Versal families of pseudoconcave Kähler-Einstein surfaces

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Abstract

G. Kiremidjian proved the existence of versal families of a certain class of Kähler-Einstein manifolds with boundary, whence versal families were shown to be parametrized by Banach analytic sets. In this paper, we will introduce special analytic subsets of finite dimension, in Banach analytic sets parametrizing versal families of a certain class of Kähler-Einstein surfaces with boundary. We show that these analytic subsets give increasing filtrations of versal families. It is expected that these filtrations will define filtrations on the space of CR-structures on the boundary, and later filtrations will measure deviation of a given CR-structure from being embeddable.

Introduction

There are two fundamental questions concerning deformation of compact complex manifolds with boundary; (1) existence of versal families (2) extension of a small deformation of the CR-structure on the boundary to a deformation of the complex structure on the whole complex manifold.

Of course these questions are not independent, because deformations of a complex manifold with boundary induces deformations of the CR-structure on the boundary. Indeed, G. Kiremidjian considered both questions. In [6] he proved the existence of versal families for Kähler-Einstein manifolds satisfying a certain condition (e.g. a large class of locally symmetric spaces). On the other hand, G. Kiremidjian [7] showed that a small deformation of the CR-structure on the boundary of a complex manifold M is always extendable to a deformation of the complex structure on the manifold M, provided that the second cohomology group with compact support $H^2_c(M, TM)$ is trivial, where TM is the holomorphic tangent bundle of M (*Kiremidjian's extension theorem*).

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Recently the extension problem for pseudo-concave complex surfaces was taken up again by C. Epstein and G. Henkin [5], where they proved an extension theorem under a weaker hypothesis. Their motivation does not only come from generalizing Kiremidjian's extension theorem, but also from an attempt to understand embeddable CR-structures on 3-manifolds in an extrinsic way, *i.e.* in terms of complex structures on pseudo-concave complex surfaces whose boundaries are given 3-manifolds.

In this paper, we are concerned with versal families of pseudo-concave surfaces, with intention of a complex-geometric description of the space of CR-structures on 3-manifolds. As we already mentioned, existence of versal families was established for Kähler-Einstein manifolds satisfying some condition, whence versal families are parametrized by Banach analytic sets.

The main result of this paper is to show the existence of natural filtrations on versal families of a certain class of Kähler-Einstein surfaces by finite dimensional analytic subsets. To be precise, let X be a algebraic surface with an isolated singularity P such that the exceptional set of a desingularization map $f : Y \to X$ is an irreducible reduced complex curve C. We suppose that a pseudo-concave complex surface $M := X \setminus V$, where V is a small neighborhood of P in X, carries a Kähler-Einstein metric of negative scalar curvature satisfying a certain condition (see (1.1) in §1).

Under this assumption, we will introduce special analytic sets A_n of finite dimension, indexed by sufficiently large positive integers. Hence the analytic set A_n parametrizes deformations of M, whose deformation tensors have a pole of at most order n at C, and A_n has a distinguished point o corresponding to the reference structure on M. The Zariski tangent space of A_n at o is identified with a subspace of $H^1(Y, TY(nC))$. We will show that the versal family of M has an increasing filtration by analytic sets $\{A_n\}$.

We remark that these filtrations are different from filtrations constructed in [14]. Though later filtrations are merely defined on the formal deformation level, they can be defined for much wider class of pseudo-concave surfaces.

We expect that the filtration constructed in this paper will play an important role in understanding non-embeddable CR-structures on 3-manifolds from the complex-geometric viewpoint. Namely, it is expected that the filtration on the versal family will induce a filtration on the space of CR-structures on the boundary, and the later filtration will measure deviation of a given CR-structure from being embeddable. We hope to discuss this problem in the subsequent paper.

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1 Preliminary

In this section we recall results of G. Kiremidjian [6]. Suppose that S is a subset of a complex Banach space B. Then S is called a *Banach analytic set* if for each $s \in S$ there exists an open neighborhood V_s of s, a Banach space C and an analytic map $f : V_s \to C$ with $S \cap V_s = f^{-1}(0)$.

Let M be a relatively compact complex n-manifold with boundary, embedded in a complex n-manifold U. We suppose that the boundary ∂M of M is a C^∞ real submanifold of U. For simplicity, we assume that U does not have boundary.

