X_\beta \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

as desired.

All that remains is to check Axiom (SM7) which is immediate since cotensors, fibrations, and weak equivalences are calculated in $\mathscr{C}$, which we assumed satisfied this axiom. □

Remark 3.9. In practice, checking the smallness conditions is relatively easy and most of the work required to apply Theorem 3.8 involves checking that applying U to a F_TJ-cell diagram yields a weak equivalence. Assuming these smallness conditions, this can be verified (see [Sch99, Lem. B2]) by showing the following two properties are satisfied.

(1) There is a 'fibrant replacement' functor $Q\colon \mathscr{C}_T \to \mathscr{C}_T$ and a natural transformation $\mathrm{Id} \to Q$ such that for all $X \in \mathscr{C}_T$, the natural map $UX \to UQX$ is a fibrant replacement.
(2) If UX is fibrant then applying U to the canonical factorization $X \to X^{\Delta^1} \to X^{\partial\Delta^1} \cong X \times X$ of the diagonal yields a weak equivalence followed by a fibration.

The second property follows from the fact that U preserves cotensors and $\mathscr{C}$ is a simplicial model category. For the first property one can sometimes show that the fibrant replacement functor $\mathscr{C}$ lifts to an endomorphism of $\mathscr{C}_T$. This is automatic if every object is fibrant in $\mathscr{C}$ which is true in topological spaces as well as several model categories of spectra. Since the two fibrant replacement functors Ex^∞ and $\mathrm{Sing}_*|-|$ on simplicial sets are product preserving we can use either of them as fibrant replacement functors for simplicial T-algebras to obtain the following proposition.

Proposition 3.10. [Sch01, Thm. 3.1] Each of the monads coming from an algebraic theory on simplicial sets (such as those in Example 2.14) is Quillen.

Proposition 3.11. Suppose that the forgetful functor $U\colon \mathscr{C}_T \to \mathscr{C}$ is a Quillen right adjoint. Then the monad T induces a monad hT on $ho\mathscr{C}$ such that the forgetful functor $ho(\mathscr{C}_T) \to ho\mathscr{C}$ factors through $ho(\mathscr{C})_{hT}$.

Proof. Quillen adjoints induce adjoints between the homotopy categories and consequently a monad action on $ho\mathscr{C}$ given by the composite. The right adjoint between the homotopy categories always lands in the category of algebras over this monad. □

3.2. **Simplicial resolutions.** To construct a spectral sequence computing the homotopy groups of the space $\mathscr{C}_T(X,Y)$ we would like to resolve X, by which we mean we replace X by a simplicial object $X_\bullet$ of $\mathscr{C}_T$ such that $\mathscr{C}_T(|X_\bullet|,Y) \simeq \mathscr{C}_T^d(X,Y)$.

If T is a monad acting on $\mathscr{C}$, then applying T levelwise to simplicial objects in $\mathscr{C}$ yields a monad, which we also denote by T, on $s\mathscr{C}$.

Definition 3.12. Suppose X is a T-algebra in $\mathscr{C}$. The *bar resolution* (also called the cotriple resolution) of X is the simplicial T-algebra

$$B_\bullet X = B_\bullet(F_T, T, UX) = B_\bullet(F_TU, F_TU, X)$$

defined as follows:

(a) $B_nX = (F_TU)^{n+1}X$.
(b) The ith face map

$$d_i\colon (F_TU)^{n+1}X \to (F_TU)^nX,\ 0 \le i \le n$$

is obtained by applying the counit in position $i+1$:

$$(F_T U)^i \, F_T U \, (F_T U)^{n-i} X \to (F_T U)^i \, \mathrm{Id} \, (F_T U)^{n-i} X.$$

(c) The ith degeneracy map

$$s_i \colon (F_T U)^{n+1} X \to (F_T U)^{n+2} X, \; 0 \le i \le n$$

is obtained by applying the unit between positions i and $i+1$:

$$F_T (U F_T)^i \, \mathrm{Id} \, (U F_T)^{n-i} U X \to F_T (U F_T)^i \, U F_T \, (U F_T)^{n-i} U X.$$

It is straightforward to check the simplicial identities from the monad structure on $T = U F_T$ and the T-algebra structure on X.

Note that the counit $F_T U X \to X$ extends to a map of simplicial T-algebras

$$\varepsilon \colon B_\bullet X \to X \tag{3.13}$$

where we regard the target as a constant simplicial object. By applying U to (3.13) and observing $U B_n X = T^{n+1}(UX)$, we obtain a map in $s\mathscr{C}$

$$\varepsilon \colon T^{\bullet+1} U X \to U X.$$

We also have a simplicial map

$$e \colon U X \to T^{\bullet+1} U X$$

by iterating the unit map $UX \to TUX$.

Definition 3.14. Let $\mathscr{C}$ be a cocomplete simplicial category. If $X_\bullet$ is a simplicial object in $\mathscr{C}$, the *geometric realization* of $X_\bullet$ is the following coequalizer:

$$\begin{aligned} |X_\bullet|_{\mathscr{C}} &= \int^{[n] \in \Delta} X_n \otimes \Delta^n \\ &= \mathrm{coeq}\left(\coprod_{[k] \to [n]} X_n \otimes \Delta^k \rightrightarrows \coprod_{[n]} X_n \otimes \Delta^n \right) \end{aligned}$$

where the two maps in the coequalizer diagram are given by applying $X_\bullet$ and $\Delta^\bullet$ respectively to each morphism $[k] \to [n]$.

For a simplicial T-algebra X, there are two relevant geometric realizations. One is realization in the category of T-algebras, and another is realization in the underlying category. We would like to have conditions under which these two notions coincide, i.e., under which U commutes with geometric realizations.

Remark 3.15. One such condition appears in [EKMM97, p. 197]: If T is given by a coend formula, then U preserves geometric realizations. More precisely, if T is given by a formula such as the one in (2.10), we will show that T commutes with geometric realization and then apply Proposition 3.16 to see that U commutes with geometric realizations. To show that T commutes with geometric realization we use the fact that geometric realization commutes with finite products and

Fubini's theorem for iterated coends as follows:

$$\begin{aligned}
T|UX_\bullet|_{\mathscr{C}} &= \int^{j\in\mathcal{F}in\mathcal{S}et} |UX_\bullet|_{\mathscr{C}}^{|j|} \otimes \mathscr{T}(j,1) \\
&\cong \int^{j\in\mathcal{F}in\mathcal{S}et} |UX_\bullet^{|j|}|_{\mathscr{C}} \otimes \mathscr{T}(j,1) \\
&= \int^{j\in\mathcal{F}in\mathcal{S}et} \left(\int^{n\in\Delta} X_n^j \otimes \Delta^n)\right)^{|j|} \otimes \mathscr{T}(j,1) \\
&\cong \int^{\Delta} \left(\int^{j\in\mathcal{F}in\mathcal{S}et} X_n^{|j|} \otimes \mathscr{T}(j,1)\right) \otimes \Delta^n) \\
&= |TX_\bullet|_{\mathscr{C}}.
\end{aligned}$$

Note that the commutation of products with geometric realization plays a key role in the above result. This is easy to verify in the case where $\mathscr{C}$ is simplicial sets, since the geometric realization of a bisimplicial set is isomorphic its diagonal. The result is non-trivial but true in the case of compactly generated weak Hausdorff spaces. From these cases one can deduce that the smash product on simplicial objects in pointed simplicial sets, compactly generated pointed spaces, or categories of spectra built from these categories also commutes with geometric realization. As a consequence similarly defined monads will also commute with geometric realization.

Proposition 3.16. Let $X_\bullet$ be a simplicial object in $\mathscr{C}_T$. If the conditions of Proposition 2.12 are satisfied, so $\mathscr{C}_T$ is a bicomplete simplicial category, and T commutes with geometric realization, then $|UX_\bullet|_{\mathscr{C}}$ is a T-algebra and $|X_\bullet|_{\mathscr{C}_T} \cong |UX_\bullet|_{\mathscr{C}}$ in $\mathscr{C}_T$. So U commutes with geometric realization.

Proof. Beginning with the canonical presentation of a simplicial T-algebra

$$\begin{tikzcd}
F_T T U X_\bullet \arrow[r, shift left] \arrow[r, shift right] & F_T U X_\bullet \arrow[l, bend right, "e"'] \arrow[r] & X_\bullet,
\end{tikzcd}$$

we now take the geometric realization in the category of T-algebras and apply U to obtain

$$\begin{tikzcd}
U|F_T TUX_\bullet|_{\mathscr{C}_T} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"'] & U|F_T UX_\bullet|_{\mathscr{C}_T} \arrow[l, bend right, "e"'] \arrow[r] \arrow[d, "\cong"] & U|X_\bullet|_{\mathscr{C}_T} \arrow[d, equal] \\
T|TUX_\bullet|_{\mathscr{C}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"'] & T|UX_\bullet|_{\mathscr{C}} \arrow[l, bend right, "e"'] \arrow[r] \arrow[d, "\cong"] & U|X_\bullet|_{\mathscr{C}_T} \arrow[d, "\cong"] \\
|TTUX_\bullet|_{\mathscr{C}} \arrow[r, shift left] \arrow[r, shift right] & |TUX_\bullet|_{\mathscr{C}} \arrow[l, bend right, "e"'] \arrow[r] & |UX_\bullet|_{\mathscr{C}}.
\end{tikzcd}$$

□

We can interpret the following result as saying that the bar resolution is indeed a resolution.

Proposition 3.17. Suppose that T is a Quillen monad acting on $\mathscr{C}$ which commutes with geometric realization. Then

$$\varepsilon\colon BX := |B_\bullet X|_{\mathscr{C}_T} \to X$$

is a weak equivalence of T-algebras.

Proof. This follows from Proposition 3.16 and the following well known lemma. □

Lemma 3.18. [May72, 9.8] Let $X \in \mathscr{C}_T$. The maps e and ε on realization

$$UX \xrightarrow{e} |T^{\bullet+1}UX|_{\mathscr{C}} \xrightarrow{\varepsilon} UX$$

exhibit UX as a strong deformation retract of $|T^{\bullet+1}UX|_{\mathscr{C}}$ in $\mathscr{C}$.

3.3. **Reedy model structure.** To construct our spectral sequence using the bar resolution of X we require that this resolution is homotopically well behaved, that is we will require it to be a Reedy cofibrant simplicial diagram. To show that this diagram is Reedy we will use a trick (Proposition 3.21) which makes use of a closely related *almost simplicial* diagram.

