COLLAPSING RIEMANNIAN MANIFOLDS TO ONES OF LOWER DIMENSION II

by

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§ 0 Introduction

The purpose of this paper is to investigate the phenomena that a sequence of Riemannian manifolds M_i converges to one lower dimension, N, with respect to the Hausdorff distance, which is introduced in [11]. We have studied this phenomena in [7] and proved there that M_i is a fibre bundle over N with infranilmanifold fibre. In this paper, we study which fibre bundle is it, and give a necessary and sufficient condition, which are stated as Theorems 0-1 and 0-7.

Theorem 0-1 Let M_i be a sequence of $n+m$-dimensional compact Riemannian manifolds and N be an n-dimensional compact Riemannian manifold. Assume

(0-2-1) M_i converges to N with respect to the Hausdorff distance,

(0-2-2) $|\text{sectional curvature of } M_i| \leq 1$.

Then, for sufficiently large i, there exists a map $\pi_i : M_i \longrightarrow N$ such that the following holds.

(0-3-1) π_i is a fibre bundle.

(0-3-2) $\pi_i^{-1}(p) = G/\Gamma$, where G is a nilpotent Lie group and Γ is a discrete group of affine transformations of G satisfying $[\Gamma : G \cap \Gamma] < \infty$. Here we put the (unique) connection on G which makes all right invariant vector field parallel, and G is regarded to be a group of affine transformations on G by right multiplication.

(0-3-3) <u>The structure group of</u> π_i <u>is contained in the skew product of</u> $C(G)/(C(G) \cap \Gamma)$ <u>and</u> Aut Γ, <u>where</u> $C(G)$ <u>denotes the center of</u> G.

<u>Remark 0-4</u> Statements (0-3-1) and (0-3-2) were proved in [7].

<u>Remark 0-5</u> [7, 0-1-3] also holds. Namely π_i is an almost Riemannian submersion in the sence stated there.

<u>Remark 0-6</u> It is well known that the group $\pi_k(\mathrm{Diff}(G/\Gamma))$ is not finitely generated in general, but $\pi_k(C(G)/(C(G) \cap \Gamma) \ltimes \mathrm{Aut}\ \Gamma)$ is always finitely generated. Therefore, there exist a lot of fibre bundles which satisfy (0-3-1) (0-3-2) but do not satisfy (0-3-3).

<u>Theorem 0-7</u> <u>Let</u> M <u>be an</u> $n+m$-<u>dimensional manifold</u>, N <u>an</u> n-<u>dimensional complete Riemannian manifold with bounded sectional curvature, and</u> $\pi : M \longrightarrow N$ <u>be a smooth map. Suppose that</u> π <u>satisfies (0-3-1), (0-3-2) and (0-3-3). Then, there exists a family of Riemannian metrics</u> g_ε <u>on</u> M <u>such that the following holds.</u>

(0-8-1) <u>The sequence of Riemannian manifolds</u> (M, g_ε) <u>converges to the Riemannian manifold</u> N, <u>with respect to the Hausdorff distance.</u>

(0-8-2) <u>The exists a constant</u> C <u>independent of</u> ε <u>such that</u>

$$|\text{sectional curvature of } (M, g_\varepsilon)| \leq C.$$

Theorems 0-1 and 0-7, combined with [9, Theorem 0-6], imply the following:

Theorem 0-9 For each m and D, there exists a positive constant $\varepsilon(n,D)$ such that the following holds. Suppose an m-dimensional Riemannian manifold M satisfies

(0-10-1) Volume of $M \le \varepsilon(m,D)$,

(0-10-2) Diameter of $M \le D$,

(0-10-3) $|$sectional curvature of $M| \le 1$,

(0-10-4) $\pi_k(M) = 1$, for $k \ge 2$.

Then, Minvol $M = 0$, where Minvol M is defined in [10] .

Theorem 0-9 is a partial answer to the following

Problem 0-11 Does there exists ε_m such that Minvol $M \le \varepsilon_m$ implies Minvol $M = 0$?

If we can remove the condition (0-10-2) and (0-10-4), we will have the affirmative answer.

The organization of this paper is as follows. § 1,..., § 5 is devoted to the proof of Theorem 0-1. The outline of these sections is in § 1. In the course of the proof, we shall prove some results on eigenfunctions of Laplace operator, which improve one of [6]. These results may have an independent interest. In § 6, we shall prove Theorem 0-7. The proof of Theorem 0-9 is in § 7. In § 8, we add some remarks concerning the case when the limit space is not a manifold.

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Notation

For a Riemannian manifold M, Vol M dentoes the volume of M, Diam M denotes the diameter of M. For a metric space X and $x \in X$ we put

$$B_D(x,X) = \{y \in X \mid d(x,y) < D\} .$$

$B(C)$ stands for $B_C(0,\mathbb{R}^n)$. For two metric spaces X,Y, $d_H(X,Y)$ denotes the Hausdorff distance between them which is defined in [11], $\lim_{i\to\infty} {}_H X_i = X$ means $\lim_{i\to\infty} d_H(X,X_i) = 0$.

§ 1 Outline of the proof

Our main Theorem 0-1 is a consequence of the following:

Theorem 1-1 Let M_i and N be as in Theorem 0-1. Then, for each sufficiently large i, there exists a fibration $\pi_i : M_i \longrightarrow N$ such that the following holds.

(1-2-1) For each $p \in N$, there exists a flat connection on $\pi_i^{-1}(p)$, which depends smoothly on p.

(1-2-2) There exists a nilpotent Lie group G and a group of affine transformations Γ of G such that $\pi_i^{-1}(p)$ is affinely diffeomorphic to G/Γ and that $[\Gamma : \Gamma \cap G] < \infty$.

Theorem 1-1 is a generalization of Ruh's result [14], which corresponds to the case when N is a point.

Theorem 0-1 is a corollary of Theorem 1-1. In fact, let $\pi_i : M_i \longrightarrow N$ be as in Theorem 1-1. Then, by (1-2-1) and (1-2-2), we can find $(U_j, \psi_{i,j})$ such that

(1-3-1) U_j, $j = 1,2,\ldots$ is an open covering of N.

(1-3-2) $\psi_{i,j}$ is a diffeomorphism between $\pi_i^{-1}(U_j)$ and $U_i \times G/\Gamma$.

(1-3-3) the restriction of $\psi_{i,j}$ to each fibre gives an affine diffeomorphism between $\pi_i^{-1}(p)$ and $\{p\} \times G/\Gamma$.

By (1-3-3), the transition function of π_i with respect to the chart $(U_j, \psi_{i,j})$ is contained $\mathrm{Aff}(G/\Gamma)$, the group of affine diffeomorphism of G/Γ. On the other hand, we have the following:

<u>Lemma 1-4</u> <u>There exists an exact sequence</u>

$$1 \longrightarrow G/_{\Gamma \cap C(G)} \longrightarrow \mathrm{Aff}(G/\Gamma) \longrightarrow \mathrm{Aut}\Gamma \longrightarrow 1$$

Here $C(G)$ denotes the center of G.

We omit the proof, which is straightforward. Let $\mathrm{Aff}'(G/\Gamma)$ be the subgroup of $\mathrm{Aff}(G/\Gamma)$ generated by $C(G)/\Gamma \cap C(G)$ and $\mathrm{Aut}\,\Gamma$. Then we have $\mathrm{Aff}(G/\Gamma)/\mathrm{Aff}'(G/\Gamma) \cong \mathbb{R}^k$. Therefore the structure group of the $\mathrm{Aff}(G/\Gamma)$ bundle $\pi_i : M_i \longrightarrow N$ can be reduced to $\mathrm{Aff}'(G/\Gamma)$. And $\mathrm{Aff}'(G/\Gamma)$ is an extension of $C(G)/\Gamma \cap C(G)$ by $\mathrm{Aut}\,\Gamma$. This implies Theorem 0-1.

The proof of Theorem 1-1 occupies §§ 2,3,4 and 5. Since it is long, we shall give an outline first. The proof uses a parametrized version of Ruh's argument in [14]. To apply it, we have to improve the result of [7] and to prove that the fibres of the fibre bundles $f_i : M_i \longrightarrow N$ obtained there are almost flat. ([7,0-1-2] implies that fibres are diffeomorphic to almost flat manifolds. But, in [7], we did not obtain the estimate of the curvatures of the fibres.) Namely we shall prove Lemma 1-6 below. As will be remarked at the beginning of § 5, we can assume, without loss of generality, that

(1-5) $$|\nabla^k R(M_i)| < C_k .$$

Here $R(M_i)$ is a curvature tensor, $|\;|$ the C^0-norm, and C_k a constant independent of i. For $x \in M_i$, we let $\exp_{x,r} : B(r) \longrightarrow M_i$ denote the exponential map at x. We fix a coordinate system $(U_j, \psi_j) : U_j \subseteq \mathbb{R}^m$, $\psi_j : U_j \longrightarrow N$.