For a holomorphic vector bundle E on U, we denote by $\mathcal{H}_k^{p,q}(M, E)$ the completion of $A^{p,q}(M, E)$ in the Sobolev k-norm $| \, |_k$, where $A^{p,q}(M, E)$ is the space of E-valued C^∞ (p, q)-forms on M, which are extendable to a neighborhood of $\bar{M} := M \cup \partial M$.

It is well-known that the set of complex structures on M, which is sufficiently close to the reference structure, is in bijective correspondence with the set of all $\phi \in A^{0,1}(M, TM)$ satisfying the integrability condition $\bar{\partial}\phi - \frac{1}{2}[\phi, \phi] = 0$, where $TM := TU|_M$, and TU is the holomorphic tangent bundle of M (cf. [10]). For $\phi \in A^{0,1}(M, TM)$ satisfying the integrability condition, we denote by M_ϕ the new complex structure on M defined by ϕ.

A *complex analytic family* of M is a triple (S, ψ, s_0), where S is a Banach analyitc set and $\psi : S \to A^{0,1}(M, TM)$ is an analytic map (*i.e.* the composition of ψ with the inclusion $A^{0,1}(M, TM) \subset \mathcal{H}_k^{0,1}(M, TM)$ is analytic) such that for each $s \in S$, $\psi(s)$ represents a complex structure $M_{\psi(s)}$ on M, and $\psi(s_0) = 0$.

A complex analytic family (S, ψ, s_0) is called a *versal family* if for any analytic family (T, φ, t_0) of M, there exists a neighborhood T' of t_0 in T, an analytic map $f : T' \to S$ and a C^∞ family of diffeomorphisms F_t of M for each $t \in T'$ so that $f(t_0) = s_0$, $F_{t_0} = id$ and $F_t^*\psi(t) = \varphi(f(t))$. Here $F_t^*\psi(t)$ is the pull-back of $\psi(t)$ by F_t. It should be remarked that a versal family of M, if exists, is unique up to isomorphism.

Now we suppose that the complex n-manifold U carries a Kähler-Einstein metric g of negative scalar curvature. Recall that a Kähler manifold (U, g) is said to be *Kähler-Einstein* if there exists a constant c such that

$$\mathrm{Ric}(g) = c\omega$$

where ω is the Kähler form on M, and $\mathrm{Ric}(g)$ is the Ricci curvature:

$$\mathrm{Ric}(g) = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}).$$

Now we consider the covariant differential operator $D = \partial_\theta + \bar\partial$ on $A^{0,1}(U, TU)$, where $\partial_\theta = \partial\phi + \theta \wedge \phi$, and θ is the connection form of the metric g.

Since g is Kähler, the following relations hold:

$$(1) \qquad \tilde\Delta = DD^* + D^*D = \Box' + \Box''$$

where $\Box' = \bar\partial\vartheta + \bar\partial\vartheta$, $\Box'' = \partial_\theta\partial_\theta^* + \partial_\theta^*\partial_\theta$ and $\partial_\theta^* = - * \partial_\theta *$.

$$(2) \qquad \tilde A = \Box' - \Box'' = \sqrt{-1}(e(s)\Lambda - \Lambda e(s))$$

where $e(s)\phi = \bar\partial\theta \wedge \phi$, and $\Lambda\phi = *^{-1}(\omega \wedge *\phi)$ with $\omega = \sqrt{-1}\sum_{\alpha,\beta} g_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$.

In addition to the condition that g is Kähler-Einstein of negative scalar curvature, we put the following assumption on g:

There exists a constant $c > 0$ such that for all $\phi \in A^{0,q}(M, TM)$ $(q = 1, 2)$,

$$(\tilde A\phi, \phi) \geq c|\phi|^2. \qquad (1.1)$$

A large class of locally symmetric spaces are examples of Kähler-Einstein manifolds satisfying these conditions (cf. [1]).

The next coercive estimate, obtained by G. Kiremidjian ([6] p.186), played an important role in his proof of the existence of the versal family of M.

