# Vector bundles and torsion free sheaves on the projective plane


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The aim of this paper is to present a method for investigaton of the topological properties and the birational geometry of the moduli spaces of vector bundles and torsion free sheaves on the projective plane $\mathbf{P}^2$. Our method is based on the theory of equivariant vector bundles and sheaves on a toric variety. One of the main points of the paper is to explain how one can work with such an extraordinary for geometry object as torsion free sheaf. The appearance of such sheaves is inevitable in any attempt to complete the moduli space of vector bundles. We use this occasion to collect all information on the subject known to us.

All the technical devices are developed for a general situation of an arbitrary non-singular projective toric variety $X$. However the main features and geometrical dificulties of the theory are clearly seen already in the simplest case $X = \mathbf{P}^2$. This is why we followto a well-established classical tradition to begin any theory from detailed treatment of the first nontrivial example.

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## 1 Introduction

Let $\mathcal{M}_r(c_1, c_2)$ be a moduli space of the Mumford-Takemoto [31] stable vector bundles $\mathcal{E}$ of rank $r$ on a projective plane $\mathbf{P}^2$ with Chern classes $c_1, c_2$. Twisting with a line bundle $\mathcal{O}(d)$

$$\mathcal{E} \rightsquigarrow \mathcal{E} \otimes \mathcal{O}(d)$$

defines an isomorphism

$$\mathcal{M}_r(c_1, c_2) \cong \mathcal{M}_r(c_1 + rd, c_2 + c_1(r-1)d + \frac{r(r-1)}{2}d^2).$$

Hence $\mathcal{M}_r(c_1, c_2)$ depends on the discriminant

$$-D := (r-1)c_1^2 - 2rc_2 \tag{1}$$

only. So we will write $\mathcal{M}_r(D)$ instead of $\mathcal{M}_r(c_1, c_2)$.

It is well known that for a stable vector bundle $\mathcal{E}$ the discriminant $D(\mathcal{E})$ is positive and the moduli space $\mathcal{M}_r(D)$ is an irreducible nonsingular quasiprojective variety of dimension $D - r^2 + 1$ (or empty)[22, 2, 10, 7]. In this paper I shall be interested in topological properties of this moduli space $\mathcal{M}_r(D)$ and of its natural compactification

$$\overline{\mathcal{M}}_r(D) := (\textit{moduli space of semistable torsion free scheaves}).$$

In this definition we have to use the Gieseker - Maruyama (semi)stability and to identify the sheaves with the same stable composition factors to obtain separable projective moduli space $\overline{\mathcal{M}}_r(D)$ [22, 11]. But in fact I can not treat semistable bundles and sheaves properly. A semistable object lies on the boundary of the moduli space and seems to escape our understanding. This leads to a singularity of the moduli space in semistable points. Some authors enjoy singularity, I do not like it however. Therefore I consider here only the case when stability is equivalent to semistability, namely

$$(D(\mathcal{E}), rank(\mathcal{E})) = 1 \tag{2}$$

In this case the fraction $c_1/rank$ is irreducible and semistability is impossible for arithmetical reasons. Hence the moduli space $\overline{\mathcal{M}}_r(D)$ is a projective nonsingular algebraic variety and all the notions of stability are equivalent.[1]

Thus God gave us two nice objects $\mathcal{M}_r(D)$ and $\overline{\mathcal{M}}_r(D)$ to study. Up to now, for general $r$ and $D$ not very much is known about their topology and geometry, save for the above mentioned results on smoothnes and irreducibility.

[1] In fact $\overline{\mathcal{M}}_r(D)$ is smooth under a slightly less strict condition $g.c.d.(c_1(\mathcal{E}), rank(\mathcal{E}), \chi(\mathcal{E})) = 1$, but in this case we would have to use some additional technical tools.

Strome [30] and Dreze [8] have determined the Picard group $Pic\overline{\mathcal{M}}_r(D)$. Its rank does not exceed two. If the rank is zero, then $\overline{\mathcal{M}}_r(D)$consists of only one point and the vector bundle $\mathcal{E}$ is called *exceptional*, [7]. The exceptional vector bundles are closely related to the moduli spaces $\overline{\mathcal{M}}_r(D)$with Picard number one, having a very explicit description [8]. Dreze knows a recurrent formula for the Betti numbers of these moduli spaces.

In the general case $rk\, Pic\,\overline{\mathcal{M}}_r(D) = 2$ . It follows that the second Betti number $b_2(\overline{\mathcal{M}}_r(D)) = 2$.

Further information on the moduli space $\overline{\mathcal{M}}_r(D)$, in particular when it is not empty, one can find in [7, 28].

In the simplest case $r = 2$ there is some additional information.

The variety $\overline{\mathcal{M}}_2(D)$ is rational [23, 24, 9] (at least in our case $(c_1, r) = 1$).