Lemma 1-6 *Let* M_i *and* N *be as in Theorem 0-1. Assume that* M_i *satisfies (1-5). Then, for sufficiently large* i, *there exists a fibration* $\pi_i : M_i \longrightarrow N$ *such that*

$$(1\text{-}7) \qquad \left| \frac{\partial^{|\alpha|}(\psi_j \circ \pi_i \circ \exp_{x,r})}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right| \leq C_\alpha$$

holds for each multiindex α. *Here* C_α *denotes a constant independent of* i.

(1 - 7) implies that the sectional curvatures of the fibres of π_i are uniformly bounded. Hence, the fibres are almost flat for sufficiently large i. Therefore, [14] shows that there exists a flat connection on each fibre satisfying (1-2-2). A little more argument is required to obtain a connection on $\pi_i^{-1}(p)$ depending smoothly on p. This is done in § 5.

The proof of Lemma 1-6 is performed in §§ 2,3 and 4. Recall that in [7] we used embeddings $M_i, N \hookrightarrow \mathbb{R}^z$ in order to construct the fibration $M_i \longrightarrow N$. The embeddings there were constructed by making use of the distance function from a point. To obtain an embedding satisfying (1-7), we have to approximate this embedding by one with bounded higher derivatives. The approximation we used in [7] is not appropriate for this purpose, because it is not of C^2-class. In this paper, we use another embedding constructed by making use of eigenfunctions of Laplace operator. This embedding is appropriate for our purpose since eigenfunctions enjoy uniform estimate of higher derivatives. In order to apply the argument of [7, §§1,2] to our embedding, we need to study the convergence of eigenfunctions. In [6], we introduce a notion, measured Hausdorff topology and proved that the k-th eigenvalue of the Laplace operator on M_i converges to that of the operator $P_{(N,\mu)}$ defined in [6, §0], if M_i converges to (N,μ) with respect to the measured Hausdorff topology. We also proved a "L^2-convergence" there. But, for our purpose, L^2-convergence is not suffice. We have to prove a "C^1-convergence". (Precise statement will be given as Theorem 3-1.) For this purpose, we shall begin with proving that eigenfunctions of $P_{(N,\mu)}$ are smooth. [6, Theorem 0.6] implies that the measure μ is a multiple of the volume element Ω_N by a continuous function χ_N. If χ_N is of C^1-class, our operator $P_{(N,\mu)}$ is written as

$$(1.8) \qquad P_{(N,\mu)}\varphi = \Delta_N\varphi - \langle d\varphi, d\chi_N\rangle/\chi_N \ .$$

Therefore, to prove that the eigenfunctions of $P_{(N,\mu)}$ are smooth, it suffices to show that χ_N is smooth. This is done in § 2. In § 3, we shall prove the "C^1-convergence". The proof of Lemma 1-6 is completed in § 4.

<u>Remark</u> In 1984, S. Gallot proposed to embedd Riemannian manifolds using heat kernels, in order to study Hausdorff convergence. The embedding we use in this paper is essentially the same as Gallot's.

§ 2 Smoothing density functions

Lemma 2-1 Let M_i be a sequence of n+m-dimensional compact Riemannian manifolds satisfying (0-2-2) and (1-5), and X be a metric space, μ a provability measure on it. Suppose M_i converges to (X,μ) with respect to the measured Hausdorff topology defined in [6, 0.2 B]. Then there exists a function χ_X on X such that

(2-2-1) $\mu = \chi_X X$ (the volume element of X),

(2-2-2) χ_X is of C^∞-class,

(2-2-3) χ_X satisfies [6, 0.7.1 and 0.7.3].

Proof In [6, 0.6], we have already proved (2-2-1) and (2-2-3). By the argument in [6, §3], it suffices to show (2-2-2) in the case when X is a compact Riemannian manifold N. Put $V_i = \mathrm{Vol}\, M_i$, $\mu_{M_i} = \Omega_{M_i}/V_i$, where Ω_{M_i} denotes the volume element of M_i. By the definition of measured Hausdorff topology, we can take ε_i-Hausdorff approximation $f_i: M_i \longrightarrow N$ such that $(f_i^\wedge)_*^\wedge(\mu_{M_i^\wedge})$ converges to μ with respect to the weak* topology. (Here $\varepsilon_i \longrightarrow 0$. The definition of the Hausdorff approximation is in [11].) In view of [7], we may assume that f_i is a fibration. Then, by [6, §3], the functions $p \longmapsto \mathrm{Vol}(f_i^{-1}(p))/V_i$ $i = 1,2,\ldots$ on N converge, with respect to the C^0-norm, to a continuous function χ_N satisfying (2-2-1) and (2-2-3). We shall prove that χ_N is of C^∞-class. Choose (not necessarily continuous) section ψ_i:

$N \longrightarrow M_i$ to f_i. Take an arbitrary point p_0 of N and put $p_i = \psi_i(p_0)$. We shall prove that χ_N is of C^∞-class at p_0. Put $B = B(1)$. Let $\mathrm{Exp}_i : B \longrightarrow M_i$ be the composition of an origin preserving isometry $B \longrightarrow T_{p_i}(M_i)$ and the exponential map $T_{p_i}(M_i) \to M_i$. Let g_i denote the Riemannian metric on B induced by Exp_i from the metric on M_i. In view of (1-5), we may assume, by taking a subsequence if necessary, that g_i converges to a metric g_0 with respect to the C^∞-topology. Now, recall the argument in [8, §3], where we constructed a sequence of local groups G_i converging to a Lie group germ G, such that

(2-3-1) G_i acts by isometry on the pointed metric space $((B,g_0),0)$,

(2-3-2) $((B,g_i),0)/G_i$ is isometric to a neighorhood of p_i in M_i,

(2-3-3) G acts by isometry on $((B,g_0),0)$,

(2-3-4) $((B,g_0),0)/G$ is isometric to a neighborhood of p_0 in N.

Let $p_i : (B,g_i) \longrightarrow M_i$, $p : (B,g_0) \longrightarrow N$ denote natural projections. (In fact, $p_i = \mathrm{Exp}_i$.) In our case, since N is a manifold, the action of G on B is free. Let $\mathfrak{g}$ denote the Lie algebra of G. Choose a basis $X_1, \ldots, X_m$ of $\mathfrak{g}$. We can regard X_i as a Killing vector field on (B,g_0). For $x \in B$, we put

$$(2\text{-}4) \qquad \tilde{\chi}(x) = |X_1(x) \wedge \ldots \wedge X_m(x)|.$$

Since X_i, $i = 1,\ldots,m$ are G-invariant, there exists a function χ on a neighborhood of p_0 such that $\chi \circ p = \tilde{\chi}$. Clearly χ is of C^∞-class. Hence, to prove Lemma 2-1, it suffices to show the following:

<u>Lemma 2-5</u> χ_N/χ <u>is a constant function on a neighborhood of</u> p_0.

<u>Proof</u> Put

(2-6-1) $G_i' = \{\gamma \in G_i \mid d_{(G,g_i)}(\gamma(0),0) < \frac{1}{2}\}$

(2-6-2) $G' = \{\gamma \in G \mid d_{(G,g_0)}(\gamma(0),0) < \frac{1}{2}\}$.

There exists a neighborhood U of p_0 in N and a C^∞-map $s:U \longrightarrow B$ such that

(2-7-1) $s(p_0) = 0$,

(2-7-2) $P \circ s = \text{identity}$,

(2-7-3) $d_{(B,g_0)}(s(q),0) = d_N(q,p_0)$ holds for $q \in N$.