Lemma 1.1 *For each $\varphi \in A^{0,q}(M, TM)$ $(q = 1, 2)$, there exists a unique $\psi \in A^{0,q}(M, TM)$ with $*\psi = *D\psi = 0$ on ∂M, $\Box'\psi = \varphi$ and $|\psi|_{k+2} \leq C|\varphi|_k$ for some constant C.*

In the sequel, we will denote ψ by $G\varphi$. Then $\Box' G\varphi = \varphi$. Moreover φ is uniquely written as $\varphi = \varphi_1 + \varphi_2$ satisfying $\varphi_1 = G\Box'\varphi$ and $\Box'\varphi_2 = 0$. Setting $H\varphi := \varphi_2$, we have a decomposition $\varphi = G\Box'\varphi + H\varphi$.

Remark 1.1 The next relations hold:

$$\bar\partial G\varphi = G\bar\partial\varphi + H\bar\partial G\varphi, \qquad \vartheta G\varphi = G\vartheta\varphi + H\vartheta G\varphi$$

However, in general $H\bar\partial G\varphi \neq 0$ and $H\vartheta G\varphi \neq 0$. Namely, G does not necessarilly commute with $\bar\partial$ and ϑ.

By Stokes' theorem, we have

$$(d\varphi, \psi) = (\varphi, d^*\psi) + \int_{\partial M} \varphi \wedge *\psi \quad \text{for} \quad \varphi \in A^{r-1}(TM), \psi \in A^r(TM),$$

where d^* is the formal adjoint of d. Let us denote $\mathcal{B}^r = \{\varphi \in A^r(M, TM) | * \psi = 0\}$. Then for $\psi \in \mathcal{B}^r$,

$$(d\varphi, \psi) = (\varphi, d^*\psi).$$

Thus Lemma 1.1 fits in the theory of harmonic integral for manifolds with boundary, and G. Kiremidjian [6] proved

Kiremidjian's Theorem *Let U be a complex Kähler-Einstein n-manifold with negative scalar curvature. Let M be a relatively compact complex n-manifold with boundary, embedded in U. If the condition (1.1) is satisfied, the versal family of M exists.*

The proof of this theorem is similar to Kuranishi's proof of the existence of versal families of compact complex manifolds *without* boundary. See [6] for details of the proof.

2 Harmonic representative

Suppose that Y is a compact complex surface *without* boundary. Let E be a holomorphic vector bundle of rank r on Y, and $g = \sum g_{\alpha\beta} dz^\alpha d\bar{z}^\beta$ be a hermitian metric on E. Fixing a trivialization of E with respect to a finite open covering $Y = \cup U_j$, we suppose that the transition functions of E are given by $\{f_{ij}\}$. For $\varphi \in A^{0,q}(Y,E)$, the trivialization of φ is written as $\varphi_j = (\varphi_j^1, \cdots, \varphi_j^r)$ on U_j. Then $\varphi_j^\alpha = \sum_\beta f_{jk\beta}^\alpha \varphi_k^\beta$ on $U_j \cap U_k$, where $f_{jk\beta}^\alpha$ is the (α,β)-entry of the matrix f_{jk}.

It is easy to check that

$$\sum_{\alpha\beta} g_{j\alpha\bar{\beta}} \varphi_j^\alpha \wedge *\bar{\psi}_j^\beta = \sum_{\alpha\beta} g_{k\alpha\bar{\beta}} \varphi_k^\alpha \wedge *\bar{\psi}_k^\beta \qquad \text{on} \quad U_j \cap U_k,$$

where $*$-operator is defined by the metric g on Y. Thus the inner product on $A^{0,q}(Y,E)$ is well-defined;

$$(\varphi, \psi) = \int_Y \sum_{\alpha\beta} g_{j\alpha\bar{\beta}} \varphi_j^\alpha \wedge *\bar{\psi}_j^\beta.$$

We define the adjoint operator ϑ_g of $\bar{\partial}$ by

$$(\bar{\partial}\varphi, \psi) = (\varphi, \vartheta_g \psi).$$

It is easy to see that for $\varphi = \{\varphi_j\} \in A^{0,q}(Y,E)$, $\vartheta_g \varphi$ is locally expressed as

$$\vartheta_g \varphi = \{\sum g_j^{\alpha\beta} \vartheta(g_{j\alpha\beta} \varphi_j^\beta)\} \in A^{0,q-1}(Y,E) \qquad (2.1),$$

where $\{g_j^{\alpha\beta}\}$ is the inverse matrix of $g_j = \{g_{j\alpha\beta}\}$.