Let Δ_0 be the subcategory of Δ with the same objects but whose morphisms are those morphisms of linearly ordered sets which preserve the minimal element. The restriction morphism

$$\mathscr{C}^{\Delta^{\mathrm{op}}} \to \mathscr{C}^{\Delta_0^{\mathrm{op}}}$$

takes a simplicial object and forgets the d_0 face maps, which are induced by the injections missing the minimal element, while retaining all of the other structure. So one can think of a Δ^{op} shaped diagram as an almost simplicial diagram, simply lacking the d_0 face maps.

Definition 3.19. Let $X_\bullet$ be in $\mathscr{C}^{\Delta^{\mathrm{op}}}$ (resp. $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$). The nth *latching object* of $X_\bullet$ is

$$L_n(X_\bullet) = \operatorname*{colim}_{[n]\to[k]} X_k,$$

where the colimit is indexed over the non-identity surjections in Δ (this is equal to the set of non-identity surjections in Δ_0).

Definition 3.20. Suppose that $\mathscr{C}$ is a model category then the *Reedy* model structure on $\mathscr{C}^{\Delta^{\mathrm{op}}}$ (resp. $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$) is determined by

(a) $f\colon X_\bullet \to Y_\bullet$ is a (Reedy) weak equivalence if $f_n\colon X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n \geq 0$.

(b) $f\colon X_\bullet \to Y_\bullet$ is a (Reedy) cofibration if the induced map

$$X_n \coprod_{L_n X_\bullet} L_n Y_\bullet \to Y_n$$

is a cofibration in $\mathscr{C}$ for all $n \geq 0$.

To show the bar resolution is Reedy cofibrant in particular cases we will use the following trick:

Proposition 3.21. Suppose $\mathscr{C}$ and $\mathscr{D}$ are model category and $L\colon \mathscr{D} \to \mathscr{C}$ is a left Quillen functor. Let $X_\bullet$ be a simplicial diagram in $\mathscr{C}$ and let $RX_\bullet \in \mathscr{C}^{\Delta_0^{\mathrm{op}}}$ denote its restriction. Suppose that there exists a Reedy cofibrant $Y_\bullet \in \mathscr{D}^{\Delta_0^{\mathrm{op}}}$ such that $LY_\bullet \cong RX_\bullet$, then $X_\bullet$ is Reedy cofibrant.

Proof. By definition $X_\bullet$ is Reedy cofibrant if for each non-negative n the latching map

$$L_n X_\bullet \to X_n$$

is a cofibration. Note that the latching object and map depend only on the restriction of $X_\bullet$ to the subcategory $\Delta^{\mathrm{op}}_{\mathrm{surj}}$, where Δ_{surj} consists of all objects $[n]$ but only surjective maps $[n+m] \to [n]$ for $m \geq 0$, in particular it only depends on the restriction to Δ_0^{op} so it suffices to show $RX_\bullet$ is Reedy cofibrant. Since L is a Quillen left adjoint it commutes with colimits and preserves cofibrations so it takes the Reedy cofibrant $Y_\bullet$ to the Reedy cofibrant diagram $LY_\bullet$. Since being cofibrant is invariant under isomorphism the result follows. □

For a T-algebra $X \in \mathscr{C}_T$, we have an Δ_0^{op}-shaped diagram

$$T^\bullet UX \colon \Delta_0^{\mathrm{op}} \to \mathscr{C}$$

where

$$(T^\bullet UX)[n] = T^n UX$$

and the maps $(T^\bullet UX)(s_i)$ and $(T^\bullet UX)(d_i)$ are defined as in the bar construction.

Definition 3.22. Suppose that T is a Quillen monad acting on $\mathscr{C}$. A T-algebra X is *resolvable* if it is weakly equivalent as T-algebras to a T-algebra $\widetilde{X}$ (called its replacement) such that $T^\bullet U\widetilde{X}$ is Reedy cofibrant in $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$.

Proposition 3.23. Let T be a Quillen monad acting on $\mathscr{C}$ and X be a resolvable T-algebra with replacement $\widetilde{X}$. Then the bar resolution $B_\bullet\widetilde{X}$ is a Reedy cofibrant simplicial T-algebra.

Proof. By assumption $T^\bullet U\widetilde{X}$ is Reedy cofibrant in $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$. We obtain the conclusion by applying the left Quillen functor $F_T\colon \mathscr{C} \to \mathscr{C}_T$ levelwise to this diagram and using Proposition 3.21. □

The remainder of this section is devoted to proving various technical results which will assist in determining when a T-algebra is resolvable.

3.3.1. *Monads on diagrams of simplicial sets.*

Lemma 3.24. Let $\mathscr{C} = s\mathcal{S}et^{\mathscr{I}}$ be the category of simplicial $\mathscr{I}$-shaped diagrams equipped with the injective model structure, i.e., a natural transformation of diagrams is a weak equivalence (resp. cofibration) if and only if it is levelwise a weak equivalence (resp. cofibration). Then any diagram $X_\bullet\colon \Delta_0^{\mathrm{op}} \to \mathscr{C}$ is Reedy cofibrant.

Proof. We will show that Δ_0 is *Eilenberg-Zilber* [BR12, Definition 4.1], i.e., Δ_0 satisfies:

(EZ1) For all surjections $\sigma\colon [n+m] \to [n]$ in Δ_0, the set of sections
$$\Gamma(\sigma) = \{\tau \in \Delta_0 \mid \sigma\tau = \mathrm{id}_{[n]}\}$$
is nonempty.

(EZ2) For any two distinct surjections $\sigma_1, \sigma_2\colon [n+m] \to [n]$, the sets of sections $\Gamma(\sigma_1)$ and $\Gamma(\sigma_2)$ are distinct.

By [BR12, 4.2], every Eilenberg-Zilber Reedy category is *elegant* ([BR12, Definition 3.5]) and by [BR12, 3.15] the injective and Reedy model structures on diagrams in elegant categories are the same. In particular, the object $X_\bullet$ will be Reedy cofibrant because the cofibrations in the injective model structure are the levelwise cofibrations and every simplicial set is cofibrant.

To verify (EZ1), we note that if σ is a surjection in Δ_0, it is also a surjection in Δ and therefore has a section $\tau \in \Delta$. Define
$$\tau'(i) = \begin{cases} 0 & \text{if } i = 0 \\ \tau(i) & \text{if } i > 0 \end{cases}$$
The map τ' is certainly order-preserving because 0 is minimal and τ is order-preserving. It is also a section of σ because $\sigma(0) = 0$ and $\sigma(\tau(i)) = i$ for $i > 0$. Moreover, $\tau'(0) = 0$ so $\tau' \in \Delta_0$.

To verify (EZ2), suppose that σ_1 and σ_2 are two distinct surjections of Δ_0. Then there is a minimal j such that $\sigma_1(j) \neq \sigma_2(j)$. Note that j must be positive. Without loss of generality, we may assume $\sigma_1(j) < \sigma_2(j)$. Define τ_2 by letting $\tau_2(i)$ be the minimal element of $\sigma_2^{-1}(i)$ for each i; this defines an order preserving section of σ_2 with $\tau_2(0) = 0$, so $\tau_2 \in \Delta_0$. By minimality of j, we must have $\sigma_1(i) = \sigma_2(i)$ for $i < j$, and hence $\tau_2(\sigma_2(j)) = j$. Therefore
$$(\sigma_1\tau_2)(\sigma_2(j)) = \sigma_1(j) \neq \sigma_2(j),$$
so τ_2 is not a section of σ_1. □

Proposition 3.25. Suppose that T is a Quillen monad acting on $s\mathcal{S}et^{\mathscr{I}}$, equipped with its injective model structure. Then any T-algebra is resolvable.

Proof. This is an immediate application of Lemma 3.24. □

3.3.2. *Cellular monads.*

Proposition 3.26. Let $\mathscr{C}$ be a cofibrantly generated model category and let $X_\bullet \in \mathscr{C}^{\Delta_0^{\mathrm{op}}}$ be an objectwise cellular diagram such that each degeneracy s_i is a subcellular inclusion. Then the latching maps of $X_\bullet$ are cellular inclusions, and therefore $X_\bullet$ is Reedy cofibrant.

Proof. The proof of [EKMM97, X.2.5], paragraph 2, demonstrates the analogue of this statement in the context of cellular spectra, and is general enough to hold for any cellular simplicial object whose degeneracies are subcell inclusions. By good fortune, their argument relies only on simplicial face maps d_k for k positive, and hence applies to $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$. □

Proposition 3.27. Let T be a Quillen monad acting on a cofibrantly generated model category $\mathscr{C}$. Suppose that for any cellular object M, TM is cellular and the natural unit map $M \to TM$ is a cellular inclusion. If X is a T-algebra and is weakly equivalent as a T-algebra to some $\widetilde{X}$ such that $U\widetilde{X}$ is cellular, then X is resolvable with replacement $\widetilde{X}$.

Proof. This is an immediate application of Proposition 3.26. □

3.3.3. *Monads whose unit maps are inclusions of summands.*

Proposition 3.28. Let $X_\bullet \in \mathscr{C}^{\Delta_0^{\mathrm{op}}}$ be a diagram in a pointed model category $\mathscr{C}$ such that X_0 is cofibrant and each degeneracy s_i is a cofibration and the inclusion of a summand. Then the latching maps of $X_\bullet$ are cofibrations and summand inclusions, and therefore $X_\bullet$ is Reedy cofibrant.

Proof. In this case X_n splits as a coproduct of the union of the degenerate simplices $L_n X_\bullet$ and the non-degenerate simplices X'_n. As a coproduct of cofibrant objects the degenerate simplices are cofibrant and X_n is cofibrant because there is a chain of degeneracies

$$\varnothing \to X_0 \to X_1 \to \cdots \to X_n$$

which are all cofibrations. As a retract of a cofibrant object X'_n is cofibrant.

Now the latching map is a coproduct of the identity map and the map from the initial object to the non-degenerate part of X_n and hence is a cofibration. □

Proposition 3.29. Let T be a Quillen monad acting on a pointed category $\mathscr{C}$. Suppose that for any cofibrant object M the natural unit map $M \to TM$ is a cofibration and inclusion of a summand. If X is a T-algebra and is weakly equivalent as a T-algebra to some $\widetilde{X}$ such that $U\widetilde{X}$ is cofibrant, then X is resolvable with replacement $\widetilde{X}$.

Proof. This is an immediate application of Proposition 3.28. □

4. The spectral sequence and examples

4.1. **Proof of Theorem 1.1.** Now we recall and prove the central theorem of this paper:

Theorem. Suppose T is a monad acting on a simplicial category $\mathscr{C}$ and X and Y are T-algebras such that:

(a) T is Quillen,
(b) T is commutes with geometric realization,
(c) and X is resolvable with replacement $\widetilde{X}$.