It is known also [4, 20] that the Euler characteristic of the moduli space of vector bundles is equal to :

$$\chi(\mathcal{M}_2(D)) = \begin{cases} 3H(D) & \text{if } D \text{ is odd} \\ 3H(D) - \frac{3}{2}\sigma_0(\frac{D}{4}) & \text{if } D \text{ is even,} \end{cases} \tag{3}$$

where $\sigma_0(n)$ is the sum of divisors, and $H(D)$ is the Hurwitz class number

$$H(D) = \left( \begin{array}{c} \textit{number of integer binary quadratic forms } Q \textit{ of} \\ \textit{discriminant -D counted with the weight } 2/Aut(Q) \end{array} \right) \tag{4}$$

## 1.1 Results

The main object of this paper is to give an exposition of a method that allows us to find all the cohomology groups of the complete moduli space $\overline{\mathcal{M}}_r(D)$ and to investigate its birational nature. By means of this method, if one has enough time to spend, it is easy to fill the following table of the Betti numbers of the moduli space of rank two sheaves $\overline{\mathcal{M}}_2(D)$ (the odd Betti numbers are equal zero):

| $D$ | $b_0$ | $b_2$ | $b_4$ | $b_6$ | $b_8$ | $b_{10}$ | $b_{12}$ | $b_{14}$ | $b_{16}$ | $b_{18}$ | $b_{20}$ | $b_{22}$ | $b_{24}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | | | | | | | | | | | | |
| 7 | 1 | 2 | 3 | 2 | 1 | | | | | | | | |
| 11 | 1 | 2 | 6 | 9 | 12 | 9 | 6 | 2 | 1 | | | | |
| 15 | 1 | 2 | 6 | 13 | 24 | 35 | 41 | 35 | 24 | 13 | 6 | 2 | 1 |

This table confirms the above mentioned results on connectness ($b_0 = 1$) and Picard number ($b_2 = 2$) of the moduli space. The method also shows that the Betti numbers $b_i(\mathcal{M}_2(D))$ stabilize as $D \to \infty$. For example, $b_4 = 6$ for $D \geq 11$.

I have not ovecome the arising combinatorial problems and therefore I can not give a general formula for a Betti number. Nevertheless the Euler characteristic of the moduli spaces $\overline{\mathcal{M}}_2(D)$ may be interpreted as a coefficient of a modular function:

$$\frac{1}{3}\sum_{n=1}^{\infty}\chi(\overline{\mathcal{M}}_2(4n-1))q^{n-\frac{1}{4}} = \frac{1}{\eta(q)^6}\sum_{n=1}^{\infty}H(4n-1)q^n, \tag{5}$$

where $\eta(q) = q^{\frac{1}{24}}\prod_1^{\infty}(1-q^n)$ is the Dedekind $\eta$-function and $H(D)$ is the Hurwitz class number (4).

There are similar formulae relating the Euler characteristic of the moduli space of stable bundles $\mathcal{M}_r(D)$ and semistable sheaves $\overline{\mathcal{M}}_r(D)$ for arbitrary rank $r$ (see below).

As a byproduct we get rationality of the moduli space of vector bundles of rank 3 in the case $(D, 3) = 1$.

The method of treatment of the topological properties and of the birational geometry of the moduli space used in this paper is the same as in [4, 20]. It based on a reduction of topological and birational questions concerning $\mathcal{M}(D)$ and $\overline{\mathcal{M}}(D)$ to the corresponding problems concerning invariant spaces $\mathcal{M}(D)^T$ and $\overline{\mathcal{M}}(D)^T$ with respect to the natural action of the maximal torus $T \subset \mathbf{PGL}_2 = Aut\,\mathbf{P}^2$. The following two observations play a crucial role.

i) Let an algebraic torus $T$ act on a smooth algebraic variety $X$. Then the topologies of $X$ and of its fix point set $X^T$ are very closely related [14, 5, 12]. For example, their Euler characteristics are equal:

$$\chi(X) = \chi(X^T) \tag{6}$$

( in fact this is the Lefschetz trace formula applied to a sufficiently general element $t \in T$ ).

If $X$ is a projective variety, then all the cohomology groups of $X$ and their Hodge decomposition may be reconstructed from the corresponding information on the connected components $(X^T)_i$ of $X^T$. More precisely, let

$\tau : G_m \to T$ be a one dimensional subtorus of $T$ such that $X^\tau = X^T$. For each connected component $(X^T)_i = (X^\tau)_i$ fix a point $x_i \in (X^T)_i$ and put

$$n_i = \begin{pmatrix} \textit{the number of weights } \chi \textit{ of the torus } T \textit{ in} \\ \textit{the tangent space } \mathcal{T}(x_i) \textit{ such that } \langle \chi, \tau \rangle > 0 \end{pmatrix}$$

Then there is a natural isomorphism

$$H^{pq}(X) = \bigoplus_i H^{p-n_i, q-n_i}(X_i^T) \tag{7}$$

This formula is due to Bialynicki-Birula [5] in the case of isolated fix points and to Ginsburg [12] in general. To prove it Bialinycki-Birula constructed a decomposition $X = \coprod X_i$ on a locally closed nonsingular $T$-invariant subscheme and morphisms $\gamma_i : X_i \to (X^T)_i$ which are $G_m$-fibrations( the action of multiplicative group $G_m$ defined by $\tau : G_m \to T$). At the present time it is known that these fibrations are in fact vector bundles [3, 21].[2] In particular the variety $X$ is birationally equivalent to a product of an invariant component $(X^T)_i$ and affine space $\mathbf{A}^n$.