Now we consider an algebraic surface X with an isolated singularity P, embedded in $\mathbf{CP}^N$. Let us denote by $f: Y \to X$ a desingularization map. We suppose that the exceptional set $f^{-1}(P)$ is an irreducible reduced curve C. Then the self-intersection number of C is negative.

The intersection of X with a small closed ball with the center P gives a small neighborhood V of P in X. For $M := X \setminus V$, let U be an open complex surface in X, such that the closure $\bar{M}$ of M is contained in U.

We may view M and U as submanifolds in Y via the desingularization map f. We assume that the pseudo-concave surface U is equipped with a Kähler-Einstein metric g of negative scalar curvature, which satisfies the condition (1.1).

Now we take a hermitian metric on Y, whose restriction to M coincides with a Kähler-Einstein metric g. This metric is also denoted by g. One can always find such a metric by using a partition of the unity.

Let $\mathcal{O}_Y(nC)$ be the line bundle defined by a divisor nC. Of course, the line bundle $\mathcal{O}_Y(nC)$ restricted to M is trivial, since C does not intersect with M. We fix a trivialization of $\mathcal{O}_Y(nC)$ on M, and we put a hermitian metric h on $\mathcal{O}_Y(C)$ such that $h|_M$ is a standard Euclidean inner product with respect to the fixed trivialization. We denote the tensor product metric $g \otimes h^n$ by $g(n) = \sum g(n)_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$, which is a hermitian metric on the holomorphic vector bundle $TY(nC) := TY \otimes \mathcal{O}_Y(nC)$.

Then $A^{0,q}(Y, TY(nC))$ is given an inner product by setting

$$(\varphi, \psi) = \int_Y \sum_{\alpha\beta} g(n)_{\alpha\bar{\beta}} \varphi^\alpha \wedge *\bar{\psi}^\beta$$

We introduce the adjoint operator $\vartheta_{g(n)}$ of $\bar{\partial}$ with respect to this inner product;

$$(\bar{\partial}\varphi, \psi) = (\varphi, \vartheta_{g(n)}\psi).$$

Let us consider $TY(nC)$-valued harmonic $(0,q)$-forms on Y:

$$\mathbf{H}^q(Y, TY(nC)) = \{\varphi \in A^{0,q}(Y, TY(nC)) | \Box_{g(n)}\varphi = 0\}$$

where $\Box_{g(n)} = \bar{\partial}\vartheta_{g(n)} + \vartheta_{g(n)}\bar{\partial}$. Of course, $\varphi \in \mathbf{H}^q(Y, TY(nC))$ if and only if $\bar{\partial}\varphi = 0$ and $\vartheta_{g(n)}\varphi = 0$.

For $\varphi \in \mathbf{H}^q(Y, TY(nC))$, we denote its restriction to M by $\tilde{\varphi}$. By (2.1), it is easy to see that $\vartheta_{g(n)}|_M$ does not depend on n, *i.e.* $\vartheta_g \equiv \vartheta_{g(n)}|_M$, whence $g = g(1)$. Here we note that $TY(nC)|_M \simeq TY|_M$.

Recall that $\mathbf{H}^q(M, TM) := \{\omega \in A^{0,q}(M, TM) | \Box'\omega = 0\}$, where $\Box'$ is defined in §1. The next lemma is obvious.

Lemma 2.1 *For $\varphi \in \mathbf{H}^q(Y, TY(nC))$, $\tilde{\varphi}$ belongs to $\mathbf{H}^q(M, TM)$.*

Note that since M has a boundary, the set $\{\omega \in A^{0,q}(M, TM) | \bar{\partial}\omega = 0, \vartheta_g\omega = 0\}$ is in general a proper subset of $\mathbf{H}^q(M, TM)$.