Then there exists an obstruction-theoretic spectral sequence satisfying:

(1) $E_1^{0,0} = ho\mathscr{C}(UX, UY)$.
(2) $E_2^{0,0} = (ho\mathscr{C})_{hT}(X, Y)$. That is, a homotopy class $[f]\colon UX \to UY$ survives to the E_2 page if and only if it is a map of hT-algebras in the homotopy category.

(3) Provided a T-algebra map $\varepsilon\colon X \to Y$ to serve as a base point, the spectral sequence conditionally converges to the homotopy of the derived mapping space

$$\pi^s\pi_t(\mathscr{C}^d(T^\bullet U\widetilde{X}, UY), \varepsilon) \Longrightarrow \pi_*(\mathscr{C}^d_T(X,Y), \varepsilon).$$

(4) In this case the differentials $d_r[f]$ provide obstructions to lifting $[f]$ to a map of T-algebras.

(5) The edge homomorphisms

$$\begin{aligned}\pi_0\mathscr{C}^d_T(X,Y) &\twoheadrightarrow E_\infty^{0,0}\\ &\hookrightarrow E_2^{0,0} = (ho\mathscr{C})_{hT}(UX,UY)\\ &\hookrightarrow E_1^{0,0} = ho\mathscr{C}(UX,UY)\end{aligned}$$

are the corresponding forgetful functors.

(6) The spectral sequence is contravariantly functorial in $X \in ho\mathscr{C}_T$ and covariantly functorial in $Y \in ho\mathscr{C}_T$ and T satisfying the hypotheses.

Proof. First, in order for the theorem to make sense there needs to be a derived mapping space of T-algebras. This follows from the assumption that T is Quillen.

The conclusions of the theorem only depend on the weak equivalence classes of X and Y, so without loss of generality we assume Y is a fibrant T-algebra and that $X = \widetilde{X}$ is a T-algebra such that $T^\bullet UX$ is a Reedy cofibrant diagram in $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$. By Proposition 3.23 the bar resolution $B_\bullet X$ is a Reedy cofibrant simplicial T-algebra. Since Y is fibrant and $\mathscr{C}$ is a simplicial model category, applying the mapping space functor $\mathscr{C}_T(-,Y)$ to a Reedy cofibrant simplicial T-algebra yields a Reedy fibrant cosimplicial space. In particular, $\mathscr{C}_T(B_\bullet X, Y)$ is Reedy fibrant.

Applying [Bou89], the totalization tower for this cosimplicial space arising from the skeletal filtration on $|B_\bullet X|$ yields an obstruction theoretic spectral sequence computing the homotopy of the totalization

$$\mathrm{Tot}(\mathscr{C}_T(B_\bullet X, Y)) \cong \mathscr{C}_T(|B_\bullet X|, Y).$$

This spectral sequence conditionally converges provided there exists a base point at which to take homotopy groups. (A list of obstructions to determining such a base point is also provided by the construction; see Remark 4.4.)

Now since $B_\bullet X$ is Reedy cofibrant and $\mathscr{C}_T$ is a simplicial model category, $|B_\bullet X|$ is a cofibrant T-algebra. Since T commutes with geometric realization, Proposition 3.17 shows that the augmentation map

$$|B_\bullet X| \to X$$

is a weak equivalence of T-algebras. It follows that $\mathscr{C}_T(|B_\bullet X|, Y)$ is a model for $\mathscr{C}^d_T(X,Y)$ and this gives the target of the spectral sequence in (3). Conclusion (4) follows immediately from the convergence of the spectral sequence.

The $E_1^{0,0}$ term of Bousfield's spectral sequence is the set

$$\pi_0\mathscr{C}_T(B_0X, Y) = \pi_0\mathscr{C}_T(F_TUX, Y) \cong \pi_0\mathscr{C}(UX, UY).$$

To prove (1) we will show the right-hand side can be identified with morphisms in the homotopy category. Since $\mathscr{C}$ is a simplicial model category this follows if UX is cofibrant and UY is fibrant. These follow from the hypotheses that X is resolvable and that T is Quillen: Because T^nUX is Reedy cofibrant, the zeroth latching map shows that UX is cofibrant. Since T is Quillen, U is a right Quillen functor and therefore UY is fibrant because Y is fibrant.

The edge homomorphism

$$\pi_0\mathscr{C}^d_T(X,Y) \to E_1^{0,0}$$

is induced by restricting along the inclusion

$$\mathrm{sk}_0|B_\bullet X| = F_TUX \to |B_\bullet X|$$

which by adjunction gives the second half of (5). The first half follows from the identification of the $E_2^{0,0}$ term in (2).

To prove (2) recall that the $E_2^{0,0}$ term of Bousfield's spectral sequence is defined to be the equalizer of the two face maps

$$\pi_0\mathscr{C}_T(B_0X,Y) \rightrightarrows \pi_0\mathscr{C}_T(B_1X,Y).$$

We again use the adjunction and the fact that $T^\bullet UX$ is Reedy cofibrant to see that the diagram above is isomorphic to

$$ho\mathscr{C}(UX,UY) \rightrightarrows ho\mathscr{C}(TUX,UY),$$

whose equalizer is, by definition, $(ho\mathscr{C})_{hT}(UX,UY)$ (see (2.2) and Proposition 3.11). In other words, a map lifts to $E_2^{0,0}$ precisely if it is a homotopy T-algebra map.

Provided a base point ϵ for the spectral sequence, or even a point that lifts to Tot^2 (see Remark 4.4), the E_1 page of this spectral sequence is given by applying π_t to the spaces $\mathscr{C}_T(B_sX,Y)$ and normalizing as in [Bou89, §2]. The E_2 term can be identified with the cohomotopy of this graded cosimplicial object which is typically denoted as follows:

$$E_2^{s,t} = \pi^s\pi_t(\mathscr{C}_T(B_\bullet X,Y),\epsilon).$$

By adjunction we have

$$\mathscr{C}_T(B_nX,Y) = \mathscr{C}(F_TT^nUX,Y) \cong \mathscr{C}(T^nUX,UY).$$

As in the previous steps, the right-hand side is a model for the derived mapping space since UY is fibrant and $T^\bullet UX$ is Reedy cofibrant. This completes the proof of (3).

To see that the spectral sequence is functorial with respect to maps in $ho\mathscr{C}_T$ it suffices to see that it is functorial with respect to maps in Y, where Y is assumed to be fibrant, and X such that $T^\bullet UX$ is Reedy cofibrant. The former is obvious and the latter follows from the naturality of the bar construction.

To check functoriality in T we suppose that we have the following diagram of adjunctions:

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, shift left, "F_{T_2}"] \arrow[dr, shift left, "F_{T_1}"] & & \mathscr{C}_{T_2} \arrow[ll, shift left, "U_2"] \arrow[dl, shift left, "U_3"] \\
& \mathscr{C}_{T_1} \arrow[ul, shift left, "U_1"] \arrow[ur, shift left, "F_{T_3}"] &
\end{tikzcd}$$

where $F_{T_2} = F_{T_3}F_{T_1}$ and $U_{T_2} = U_1U_3$. We assume that all of the adjunctions are simplicial Quillen adjunctions and their associated monads satisfy the hypotheses of the theorem. Moreover we suppose that X has Reedy cofibrant resolutions with respect to T_1 and T_2 simultaneously. To obtain a map between the spectral sequences corresponding to the map of mapping spaces:

$$\mathscr{C}_{T_2}(X,Y) \xrightarrow{U_3} \mathscr{C}_{T_1}(U_3X,U_3Y)$$

we apply U_3 to the T_2 bar construction for X and our assumption that T_3 and hence U_3 commutes with geometric realizations we see

$$\begin{aligned}
U_3B(F_{T_2},T_2,U_2X) &= U_3B(F_{T_3}F_{T_1},U_1U_3F_{T_3}F_{T_1},U_1U_3X) \\
&= B(T_3F_{T_1},U_1T_3F_{T_1},U_1U_3X) \\
&\leftarrow B(F_{T_1},U_1F_{T_1},U_1U_3X)
\end{aligned}$$

where the last map is induced by the unit map $\mathrm{Id}_{\mathscr{C}_{T_2}} \to T_3$. □

We highlight two immediate corollaries of Theorem 1.1.

Corollary 4.1. The forgetful functor taking a non-empty $ho(\mathscr{C}_T)(X,Y)$ to $(ho\mathscr{C})_{hT}(X,Y)$ is surjective if and only if the differential d_r on $E_r^{0,0}$ is trivial for all $r \geq 2$.

Corollary 4.2. Suppose the portion of the spectral sequence computing $\pi_0\mathscr{C}_T(X,Y)$ converges, i.e., there exists a base point and the associated $\lim^1$ term vanishes. Then the forgetful functor taking $ho(\mathscr{C}_T)(X,Y)$ to $(ho\mathscr{C})_{hT}(X,Y)$ is injective if and only if $E_\infty^{t,t} = 0$ for $t > 0$.

Remark 4.3. As stated in Bousfield every entry in the spectral sequence above should consist of based sets. We have chosen to omit the distinguished point $[\epsilon]$ in $E_1^{0,0}$ and $E_2^{0,0}$ to simplify the statement of Theorem 1.1.

Remark 4.4. There are, in fact, a variety of obstruction sequences whose vanishing can give a lift of ε through the totalization tower. For additional details see [Bou89, §§2.4, 2.5, 5.2]. Let $X_\bullet$ be a cosimplicial object in a simplicial category $\mathscr{D}$.

(1) The rth spectral sequence page $E_r^{p,q}$ is defined if there is an element $\varepsilon_{r-1} \in \mathrm{Tot}^{r-1}\mathscr{D}(X_\bullet,Y)$ which lifts to $\mathrm{Tot}^{2r-2}\mathscr{D}(X_\bullet,Y)$, and the page depends naturally on ε_{r-1}.
(2) Let $\varepsilon_p \in \mathrm{Tot}^p\mathscr{D}(X_\bullet,Y)$, and let ε_k be the projection of ε_p to $\mathrm{Tot}^k\mathscr{D}(X_\bullet,Y)$, where $p/2 \leq k \leq p$. Then there is an obstruction element lying in $E_{p-k+1}^{p+1,p}$ which vanishes if and only if ε_k lifts to $\mathrm{Tot}^{p+1}\mathscr{D}(X_\bullet,Y)$.