Ginsburg's proof uses essentially the same stratification but in the general situation of simplectic geometry.

To sum up we may say that the invariant space $X^T$ together with the torus representations in the fibers of its normal bundle is a skeleton on which the whole body of $X$ is spanned and from which the topology and biratinal geometry of $X$ may be reconstructed.

**ii)** The second observation is that for the above moduli spaces $\mathcal{M}_r(D)$ and $\overline{\mathcal{M}}_r(D)$ and a maximal torus $T \subset Aut\mathbf{P^2} = \mathbf{PGL_2}$ the corresponding invariant spaces $\mathcal{M}(D)^T$ and $\overline{\mathcal{M}}(D)^T$ have a natural interpretation as moduli spaces of $T$-equivariant vector bundles or torsion free sheaves $\mathcal{E}$on $\mathbf{P^2}$:

$$\mathcal{M}_r(D)^T = \begin{pmatrix} \textit{moduli space of } T\textit{-equivariant vector bundles } \mathcal{E} \textit{ of rank} \\ r \textit{ up to twisting with character } \mathcal{E} \mapsto \mathcal{E} \otimes \chi, \ \chi \in \hat{T} \end{pmatrix}$$

An interpretation of $\overline{\mathcal{M}}(D)^T$ is similar to that of $\mathcal{M}(D)^T$. Thus to apply i) to the moduli spaces $\mathcal{M}_r(D)$ and $\overline{\mathcal{M}}_r(D)$ we need:

- Description of the moduli space of the $T$-equivariant vector bundles and sheaves.

[2] The author is indebted to H.Kraft who has pointed out to him these papers and explained the result.

- The torus representation in the tangent space of the moduli, which is known to be equal to $Ext^1(\mathcal{E},\mathcal{E})$ [23].

We may replace $\mathbf{P}^2$ by an arbitrary nonsimgular complete toric variety $X$, our approach being applicable in this mor general situation as well. In the next two sections we give an exposition of the general theory of equivariant vector bundles and torsion free sheaves on toric varieties.

## 1.2 Toric vector bundles

Here we give a digest of the theory of toric vector bundles as it has been developed in [18, 19, 20]. See also earlier papers of Kaneyama [16, 17], used by Bertin and Elengzwajg in [4].[3]

Let $X$ be a complete nonsingular toric variety [6, 26]. This means that an action of an algebraic torus $T$ is defined on $X$ and that $X$ contains an open orbit on which this action is free. An *equivariant* or *toric* vector bundle on $X$ is a vector bundle $p : \mathcal{E} \to X$ together with an equivariant $T$-structure,i.e. with an action of the torus $T$ on $\mathcal{E}$ which makes the following diagram commute for all $t \in T$:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{t} & \mathcal{E} \\ p\downarrow & & p\downarrow \\ X & \xrightarrow{t} & X \end{array}$$

Recall that a toric variety $X$ is defined by a fan $\Sigma = \Sigma(X)$ in the lattice $\hat{T}^{\circ}$ dual to the lattice of characters $\hat{T}$ of $T$. The cones $\sigma \subset \Sigma$ are in one to one correspondence with the orbits $O_\sigma \subset X$, in such a way that $\tau \subset \sigma \Leftrightarrow O_\sigma \subset \overline{O}_\tau$; $\dim\sigma = \text{codim} O_\sigma$. In particular the orbits of codimention 1 correspond to the one dimensional cones of $\Sigma$. We denote by $|\Sigma|$ the set of primitive generators of this cones. For $\sigma \in \Sigma$ we let $|\sigma| = \sigma \cap |\Sigma|$.

**Definition 1.2.1** *A family of filtrations $E^\alpha, \alpha \in A$ of a vector space $E$ is called split if the multifiltered space $(E; E^\alpha, \alpha \in A)$ can be decomposed in a direct sum of multifiltered spaces of dimension one.*

[3] Unfortunately while working on my papers [18, 19, 20] I was not aware of these articles. I use this opportunity to apologize to their authors. The overlap with [16, 17] both in methods and results seems to be minor; my note [20] may be considered as a supplement to [4].

In this definition, and henceforth, the filtrations are assumed to be decreasing and full: $E^\alpha(i) = 0, i \gg 0$ ; and $E^\alpha(i) = E, i \ll 0; i \in \mathbf{Z}$.