Put

(2-8-1) $E_i(q,\delta) = \{x \in B \mid$ there exists $\gamma \in G_i'$ such that

$$d_{(B,g_i)}(x,\gamma s(q)) < \delta\} ,$$

(2-8-2) $E_0(q,\delta) = \{x \in B \mid$ there exists $\gamma \in G'$ such that

$$d_{(G,g_0)}(x,\gamma s(q)) < \delta\} .$$

<u>Sublemma 2-9</u> <u>There exists a positive number</u> C <u>independent of</u> q <u>such that</u>

$$\lim_{\delta\to 0}\ \lim_{i\to\delta} \left| \frac{\mathrm{Vol}(E_i(q,\delta))}{\#G_i' \cdot \delta^n \cdot \mathrm{Vol}(f_i^{-1}(q))} - C \right| = 0 \quad .$$

The proof of the sublemma will be given at the end of this section. Next we see that

$$(2\text{-}10)\qquad \lim_{i\to\infty}\ \sup_{q\in U}\left|\frac{\mathrm{Vol}(E_i(q,\delta))}{\mathrm{Vol}(E_0(q,\delta))}-1\right| = 0$$

holds for each $\delta > 0$. Thirdly, we put

$$G'(q) = \{\gamma s(q) \mid \gamma \in G'\}.$$

Then, clearly we have

$$(2\text{-}11)\qquad \lim_{\delta\to 0}\ \mathrm{Vol}\ ((E_0(q,\delta))/\delta^n = W_n \mathrm{Vol}(G'(q)) \ ,$$

$$(2\text{-}12)\qquad \frac{\mathrm{Vol}(G'(q)}{\chi(q)} = \frac{\mathrm{Vol}(G'(q'))}{\chi(q')} \ ,$$

for $q,q'\in U$, Here $n = \dim N$, $W_n = \mathrm{Vol}\ B^n(1)$.

(2-11) and (2-12) imply

$$(2\text{-}13)\qquad \lim_{\delta\to 0}\ \frac{\mathrm{Vol}(E_0(q,\delta)).\chi(q')}{\mathrm{Vol}(E_0(q',\delta)).\chi(q)} = 1 \ .$$

From Sublemma 2-9, Formula (2-10), (2-13), we conclude

$$\lim_{i\to\infty}\ \frac{\mathrm{Vol}(f_i^{-1}(q))\chi(q')}{\mathrm{Vol}(f_i^{-1}(q'))\chi(q)} = 1 \ .$$

On the other hand, we have

$$\lim_{i\to\infty}\ \sup_{q,q'\in N}\left|\frac{\mathrm{Vol}(f_i^{-1}(q).\chi_N(q')}{\mathrm{Vol}(f_i^{-1}(q'))\chi_N(q)}-1\right| = 0 \ .$$

Therefore,

$$\frac{\chi_N(q)\chi(q')}{\chi_N(q')\chi(q)} = 1 .$$

This implies Lemma 2-5.

Proof of Sublemma 2-9 Put $s_i = P_i \circ s : U \longrightarrow M_i$. Choose an open subset $V_i(\delta)$ of B such that the following holds.

(2-14-1) If $\gamma \in G_i'$, $\gamma \neq 1$, then $\gamma V_i(\delta) \cap V_i(\delta) = \phi$.

(2-14-2) $P_i(V_i(\delta))$ is a dense subset of $B_\delta(s_i(q), M_i)$.

(2-14-3) $V_i(\delta) \subset B_\delta(s(q), (B, g_i))$ and if $x \in V_i(\delta)$, $\gamma \in G_i'$, then $d(\gamma(x), s(q)) \geq d(x, s(q))$.

Put $E_i'(q,\delta) = \{\gamma(x) \mid \gamma \in G_i' , x \in V_i(\delta)\}$. Then, by the definition of $V_i(\delta)$ and $E_i(q,\delta)$, we have $\overline{E_i'(q,\delta)} = \overline{E_i(q,\delta)}$. Hence, by (2-14-1), we have

(2-16) $$\mathrm{Vol}(V_i(\delta)) = \frac{\mathrm{Vol}(E_i(q,\delta))}{\#G_i'} .$$

On the other hand, put

$$c_i = \sup_{p \in U} d(s_i(p), p_i)$$

$$d_i = \sup_{p \in U} \mathrm{Diam}\, f_i^{-1}(p) .$$

Then, $\lim_{i\to\infty} c_i = \lim_{i\to\infty} d_i = 0$. It is easy to see

$$(2\text{-}17)\qquad f_i^{-1}(B_{\delta-d_i-c_i}(q,\ N)) \subset B_\delta(s_i(q),\ M_i) \subset f_i^{-1}(B_{\delta+d_i+c_i}(q,\ N))\ ,$$

(2-15, (2-16), and (2-17) imply

$$(2\text{-}18)\qquad \lim_{i\to\infty} \frac{\#G_i' \cdot \int_{p\in B_\delta(q,N)} \mathrm{Vol}(f_i^{-1}(p))\cdot\Omega_N}{\mathrm{Vol}(E_i(q,\delta))} = 1$$

where Ω_N is the volume element of N . Since the family of functions $p \longmapsto \log(\mathrm{Vol}(f_i^{-1}(p)))$ $i = 1,2,\ldots$ is equicontinuous ([6, Lemma 3.2]), it follows that

$$(2\text{-}19)\qquad \lim_{\delta\to 0}\ \sup_{i=1,2,\ldots} \left| \frac{\int_{p\in B_\delta(q,N)} \mathrm{Vol}(f_i^{-1}(p))\cdot\Omega_N}{\delta^n w_n\ \mathrm{Vol}(f_i^{-1}(q))} - 1 \right| = 0\ .$$

The sublemma follows immediately from (2-18) and (2-19).

Q.E.D.

§ 3 C^1-convergence of eigenfunctions

Theorem 3-1 Let M_i and (X,μ) be as in Lemma 2-1. Then, there exists smooth maps $f_i : M_i \longrightarrow X$ such that the following holds.

(3-2-1) f_i satisfies [17, (0-1-1), (0-1-2), (0-1-3)] .

(3-2-2) $(f_i)_*(\mu_{M_i})$ converges to μ with respect to the weak* topology, where $\mu_{M_i} = \Omega_{M_i}/\mathrm{Vol}(M_i)$.

(3-2-3) Let $\varphi_{i,k}$ be a k-th eigenfunction of the Laplace operator on M_i satisfying $\sup_{x\in M_i}|\varphi_{i,k}(x)| = 1$. Then there exist functions $\varphi'_{i,k}$ on X such that

(a) $\varphi'_{i,k}$ is a k-th eigenfunction of $P_{(X,\mu)}$,

(b) for each $p_i \in M_i$, we have

$$|\varphi_{i,k}(P_i) - \varphi'_{i,k}(f_i(P_i))| < \varepsilon_i(k) ,$$

(c) for each vector $V_i \in T(M_i)$, we have

$$|V_i(\varphi_{i,k}) - (f_i)_*(V_i)(\varphi'_{i,k})| < \varepsilon_i(k)\cdot|V_i| ,$$

where $\varepsilon_i(k)$ denotes positive numbers depending only on i and k and satisfying $\lim_{i\to\infty} \varepsilon_i(k) = 0$.

Remark In the case when X is a manifold, (3-2-1) means that f_i is a fibration with infranilmanifold fibre.

First, we shall prove C^0-convergence, (b) . We begin with the following Ascoli-Alzera type Lemma.

<u>Lemma 3-3</u> <u>Let</u> X_i <u>and</u> X <u>be compact metric spaces</u>, $\psi_i : X_i \longrightarrow X$ ε_i-<u>Hausdorff approximation</u>, $\lim \varepsilon_i = 0$, <u>and</u> φ_i <u>be continuous functions on</u> X_i . <u>Assume</u>

(3-4-1) φ_i , <u>$i = 1,2,3\ldots$ are uniformly bounded</u>

(3-4-2) φ_i , <u>$i = 1,2,3\ldots$ are equi-uniformly continuous. Namely</u>, <u>for each</u> $\varepsilon > 0$, <u>there exists</u> $\delta > 0$ <u>independent of</u> i,x <u>and</u> y <u>such that</u> $d(x,y) < \delta$, $x,y \in X_i$ <u>implies</u> $|\varphi_i(x) - \varphi_i(y)| < \varepsilon$.

<u>Then, there exists a subsequence</u> i_j <u>and a continuous function</u> φ <u>on</u> X <u>such that</u>

$$\lim_{j\to\infty} \sup_{x\in X} |\varphi(x) - \varphi_{i_j} \circ \psi_{i_j}(x)| = 0 .$$

The proof is an obvious analogue of that of Ascoli-Alzera's Theorem, and hence is omitted. Next we need the following:

<u>Lemma 3-5</u> $\varphi_{i,k}$ <u>$i = 1,2,3\ldots$ is equi-uniformly continuous for each</u> k .

<u>Proof</u> By [6, 4.3], we have

$$|V(\varphi_{i,k})| < k\cdot|V| \, \|\varphi_{i,k}\|_{L^2} / \mathrm{Vol}(M_i)^{1/2}$$

for each $V \in T(M_i)$. The lemma follows immediately.

Q.E.D.