If Whitehead products vanish in each $\mathscr{D}(X_s,Y)$ (such as, e.g., when the mapping spaces of $\mathscr{D}$ are H-spaces), then the range in which the obstruction classes are defined can be extended as follows:

(1′) The rth spectral sequence page $E_r^{p,q}$ is defined if there is an element $\varepsilon_{r-2} \in \mathrm{Tot}^{r-2}\mathscr{D}(X_\bullet,Y)$ which lifts to $\mathrm{Tot}^{2r-3}\mathscr{D}(X_\bullet,Y)$, and the page depends naturally on ε_{r-2}.
(2′) Let $\varepsilon_p \in \mathrm{Tot}^p\mathscr{D}(X_\bullet,Y)$, and let ε_k be the projection of ε_p to $\mathrm{Tot}^k\mathscr{D}(X_\bullet,Y)$, where $(p-1)/2 \leq k \leq p$. Then there is an obstruction element lying in $E_{p-k+1}^{p+1,p}$ which vanishes if and only if ε_k lifts to $\mathrm{Tot}^{p+1}\mathscr{D}(X_\bullet,Y)$.

Taking $p = 1$ and $k = 0$ in (2′) from Remark 4.4, we obtain the following useful refinement of Theorem 1.1:

Theorem 4.5. [Compare [GH05, Cor. 2.4.15]] Suppose T is a monad acting on a simplicial category $\mathscr{C}$ and X and Y are T-algebras satisfying the conditions of Theorem 1.1. Moreover suppose that the derived mapping spaces $\mathscr{C}^d(T^nU\widetilde{X},UY)$ have the homotopy type of H-spaces. Then the T-algebra spectral sequence of Theorem 1.1 exists, its E_2 term is always defined, and there is a series of successively defined obstructions to realizing a map

$$[f] \in E_2^{0,0} = (ho\mathscr{C})_{hT}(UX,UY)$$

in the groups

$$\pi^{s+1}\pi_s(\mathscr{C}^d(T^\bullet U\widetilde{X},UY),f)$$

for $s \geq 1$. In particular, if these groups are all zero, then the map induced by the forgetful functor

$$ho(\mathscr{C}_T)(X,Y) \to (ho\mathscr{C})_T(UX,UY)$$

is surjective. If, in addition

$$\pi^s\pi_s(\mathscr{C}^d(T^\bullet U\widetilde{X},UY),f) = 0$$

for all $s \geq 1$, then this map is a bijection.

We will primarily make use of this theorem in the following form:

Corollary 4.6. Suppose T is a monad on $\mathcal{S}pectra$ and $X,Y \in \mathcal{S}pectra_T$ satisfy the hypotheses of Theorem 1.1. Then the T-algebra spectral sequence of Theorem 1.1 exists, its E_2 term is always defined, and there is a series of successively defined obstructions to realizing a map

$$[f] \in E_2^{0,0} = (ho\mathcal{S}pectra)_{hT}(UX,UY)$$

in the groups

$$\pi^{s+1}\pi_s(\mathcal{S}pectra^d(T^\bullet U\widetilde{X}, UY), f)$$

for $s \geq 1$. In particular, if these groups are all zero, then the map induced by the forgetful functor

$$ho(\mathcal{S}pectra_T)(X,Y) \to (ho\mathcal{S}pectra)_T(UX,UY)$$

is surjective. If, in addition

$$\pi^s\pi_s(\mathcal{S}pectra^d(T^\bullet U\widetilde{X}, UY), f) = 0$$

for all $s \geq 1$, then this map is a bijection.

4.2. **Observations on E_1.** Provided all of the terms in $E_1^{s,t}$ for $t > 0$ are abelian groups, for example if the mapping spaces $\mathscr{C}(T^nUX, UY)$ have the homotopy type of H-spaces, then we can avoid using the normalized cocomplex in [Bou89] and instead use Moore cochains. We then have a reinterpretation of the terms in $E_1^{s,t}$, via the tensor-cotensor adjunction:

$$E_1^{s,t} = \pi_t\big(\mathscr{C}(T_sUX, UY), \varepsilon\big) \cong \pi_0\big(\mathscr{C}(T^sUX, UY^{S^t}), \varepsilon\big).$$

This displays $E_1^{s,t}$ as a set of homotopy classes of lifts in the diagram below, with homotopies fiberwise over ε:

$$\begin{tikzcd} & UY^{S^t} \arrow[d] \\ T^sUX \arrow[ur, dashed] \arrow[r, "\varepsilon"] & UY \end{tikzcd}$$

Thus the other terms in the spectral sequence are the elements of

$$ho\mathscr{C}_{\downarrow UY}(T^sUX, UY^{S^t})$$

where UX is an object over UY by the map $\varepsilon\colon UX \to UY$.

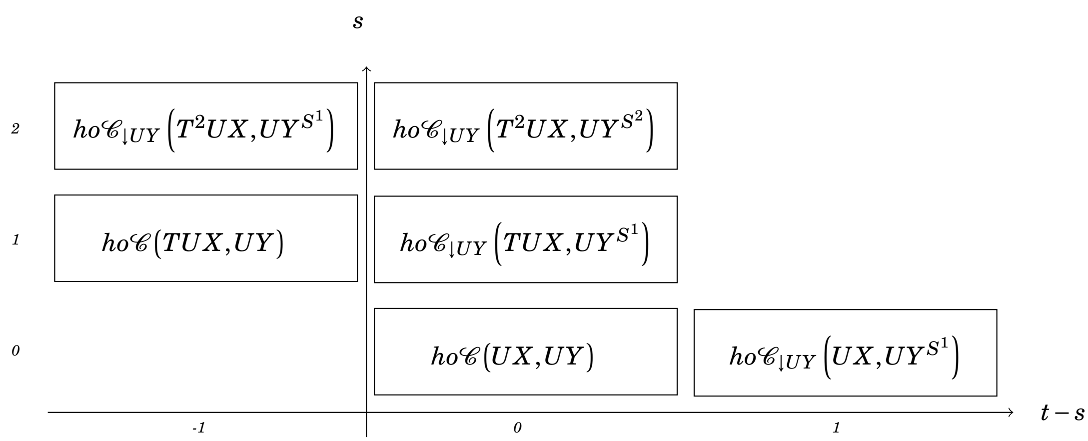

FIGURE 4.7. Low-degree terms on the E_1 page of the spectral sequence, interpreted as homotopy classes of lifts.

4.3. **Examples.** This section will be devoted to examples satisfying the hypotheses of Theorem 1.1.

4.3.1. *Simplicial algebraic theories.*

Theorem 4.8. Let T be a monad on $s\mathcal{S}et$ associated to any algebraic theory as in Section 2.2, then the T-algebra spectral sequence of Theorem 1.1 can be applied to any $X, Y \in s\mathcal{S}et_T$.

Proof. By Proposition 3.10 we see that T is Quillen. Remark 3.15 shows that T commutes with geometric realizations. Finally Proposition 3.25 shows that any X is resolvable. □

4.3.2. *G-actions.*
For the following result one can use any of the standard cofibrantly generated models for the category of spectra which is enriched in spaces and such that the tensor product of a cellular space with a cellular spectrum is naturally a cellular spectrum.

Proposition 4.9. Let G be a topological group admitting a cellular structure such that the unit is the inclusion of a sub-complex. Let $TX = G_+ \wedge X$ be the monad on based spaces/spectra whose algebras are G-spaces/spectra. Then the T-algebra spectral sequence of Theorem 1.1 can be applied to any X, Y in these categories.

Proof. It is well known and straightforward to show using Theorem 3.8 that T is Quillen. Since geometric realization commutes with smash products in either of these categories we see that T commutes with geometric realization. Since the unit transformation applied to cellular spectra gives an inclusion of subcomplexes by Proposition 3.27 we see that every X is resolvable. □

In the case of G-spaces or G-spectra the T-algebra spectral sequence of Theorem 1.1 takes a familiar form. The bar resolution applied to X is the standard cofibrant replacement $EG_+ \wedge X \to X$ in the naive model structure. The skeletal filtration on the bar resolution corresponds to the bar filtration on EG and our spectral sequence computing the homotopy groups of the space of G-maps between X and Y becomes the homotopy fixed point spectral sequences computing the homotopy groups of $F(X,Y)^{hG}$ where $F(X,Y)$ is the corresponding G-space of maps.

Remark 4.10. As expected, the homotopy G-spaces/spectra (i.e., the homotopy T-algebras for T as above) will correspond to those spaces/spectra which admit a G-action in the homotopy category. Morphisms of homotopy G-spaces/spectra are maps in the homotopy category which commute with the G-action. In particular, any G-map which is non-equivariantly null-homotopic is necessarily trivial in the category of homotopy G-spaces (see Section 5.1).

4.3.3. *Algebras over operads.*

Proposition 4.11. Suppose T is a monad arising from a cofibrant admissible operad (see [BM03]) acting on a symmetric monoidal simplicial model category $(\mathscr{C}, \otimes, I)$. In addition suppose that geometric realization commutes with the symmetric monoidal structure on $\mathscr{C}$.

Moreover if one of the following conditions holds:

(a) The underlying category $\mathscr{C}$ is $s\mathcal{S}et^{\mathscr{I}}$ and is endowed with the injective model structure.
(b) For every T-algebra Y which is cellular in $\mathscr{C}$, the unit map $Y \to TY$ is a cellular inclusion
(c) For every T-algebra Y which is cofibrant in $\mathscr{C}$, the unit map $Y \to TY$ is the cofibrant inclusion of a summand and $\mathscr{C}$ is pointed.

Then the T-algebra spectral sequence of Theorem 1.1 can be applied to any $X, Y \in \mathscr{C}_T$.

Proof. Essentially the definition of admissibility is that Theorem 3.8 can be applied to show that T is Quillen. Remark 3.15 and the assumption that geometric realization commutes with the monoidal structure shows that T commutes with geometric realization.

Since our operad is cofibrant we can replace any T-algebra by one which is cellular or cofibrant in $\mathscr{C}$ by [BM03, Thm. 3.5 (b)]. Finally by the remaining hypothesis we can apply either Proposition 3.25, Proposition 3.27, or Proposition 3.29 to see that any T-algebra is resolvable. □

Lemma 4.12. Suppose that $\mathscr{O}$ is an operad such that $\mathscr{O}(0) = \mathscr{O}(1) = *$. Then $W\mathscr{O}(1) = *$.

Proof. Using the construction of $W\mathscr{O}$ given in [BM06], we observe that all of the maps in the sequential colimit

$$W\mathscr{O}(n) = \operatorname{colim}\left(\mathscr{O}(n) = W_0(H,\mathscr{O})(n) \to W_1(H,\mathscr{O})(n) \to \cdots\right)$$

are isomorphisms when $n = 1$ (H is the unit interval here). To see this, one observes that the right-hand (and therefore left-hand) vertical maps in the pushout [BM06, (13)] are isomorphisms for $n = 1$:

For trees G with a single input edge, the objects $\underline{\mathscr{O}}(G)$ and $\underline{\mathscr{O}}^-(G)$ are equal (all vertices of G are univalent, and if $\mathscr{O}(1) = *$ then $\underline{\mathscr{O}}_c(G) = \underline{\mathscr{O}}(G)$ for any subset of univalent vertices c). As an aside, note that this implies the vertical arrows in the pushout diagram at the end of [BM06, §3] are isomorphisms for $n = 1$, and hence $\mathscr{F}_*(\mathscr{O})(1) = \mathscr{O}(1) = *$.