**Theorem 1.2.2** ([18]) *The category of toric bundles on a toric variety $X = X(\Sigma)$ is equivalent to the category of multifiltered vector spaces $(E; E^\alpha, \alpha \in |\Sigma|)$, satisfying the following compatibility condition:*

$$\forall \sigma \in \Sigma, \textit{the family of filtrations } (E^\alpha; \alpha \in |\sigma|) \textit{ is split.} \tag{8}$$

The equivalence of the categories is established by associating to each bundle $\mathcal{E}$ the fiber $E = \mathcal{E}(x_0)$ at a fixed point $x_0$ of the open orbit. The filtrations on $E$ are formed as follows. For every orbit $O_\alpha, \alpha \in |\Sigma|$, of codimension 1, choose a point $x_\alpha \in O_\alpha$ and put

$$E^\alpha(i) = \{e \in E | \exists \lim_{tx_0 \to x_\alpha} f(te)(te); t \in T\} \tag{9}$$

where f(x) is a rational function on $X$ with a pole of order $i$ at $\overline{O}_\alpha$. It turns out that filtrations $E^\alpha(i)$ satisfy the compatibility condition 8) and uniquely determine the vector bundle $\mathcal{E}$.

**Example 1.2.3** *For the projective plane this theorem gives us an equivalence of categories*

$$\begin{pmatrix} \textit{Toric vector} \\ \textit{bundles on } \mathbf{P}^2 \end{pmatrix} \leftrightarrow \begin{pmatrix} \textit{Vector spaces } E \textit{ with a triplet} \\ \textit{of filtrations } E^\alpha, E^\beta, E^\gamma \end{pmatrix} \tag{10}$$

It is easy to see that a pair of filtrations is always split. Hence the compatibility condition (8) in this example is automatically satisfied.

It follows from the theorem that all the geometric properties and invariants of toric vector bundles can be expressed in terms of filtrations. Below we give a small glossary for translation from one language to another.

## Glossary

*Notations.* Let $\mathcal{E}$ be an equivariant vector bundle on a toric variety $X = X(\Sigma)$ and let $(E; E^\alpha, \alpha \in |\Sigma|)$ be the corresponding multifiltered space of theorem 1.2.2. For any $\sigma \in \Sigma$ and $\chi \in \hat{T}$ we define the vector spaces

$$E^\sigma(\chi) = \bigcap_{\alpha \in |\sigma|} E^\alpha(\langle \chi, \alpha \rangle),$$

$$E^{[\sigma]}(\chi) = E^\sigma(\chi) / \sum_{i_\alpha} \bigcap_{\alpha \in |\sigma|} E^\alpha(i_\alpha),$$

where in the second formula the sum is taken over all the integers $i_\alpha; \alpha \in |\sigma|$ such that $i_\alpha \geq \langle \chi, \alpha \rangle$ and at least for one $\alpha \in |\sigma|$ the inequality is strict. Sometimes it is convenient to use the detailed notation:

$$E^\sigma(\chi) := E^{(\alpha_1, \dots, \alpha_k)}(i_1, \dots, i_k),$$

$$E^{[\sigma]}(\chi) := E^{[\alpha_1, \dots, \alpha_k]}(i_1, \dots, i_k),$$

where $\sigma = \langle \alpha_1, \dots, \alpha_k \rangle$, $\alpha_i \in |\Sigma|$ and $i_p = \chi(\alpha_p)$.

For example for an one-dimensional cone $E^{[\alpha]}(i) = E^\alpha(i)/E^\alpha(i+1)$.

**Representations in fibers.** Let $O_\sigma \subset X, \sigma \in \Sigma$ be an orbit of a torus $T$ on $X$ and let $T_\sigma \subset T$ be the stabiliser of a point $x_\sigma \in O_\sigma$. Then $T_\sigma$ acts in the fiber $\mathcal{E}(x_\sigma)$ and

$$m(\chi, \mathcal{E}(x_\sigma)) := \begin{pmatrix} \textit{multiplicity of character} \\ \chi \in \hat{T}_\sigma \textit{ in the fiber } \mathcal{E}(x_\sigma) \end{pmatrix} = \dim E^{[\sigma]}(\chi) \qquad (11)$$

(strictly speaking on the right hand side instead of the character $\chi \in \hat{T}_\sigma$ should be one of its extensions on $T$).

**Characteristic classes.** Let $X_\alpha = \overline{O}_\alpha, \alpha \in |\Sigma|$ denote the closure of an orbit of codimension one and at the same time its class in the Chow or in the cohomology ring. Then the following formulae for the Chern character and for the full Chern class hold:

$$ch(\mathcal{E}) = \sum_{\sigma \in \Sigma, \chi \in \hat{T}_\sigma} (-1)^{codim \sigma} \dim E^{[\sigma]}(\chi)\, e^{\sum_{\alpha \in |\sigma|} \langle \chi, \alpha \rangle X_\alpha},$$

$$c(\mathcal{E}) = \prod_{\sigma \in \Sigma, \chi \in \hat{T}_\sigma} (1 + \sum_{\alpha \in |\sigma|} \langle \chi, \alpha \rangle X_\alpha)^{(-1)^{codim \sigma} \dim E^{[\sigma]}(\chi)}. \qquad (12)$$