3 Filtration of infinitesimal deformations

We use the same notation as in §2; X is an algebraic surface with an isolated singularity P, and $f : Y \to X$ is a desingularization map. We suppose that the exceptional set $f^{-1}(P)$ is an irreducible reduced curve C. Let $M := X \setminus V$, with V a small closed neighborhood of P in X. We let U be an open complex surface in X, with $\bar{M} \subset U$. We assume that the pseudo-concave surface U is equipped with a Kähler-Einstein metric g of negative scalar curvature, which satisfies the condition (1.1) in §1. We sometimes identify M and U with submanifolds of Y via the desingularization map f. For the rest of this paper, we fix these notations.

Now we consider the homomorphism $s_n : H^1(Y, TY(nC)) \to H^1(M, TM)$ induced by restriction. Recall that $TY(nC)$ is a sheaf of meromorphic vector fields on Y, which have a pole of at most order n at C. We need the following lemma.

Lemma 3.1 *The homomorphism s_n is injective for all sufficiently large n.*

Proof. Let us take a hyperplane section H on X, which does not pass through the isolated singularity P. By Nakai-Moishezon's criterion of ampleness, it is easy to check that $mf^*H - C$ is ample for any sufficiently large m. We take a curve D_{nm} on Y, which is linearly equivalent to nmf^*H, and D_{nm} does not intersect with C.

¿From the short exact sequence

$$0 \longrightarrow TY(nC - nmf^*H) \longrightarrow TY(nC) \longrightarrow TY(nC)|_{D_{nm}} \longrightarrow 0,$$

we obtain the long exact sequence of cohomologies

$$\longrightarrow H^1(Y, TY(nC - nmf^*H)) \longrightarrow H^1(Y, TY(nC)) \xrightarrow{r_n} H^1(Y, TY(nC)|_{D_{nm}}) \longrightarrow .$$

Hence we have $H^1(Y, TY(nC - nmf^*H)) = 0$, because $n(mf^*H - C)$ is very ample for any sufficiently large n. Thus the homomorphism r_n is injective. It follows from the commutativity of the following diagram that s_n is also injective.

$$\begin{array}{ccccc} H^1(Y, TY(nC)) & & \overset{r_n}{\longrightarrow} & & H^1(TY|_{D_{nm}}) \\ & s_n \searrow & & \nearrow \text{restriction} & \\ & & H^1(M, TM) & & \end{array}$$

■

Now we fix a section s of $H^0(Y, \mathcal{O}_Y(C))$, which vanishes precisely at C. Let us consider the exact sequence of cohomologies

$$\longrightarrow H^0(TY((n+1)C)|_C) \longrightarrow H^1(Y, TY(nC)) \xrightarrow{t_n} H^1(Y, TY(n+1)C) \longrightarrow,$$

which is induce by

$$0 \longrightarrow TY(nC) \xrightarrow{\otimes s} TY((n+1)C) \longrightarrow TY((n+1)C)|_C \longrightarrow 0.$$

Hence $H^0(TY((n+1)C)|_C)$ vanishes for any sufficiently large n, because the self-intersection number of C is negative. Thus we have

Lemma 3.2 *The homomorphism t_n is injective for any sufficiently large n.*

The next lemma follows easily from the Riemann-Roch formula.

Lemma 3.3 *The dimension of $H^1(TY(nC)|_{nC})$ grows at least quadratically in n.*

Now we can show

Lemma 3.4 *$h^1(Y, TY(nC))$ is at least quadratic in n for all sufficiently large n.*

Proof. ¿From the exact sequence

$$0 \longrightarrow TY \longrightarrow TY(nC) \longrightarrow TY(nC)|_{nC} \longrightarrow 0$$

we obtain the long exact sequence of cohomologies

$$\cdots \longrightarrow H^1(Y, TY(nC)) \xrightarrow{\alpha} H^1(TY(nC)|_{nC}) \xrightarrow{\beta} H^2(Y, TY) \longrightarrow \cdots$$

Hence $h^1(TY(nC)|_{nC})$ =dim Imα+dim Imβ. Notice that the dimension of Imβ is bounded by a constant $h^2(Y, TY)$. We have seen that $h^1(TY(nC)|_{nC})$ is at least quadratic in n for all sufficiently large n. Therefore dim Imα must grow at least quadratically with n, too. In particular, the same is true for $h^1(Y, TY(nC))$. ■

4 Filtrations on versal families

In this section, we will prove the main result of this paper. Namely, we shall define special analytic subsets of finite dimension, in the Banach analytic set which parametrizes the versal family of M. It will be shown that these analytic sets give an increasing filtration on the Banach analytic set.