Moreover, this implies $(H \otimes \mathscr{O})^-(G) = H(G) \otimes \mathscr{O}^-(G)$. Therefore the vertical maps in [BM06, (13)] are isomorphisms and $W(H,\mathscr{O})(1) = W_0(H,\mathscr{O})(1) = \mathscr{O}(1)$. □

Proposition 4.13. Suppose T is a monad arising from an admissible operad $W\mathscr{O}$ (see [BM03]) acting on a symmetric monoidal simplicial model category $(\mathscr{C}, \otimes, I)$. Here $W\mathscr{O}$ is the Boardman-Vogt (see [BM06]) cofibrant replacement of an operad $\mathscr{O}$, such that $\mathscr{O}(0) = \mathscr{O}(1) = *$. In addition suppose that geometric realization commutes with the symmetric monoidal structure on $\mathscr{C}$ and that $\mathscr{C}$ is pointed. Then the T-algebra spectral sequence of Theorem 1.1 can be applied to any $X, Y \in \mathscr{C}_T$.

Proof. We will apply Proposition 4.11 using the hypothesis that the unit map is the inclusion of a summand. As shown in [BM06] the Boardman-Vogt construction yields a functorial cofibrant replacement of our operad. Moreover by construction, $(W\mathscr{O})(1) = *$ so the unit map $X \to TX$ is always the inclusion of a summand.

Since $W\mathscr{O}$ is cofibrant, we can assume a given T-algebra is cofibrant in $\mathscr{C}$ by [BM03, Thm. 3.5 (b)]. Since $\mathscr{C}$ is a symmetric monoidal model category it is straightforward to apply the pushout-product axiom and induction on n to see that $X^{\otimes n}$ is cofibrant. Finally since our cofibrant operad is Σ-cofibrant we see that $W\mathscr{O}(n) \otimes X^{\otimes n}$ is a retract of a cellular complex built with free Σ_n-cells. It follows that $W\mathscr{O}(n) \otimes_{\Sigma_n} X^{\otimes n}$ is cofibrant which in turn implies TX is cofibrant. □

Corollary 4.14. If $\mathscr{C}$ is the category of R-modules for a commutative cofibrant ring spectrum R and T is the monad associated to the Boardman-Vogt replacement of either the associative or the commutative operad (so it is an A_∞ or E_∞ operad) then the T-algebra spectral sequence of Theorem 1.1 can be applied to any $X, Y \in \mathscr{C}_T$.

5. Computations

5.1. ***G*-actions.** The next two examples provide, respectively, an example of a non-trivial G-map which is trivial as a homotopy G-map and an example of a homotopy G-map (necessarily non-trivial) which does not lift to a G-map.

Example 5.1. Regard $\mathbb{R}$ as a C_2-space via the sign action. Then applying one point compactification to the inclusion

$$\{0\} \to \mathbb{R}$$

yields an essential map

$$e_\sigma \colon S^0 \to S^\sigma$$

of pointed C_2-spaces.

Taking the trivial map as our base point of $\mathcal{T}op_{C_2}(S^0, S^\sigma)$ and applying the spectral sequence of Theorem 1.1 we have [2]:

$$E_2^{s,t} = H^s(C_2; \pi_t S^\sigma) \implies \pi_{t-s}(S^\sigma)^{hC_2}.$$

As noted in Remark 4.10, e_σ must represent the trivial map in the category of homotopy C_2-spaces. The spectral sequence confirms this since $E_2^{0,0} = (\pi_0 S^\sigma)^{C_2} = 0$. In fact, since the homotopy groups of S^1 are concentrated in degree 1, this spectral sequence is concentrated on the line $t = 1$ and necessarily collapses at E_2. The only non-zero contribution is from $E_2^{1,1} = \mathbb{Z}/2$ which detects the map e_σ above.

Example 5.2. Let C_2 act on KU via complex conjugation. The C_2-action on $\pi_* KU$ is trivial precisely on those homotopy groups generated by even powers of the Bott map. In particular, if we regard S^4 as having a trivial C_2 action we obtain a non-trivial map

$$\beta^2 : S^4 \to KU$$

in the category of homotopy C_2-spectra.

The spectral sequence of Theorem 1.1 computing the homotopy groups of the space of C_2-equivariant maps from S^0 to KU is (the connective cover) of the homotopy fixed point spectral sequence. After 2-completion this spectral sequence converges to the homotopy of KO and there is a well-known differential $d_3(\beta^2) = \eta^3$ in this spectral sequence forced by the relation $\eta^4 = 0 \in \pi_* S$. Since the T-algebra spectral sequence computing $\pi_* \mathcal{S}pectra_{C_2}(S^4, KU)$ is just a shift of this spectral sequence we see that the element $\beta^2 \in E_2^{0,0}$ supports a d_3 and does not lift to a map of C_2-spectra.

5.2. **Methodology for calculating the E_2-term.** To obtain a computationally useful, i.e. algebraic, description of the E_2 term from Theorem 1.1 we would like to verify the following:

(1) There is a functor
$$\pi_* : ho\mathcal{C} \to \mathcal{D}.$$
(2) The associated Hurewicz map
$$\pi_* \mathcal{C}(X,Y) \to \mathcal{D}(\pi_* X, \pi_* Y^{S^t})$$
is an isomorphism for general X and Y.
(3) There is a natural isomorphism $\pi_* TX \cong T_{alg} \pi_* X$ for a monad T_{alg}.
(4) The category of algebras $\mathcal{D}_{T_{alg}}$ is closed under finite limits and has enough projectives (see [Qui69]).
(5) $\pi_* Y^{S^t}$ for $t \geq 1$ is naturally an abelian group object in the category of T_{alg}-algebras over $\pi_* Y$.

When these conditions are satisfied, we can apply the Hurewicz homomorphism levelwise to the cosimplicial space defining the T-algebra spectral sequence and obtain an analogous cosimplicial graded object in the (computationally accessible) category $\mathcal{D}$. The associated E_2 term is the cohomotopy of this graded object as in [Bou89]. The $E_2^{0,0}$ term is always $\mathcal{D}_{T_{alg}}(\pi_* X, \pi_* Y)$ which, for well-chosen $\mathcal{D}$, gives a purely algebraic description of the homotopy T-algebra maps. The remaining terms will be cotriple cohomology groups as in [Qui69, II.5.(2)].

When the category of algebras $\mathcal{D}_{T_{alg}}$ has enough projectives and is closed under finite limits we can apply [Qui70, §1] to see that the category $sT_{alg,\pi_* Y}$ of simplicial T-algebras over $\pi_* Y$ and its subcategory of abelian group objects admit model structures satisfying the conditions of

[2]Normally instability, e.g., actions of the fundamental group, prevents getting such a simple description of the E_2 term, however in this case S^σ is non-equivariantly an Eilenberg-MacLane space for $\mathbb{Z}$ and so the second half of the refinements in Remark 4.4 apply.

[Qui69, II.5.(1),(2),(4)]. We can use this to identify the remaining groups on the E_2 page with André-Quillen cohomology groups. More explicitly we obtain:

$$E_2^{s,t} \cong H^s_{AQ,\pi_*Y}(\pi_*X;\pi_*Y^{S^t})$$

for $t>0$ by [Qui69, II.5.Theorem 5]. The cohomology groups on the right are the associated André-Quillen cohomology groups of our T_{alg}-algebra π_*X viewed as an algebra over π_*Y via a choice of an element in $E_2^{0,0}$ [Qui69, II.5].

5.3. **Algebras over an operad in spectra.** For one example of applying the methodology in Section 5.2, let k be a field and let T be the monad

$$X \mapsto TX = \bigvee_{n\geq 0} K_n \otimes X^{\wedge_{Hk} n}$$

on Hk-module spectra associated to the A_∞ operad. In Section 5.5 we will verify the conditions above where

$$T_{alg}\pi_*X \cong \bigoplus_{n\geq 0} (\pi_*X)^{\otimes_k n}$$

is the monad on graded k-modules whose algebras are graded associative k-algebras.

In the category of objects over π_*Y, the abelian group objects are the square zero extensions of π_*Y such as $\pi_*Y^{S^t} \cong \pi_*Y \oplus \pi_*\Sigma^{-t}Y$ and hence condition (5) is satisfied. The category of simplicial associative algebras is one of the classical examples studied in [Qui70] and satisfies the conditions necessary to define the associated cohomology groups.

So we obtain a spectral sequence

$$E_2^{s,t} \Rightarrow \pi_{t-s}A_\infty Hk\text{-}\mathcal{Alg}(X,Y)$$

such that

$$E_2^{0,0} = k\text{-}\mathcal{Alg}(\pi_*X,\pi_*Y)$$

and

$$E_2^{s,t} = H^s_{AQ,\pi_*Y}(\pi_*X,\pi_*Y^{S^t}) \quad \text{for } t>0,$$

where the cohomology groups are calculated in the category of graded associative k-algebras over π_*Y. For $s=0$ these can be identified with the derivations of π_*X into $\pi_{*+t}Y$ and for $s>0$ these are the $s+1$st Hochschild cohomology groups

$$HH^{s+1}(\pi_*X;\pi_{*+t}Y) \cong \mathrm{Ext}^{s+1}_{\pi_*X\otimes_k(\pi_*X)^{\mathrm{op}}}(\pi_*X,\pi_{*+t}Y)$$

of π_*X with coefficients in $\pi_{*+t}Y$ [Qui70, Prop. 3.6].

As shown in Section 5.5, we obtain a similar result where T is the monad on Hk-modules whose algebras are the E_∞-algebras in this category. Here T_{alg} is the monad on graded k-modules whose algebras are Dyer-Lashof algebras [BMMS86]. If k is a field of characteristic 0 these are just the graded commutative k-algebras.