For example

$$c_1(\mathcal{E}) = \sum_{i\in\mathbf{Z},\alpha\in|\Sigma|} i \dim E^{[\alpha]}(i) X_\alpha. \qquad (13)$$

**Cohomology.** Let $H^*(X,\mathcal{E})_\chi$ be the isotypical component of the character $\chi \in \hat{T}$ in cohomology group $H^*(X,\mathcal{E})$. It can be evaluated by means of the following complex $C_*(\mathcal{E},\chi)$:

$$0 \leftarrow E \leftarrow \bigoplus_{\dim\sigma=1} E^\sigma(\chi) \leftarrow \bigoplus_{\dim\sigma=2} E^\sigma(\chi) \leftarrow \cdots \leftarrow \bigoplus_{\dim\sigma=n} E^\sigma(\chi) \leftarrow 0, \qquad (14)$$

with a differential given by the formula

$$d^k = \sum_{\dim\sigma=k} d^\sigma; \quad d^\sigma = \sum_i (-1)^i \varepsilon_i$$

where $\varepsilon_i : E^{\sigma_i} \hookrightarrow E^\sigma$ is a natural inclusion corresponding to the i-th face $\sigma_i \subset \sigma$.

In this notations we have a natural isomorphism:

$$H^p(X,\mathcal{E})_\chi = H_{n-p}(C_*(\mathcal{E},\chi)); \; n = \dim X.$$

For the zero- and the highest degree cohomology one has the followings formulae:

$$H^0(X,\mathcal{E})_\chi = \bigcap_{\alpha\in|\Sigma|} E^\alpha(\chi),$$

$$H^n(X,\mathcal{E})_\chi = \frac{E}{\sum_{\alpha\in|\Sigma|} E^\alpha(\chi)}.$$

All the cohomology groups for the projective space $\mathbf{P^n}$ may be written explicitly:

$$H^{n-p}(\mathbf{P^n},\mathcal{E})_\chi = \frac{E^0(\chi)\cap\ldots\cap E^{p-1}(\chi)\cap\sum_{k\geq p} E^k(\chi)}{\sum_{k\geq p} E^0(\chi)\cap\ldots\cap E^{p-1}(\chi)\cap E^k(\chi)},$$

where $E^\alpha; \alpha = \overline{0,n}$ are the filtrations, defining the bundle $\mathcal{E}, 0 < p < n$.

**Euler characteristic and a trace formula.** The complex (14) gives the following equality for the Euler characteristic:

$$\sum_p (-1)^p \dim H^p(X,\mathcal{E})_\chi = \sum_{\sigma\in\Sigma} (-1)^{codim\sigma} \dim E^\sigma(\chi).$$

This equality may be written also as a *trace formula* for $t \in T$:

$$\sum_p (-1)^p Tr(t|H^p(X,\mathcal{E})) = \sum_\Delta \frac{Tr(t|\mathcal{E}(x_\Delta))}{\prod_{\omega\in\Delta^*}(1-\omega^{-1}(t))}, \tag{15}$$

where $\Delta \in \Sigma^{(n)}$ ranges over the cones of maximal dimention $n = \dim X$; $x_\Delta \in X^T$ is a fixed point corresponding to $\Delta$; $\Delta^*$ is the basis of the character group $\hat{T}$ dual to the basis $|\Delta|$ of the lattice $\hat{T}^\circ$.

The trace on the right hand side may be expressed in terms of the filtrations $E^\alpha$ as explained in the first item of the Glossary.

**Other structure groups.** For a toric vector bundle $\mathcal{E}$ with a structure group $G$ different from $GL(E)$, theorem 1.2.2 requires some modifications. Up to discrete parameters the filtration $E^\alpha$ is determined by the parabolic subgroup

$$P^\alpha := \{g \in GL(E)|\ g(E^\alpha) = E^\alpha\}. \tag{16}$$

The disrete parameters of the filtration $E^\alpha$ (that is a numbering of the subspaces) defines a character

$$\omega^\alpha : P^\alpha \to G_m$$

$$\omega^\alpha(\mathrm{diag}(x_1,\ldots,x_n)) = {x_1}^{a_1}\cdots {x_n}^{a_n}, \tag{17}$$

where $a_k \in \mathbf{Z}$ are the jump points of a function $\dim E^\alpha(i)$,

$$\dim E^\alpha(a_k+1) < k \le \dim E^\alpha(a_k).$$

In this language the compatibility condition (8) of theorem 1.2.2 means that

$$\forall \sigma \in \Sigma, \quad \bigcap_{\alpha\in|\sigma|} P^\alpha \ \textit{contains a maximal torus}\ . \tag{18}$$

**Theorem 1.2.4** *Let $X$ be a nonsingular toric variety and let $G$ be a reductive group. Then principal toric $G$-bundles on $X$ are parametrised by the families $(P^\alpha,\omega^\alpha)$, $\alpha \in |\Sigma|$, of parabolic subgroups $P^\alpha \in G^\circ$ of the dual group $G^\circ$ and its characters $\omega^\alpha$ satisfying the compatibility condition (18).*

This theorem allows one to formulate the main problems on toric vector bundles in the framework of the theory of reductive groups. A resonable generalisation often simplifies a mathematical problem. In any case the "highest vectors" $\omega^\alpha$ from (17) play an essential role in the study of stability.