Recall that $H^1(Y, TY(nC))$ is isomorphic to $TY(nC)$-valued harmonic (0.1)-forms $\mathbf{H}^1(Y, TY(nC))$. Now we fix a sufficiently large n such that the homomorphism s_n is injective (Lemma 3.1), and also fix a basis $\{\beta_\lambda(n)\}_{\lambda=1,\cdots,m}$ of $\mathbf{H}^1(Y, TY(nC))$. For the sake of brevity, we sometimes write $\beta_\lambda(n)$ as β_λ.

For $\beta \in \mathbf{H}^1(Y, TY(nC))$, we denote the restriction of β to M by $\tilde{\beta}$. Then, of course, $\tilde{\beta}$ is extendable to a neighborhood of $\bar{M}$ in U, simply because $\tilde{\beta}$ is obtained by restriction. Now we consider the following differential equation:

$$\phi(t) = \phi_1(t) + \frac{1}{2}\vartheta G[\phi(t), \phi(t)] \qquad (4.1)$$

where $t = (t_1, \cdots, t_m)$, and $\phi_1(t) := \sum_{\lambda=1}^m \tilde{\beta}_\lambda t_\lambda$. We shall construct a formal solution $\phi(t) = \sum_{i=1}^\infty \phi_\mu(t)$ of (4.1) inductively, where ϕ_μ is the homogeneous component of degree μ in t. We define $\phi_\mu(t)$ $(\mu \geq 2)$ inductively by

$$\phi_\mu(t) = \frac{1}{2}\vartheta G(\sum_{i=1}^{\mu}[\phi_i(t), \phi_{\mu+1-i}(t)]) \qquad (4.2)$$

Then it is easy to see that a formal series $\phi(t)$ thus obtained satisfies the differential equation (4.1).

Now we proceed to show the convergence of the formal solution $\phi(t)$. The proof is fairly standard. First, we introduce some notations. Let $P(t) = \sum P_{\nu_1\cdots\nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m}$ and $Q(t) = \sum Q_{\nu_1\cdots\nu_m} t_1^{\nu_1} \cdots t_m^{\nu_m}$ be power series in $t = (t_1, \cdots, t_m)$ with complex coefficients. We write $P(t) << Q(t)$ if $|P_{\nu_1\cdots\nu_m}| \leq |Q_{\nu_1\cdots\nu_m}|$ holds for any $(\nu_1, \cdots, \nu_m)$. Now let us consider a power series

$$A(t) = \frac{b}{16c}\sum_{i=1}^{\infty}\frac{c^i}{i^2}(t_1 + \cdots + t_m)^i.$$

Here $b, c > 0$ are some constants, which will be determined in the course of the proof of Lemma 4.2 below. Notice that the power series $A(t)$ converges absolutely in the domain $|t| < \frac{1}{mc}$. The following lemma is well-known (e.g. [8] p.279).

Lemma 4.1 $A(t)^2 << \frac{b}{c}A(t)$.

Let us denote by $|\,|_k$ the Sobolev k-norm, and set $a_\mu(t) := \sum_{i=1}^{\mu}\phi_i(t)$. We fix a sufficiently large k. Now we have

Lemma 4.2 $|\phi(t)|_k << A(t)$.

Proof. It is sufficient to show that the following inequality holds for any $\mu \geq 1$.

$$|a_\mu(t)|_k << A(t) \qquad (4.3)$$

We prove (4.3) by induction in μ. First, if we choose b in the definition of $A(t)$ in such a way that $b \geq 16\max_\lambda|\beta_\lambda|_k$ holds, then it is easy to see that (4.3) is true for $\mu = 1$. Now assuming $(4.3)_\mu$, we shall verify $(4.3)_{\mu+1}$. Hence we have

$$\begin{aligned} |\phi_{\mu+1}(t)|_k &<< \frac{1}{2}|\vartheta G[a_\mu(t), a_\mu(t)]|_k \\ &<< \frac{1}{2}C_1|G[a_\mu(t), a_\mu(t)]|_{k+1} \end{aligned}$$