Example 5.3. In the category of Hk-modules, consider the A_∞-algebras $Hk\wedge\Sigma^\infty_+\Omega SU(n+1)$. The homotopy of these algebras is a polynomial algebra $R=R_n$ on generators $\{x_i\}_{1\leq i\leq n}$ where the $|x_i|=2i$. To compute the A_∞ self-maps we apply our spectral sequence and the discussion above to compute

$$\begin{aligned}
E_2^{0,0} &\cong k\text{-}\mathcal{Alg}(R,R) \cong \prod_{1\leq i\leq n}(R)_{2i}\\
E_2^{0,t} &\cong Der(R;\Sigma^{-t}R) \cong \prod_{1\leq i\leq n}(R)_{2i+t}\\
E_2^{s,t} &\cong \mathrm{Ext}^{s+1}_{R\otimes_k R^{\mathrm{op}}}(R,\Sigma^{-t}R) \quad \text{for } s>0
\end{aligned}$$

In particular, these groups are zero for t odd, hence $E_{2i} = E_{2i+1}$. The Hochschild cohomology groups can be calculated by first pulling back the $R \otimes_k R^{\mathrm{op}}$ action to an $R \otimes_k R$ action via the isomorphism defined by

$$x_i \otimes 1 \mapsto x_i \otimes 1, \quad 1 \otimes x_i \mapsto x_i \otimes 1 - 1 \otimes x_i.$$

Since the second copy of R acts trivially on the source we obtain an $(R \otimes_k R)$-free resolution of R by tensoring the Koszul resolution of k by right R modules, on the left with R:

$$(\Lambda_k[\sigma x_1, \cdots, \sigma x_n] \otimes_k R \to k) \xrightarrow{R \otimes_k -} (R \otimes_k \Lambda_k[\sigma x_1, \cdots, \sigma x_n] \otimes_k R \to R)$$

Here σx_i has bidegree $(1, 2i)$ and $d(\sigma x_i = x_i)$. Using this resolution we see that

$$\mathrm{Ext}^*_{R \otimes_k R^{\mathrm{op}}}(R, R) \cong (\Lambda[\sigma x_1, \cdots, \sigma x_n])^* \otimes R.$$

So the Hochschild cohomology groups vanish above cohomological degree n so our spectral sequence is concentrated on the first $n-1$ lines and must collapse at E_n for $n \geq 2$. In particular, if $n = 1$ then the spectral sequence collapses at E_2 onto the 0-line.

Hence there are no obstructions to lifting a map of k-algebras

$$H_* \Omega SU(n+1) \to H_* \Omega SU(n+1)$$

to a map of A_∞-algebras if $n \leq 3$ and such a map is homotopically unique if $n \leq 2$. For $n = 1$ this result is expected since $\Omega SU(2) \cong \Omega \Sigma S^2$ is stably a free A_∞-algebra.

The previous computation did not depend on the A_∞ algebra $Hk \wedge \Sigma_+ \Omega SU(n+1)$ so much as the fact that its ring of homotopy groups is polynomial on generators in even degrees. In particular, if V is an A_∞-algebra in Hk-modules with no more than three such polynomial generators (e.g., $V = HR_n$, $n \leq 3$), then any morphism of k-algebras

$$\pi_* V \to R_n$$

lifts to an A_∞ map

$$V \to Hk \wedge \Omega SU(n+1).$$

This proves the following result:

Proposition 5.4. For $n \leq 3$, there is a unique Hk-algebra V up to homotopy such that $\pi_* V$ is polynomial algebra on n generators in even degrees. In particular, all such algebras admit a commutative Hk-algebra structure.

Example 5.5. If we allow n to go to infinity in the previous example then $\Omega SU \simeq BU$ is an infinite loop space and consequently $Hk \wedge \Sigma^\infty_+ \Omega SU$ is an E_∞-algebra in Hk-modules.

When the characteristic of k is zero, the results in Section 5.5 enable us to compute the space of E_∞ self maps. We have the following identification of the E_2-term, where

$$R = H_*(\Omega SU) \cong k[x_i]_{i \geq 1}$$

and k-$\mathcal{CAlg}$ is the category of commutative k-algebras:

$$E_2^{0,0} \cong k\text{-}\mathcal{CAlg}(R, R) \cong \prod_{i \geq 1} (R)_{2i}$$
$$E_2^{0,t} \cong Der(R, \Sigma^{-t} R) \cong \prod_{i \geq 1} (R)_{2i+t}$$
$$E_2^{s,t} \cong H^s_{AQ,R}(R; \Sigma^{-t} R) \quad t > 0$$

Since R is a polynomial algebra, it is smooth and all higher André-Quillen cohomology groups vanish [Qui70]. As a consequence we see the spectral sequence collapses at E_2 onto the 0 line. Hence every map of homology rings lifts to a homotopically unique map of E_∞-algebras in Hk-modules.

Alternatively one could deduce the conclusion from the previous example in a more direct fashion. There is a map $\mathbb{C}P^{\infty} \to BU$ which maps the reduced homology of $\mathbb{C}P^{\infty}$ isomorphically onto indecomposable generators for the homology of BU. Since BU is a based E_∞ space this canonically extends to a map of E_∞-algebras

$$T\mathbb{C}P^{\infty} \to \Sigma^{\infty}_{+}BU.$$

Localizing rationally this map is an equivalence and as a consequence $\Omega SU \simeq BU$ is rationally a free E_∞ spectrum. By the following example the spectral sequence of Theorem 1.1 always collapses at the E_2 term onto the 0 line when the source is such a spectrum. We see that $Hk \wedge \Omega SU$ is equivalent to a free E_∞-algebra in Hk-modules so the spectral sequence in Example 5.5 collapses.

Since rational localization is smashing, the extension functor from E_∞ algebras to E_∞ algebras in $H\mathbb{Q}$ modules is an equivalence for every rational E_∞ ring spectrum. From this we obtain for any rational E_∞ ring spectra X and Y

$$E_\infty(X,Y) \simeq E_\infty H\mathbb{Q}\text{-}\mathcal{Alg}(H\mathbb{Q} \wedge X, Y) \simeq E_\infty H\mathbb{Q}\text{-}\mathcal{Alg}(X,Y).$$

So there is no difference homotopically between the space of E_∞ maps between two rational E_∞ rings and the space of E_∞-algebra maps in $H\mathbb{Q}$-modules.

Example 5.6. If $X = TM$ is a free E_∞ ring spectrum then the unit map $X \to TM$ is a map of E_∞ ring spectra and defines a section of the bar resolution. Consequently the spectral sequence of Theorem 1.1 computing the homotopy of $E_\infty(X,Y)$ collapses at E_2 onto the 0 line. So in this case the edge homomorphism $\pi_0 E_\infty(X,Y) \to H_\infty(X,Y)$ is an isomorphism. Moreover there is a homotopy equivalence $E_\infty(X,Y) \simeq \mathcal{Spectra}(M,Y)$.

Example 5.7. We will now construct infinitely many E_∞ maps that all induce the same H_∞-map. For a space X, recall that the cotensor $H\mathbb{Q}^X$ is an E_∞ ring spectrum satisfying $\pi_* H\mathbb{Q}^X \cong H^{-*}(X)$. Now to calculate the homotopy groups of $E_\infty(H\mathbb{Q}^{S^2}, H\mathbb{Q}^{S^3})$ we apply the spectral sequence from Theorem 1.1 and the identification of the E_2-term above. As a base point we will take a 'trivial' map ϵ of E_∞-rings induced by a map $S^3 \to * \to S^2$.

To calculate the E_2 term we have

$$E_2^{0,0} \cong \mathbb{Q}\text{-}\mathcal{CAlg}\left(\pi_* H\mathbb{Q}^{S^2}, \pi_* H\mathbb{Q}^{S^3}\right) \cong \mathrm{Ind}_{-3}\left(\pi_* H\mathbb{Q}^{S^2}\right) = 0 = \epsilon.$$

For $t > 0$ we use the map ϵ above to regard $\pi_* H\mathbb{Q}^{S^2}$ as a commutative algebra over $\pi_* H\mathbb{Q}^{S^3}$ and obtain

$$E_2^{s,t} \cong H^s_{AQ}\left(\pi_* H\mathbb{Q}^{S^2}; \pi_{*+t} H\mathbb{Q}^{S^3}\right) \cong \mathbb{Q}\text{-}\mathcal{CAlg}^d_{\pi_* H\mathbb{Q}^{S^3}}\left(\pi_* H\mathbb{Q}^{S^2}, \pi_* H\mathbb{Q}^{S^3} \oplus \Sigma^s_{\mathbb{Q}} \pi_{*+t} H\mathbb{Q}^{S^3}\right)$$

where the right-hand side is the derived homomorphisms of simplicial commutative $\mathbb{Q}$-algebras over the constant simplicial algebra $\pi_* H\mathbb{Q}^{S^3}$ into the square-zero extension of this algebra by the sth suspension of $\pi_{*+t}(H\mathbb{Q}^{S^3}$ in simplicial $\pi_* H\mathbb{Q}^{S^3}$-modules. To calculate these derived homomorphisms we first construct a cofibrant replacement of the source.

We construct a cofibrant replacement of the exterior algebra $\pi_* H\mathbb{Q}^{S^2}$ via a homotopy pushout diagram of cellular algebras. Let e_{-2} denote a generator of a one-dimensional $\mathbb{Q}$-module in dimension -2 and $T(e_{-2})$ the free simplicial commutative $\mathbb{Q}$-algebra on this module. A non-zero map

$$e_{-2} \to \pi_{-2} H\mathbb{Q}^{S^2}$$

canonically extends to a map of simplicial commutative $\mathbb{Q}$-algebras

$$T(e_{-2}) \to \pi_* H\mathbb{Q}^{S^2}.$$

Both of these algebras are constant and the kernel of this surjective map is (e_{-2}^2). We can similarly construct a map

$$T(f_{-4}) \to T(e_{-4})$$

which surjects onto this kernel. Let $C(f_{-4})$ denote a cone on the one dimensional $\mathbb{Q}$-module in degree -4, which sits in a factorization of the 0 map

$$\mathbb{Q}\{f_{-4}\} \to C(f_{-4}) \to 0$$

by a cofibration followed by an acyclic fibration in simplicial graded $\mathbb{Q}$-modules. Since $f_{-4} \mapsto 0 \in \pi_* H\mathbb{Q}^{S^2}$ we obtain a map of simplicial $\mathbb{Q}$-modules $C(f_{-4}) \to \pi_* H\mathbb{Q}^{S^2}$.

We now obtain a homotopy pushout diagram

$$\begin{tikzcd} T(f_{-4}) \arrow[r, "e_{-2}^2"] \arrow[d] & T(e_{-2}) \arrow[d] \\ T(Cf_{-4}) \arrow[r] & T(Cf_{-4}) \otimes_{T(f_{-4})} T(e_{-2}) \end{tikzcd}$$

Now we have an induced map from this pushout to $\pi_* H\mathbb{Q}^{S^2}$ which is a quasi-isomorphism.