It follows from this theorem that up to some discrete parameters the space of toric vector bundles on $X = X(\Sigma)$ with a reductive structure group $G$ is parametriesed by the simplicial maps of the fan $\Sigma$ to the complex $\mathcal{P}(G)$ whose vertices are the parabolic subgroups $P \subset G$, while the simplices are formed by the families of subgroups $P^\alpha, \alpha \in A$, containing a common maximal torus. So the complex $\mathcal{P}(G)$ may be considered as a classifying space for toric bundles. Some of its properties one may find in [18, 19].

**Stability.** The following conditions on a toric vector bundle $\mathcal{E}$ determined by a compatible family of filtrtions $(E^\alpha | \alpha \in |\Sigma|)$ of vector space $E$ are equvalent:

1. $\mathcal{E}$ is Mumford - Takemoto stable [31, 27];
2. The family of subspaces $E^\alpha(i) \subset E$ is Mumford stable [25] with respect to the action of the group $GL(E)$;
3. For any subspace $F \subset E$ with $0 < \dim F < \dim E$, the following inequality holds

$$\frac{1}{\dim F} \sum_{\alpha,i} \dim F^\alpha(i) < \frac{1}{\dim E} \sum_{\alpha,i} \dim E^\alpha(i), \tag{19}$$

   where $F^\alpha(i) = F \cap E^\alpha(i)$ and the sums are taken over all $\alpha \in |\Sigma|$ and $i > N$; $N \ll 0$.

It is easy to see that inequality (19) is independent of the choice of the sufficiently small N, but we have to fix it to make the sums finite. The same problem appears in the second item as well: the family of subspaces $E^\alpha(i)$ is infinite but almost all of its members are equal to 0 or to $E$ and do not influence stability.

**Example 1.2.5** *Toric vector bundles on the projective plane.*

As we have seen in example 1.2.3 toric vector bundles $\mathcal{E}$ on $\mathbf{P}^2$ are determined by a triplet of filtrtions $E^\alpha$, $E^\beta$, $E^\gamma$ of a vector space $E$. To evaluate the discriminant of $\mathcal{E}$ one can use formulae (12) for the characteristic classes: of $\mathcal{E}$

$$-D(\mathcal{E}) = (r-1){c_1}^2 - 2rc_2 =$$

$$r^2(D(i_\alpha) + D(i_\beta) + D(i_\gamma) + 2cov(i_\alpha, i_\beta) + 2cov(i_\beta, i_\gamma) + 2cov(i_\gamma, i_\alpha)) \tag{20}$$

where the dispersion $D(i_\alpha)$ and the covariance $cov(i_\alpha, i_\beta)$ are defined by the formulae

$$D(i_\alpha) = \frac{1}{r}\sum_i i^2 \dim E^{[\alpha]}(i) - \left(\frac{1}{r}\sum_i i \dim E^{[\alpha]}(i)\right)^2,$$

$$cov(i_\alpha, i_\beta) =$$

$$\frac{1}{r}\sum_{i,j} ij \dim E^{[\alpha,\beta]}(i,j) - \left(\frac{1}{r}\sum_i i \dim E^{[\alpha]}(i)\right)\left(\frac{1}{r}\sum_j j \dim E^{[\beta]}(j)\right).$$

It follows from the formula (20) that the discriminant $D(\mathcal{E})$ depends on the pairwise relative positions of the filtrations $E^\alpha, E^\beta, E^\gamma$ or of the corresponding parabolic subgroups $P^\alpha, P^\beta, P^\gamma$ only (cf.(16)). The relative position of $P^\alpha$ and $P^\beta$ is determined by an element $\pi$ of the Weyl group $W$ (in our case the symmetric group $S_r$) such that the intersection $(P^\alpha)^\pi \cap P^\beta$ contains a Borel subgroup. There is only one such an element $\pi_{\alpha\beta} = \pi(P^\alpha, P^\beta)$ of minimal length which will be refered to as the relative position of the parabolic subgroups $P^\alpha$ and $P^\beta$ or the corresponding filtrations $E^\alpha$ and $E^\beta$. In terms of the relative positions $\pi_{\alpha\beta}$ of filtrations the discriminant may be written as follows

$$-D(\mathcal{E}) = r({\omega_\alpha}^2 + {\omega_\beta}^2 + {\omega_\gamma}^2 + 2({\omega_\alpha}^{\pi_{\alpha\beta}}, \omega_\beta) + 2({\omega_\beta}^{\pi_{\beta\gamma}}, \omega_\gamma) + 2({\omega_\gamma}^{\pi_{\gamma\alpha}}, \omega_\alpha)),$$

where $\omega_\alpha, \omega_\beta, \omega_\gamma$ are the dominant weights defined by (17) and $(*,*)$ is the standard scalar product on the weight lattice (squares of roots are equal to two).