$$
\begin{aligned}
&<< \frac{1}{2}C_1C_2|[a_\mu(t),a_\mu(t)]|_{k-1}\\
&<< C_1C_2C_3|a_\mu(t)|_k|a_\mu(t)|_k\\
&<< C_1C_2C_3A(t)^2\\
&<< C_1C_2C_3\frac{b}{c}A(t),
\end{aligned}
$$

where C_i $(i=1,2,3)$ are positive constants. Here we utilized Lemma 1.1 in the third inequality. In the fourth inequality, we used the fact that there is a constant $C>0$ such that $|uv|_k < C|u|_k|v|_k$, which follows from Sobolev's lemma (e.g. [9] p.146).

Note that C_i $(i=1,2,3)$ do not depend on μ. Thus in the definition of $A(t)$, if c is taken to be sufficiently large such that $\frac{C_1C_2C_3b}{c}<1$, then we have $|\phi_{\mu+1}(t)|_k << A(t)$. Consequently $|a_{\mu+1}(t)|_k << A(t)$. This completes the induction step. ■

Therefore $\phi(t)$ converges for sufficiently small t, because $A(t)$ converges absolutely for $|t|<\frac{1}{mc}$.

Next we shall show that the solution is C^∞-differentiable. Applying ϑ-operator to the both side of (4.1), we see that

$$\vartheta\phi(t)=0 \quad (4.4).$$

Similarly we apply the $\bar\partial$-operator to the both side of (4.1), and obtain

$$
\begin{aligned}
\bar\partial\phi(t) &= \frac{1}{2}\bar\partial\vartheta G[\phi(t),\phi(t)]\\
&= \frac{1}{2}(-\vartheta\bar\partial G[\phi(t),\phi(t)]+[\phi(t),\phi(t)]) \qquad (4.5)
\end{aligned}
$$

where we used the relation $[\phi,\phi]=\Box' G[\phi,\phi]$ (see §1). We apply the ϑ-operator to (4.5), and obtain

$$\vartheta\bar\partial\phi(t)=\frac{1}{2}\vartheta[\phi(t),\phi(t)]. \qquad (4.6)$$

Together with (4.4), we see

$$\Box'\phi(t)-\frac{1}{2}\vartheta[\phi(t),\phi(t)]=0.$$

Therefore the regularity theorem of elliptic PDE of the second order implies that $\phi(t)$ is C^∞-differentiable.

Now we turn to the question when the solution of (4.1) satisfies the integrability condition of the complex structure. We apply $\bar\partial$-operator to the both side of (4.1), and obtain

$$
\begin{aligned}
\bar\partial\phi &= \frac{1}{2}\bar\partial\vartheta G[\phi,\phi]\\
&= \frac{1}{2}(\Box' G[\phi,\phi]-\vartheta\bar\partial G[\phi,\phi])\\
&= \frac{1}{2}([\phi,\phi]-\vartheta\bar\partial G[\phi,\phi]).
\end{aligned}
$$

Thus we have

Lemma 4.3 *The formal solution* $\phi(t)$ *of* (4.1) *satisfies*

$$\bar{\partial}\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] = 0 \qquad (4.7)$$

if and only if $\vartheta\bar{\partial}G[\phi(t), \phi(t)] = 0 \quad (4.8)$.

Remark 4.1 In case M is a compact complex manifold *without* boundary, the equation (4.8) for sufficiently small t is equivalent to the equation $H[\phi(t), \phi(t)] = 0$ (see Kuranishi [9]). However this is not the case for a compact complex manifold M with *boundary*, because the Green operator G does not commute with $\bar{\partial}$ and ϑ (see Remark 1.1).

Now we can construct a filtration on the Banach analytic set, which parametrizes the versal family of M. Let S_n be the set of solutions of (4.1) satisfying (4.8). Then the set S_n parametrizes deformations of M, whose deformation tensors have a pole of at most order n at C.