Now we shall prove (3-2-1), (3-2-2) and (3-2-3) (a) and (b). We constructed, in [7, Theorem 0-1], the map f_i satisfying (3-2-1) and (3-2-2). Suppose that we can not find f_i satisfying (3-2-3) (a) and (b). Then, there exist $\Theta > 0$ and a subsequence i_j such that

$$\text{(3-6)} \quad \sup_{x \in M_{i_j}} |\varphi_{i_j,k}(x) - \varphi \circ f_{i_j}(x)| > \Theta$$

holds for each j and each k-th eigenfunction φ of $P_{(X,\mu)}$. On the other hand, Lemma 3-3 and 3-5 imply that we may assume, by taking a subsequence if necessary, the existence of a continuous function φ_∞ on X such that

$$\text{(3-7)} \quad \lim_{j\to\infty} \sup_{x \in M_{i_j}} |\varphi_{i_j,k}(x) - \varphi_\infty \circ f_{i_j}(x)| = 0 .$$

Moreover, [6, Theorem 0.4] implies that the L^2-distance between $\varphi_{i_j} \circ \psi_j$ and the k-th eigenspace of $P_{(X,\mu)}$ converges to 0, where $\psi_j : X \longrightarrow M_{i_j}$ is a measurable map satisfying $f_{i_j} \circ \psi_j$ = identity. Therefore, (3-7) implies that φ_∞ is a k-th eigenfunction of $P_{(X,\mu)}$. This contradicts (3-6).

<u>Remark</u> We have not yet used Assumption 1-5.

To prove (3-2-3) (c), we first remark the following elementary inequality.

<u>Lemma 3-8</u> <u>Let</u> $\varphi : (a-\varepsilon, b+\varepsilon) \longrightarrow \mathbb{R}$ <u>be a</u> C^2-<u>function satisfying</u>

$$\sup_{t\in[a,b]} \left| \frac{d^2\varphi}{dt^2} \right| \leq C .$$

Then we have

$$\left| \frac{d\varphi}{dt}(a) - \frac{\varphi(b) - \varphi(a)}{b - a} \right| < C\cdot(b-a) .$$

Secondly, [6, 4.3.2] implies the following.

Lemma 3-9 *There exists a constant* C_k *independent* i *such that the following holds. Let* $\ell : [0,1] \longrightarrow M_i$ *be a geodesic with unit speed. Then*

$$\sup_{t\in[0,1]} \left| \frac{d^2(\varphi_{i,k}\circ\ell)}{dt^2} \right| < C_k .$$

By a method similar to [6, § 7], we may assume that X is a manifold, N . Then, since the k-th eigenspace of $P_{(N,\mu)}$ is finite dimensional and consists of smooth functions, it follows that

$$(3\text{-}10) \quad \sup_{t\in[0,1]} \left| \frac{d^2(\varphi'_{i,k}\circ\ell)}{dt^2} \right| < C'_k$$

holds for each geodesic $\ell : [0,1] \longrightarrow N$ with unit speed.

Now let $V_i \in T(M_i)$ be a unit vector. We put $\ell_i(t) = \mathrm{ext}(t\cdot V_i)$, $\ell'_i(t) = \mathrm{ext}(t\cdot(f_i)_*(V_i)/|(f_i)_*(V_i)|)$. Then, by [7, § 4], we have

$$(3\text{-}11) \quad \lim_{i\to\infty} \sup_{t\in[0,1]} d(f_i\ell_i(t), \ell'_i(t)) = 0 ,$$

$$(3\text{-}12) \quad \lim_{i\to\infty} |(f_i)_*(V_i)| = 1 .$$

Let δ be an arbitrary small positive number. Lemmae 3-8 and 3-9 imply

$$(3\text{-}13)\quad \left| V_i(\varphi_{i,k}) - \frac{\varphi_{i,k}\circ \ell_i(\delta) - \varphi_{i,k}\circ \ell_i(0)}{\delta} \right| \leq C_k \cdot \delta .$$

On the other hand, by Lemma 3-8, Formulae (3-10), (3-12), we have

$$(3\text{-}14)\quad \limsup_{i\to\infty} \left| (f_i)_*(V_i)(\varphi'_{i,k}) - \frac{\varphi_{i,k}\circ \ell_i(\delta) - \varphi'_{i,k}\circ \ell'_i(0)}{\delta} \right| \leq C'_k \cdot \delta .$$

Furthermore (3-2-3) (b) and (3-11) imply

$$(3\text{-}15)\quad \lim_{i\to\infty} \sup_{t\in[0,1]} |\varphi_{i,k}\circ \ell_i(t) - \varphi'_{i,k}\circ \ell'_i(t)| = 0 .$$

From Formulae (3-13), (3-14), (3-15), we conclude

$$\lim_{i\to\infty} |V_i(\varphi_{i,k}) - (f_i)_*(V_i)(\varphi'_{i,k})| \leq (C_k + C'_k)\delta .$$

Q.E.D

§ 4 Estimating derivatives of the fibration

In this section we shall prove Lemma 1-6. Let M_i and N be as in Theorem 0-1. By [1], we obtain, for each $\delta > 0$, metrics $g_{i,\delta}$ on M_i such that

(4-1-1) $|g_{i,\delta} - g_i| < \tau(\delta)$,

(4-1-2) $|\nabla^k R(M_i, g_{i,\delta})| < C(k,\delta)$.

Here g_i denotes the original Riemannian metric on M_i , and $\tau(\delta)$, $C(k,\delta)$ are positive numbers independent of i and satisfying $\lim_{\delta\to 0} \tau(\delta) = 0$. By taking a subsequence if necessary, we may assume $(M_i, g_{i,\delta})$ $i = 1,2,\ldots$ converge to a metric space N_δ with respect to the Hausdorff distance. Then, [8, Lemma 2-3] implies that N_δ is diffeomorphic to N and

(4-2) $\lim_{\delta\to 0} d_L(N,N_\delta) = 0$,

where d_L denotes the Lipschitz distance defined in [11]. Therefore, it suffices to show Lemma 1-6 for $M_{i,\delta}$ and N_δ . Hereafter we shall write M_i and N in place of $M_{i,\delta}$ and N_δ . Thus, we verified that we can assume (1-5) while proving Lemma 1-6.

By [6, Corollary 2-11], we may assume, by taking a subsequence if necessary, that M_i converges to $(N, \chi_N\Omega_N)$ with respect to the measured Hausdorff topology. Then, Lemma 2-1 implies that χ_N is smooth. Hence the operator $P_{(N,\ \chi_N\Omega_N)}$ is elliptic with smooth coefficient. It follows the following:

Lemma 4-3 *There exists* J *such that the map* $I_0 : N \longrightarrow \mathbb{R}^J$ *defined by* $I_0(P) = (\varphi_1(P), \ldots, \varphi_J(P))$ *is a smooth embedding. Here* φ_k *denotes a* k-th *eigenfunction of* $P_{(N,\ \chi_N \Omega_N)}$.

Next, we apply Theorem 3-1 to obtain eigenfunctions $\varphi_{i,k}$ and $\varphi'_{i,k}$ satisfying (3-2-3).

Put

$$I'_i(x) = (\varphi_{i,1}(x), \ldots\ldots, \varphi_{i,J}(x)) .$$

Then, there exists a sequence of isometries L_i of $\mathbb{R}^J$ such that $L_i \circ I'_i$ converges to I_0 with respect to the C^1-topology. We have the following:

Lemma 4-4 *There exists smooth maps* $I_i : M_i \longrightarrow \mathbb{R}^J$, $I_0 : N \longrightarrow \mathbb{R}^J$ *such that*

(4-5-1) I_0 *is an embedding,*

(4-5-2) $\lim_{i\to\infty} \sup_{x \in M_i} |I_i(x) - I_0 \circ f_i(x)| = 0$,

(4-5-3) $\lim_{i\to\infty} \sup_{V \in T(M_i)} |(I_i)_*(V) - (I_0 \circ f_i)_*(V)| = 0$,

(4-5-4) $|\Delta^k I_i| \le C^k |I_i|$.

Here $f_i : M_i \longrightarrow N$ *is a fibration of § 3, and* C *is a constant independent of* i *and* k .

Proof Put $I_i = L_i \circ I'_i$. We have already proved (4-5-1),...,(4-5-3). Formula (4-5-4) follows from the definition of I_i and the estimate of the eigenfunctions of Laplace operators (see [6]).

Q.E.D.