Let us consider the stability condition 3) for a vector bundle of small rank.

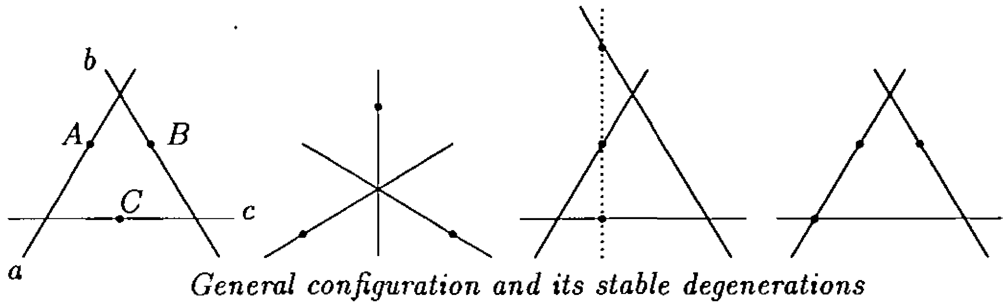
i) $rk\,\mathcal{E} = 2$, i.e. $\dim E = 2$. Put $a_\alpha = \#\{i| \dim E^\alpha(i) = 1\}$. Then from the stability condition 3) one easily deduces that

$$\mathcal{E} \textit{ is stable} \Leftrightarrow \begin{pmatrix} \textit{the filtrations } E^\alpha, E^\beta, E^\gamma \textit{ are in general position and} \\ \textit{the numbers } a_\alpha, a_\beta, a_\gamma \textit{ satisfy the triangle inequalities} \end{pmatrix}.$$

ii) $rk\,\mathcal{E} = 3$. Let

$$a = \#\{i| \dim E^\alpha(i) = 2\},\ A = \#\{i| \dim E^\alpha(i) = 1\},$$

and define $b, B; c, C$ in a similar way using $E^\beta$ and $E^\gamma$. Then we may represent the filtrations $E^\alpha, E^\beta, E^\gamma$ as a configuration of points and lines with multiplicities on the projective plane $\mathbf{P}(E)$:




*General configuration and its stable degenerations*

*Stable configurations with a noncomplete flag*

The picture contains all types of configurations with the trivial group of automorphisms (it is a necessary condition for stability). The precise stability criterion depends on the type of the configuration in the following way. Put $s = a + b + c$ and let $S = A + B + C$. Our notations are adjusted in such a way that if we replace a bundle $\mathcal{E}$ by its dual $\mathcal{E}^*$ the upper and the lower case letters will interchange.

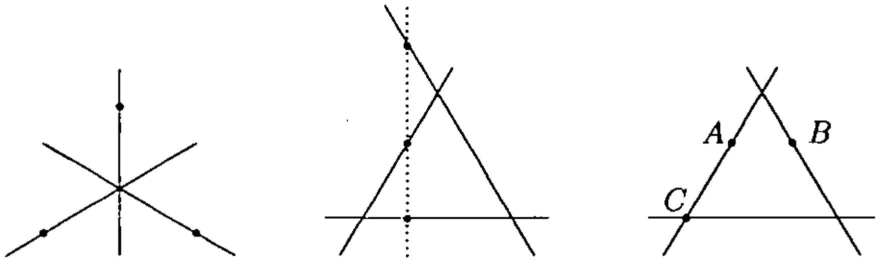
1) If the filtrations are in a general position then stability is equivalent to the following inequalities

$$a > \frac{s-S}{3};\ a + A < \frac{s+2S}{3} \text{ if } s \geq S$$

$$A > \frac{S-s}{3};\ A + a < \frac{S+2s}{3} \text{ if } s \leq S$$

combined with the corresponding inequalities for $b, B$ and $c, C$. A general configuration is determined by the complex parameter $z \neq 0, 1, \infty$ (cross ratio).

2) If $(c_1(\mathcal{E}),3) = 1 (\Leftrightarrow s \not\equiv S \bmod 3)$ then a general stable configuration has the following stable degenerations:




if $S > s$ if $S < s$ $a - B < \frac{1}{3}(s - S)$

$b - C > \frac{1}{3}(s - S)$

Exactly one of the first two pictures is stable; from the six configurations of the third type two are stable. Thus we obtain three stable configurations which fill the three gaps $0, 1, \infty$ in 1). The full space of stable configurations in our case $(c_1, 3) = 1$ is $\mathbf{P}^1$.

This example shows the spirit of the main problem arising in our reduction of the full moduli space $\mathcal{M}_r(D)$to its toric part $\mathcal{M}_r(D)^T$.