Since $\vartheta\bar{\partial}G[\phi(t), \phi(t)])$ is an analytic function in $t = (t_1, \cdots, t_m)$, the set S_n is a germ of an analytic set in $\mathbf{C}^m$ at the origin. Remember that $m = m(n)$, the dimension of $\mathbf{H}^1(Y, TY(nC))$, grows at least quadratically in n (Lemma 3.4). We consider the following commutative diagram.

$$\begin{tikzcd} H^1(Y, TY(nC)) \arrow[rr, "t_n"] \arrow[dr, "s_n"'] & & H^1(Y, TY((n+1)C))) \arrow[dl, "s_{n+1}"] \\ & H^1(M, TM) & \end{tikzcd}$$

For any sufficiently large n, the homomorphisms s_n, s_{n+1} and t_n are injective by Lemma 3.1 and Lemma 3.2. Thus we can take a basis $\{\beta_\lambda(n)\}_{\lambda=1,\cdots,m(n)}$ of $\mathbf{H}^1(Y, TY(nC))$, and $\{\beta_\lambda(n+1)\}_{\lambda=1,\cdots,m(n+1)}$ of $\mathbf{H}^1(Y, TY((n+1)C))$ such that the image of $\{\beta_\lambda(n)\}_{\lambda=1,\cdots,m(n)}$ by the homomorphism t_n constitutes a subset of the basis $\{\beta_\lambda(n+1)\}_{\lambda=1,\cdots,m(n+1)}$. Therefore there is a natural inclusion $S_n \subset S_{n+1}$. Now let B be a Banach analytic set, which parametrizes the versal family of M. Since each S_n parametrizes an analytic family of M, it follows from the versality that there is an analytic map $h_n : S_n \to B$, whose pull back is nothing other than the given analytic family on S_n. Writing the image of h_n as A_n, we obtain the main result;

Theorem 4.1 *There is an increasing filtration of the Banach analytic set* B *by finite dimensional analytic subsets* A_n*, indexed by sufficiently large positive integers. Hence* A_n *parametrizes deformations of* M*, whose deformation tensors have a pole of at most order* n *at* C*.*

Finally, it is natural to ask the following question; given an algebraic surface with an isolated singularity, does the complement of a small neighborhood of the isolated singularity carry a Kähler-Einstein metric of negative scalar curvature satifying (1.1)?

For example, let L be a negative line bundle over a complex curve. By adding a complex curve at infinity, we can compactify L to obtain an algebraic surface Y. Contracting a negative curve on Y corresponding to the zero-section of L, we get an algebraic surface with an isolated singularity. This particular case is very interesting, because if the above question is answered affirmatively for this case, then our results may be applied to a complex-geometric description of the space of CR-structures on circle bundles over complex curves (cf. [4]).

References

[1] Andreotti, A. and Vessentini, E. *On deformations of discontinuous groups* Acta Math. 112 (1964) 249-298

[2] Calabi, E. and Vessentini, E. *On compact locally symmetric Kähler manifolds* Ann. of Math. 71 (1960) 472-507

[3] Epstein, C. *A relative index for embeddable CR-structures, I, II* , I.H.E.S. preprints (1995)

[4] Epstein, C. CR-structures on three dimensional circle bundles, Inv. Math. 109 (1992) 351-403

[5] Epstein, C. and Henkin, G. *Extension of CR-structures for 3-dimensional pseudoconcave manifolds,* Contemporary Math. 205 (1997) 51-67

[6] Kiremidjian, G. *On deformations of complex compact manifolds with boundary,* Pacific J. Math. 54 No. 1 (1974) 177-190

[7] Kiremidjian, G. *A direct extension method for CR-structures,* Math. Ann. 242 (1979) 1-19

[8] Kodaira, K. *Complex manifolds and deformation of complex structures,* Springer (1985)

[9] Kuranishi, M. *New proof for existence of locally complete families of complex structures,* Proc. of the conference on complex analysis (Minneapolis), Springer (1965) 142-154

[10] Kuranishi, M. Deformations of complex analytic structures on compact manifolds, Montreal (1969)

[11] Lempert, L. *On three dimensional Cauchy-Riemann manifolds,* Jour. A.M.S. (1992) 1-50

[12] Lempert, L. *Embeddings of three dimensional Cauchy-Riemann manifolds,* Math. Ann. 300 (1994) 1-15

[13] Takamura, S. *Infinitesimal deformations of complex surfaces with boundary,* Math. Ann. 306 (1996) 195-204

[14] Takamura, S. *Laurent deformations of algebraic surfaces with an isolated singularity,* Preprint (1997)

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