Now, put

$$B_\delta N(N) = \left\{ (p,u) \in \mathbb{R}^J \;\middle|\; |u| < \delta,\ u \text{ is perpendicular to } (I_0)_*(T_p(N)). \right\}$$

Let $E : B_\delta N(N) \longrightarrow \mathbb{R}^J$ denote the map $E(p,u) = I_0(p) + u$. Then, by (4-5-1), we can choose δ such that $E : B_\delta N(N) \longrightarrow \mathbb{R}^J$ is a diffeomorphism to its image. Then, by (4-5-2), we see that, for sufficiently large i, we have $I_i(M_i) \subset E(B_\delta N(N))$. Thus, the map $\pi_i = P \circ E^{-1} \circ I_i$ is well defined, ($P : E(B_\delta N(N)) \longrightarrow N$ is defined by $P(p,u) = p$). As in [7, § 2], the fact (4-5-3) implies that π_i is a fibration. Facts (4-5-4) and (4-1-2) imply that π_i satisfies (1-7). The proof of Lemma 1-6 is now complete.

§ 5 The construction of a smooth family of connections

In this section, we shall complete the proof of Theorem 1-1. Then, Lemma 1-6 implies the following:

Lemma 5-1 Let $\pi_i : M_i \longrightarrow N$ be as in Lemma 1-6. Then, there exists a constant C independent of i, such that

$$|\text{the second fundamental form of } \pi_i^{-1}(p)| < C .$$

On the other hand, we have

(5-2) $$\lim_{i\to\infty} \sup_{p\in N} \mathrm{Diam}(\pi_i^{-1}(p)) = 0 .$$

Hence, by [14], we can construct, for each i and $p \in N$, a flat connection on $\pi_i^{-1}(p)$ such that $\pi_i^{-1}(p)$ is affinely diffeomorphic to G/Γ, where G and Γ are as in Theorem 1-1. Hence it suffices to modify these connections so that they depends smoothly on p. If the flat connection constructed in [14] was canonical, then there would be nothing to show. But, unfortunately, the connection there depends on the choice of the base point on an almost flat manifold. Therefore, we should check carefully the construction there. In [14], the construction of the connection is devided into three steps. In the first step, a flat connection ∇' with small torsion tensor is constructed. The connection ∇' is used, in the second step, to construct a flat connection with parallel torsion tensor. In the third step, it is shown that almost flat manifolds equipped with a flat connection with parallel torsion tensor is affinely diffeomorphic to G/Γ. Roughly speaking, we do not have to modify the arguments in the second and the third steps, because connections constructed there depends smoothly on the deta given in the first step.

Now, we shall present the parametrized version of the first step. First we change the normalization of the metric of the fibres. (Our normalization so far was $|\text{curvature}| \leq 1$, Diameter $\longrightarrow 0$. The normalization in [14] was Diameter $= 1$, $|\text{curvature}| \longrightarrow 0$.)

Lemma 5-3 Let $\pi_i : M_i \longrightarrow N$ be as in Lemma 1-6. Then, there exists a smooth family of Riemannian metrics $g_i(p)$ on $\pi_i^{-1}(p)$ such that

(5-4-1) $\operatorname{Diam}(\pi_i^{-1}(p), g_i(p)) = 1$,

(5-4-2) $|\nabla^k R(g_i(p))| \leq \varepsilon_{i,k}$,

where $\lim_{i\to\infty} \varepsilon_{i,k} = 0$.

Secondly, we introduce the C^k-norm on $\pi_i^{-1}(p)$ as follows. Take $x \in \pi_i^{-1}(p)$ and let $\mathrm{Exp}_x : B(100) \longrightarrow \pi_i^{-1}(p)$ be the exponential map. Let A be a tensor on $f_i^{-1}(p)$. We define $|A|_{C^k}$ to be the C^k-norm of the coefficients of $E^*(A)$. This definition is independent of x modulo constant multiple. Then (5-4-2) implies

(5-4-3) $|R(g_i(p))|_{C^k} \leq \varepsilon_{i,k}$.

Thirdly we put $p_j \in N$, $V_j = B_\mu(p_j,N)$, $U_j = B_{2\mu}(p_j,N)$, where μ is the one third of the injectivity radius of N . Assume $\cup V_j = N$. Let $s_{i,j} : U_j \longrightarrow M_i$ be smooth sections to π_i . Then, using $s_{i,j}(p)$ as a base point of $\pi_i^{-1}(p)$, we can follow the argument of [14, P5, P6] and obtain the following:

Lemma 5-5 For each i and j, there exists a smooth family of connections $\nabla^{(i,j)}(p)$ on $\pi_i^{-1}(p)$ $(p \in U_j)$ such that

(5-6-1) $\nabla^{(i,j)}(p)$ is flat,

(5-6-2) $|T^{(i,j)}(p)|_{C^k} < \varepsilon_{i,k}$, where $T^{(i,j)}(p)$ is the torsion tensor of $\nabla^{(i,j)}(p)$.

(5-6-3) $\nabla^{(i,j)}(p)$ is a metric connection with respect to the metric $g_i(p)$.

Fourthly, we shall estimate the tensor $\nabla^{(i,j)}(p) - \nabla^{(i,j')}(p)$, and prove

(5-6-4) $|\nabla^{(i,j)}(p) - \nabla^{(i,j')}(p)|_{C^k} < \varepsilon_{i,k}$.

By the construction of $\nabla^{(i,j)}(p)$ (which is presented in [14, P5, P6]), it suffices to estimate the parallel transform. (Sublemma 5-7). Let $\tilde{g}_{i,j}(p)$ be the metric on $B(100)$ induced by the exponential map $\mathrm{Exp}_{s_{i,j}(p)} : T_{s_{i,j}(p)}(\pi_i^{-1}(p)) \longrightarrow \pi_i^{-1}(p)$. For $x \in B(100)$, we identify $\mathbb{R}^n$ and $T_x(B(100))$ in an obvious way. Then, for $x, y \in B(100)$, the parallel translation along the shortest geodesic $P^{i,j,p}_{x,y} : T_x(B(100)) \longrightarrow T_y(B(100))$ with respect to the metric $\tilde{g}_{i,j}(p)$, can be regarded as an element of $Gl(n, \mathbb{R})$. Put

$$Q^{i,j,p}_{x,y}(z) = P^{i,j,p}_{x,z} - P^{i,j,p}_{y,z} .$$

$Q^{i,j,p}_{x,y}$ is a matrix valued function. Now, (5-6-4) follows from the following:

<u>Sublemma 5-7</u> <u>There exists</u> $\varepsilon_k(\delta)$ <u>independent of</u> i,j,p <u>such that</u> <u>if</u> $|x-y| < \delta$ <u>then</u> $|Q^{i,j,p}_{x,y}(Z)|_{C^k} < \varepsilon_k(\delta)$. <u>Here</u> $\lim_{\delta\to 0} \varepsilon_k(\delta) = 0$.

<u>Proof</u> If sublemma does not hold, there exist x_ℓ, y_ℓ, $z^{(0)}_{(\ell)} \in B(100)$, i_ℓ, j_ℓ, $\theta > 0$ and a multiindex α such that

(5-8-1) $$\left|\frac{\partial^{|\alpha|}\left(P^{i_1,j_1}_{x_1,z}\right)}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}} - \frac{\partial^{|\alpha|}\left(P^{i_1,j_1}_{y_1,z}\right)}{\partial z_1^{\alpha_1}\cdots\partial z_n^{d_n}}\right|_{z = z^{(0)}_{(\ell)}} > \theta ,$$

(5-8-2) $$\lim_{\ell\to\infty} d(x_\ell, y_\ell) = 0 .$$

By taking a subsequence, we may assume that $\lim x_\ell = \lim y_\ell = W$, $\lim z^{(0)}_{(\ell)} = z^{(0)}$ and $\tilde{g}_{i_\ell,j_\ell}(p)$ converges to g_∞ with respect to the C^∞-topology. Then we have

(5-9) $$\lim_{\ell\to\infty}\left(\left.\frac{\partial^{|\alpha|}P^{i_\ell j_\ell}_{x_\ell,z}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\right|_{z' = z^{(0)}_{(\ell)}}\right)$$

$$= \left.\frac{\partial^{|\alpha|}P^{\infty}_{w,z}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\right|_{z = z^{(0)}}$$

$$= \lim_{\ell\to\infty}\left(\left.\frac{\partial^{|\alpha|}P^{i_\ell,j_\ell}_{Y_\ell,z}}{\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}}\right|_{z = z^{(0)}_{(\ell)}}\right) ,$$

where P^∞ denotes the parallel translation with respect to g_∞ . (5-9) contradicts (5-8-1).

Q.E.D.

Thus, we have verified (5-6-4). Finally we shall prove the following:

Lemma 5-10 <u>There exists a smooth family of connections</u> $\nabla_i'(p)$ <u>on</u> $\pi_i^{-1}(p)$ $(p \in N)$ <u>such that</u>

(5-11-1) $\nabla_i'(p)$ <u>is flat</u>,

(5-11-2) $|T_i'(p)|_{C^k} \leq \varepsilon_{i,k}$, <u>where</u> $T_i'(p)$ <u>is the torsion tensor of</u> $\nabla_i'(p)$,

(5-11-3) $\nabla_i'(p)$ <u>is a metric connection with respect to the metric</u> $g_i(p)$.