We can now map out of this pushout diagram into the square zero extension $\pi_* H\mathbb{Q}^{S^3} \oplus \pi_{*+t} H\mathbb{Q}^{S^3}$ and obtain a long exact sequence. Note since

$$\mathbb{Q}\text{-}\mathcal{CAlg}_B^d(TM, B \oplus \Sigma_{\mathbb{Q}}^s N) \cong B\text{-}\mathcal{Mod}(B \otimes M, \Sigma_{\mathbb{Q}}^s N)$$

this is a long exact sequence of Ext groups.

From this long exact sequence we now obtain the following E_2-term:

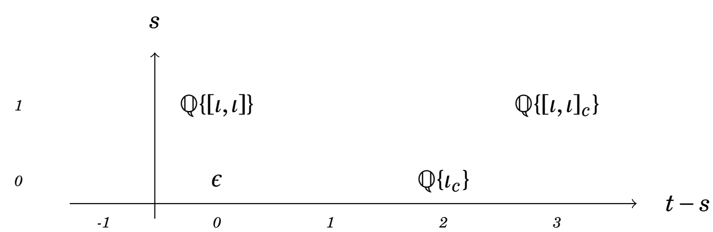

FIGURE 5.8. T-algebra spectral sequence for E_∞ maps $H\mathbb{Q}^{S^2} \to H\mathbb{Q}^{S^3}$.

All other entries are trivial so the spectral sequence collapses at E_2. The $\mathbb{Q}$ in $E_2^{1,1}$ detects an infinite family of E_∞ maps which, because they land in positive filtration, induce the same H_∞ map ϵ. It can be shown that this infinite family is generated by the morphism of E_∞ rings induced by the Hopf map $S^3 \to S^2$.

In the previous example, the spectral sequence vanished above the 1-line guaranteeing collapse of the spectral sequence and an algebraic description of the space of E_∞-maps. This is because the map $\mathbb{Q} \to \pi_* H\mathbb{Q}^{S^2}$ is a local complete intersection morphism and hence the higher André-Quillen cohomology groups vanish. We will say a morphism of $A \to B$ of graded commutative rings is a local complete intersection, resp. smooth, resp. étale, if the relative cotangent complex $L_{B/A}$ [Qui70] has projective dimension ≤ 1, resp. 0, resp. is 0.

Proposition 5.9. Suppose $f : k \to R$ and $k \to S$ are morphisms of rational E_∞-rings. Suppose the spectral sequence of Theorem 1.1 computing the space of E_∞-ring maps under k between R and S has a well-defined E_2-term (e.g., there is a map to serve as the base point). Then if the morphism f on homotopy groups is

(1) *a local complete intersection* then the spectral sequence collapses at the E_2 page onto the 0 and 1 lines and every H_∞ map can be realized by an E_∞-map, although possibly non-uniquely.

(2) *smooth* then the spectral sequence collapses at the E_2 page onto the 0 line and every H_∞ map can be realized, uniquely up to homotopy, by an E_∞-map.
(3) *étale* then the spectral sequence collapses at the E_2 page and $E_2^{s,t} = 0$ if $t > 0$. As a consequence, the mapping space is homotopically discrete and every H_∞ map can be realized, up to a contractible space of choices, by an E_∞-map.

Proof. All of the results follow from the vanishing of the relevant André-Quillen cohomology groups [Qui70, Thm. 5.4] and our identification of $E_2^{0,0}$ with the set of H_∞ maps. □

Example 5.10. We now construct examples of H_∞ ring maps that do not lift to E_∞ ring maps. The argument below does not make explicit use of the spectral sequence beyond the identification of the H_∞ maps, although it does have consequences for the behavior of the spectral sequence.

Let M be the Heisenberg 3-manifold: the quotient of the group of uni-upper triangular 3×3 real matrices by the subgroup with all integer entries. Since M is a quotient of a contractible group by a discrete subgroup it is a $K(\pi,1)$. The commutator subgroup of π is free abelian of rank one and π fits into the short exact sequence of groups

$$1 \to \mathbb{Z} \to \pi \to \mathbb{Z}\times\mathbb{Z} \to 1.$$

In particular M is a nilpotent space.

Applying the classifying space functor to the above exact sequence we see that up to homotopy, M can also be realized as the total space of an S^1 bundle over the torus T^2. This S^1 bundle is classified by the generator of $\mathbb{Z} \cong H^2(T^2;\mathbb{Z}) \cong [T^2, BS^1]$.

A computation with the Serre spectral sequence shows $\pi_* H\mathbb{Q}^M$ is generated by exterior classes x and y in degree -1, polynomial classes α and β in degree -2, and satisfy

$$xy = \alpha^2 = \beta^2 = \alpha\beta = x\alpha = y\beta = x\beta + y\alpha = 0$$

As a consequence we see:

$$H_\infty(H\mathbb{Q}^M, H\mathbb{Q}^{S^2}) \cong E_2^{0,0} \cong \mathrm{Ind}_{-2}(\pi_* H\mathbb{Q}^M) = \mathbb{Q}\{\alpha,\beta\}. \tag{5.11}$$

There are also Massey product relations $\alpha \in \langle x,x,y\rangle$ and $\beta \in \langle y,y,x\rangle$ with no indeterminacy.

Any map from $H\mathbb{Q}^M$ to $H\mathbb{Q}^{S^2}$ sends x and y to zero for degree reasons. Now α and β are Massey products in x and y and Massey products in $H\mathbb{Q}^*M$ correspond to Toda brackets in $H\mathbb{Q}^M$. Since E_∞ maps preserve Toda brackets, they must also send α and β to zero. So α and β must support differentials and correspond to H_∞-maps which do not lift to E_∞ maps.

5.4. Coker J **and maps of** E_∞ **ring spectra.** The following example is a joint result of the second author and Nick Kuhn.

For this example we will need to recall the definitions of some classical infinite loop spaces/connective spectra (cf. [HS78, p.271]). Let $SL_1S^0 = GL_1S^0\langle 0\rangle$ denote the 1-component of QS^0. At an odd prime p let q be an integer generating $(\mathbb{Z}/p^2)^\times$, the choice does not matter. Define J to be the fiber of the map

$$BU^\otimes \xrightarrow{\psi^q/\psi^1} BU^\otimes$$

where $BU^\otimes$ is the 1-component of p-local K-theory and ψ^q is the qth Adams operation. The d-invariant defines a map $S^0 \to KU$ which restricts to a map $SL_1S^0 \xrightarrow{D} BU^\otimes$ which in turn lifts to a map $SL_1S^0 \xrightarrow{D} J$. Let Coker J be the fiber of this last map.

At the prime 2 there are several possible definitions of J and consequently several possible definitions of Coker J. A perfectly reasonable approach is to set J to be the fiber of the map

$$BO^\otimes \xrightarrow{\psi^3/\psi^1} BO^\otimes. \tag{5.12}$$

However this introduces some homotopy groups in low degrees that are not in the image of D. To rectify this there are variations where one replaces one or both copies of BO by either its 1 or 2-connected cover. Rather than go through all the variations we note that all possible choices will yield the same definition of J after taking 1-connected covers. So we define J to be the 1-connected cover of the fiber of the map in (5.12). We then set $\operatorname{Coker} J$ to be the fiber of the map $SL_1S^0\langle 1\rangle \xrightarrow{D} J$.

It is a non-trivial fact that all spaces and maps in sight are infinite loop maps [May77, HS78]. We will follow tradition and denote their associated spectra with lower case letters.

Example 5.13. Let $X = \Sigma^\infty_+ \operatorname{Coker} J$ be the suspension spectrum of the infinite loop space $\operatorname{Coker} J$ and R any E_∞ ring spectrum.

The T-algebra spectral sequence of Theorem 1.1 computing the homotopy of $E_\infty(X, L_{K(2)}R)$ collapses at the E_2 page onto the 0-line. So in this case the edge homomorphism

$$\pi_0 E_\infty(X, L_{K(2)}R) \to H_\infty(X, L_{K(2)}R)$$

is an isomorphism. Moreover there is a homotopy equivalence of spaces

$$E_\infty(X, L_{K(2)}R) \simeq \Omega^\infty L_{K(2)}R.$$

This result follows from the previous example and the following result.

Theorem 5.1. [Kuhn-Noel] There is a $K(2)$-equivalence of E_∞ ring spectra

$$TS^0 \simeq \Sigma^\infty_+ \operatorname{Coker} J$$

where T is the monad whose algebras are E_∞ ring spectra.

Proof. A consequence of the main result of [Kuh06, Thm. 2.21], for any spectrum X there is a natural map of E_∞ ring spectra

$$TX \to L_{K(2)}\Sigma^\infty_+\Omega^\infty X \tag{5.14}$$

which is an equivalence if $\pi_i X = 0$ for $i \le 2$, torsion for $i = 3$, and $K(1)_*\Omega^\infty X$ is trivial.

First we consider the p-local case for an odd prime p. In this case the D-invariant

$$SL_1S^0 \xrightarrow{D} J$$

is at least $2p-1$ connected, hence $\operatorname{Coker} J$ is at least $2p-1 > 3$-connected.

To see that $K(1)_* \operatorname{Coker} J$ is trivial consider the defining fibration sequence of infinite loop spaces

$$\operatorname{Coker} J \to SL_1S^0 \xrightarrow{D} J. \tag{5.15}$$

By [HS78, Thm. 2.5] the D is a $K(1)$-equivalence. Applying the $K(1)$-Serre spectral sequence for this fibration, we see that the local coefficient system is trivial, the edge homomorphism is an isomorphism, and the spectral sequence collapses forcing $\operatorname{Coker} J$ to be $K(1)$-acyclic. Hence (5.14) is an equivalence for $X = \operatorname{coker} j$.

Delooping (5.15) we obtain an exact triangle

$$\operatorname{coker} j \to sl_1S^0 \xrightarrow{d} j.$$

Since j is $K(2)$-acyclic we have a $K(2)$-equivalence $\operatorname{coker} j \to sl_1S^0$. There is also a homotopy equivalence $SL_1S^0 \simeq QS^0_0$ between the 1 and 0 components of QS^0. Although this is not a map of infinite loop spaces, applying the Bousfield-Kuhn functor ϕ_2 to this equivalence does yield an equivalence $L_{K(2)}sl_1S^0 \simeq L_{K(2)}S^0\langle 0\rangle$.

Since Eilenberg-MacLane spectra are $K(n)$-acyclic [RW80], the defining exact triangle for the 0-connected cover

$$S^0\langle 0\rangle \to S^0 \to H\mathbb{Z}$$

shows that $L_{K(2)}S^0\langle 0\rangle \simeq L_{K(2)}S^0$. Finally we use naturality of the spectral sequence

$$H_*(\Sigma_n;K(2)_*(X)^{\otimes_{K(2)_*} n}) \implies K(2)_*((E\Sigma_n)_+ \wedge_{\Sigma_n} X^n)$$

to see the functor T preserves $K(2)$-equivalences.