<u>Proof</u> For simplicity, we assume $V_1 \cup V_2 = N$. First we shall find a gauge transformation $O_{p,i}$ such that $\nabla^{(i,1)}(p) = O_{p,i}^{-1} \circ \nabla^{(i,2)}(p) \circ O_{p,i}$ holds for $P \in U_1 \cap U_2$. Here $O_{p,i}$ is a section of the fibre bundle $\mathrm{Aut}(F(\pi_i^{-1}(p))) = F(\pi_i^{-1}(p)) \times_{Ad} O(m)$, where $F(\pi_i^{-1}(p))$ is the frame bundle and $m = \dim \pi_i^{-1}(p)$. We have two monodromy representations $\tilde{\rho}_1{}^{(p,i)}$, $\tilde{\rho}_2{}^{(p,i)}$: $\Gamma \longrightarrow O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ with respect to the flat connections $\nabla^{(i,1)}(p)$ and $\nabla^{(i,2)}(p)$, respectively. (Here we recall $\pi_i^{-1}(p) = G/\Gamma$. And $O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ denotes the set of linear isometries of $T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$. By the construction of $\nabla^{(i,j)}(p)$ presented in [14, P5, P6] we see $\tilde{\rho}_1^{(p,i)}(\Gamma \cap G) = \tilde{\rho}_2^{(p,i)}(\Gamma \cap G) = 1$. Hence there exist a projection $P : \Gamma \longrightarrow \Lambda$ to a finite group Λ and representations $\rho_1^{(p,i)}$, $\rho_2^{(p,i)}$: $\Lambda \longrightarrow O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ such that $\rho_1^{(p,i)} \circ P = \tilde{\rho}_1^{(p,i)}$, $\rho_2^{(p,i)} \circ P = \tilde{\rho}_2^{(p,i)}$. Then, since $\# \Lambda < \infty$ and $\rho_1^{(p,i)}$ and $\rho_2^{(p,i)}$ are close to each other, there

exists $\alpha_i(p) \in O(T_{s_{i,1}(p)}(\pi_i^{-1}(p)))$ depending smoothly on p such that $\rho_2^{(p,i)}(\gamma) = \alpha_i(p)^{-1} \cdot \rho_1^{(p,i)}(\gamma) \cdot \alpha_i(p)$, and $\alpha_i(p)$ converges to identity with respect to the C^∞-topology when i tends to ∞. Now we define $O_{p,i}(x) : T_x(\pi_i^{-1}(p)) \longrightarrow T_x(\pi_i^{-1}(p))$, for $x \in \pi_i^{-1}(p)$, as follows. Let $\ell : [0,1] \longrightarrow \pi_i^{-1}(p)$ be an arbitrary curve connecting x to $s_{i,\ell}(p)$, and $P_1, P_2 : T_x(\pi_i^{-1}(p)) \longrightarrow T_{s_{i,\ell}(p)}(\pi_i^{-1}(p))$ denote the parallel translations along ℓ with respect to the connections $\nabla^{(i,1)}(p)$ and $\nabla^{(i,2)}(p)$, respectively. We put

$$O_{p,i}(x)(V) = P_2^{-1}(\alpha_i(p)^{-1} \cdot P_1(V) \cdot \alpha_i(p)) .$$

Using $\alpha_i(p)^{-1} \cdot \tilde{\rho}_1^{(p,i)} \cdot \alpha_i(p) = \tilde{\rho}_2^{(p,i)}$, it is easy to verify that $O_{p,i}(x)$ does not depend on the choice of ℓ. The equality $\nabla^{(i,1)}(p) = O_{p,i} \circ \nabla^{(i,2)}(p) \circ O_{p,i}^{-1}$ is also obvious from the definition. By construction, $O_{p,i}$ converges to the identity with respect to the C^∞-topology. Therefore, the section $\log O_{p,i}$ of $F(\pi_i^{-1}(p)) \times_{ad} \mathfrak{o}(m)$ is well defined, (where $\mathfrak{o}(m)$ is the Lie algebra of $O(m)$ and $m = \dim \pi_i^{-1}(p))$, and $\log O_{p,i}$ satisfies

$$\text{(5-13)} \quad |\log O_{p,i}|_{C^k} \leq \varepsilon_i(k) .$$

Take a smooth function $\psi : N \longrightarrow [0,1]$ such that $\psi \equiv 1$ on a neighborhood of $\overline{V_1 \setminus U_2}$ and that $\psi \equiv 0$ on a neighborhood of $\overline{V_2 \setminus U_1}$. Put $O'_{p,i} = \exp(\psi(p) \log O_{p,i})$, for $p \in U_1 \cap U_2$. We define $\nabla'_i(p)$ by

$$\nabla'_{(i)}(p) \begin{cases} = O'^{-1}_{p,i} \circ \nabla^{(i,2)}(p) \circ O'_{p,i} & p \in U_1 \cap U_2 \\ = \nabla^{(i,2)}(p) & p \in V_2 \\ = \nabla^{(i,1)}(p) & p \in V_1 \ . \end{cases}$$

(5-12) implies that $\nabla'_i(p)$ depends smoothly on p . (5-13) implies (5-11-2). Facts (5-11-1) and (5-11-3) are obvious from the construction.

Q.E.D.

Thus we have proved the parametrised version of the first step in [14]. The rest of the argument is completely parallel to [14]. We use Newton's method to obtain a sequence of flat connections $\nabla'_{i,k}(p)$ and a connection $\nabla_i(p)$ such that

(5-14-1) $\nabla'_{i,0}(p) = \nabla'_i(p)$,

(5-14-2) $\lim\limits_{k\to\infty} |\nabla'_{i,k}(p) - \nabla_i(p)|_{C^2} = 0$,

(5-14-3) $\nabla_i(p)(T_i(p)) = 0$, where $T_i(p)$ is the torsion tensor of $\nabla_i(p)$.

(In [14] the convergence of $\nabla'_{i,k}(p)$ to ∇_i is the C^0-convergence. But, in our case, we can proof the C^k-convergence for an arbitrary k , thanks to (5-11-3).) By (5-14-2) $\nabla_i(p)$ is a C^2-family of connections. It is easy to modify it to a C^∞-family. Then (5-14-3) implies, as in [14P13], that $\nabla_i(p)$ is the connection we have been looking for. The proof of Theorem 1-1 is now completed.

§ 6 The construction of a collapsing family of metrics

In this section, we shall prove Theorem 0-7. Let $\pi : M \longrightarrow N$ be a fibre bundle satisfying (0-3-1), (0-3-2), (0-3-3). T denotes the structure group of the fibration π. Then T is an extension of a torus T_0 by a discrete group Λ contained in $\mathrm{Aut}\,\Gamma$, where Γ and G are as in (0-3-2). Choose a T connection of π. It gives a decomposition of $T_x(M)$ to its horizontal subspace $H_x(M)$ and vertical subspace $V_x(M) = T_x(\pi^{-1}\pi(x))$. We put

(6-1-1) $\quad g_\varepsilon(V,W) = g_N(\pi_*(V), \pi_*(W))$, if $V,W \in H_x(M)$.

(6-1-2) $\quad g_\varepsilon(V,W) = 0$, if $V \in H_x(M)$, $W \in V_x(M)$.

Here g_N denotes the Riemannian metric of N. We shall define $g_\varepsilon(V,W)$ for $V,W \in V_x(M)$.

Let $\pi_1 : P_1 \longrightarrow N$ be the principal T-bundle associated to π, and $\pi_2 : P_2 \longrightarrow N$ be the principal Λ-bundle induced from π_1. (Namely $P_2 = P_1/T_0$.) Let $\mathfrak{g}$ be the Lie algebra of G. Put $\mathfrak{g}'_0 = \mathfrak{g}$, $\mathfrak{g}'_{k+1} = [\mathfrak{g}'_k, \mathfrak{g}]$, and $\mathfrak{g}_k = \mathfrak{g}'_k +$ (center of $\mathfrak{g}$) if $\mathfrak{g}'_k \neq 0$, $\mathfrak{g}_k = 0$ if $\mathfrak{g}'_k = 0$. We have $[\mathfrak{g}, \mathfrak{g}_k] \subset \mathfrak{g}_{k+1}$. And if $\mathfrak{g}_K = 0$, $\mathfrak{g}_{K-1} \neq 0$ then $\mathfrak{g}_{K-1}$ = center of $\mathfrak{g}$. Since $\Lambda \subset \mathrm{Aut}\,\Gamma$, Malcev's rigidity Theorem (see [13, P34]) implies $\Lambda \subset \mathrm{Aut}\,G$. Hence Λ acts on $\mathfrak{g}$ by isometry. It follows that Λ preserves the filtration $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \dots \supset \mathfrak{g}_K = 0$. Put $E = P_2 \times_\Lambda \mathfrak{g}, \dots$ $E_k = P_2 \times_\Lambda \mathfrak{g}_k$. Then $\pi_0 = E \longrightarrow N$, $\pi_k : E_k \longrightarrow N$ are vector bundles. Fix a metric h_1 on E and let F_k be the intersection of E_{k-1} and the orthogonal complement of E_k. Then, F_k $k = 1,2\dots$ are orthogonal to each other and $\oplus F_k = E$. We define h_ε by

$$(6\text{-}2)\qquad h_\varepsilon(V,W) = (\varepsilon^{2^k})^2 h_1(V,W)\ ,$$

for $V \in F_k$, $W \in F_{k'}$. Let $U_i \subset N$, $\psi_i : \pi^{-1}(U_1) \longrightarrow U_i \times G/_\Gamma$ be a coordinate chart and $s_{i,j}(p) \in T$ $(p \in U_i \cap U_j)$ be the transition function. Namely, if $\psi_i(p) = (p,g)$ then $\psi_j(p) = (p, s_{j,i}(p)\cdot g)$. Let $\psi_i' : \pi_0^{-1}(U_i) \longrightarrow U_i \times \mathfrak{g}$ be a coordinate chart. By definition we can take ψ_i' so that the transition function of this chart is $P(s_{i,j})$, where $P : T \longrightarrow \Lambda = T/T_0$ is the natural projection. Namely

$$(6\text{-}3)\qquad \psi_i'(u) = (p, P(s_{i,j}(p))\cdot a),\ \text{if}\ \ \psi_j'(u) = (p,a)\ .$$

For $V,W \in \mathfrak{g}$, $p \in U_i$, we put

$$h_{\varepsilon,i}(p)(V,w) = h_\varepsilon(\psi_i'^{-1}(p,V),\ \psi_i'^{-1}(p,W))\ .$$

The quadratic form $h_{\varepsilon,i}(p)$ gives a right invariant metric $\tilde{g}_{\varepsilon,i}(p)$ on G . Hence it induces a Riemannian metric on $G/(G \cap \Gamma)$. By Lemma 1-4, $\Gamma/(G \cap \Gamma)$ is a finite subgroup of $\mathrm{Aut}(G)$. Therefore, we can choose h_1 so that $h_{\varepsilon,i}(p)$ is preserved by $\Gamma/(G \cap \Gamma) \subset \mathrm{Aut}(\mathfrak{g})$. Then, $\tilde{g}_{\varepsilon,i}(p)$ induces a Riemannian metric on $\{p\} \times G/\Gamma$. This metric, together with (6-1-1) and (6-1-2), determines a Riemannian metric $g_{\varepsilon,i}$ on $U_i \times G/\Gamma$. Then, using (6-3) and the fact that T_0 is contained in the center of G , we can easily verify that $g_{\varepsilon,i}$ can be patched together and gives a Riemannian metric g_ε on M . The equality $\lim_{\varepsilon\to 0}{}_H(M, g_\varepsilon) = N$ is obvious. Thus, we are only to show that the sectional curvatures of g_ε have an upper and a lower bound independent of ε . Since the problem is local, we have only to study $U_i \times G/\Gamma$, and also it suffices to obtain an estimate of sectional curvatures of $(U_i \times G\ ,\ \tilde{g}_{\varepsilon,i})$. (Hereafter we omit the index i .). Now, let $e'_1,\dots,e'_n$ be an orthonormal frame of vector fields on U , and $e_1,\dots,e_n$ denote their horizontal lifts to $U \times G$. Choose an orthonormal basis $X_1(p),\dots,X_m(p)$ of $(\mathfrak{g}, h_1(p))$, such that there exists a nondecreasing map $O : \{1,\dots,m\} \longrightarrow \mathbf{Z}^+$ satisfying $X_i(p) \in F_{O(i)}(p)$, where $F_k(p)$ denotes the orthogonal complement of $\mathfrak{g}_k$ in $(\mathfrak{g}_{k-1}, h_1(p))$. We may assume that $X_i(p)$ depends smoothly on p . These elements $X_i(p)$ determine, through the right action of G , a vector field on $\{p\} \times G$. Thus, we obtain a vector field f_i on $U \times G$. Then, $(e_1,\dots,e_n , f_1,\dots,f_m)$ is an orthonormal frame of vector fields on $(U \times G, \tilde{g}_1)$ and $(e_1,\dots,e_n , \varepsilon^{-2^{O(i)}} f_1,\dots,\varepsilon^{-2^{O(m)}} f_m)$ is one on $(U \times G, \tilde{g}_\varepsilon)$. We shall calculate commutators of those vector fields. First, since our connection of π is a T-connection, it follows that

$$(6\text{-}4\text{-}1) \quad [e_i,e_j] = \sum_{k=1}^{n} a^k_{i,j} e_k + \sum_{O(k)=O(m)} b^k_{i,j} f_k ,$$

where $a^k_{i,j}$ and $b^k_{i,j}$ are functions on U . Secondly, since $[\mathfrak{g}_k,\mathfrak{g}] \subset \mathfrak{g}_{k+1}$, we have

$$(6\text{-}4\text{-}2) \quad [f_i,f_j] = \sum_{\substack{O(k)>O(i)\\ O(k)>O(j)}} c^k_{i,j} \cdot f_k ,$$

where $c^k_{i,j}$ are functions on U . Next we shall calculate $[f_i,e_j]$. Let $Y_1,\dots,Y_m$ be a basis of $\mathfrak{g}$. The element Y_i of $\mathfrak{g}$, through the right action of G , induces a vector fields f^*_i on $U \times G$. Since our connection of π is a T-connection hence in particular is a G-connection, it follows that the horizontal lift is invariant by the right action of G . Therefore

(6-5) $\quad [e_i, f_j^*] = 0$.

On the other hand there exist functions $\alpha_{i,j}$ on U such that

(6-6) $\quad f_i(p,g) = \sum_{O(j) \geq O(i)} \alpha_{i,j}(p) \cdot f_j^*(p,g)$.

We regard U as an open subset of $\mathbb{R}^n$, and put

(6-7) $\quad e_i'(p) = \sum_{j=1}^{n} \beta_{i,j}(p) \frac{\partial}{\partial p^j}$.

Then, (6-5), (6-6) and (6-7) imply

$$[e_i, f_j](p,g) = \sum_{\substack{1 \leq k \leq n \\ O(\ell) \geq O(i)}} \beta_{j,k}(p) \frac{\partial \alpha_{i,\ell}}{\partial p^j} f_k^*(p,g) .$$

Therefore, we have

(6-4-3) $\quad [e_i, f_j] = \sum_{O(k) \geq O(i)} d_{i,j}^k f_k$,

where $d_{i,j}^k$ are functions on U .

Now, let $e^1, \ldots, e^n$, $f_\varepsilon^1, \ldots, f_\varepsilon^m \in \Lambda'(U \times G)$ be the dual base of $(e_1, \ldots, e_n , \varepsilon^{-2^{O(1)}} f_1, \ldots, \varepsilon^{-2^{O(m)}} f_m)$. Then, by (6-4-1), (6-4-2), (6-4-3), we have

(6-8-1) $\quad de^i = \sum_{j,k} a_{jk}^i e^j \wedge e^k$,

(6-8-2) if $O(i) \neq O(m)$, then

$$df_\varepsilon^i = \sum_{\substack{O(i) > O(j) \\ O(i) > O(k)}} c_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(j)} - 2^{O(k)}} \cdot f_\varepsilon^j \wedge f_\varepsilon^k$$

$$+ \sum_{O(i) \geq O(k)} d_{jk}^i \cdot \varepsilon^{2^{O(i)} - 2^{O(k)}} e^j \wedge f_\varepsilon^k ,$$

(6-8-3) if $O(i) = O(m)$, then

$$df^i_\varepsilon = \sum_{\substack{O(i)>O(j)\\ O(i)>O(k)}} c^i_{jk} \cdot \varepsilon^{2^{O(i)}-2^{O(j)}-2^{O(k)}} \cdot f_\varepsilon \wedge f^k_\varepsilon$$

$$+ \sum_{O(i)\geq O(k)} d^i_{jk} \cdot \varepsilon^{2^{O(i)}-2^{O(k)}} e^j \wedge f^k_\varepsilon$$

$$+ \sum b^i_{jk} \cdot \varepsilon^{2^{O(i)}} \cdot e^j \wedge e^k \ .$$

We see that the coefficients a^i_{jk}, $c^i_{jk} \cdot \varepsilon^{2^{O(i)}-O(j)-2^{O(k)}}$, $d^i_{jk} \cdot \varepsilon^{2^{O(i)}-2^{O(j)}}$, $b^i_{jk} \varepsilon^{2^{O(i)}}$ are bounded, with respect to the C^k-norm, while ε tends to 0 . Therefore, we can prove that the sectional curvatures of g_ε are uniformly bounded thanks to the well known formula which expresses the curvature tensor in terms of these coefficients. The proof of Theorem 0-7 is now complete.