Assembling these results, we obtain a zig-zag of equivalences of E_∞ ring spectra in the $K(2)$-local category

$$TS^0 \leftarrow T(S^0\langle 0\rangle) \leftarrow Tsl_1S^0 \leftarrow T\operatorname{coker} j \rightarrow \Sigma^\infty_+ \operatorname{Coker} J.$$

At the prime 2 our defining fibration sequence is

$$\operatorname{Coker} J \rightarrow SL_1S^0\langle 1\rangle \xrightarrow{D} J.$$

Here D is 3-connected so $\operatorname{Coker} J$ is sufficiently connected. Again the map D is a $K(1)$-equivalence and consequently $\operatorname{Coker} J$ is $K(1)$-acyclic. The rest of the argument proceeds as before to obtain a zig-zag of $K(2)$-local equivalences of E_∞ ring spectra

$$TS^0 \leftarrow T(S^0\langle 1\rangle) \leftarrow T(sl_1S^0\langle 1\rangle) \leftarrow T\operatorname{coker} j \rightarrow \Sigma^\infty_+ \operatorname{Coker} J.$$

□

5.5. **Computational lemmas.** One of the key steps to obtaining a calculational description of the E_2 term is condition (3) from Section 5.2. That is we need to find a monad T_{alg} such that there is a natural isomorphism

$$\pi_*T \cong T_{alg}\pi_*.$$

In the examples below the category of T_{alg}-algebras will be equivalent to graded associative or commutative k-algebras which fit into the classical work [Qui69, Qui70] of Quillen so they will always satisfy condition (4) which guarantees a definition of André-Quillen cohomology groups which agrees with the cotriple cohomology groups.

If R is a cofibrant commutative S-algebra (see [EKMM97]) then we consider this question when T is a monad on R-module spectra whose category of algebras is equivalent to the category of A_∞ or E_∞ algebras in the category of R-modules.

In both of these examples we see that our monad takes the form

$$TM = \bigvee_{n\geq 0} K_n \otimes_{\Sigma_n} M^{\wedge_R}.$$

In the E_∞ case K_n is equivariantly contractible while in the A_∞ case it is equivariantly weakly equivalent to Σ_n. To determine π_*TM as a functor of π_*M we will use a sequence of spectral sequence arguments that will require increasingly strong assumptions. These assumptions are clearly satisfied in the examples coming from Eilenberg-MacLane spectra in Section 5.3.

Lemma 5.16. If M and N are R-modules such that either π_*M or π_*N is flat as a π_*R module then

$$\pi_*(M \wedge_R N) \cong \pi_*M \otimes_{\pi_*R} \pi_*N$$

Proof. The Tor spectral sequence of [EKMM97, IV.4.1] collapses. □

Lemma 5.17. If M and N are R-modules such that π_*M is projective as a π_*R module then

$$\pi_t\big(R\text{-}\mathcal{M}od\,(M,N)\big) \cong \pi_*R\text{-}\mathcal{M}od\,(\pi_*M,\pi_{*+t}N).$$

Proof. The Ext spectral sequence of [EKMM97, IV.4.1] collapses. □

From these lemmas we easily deduce the following proposition which shows that condition (2) from Section 5.2 is satisfied.

Proposition 5.18. If M and N are R-module spectra and π_*R is a graded field then

$$\pi_t\big(R\text{-}\mathcal{M}od\,(M,N)\big) \cong \pi_*R\text{-}\mathcal{M}od\,(\pi_*M,\pi_{*+t}N).$$

We can also chain together the above lemmas to see

Proposition 5.19. If X is an R-module spectrum, π_*R is a graded field, and T is the monad on R-module spectra whose category of algebras is the category of A_∞ algebras in R-module spectra then there is a natural isomorphism

$$\pi_*TM \cong T_{alg}\pi_*M := \bigoplus_{n\geq 0}(\pi_*M)^{\otimes_{\pi_*R}n}.$$

Here T_{alg} is the monad on π_*R-modules whose algebras are the associative algebras in that category.

We now see that condition (3) from Section 5.2 is satisfied in the case of A_∞-algebras in R-module spectra where π_*R is a graded field.

For the E_∞ case we need the following:

Lemma 5.20. If $E\Sigma_n$ is a contractible Σ_n-CW-complex and M is an R-module such that $n!$ is a unit in π_0R then

$$\pi_*E\Sigma_n \otimes_{\Sigma_n} M^{\wedge_R n} \cong \pi_*(M^{\wedge_R n})/\Sigma_n.$$

Proof. The homotopy orbit spectral sequence

$$H_s(\Sigma_n;\pi_t(M^{\wedge_R n}) \implies \pi_{s+t}((E\Sigma_n)_+ \wedge_{\Sigma_n} M^{\wedge_R})$$

collapses by a standard transfer argument since $n!$ acts invertibly on the coefficients. □

We now see that condition (3) from Section 5.2 is satisfied in the case of rational E_∞-algebras from the following:

Proposition 5.21. If X is an R-module spectrum, π_*R is a graded field, π_0R is a field of characteristic 0, and T is the monad on R-module spectra whose category of algebras is the category of E_∞ algebras in R-module spectra then there is a natural isomorphism

$$\pi_*TM \cong T_{alg}\pi_*M := \bigoplus_{n\geq 0}(\pi_*M)^{\otimes_{\pi_*R}n}/\Sigma_n.$$

Here T_{alg} is the monad on π_*R-modules whose algebras are the commutative algebras in that category.

References

[AHR06] M. Ando, M. Hopkins, and C. Rezk, *Multiplicative orientations of KO-theory and of the spectrum of topological modular forms*, Available at `http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf`, Fall 2006.

[AHS01] M. Ando, M. J. Hopkins, and N. P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube.*, Invent. Math. **146** (2001), no. 3, 595–687.

[AHS04] M. Ando, M. J. Hopkins, and N. P. Strickland, *The sigma orientation is an H_∞ map*, Amer. J. Math. **126** (2004), no. 2, 247–334. MR MR2045503 (2005d:55009)

[And92] M. Ando, *Operations in complex-oriented cohomology theories related to subgroups of formal groups*, Ph.D. thesis, Massacusetts Institute of Technology, 1992.

[And95] ———, *Isogenies of formal group laws and power operations in the cohomology theories E_n*, Duke Math. J. **79** (1995), no. 2, 423–485. MR MR1344767 (97a:55006)

[Ang08] V. Angeltveit, *Topological Hochschild homology and cohomology of A_∞ ring spectra*, Geom. Topol. **12** (2008), no. 2, 987–1032. MR 2403804 (2009e:55012)

[ARV10] J. Adámek, J. Rosický, and E. M. Vitale, *What are sifted colimits?*, Theory Appl. Categ. **23** (2010), No. 13, 251–260. MR 2720191 (2011g:18002)

[BM03] C. Berger and I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831. MR 2016697 (2004i:18015)

[BM06] ———, *The Boardman-Vogt resolution of operads in monoidal model categories*, Topology **45** (2006), no. 5, 807–849. MR 2248514 (2008e:18016)

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986. MR MR836132 (88e:55001)

[Bor94a] F. Borceux, *Handbook of categorical algebra*, vol. 1, Cambridge University Press, 1994.

[Bor94b] ______, *Handbook of categorical algebra*, vol. 2, Cambridge University Press, 1994.

[Bor94c] ______, *Handbook of categorical algebra*, vol. 3, Cambridge University Press, 1994.

[Bou89] A. K. Bousfield, *Homotopy spectral sequences and obstructions*, Israel J. Math. **66** (1989), no. 1-3, 54–104. MR MR1017155 (91a:55027)

[BR12] J. Bergner and C. Rezk, *Reedy categories and the θ-construction*, arXiv preprint arXiv:1110.1066v2 (2012).

[Dub81] E. J. Dubuc, *C^∞-schemes*, Amer. J. Math. **103** (1981), no. 4, 683–690. MR 623133 (83a:58004)

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR MR1417719 (97h:55006)

[GH04] P. Goerss and M. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200. MR MR2125040 (2006b:55010)

[GH05] ______, *Moduli problems for structured ring spectra*, Available at `http://www.math.northwestern.edu/~pgoerss/spectra/obstruct.pdf`, 2005.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR 1650134 (99h:55031)

[HS78] L. Hodgkin and V. Snaith, *The K-theory of some more well-known spaces*, Illinois J. Math **22** (1978), 270–278.

[JN10] N. Johnson and J. Noel, *For complex orientations preserving power operations, p-typicality is atypical*, Topology and its applications **157** (2010), no. 14, pp. 2271–2288, `arXiv:0910.3187`. MR 2670503 (2011g:55009)

[Kel05] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. MR 2177301

[Kuh06] N. J. Kuhn, *Localization of André-Quillen-Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. **201** (2006), no. 2, 318–378. MR MR2211532 (2007d:55006)

[Law63] F. W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872. MR 0158921 (28 #2143)

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR MR2522659

[Lur12] J. Lurie, *Higher algebra*, Available at `http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`, 2012.

[May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, 1972.

[May77] ______, *E_∞-ring spaces and E_∞-ring spectra*, Lecture Notes in Math., vol. 577, Springer-Verlag, 1977.

[MR91] I. Moerdijk and G. E. Reyes, *Models for smooth infinitesimal analysis*, Springer-Verlag, New York, 1991. MR 1083355 (91m:18017)

[Qui67] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. MR MR0223432 (36 #6480)

[Qui69] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR MR0258031 (41 #2678)

[Qui70] ______, *On the (co-) homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87. MR MR0257068 (41 #1722)

[Rez96] C. Rezk, *Spaces of algebra structures and cohomology of operads*, Ph.D. thesis, Massachusetts Institute of Technology, 1996.

[Rez97] ______, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), Contemp. Math., vol. 220, Amer. Math. Soc., 1997, pp. 313–366. MR MR1642902 (2000i:55023)

[RW80] D. C. Ravenel and W. S. Wilson, *The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture*, Amer. J. Math. **102** (1980), no. 4, 691–748. MR MR584466 (81i:55005)

[Sch99] S. Schwede, *Stable homotopical algebra and Γ-spaces*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 2, 329–356. MR MR1670249 (2000b:55005)

[Sch01] S. Schwede, *Stable homotopy of algebraic theories*, Topology **40** (2001), no. 1, 1–41. MR MR1791267 (2002d:55026)

[Sch07] ______, *An untitled book project about symmetric spectra*, Available at `http://www.math.uni-bonn.de/~schwede/SymSpec.pdf`, July 2007.

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