§ 7 An application

In this section we shall prove Theorem 0-9, by contradiction. We assume that there exists a sequence of n-dimensional Riemannian manifolds M_i such that

(7-1-1) $\mathrm{Diam}\, M_i \leq D$,

(7-1-2) $\mathrm{Vol}\, M_i \leq 1/i$,

(7-1-3) $|\text{sectional curvature of } M_i| \leq 1$,

(7-1-4) $\mathrm{Minvol}\, M_i \geq \varepsilon > 0$,

where ε is independent of i . Using [9, Theorem 0-6], we can find a subsequence M_{k_i} , and an aspherical Riemannian orbifold X/Γ such that

(7-1-5) $\lim_{i\to\infty}{}_H\, M_{k_i} = X/\Gamma$,

where an aspherical Riemannian orbifold stands for the quentient X/Γ of a contractible Riemannian manifold X by a properly discontinuous action of a group Γ consisting of isometries of X . By a modification of the argument in §§ 1 ... 5, we can generalize Theorem 0-1 to the case when the limit space is an orbifold. Hence we obtain a fibration $\pi_{p_i} : M_{k_i} \longrightarrow X/\Gamma$ whose fibre is G/Γ and whose structure group is the extension of $C(G)/_{(C(G)\,\cap\,\Gamma)}$ by $\mathrm{Aut}\,\Gamma$, where G and Γ are as in (0-3-2). Hence, Theorem 0-7 (more precisely its generalization to orbifold case) implies that there exist metrics g_ε on M_{k_i} such that

(7-2-1) $\lim_{\varepsilon \to 0} {}_H(M_{k_i}, g_\varepsilon) = X/\Gamma$

(7-2-2) $|\text{sectional curvature of } g_\varepsilon| \leq C$,

where C is a number independent of ε . On the other hand, (7-1-2) and [11, 8.30] imply $\dim X/\Gamma \lneqq \dim M_{k_i}$. Hence, by (7-2-1) we have

(7-2-3) $\lim_{\varepsilon \to 0} \mathrm{Vol}(M_{k_i}, g_\varepsilon) = 0$,

(7-2-2) and (7-2-3) contradict (7-1-4).

Q.E.D.

§ 8 The case when the limit space is not a manifold

So far, we have studied sequences of Riemannian manifolds converging to a manifold. In [8] we have studied more general situation. The method of this paper can be joined with one in [8] to prove the following:

Theorem 8-1 Let M_i be a sequence of $n+m$-dimensional Riemannian manifold satisfying (0-2-2) which converges to a metric space X with respect to the Hausdorff distance. Then, there exists a $C^{1,\alpha}$-manifold Y and $\pi_i : FM_i \longrightarrow Y$, such that the following holds. (Here FM_i denotes the frame bundle.)

(8-2-1) $O(n+m)$ acts by isometry to Y. We have $X = Y/O(n+m)$.

(8-2-2) $\tilde{\pi}_i$ satisfies (0-3-1), (1-2-1), (1-2-2).

(8-2-3) $\tilde{\pi}_i$ is an $O(m+n)$-map, and the diagram

$$\begin{tikzcd} FM_i \arrow[r, "\tilde{\pi}_i"] \arrow[d] & Y \arrow[d] \\ M_i \arrow[r, "\pi_i"] & X \end{tikzcd}$$

commutes.

(8-2-4) Let $g \in O(n+m)$, $p \in Y$. Then the map $g : \tilde{\pi}_i^{-1}(p) \longrightarrow \tilde{\pi}_i^{-1}(g(p))$ preserves affine structures.

We omit the proof.

Unfortunately, our method in § 6 does not give the converse to Theorem 8-1. In other words, it seems that (8-2-1),...,(8-2-4) is not a sufficient condition for the existence of a family of metrics g_ε on M_i and that $\lim_{\varepsilon \to 0} {}_H(M_i, g_\varepsilon) = X$ and that $|\text{sectional curvatures of } g_\varepsilon| \leq C$.

In [2] and [3], Cheeger and Gromov developed another approach to study collapsing. They introduced the notion , F-structure there. Our Theorem 8-1 implies the following:

Corollary 8-3 There exists a positive number $\varepsilon(n,D)$ such that the following holds. Suppose an n-dimensional Riemannian manifold M satisfies

(8-4-1) $\mathrm{Vol}(M) \leq \varepsilon(n,D)$,

(8-4-2) $\mathrm{Diam}(M) \leq D$,

(8-4-3) $|\text{sectional curvature of } M| \leq |$.

Then M admits a pure F-structure of positive dimension.

Remark 8-5 The assumption of Cheeger and Gromov in [3] is less restrictuve than ours in the point that they do not assume the uniform bound of the diameter. Our conclusion is a little stronger. (In [3], the existence of F-structure is proved.)

Remark 8-6 The converse to Theorem 8-3 is false. A counter example is given in [2, Example 1.9].

Proof of Corollary 8-3 We prove by contradiction. Assume M_i satisfies (8-4-2), (8-4-3) and $\lim_{i \to 0} \mathrm{Vol}(M_i) = 0$, but M_i does not admit pure F-structure of positive dimension. By taking a subsequence if necessary, we may assume that M_i converges to a metric space X with respect to the Hausdorff distance. Therefore, by Theorem 8-1, we have Y, $\tilde{\pi}_i$, π_i satisfying (8-2-1),...,(8-2-4). Let $G/\Gamma = \tilde{\pi}_i(P)$. Then $C(G)/_{(\Gamma \cap ((G))}$ acts on each fibre. In view of (0-3-3), this action determines a pure (polarized) F-structure on FM_i. Then, (8-2-4) implies that this F-structure induces a pure F-structure on M_i. We shall prove that this F-structure is of positive dimension. Remark that we can assume (1-5). Let $x \in X$, $p_i \in \pi_i^{-1}(x) \subseteq M_i$. We recall the argument in [8, § 3]. We have metrics g_i, g_∞ on $B = B(1)$, local groups H_i, and a Lie group germ H such that

(8-7-1) H_i acts by isometry on the pointed metric space $((B,g_i),0)$,

(8-7-2) $(B,g_i)/H_i$ is isometric to a neighborhood of p_i on M_i,

(8-7-3) H acts by isometry on the printed metric space $((B,g_\infty),0)$,

(8-7-4) $(B,g_\infty)/H$ is isometric to a neighborhood of x in X,

(8-7-5) g_i converges to g_∞ with respect to the C^∞-topology.

Let $C(H_i)$ and $C(H)$ denote the centers of H_i and H, respectively. By construction, the dimension of the orbit through p_i of our F-structure on M_i is equal to the dimension of the orbit $C(H)(0)$. We shall prove $\dim C(H)(0) \gneq 0$. If 0 is not a fixed point of $C(H)$, there is nothing to show. We assume that there exists $\gamma \in C(H)\setminus\{1\}$ such that $\gamma(0) = 0$. Take $\gamma_i \in C(H_i)$ such that $\lim \gamma_i = \gamma$. We have

(8-8) $\lim_{i\to\infty} d(\gamma_i(0),0) = 0$.

Let δ be an arbitrary small positive number. Then (8-8) and the fact that the action of H_i is free imply the existence of n_i such that

(8-9) $\delta \geq \lim_{i\to\infty} d(\gamma_i^{n_i}(0),0) \gneq 0$.

We can take a subsequence $k(i)$ such that $\lim_{i\to\infty} \gamma_{k(i)}^{n_{k(i)}}$ converges to an element γ' of $C(H)$. Then by (8-9) we have

(8-10) $\delta \geq d(\gamma'(0),0) \gneq 0$.

Since δ is arbitrary small, (8-10) implies $\dim (C(H)(0)) \gneq 0$.

Thus we have constructed a pure F-structure on M_i for a sufficiently large i . This contradicts our choice of M_i .

Q.E.D.

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