**Problem 1.2.6** *Describe as precise as posssible the moduli space of stable triplets of filtrations $E^\alpha, E^\beta, E^\gamma$ of a vector space $E$.*

We denote this moduli space by $\mathcal{F}_r(\omega) = \mathcal{F}_r(\omega_\alpha, \omega_\beta, \omega_\gamma)$ where $r = \dim E$ and $\omega = (\omega_\alpha, \omega_\beta, \omega_\gamma)$ are the dominant weights (16) which fix dimensions of the filtrations $E^\alpha$, $E^\beta$, $E^\gamma$. This space has a natural stratification

$$\mathcal{F}_r(\omega) = \bigsqcup_{\pi} \mathcal{F}_r(\omega; \pi), \tag{21}$$

where $\mathcal{F}_r(\omega; \pi)$ is a moduli space of stable filtrations $E^\alpha$, $E^\beta$, $E^\gamma$ of a given type $\omega = (\omega_\alpha, \omega_\beta, \omega_\gamma)$ and given pairwise relative positions $\pi = (\pi_{\alpha\beta}, \pi_{\beta\gamma}, \pi_{\gamma\alpha})$.

If the "first Chern class" of the triplet

$$c_1(\omega) = \sum_i i(\dim E^{[\alpha]}(i) + \dim E^{[\beta]}(i) + \dim E^{[\gamma]}(i)) = (\omega_\alpha + \omega_\beta + \omega_\gamma, \rho)$$

is coprime to $r = \dim E$ then $\mathcal{F}_r(\omega_\alpha, \omega_\beta, \omega_\gamma)$ is a projective nonsingular variety (here $\rho$ is the sum of the highest weights of the fundamental representations $\bigwedge^k E$). The following theorems contain almost all the known facts about its structure.

**Theorem 1.2.7** *If $(c_1, r) = 1$ then the moduli space $\mathcal{F}_r(\omega)$ is a projective nonsingular irreducible variety and all its stratas $\mathcal{F}_r(\omega; \pi)$ are irreducible nonsingular of dimension*

$$\ell(\pi_{\alpha,\beta}) + \ell(\pi_{\beta,\gamma}) + \ell(\pi_{\gamma,\alpha}) - r^2 + 1 = \dim \mathcal{F}_r(\omega; \pi) \quad .$$

*( $\ell(\pi)$ is the length of $\pi$).*

The proof is based on the following calculations. Smoothness of the moduli space $\mathcal{M}_r(D)$implies nonsingularity of $\mathcal{M}_r(D)^T$ and hence $\mathcal{F}_r(\omega; \pi)$. The tangent space of the last variety is equal to

$$Ext^1(\mathcal{E}, \mathcal{E})^T = H^1(\mathbf{P}^2, \mathcal{E}nd\,\mathcal{E})^T$$

. Moreover stability and smoothness imply

$$H^0(\mathbf{P}^2, \mathcal{E}nd\,\mathcal{E}) = \mathbf{C};\ H^2(\mathbf{P}^2, \mathcal{E}nd\,\mathcal{E}) = 0.$$

All this allows us to find $\dim \mathcal{F}_r(\omega; \pi) = \dim H^1(\mathbf{P}^2, \mathcal{E}nd\,\mathcal{E})^T$ from the trace formula. It depends only on the pairwise positions of the filtrations. Thus all the components of $\mathcal{F}_r(\omega; \pi)$ have the same dimention.

To prove that $\mathcal{F}_r(\omega; \pi)$ is irreducible we determine the number of its rational points over the finite field $\mathbf{F_q}$. This may be done by calculations in the Hecke algebra $H_q$ given by the generators $T(\pi), \pi \in W$ and the relations

$$\begin{array}{ll} T(\pi_1)T(\pi_2) = T(\pi_1\pi_2), & \textit{if } \ell(\pi_1\pi_2) = \ell(\pi_1)\ell(\pi_2); \\ T(s^2) = q + (q-1)T(s), & s \textit{ is a fundamental reflection.} \end{array}$$

It is essential for us that $H_q$ has the same multiplication table as the algebra of the double classes $B\backslash G/B$ over $\mathbf{F_q}$. It follows that the following equality holds

$$\begin{pmatrix} \textit{number of flags } F^\alpha, F^\beta, F^\gamma \textit{ in} \\ \textit{pairwise positions } \pi_{\alpha\beta}, \pi_{\beta\gamma}, \pi_{\gamma\alpha} \end{pmatrix} = |Flag(\mathbf{F_q})| \begin{pmatrix} \textit{coefficient at 1 in} \\ T(\pi_{\alpha\beta})T(\pi_{\beta\gamma})T(\pi_{\gamma\alpha}) \end{pmatrix}$$

It is easy to see that the coefficient at 1 in the product $T(\pi_{\alpha\beta})T(\pi_{\beta\gamma})T(\pi_{\gamma\alpha})$ is a polinomial of $q$ with the leading coefficient equal to 1. This means that the variety $\mathcal{F}_r(\omega; \pi)$ has only one component of the highest dimension. But as we have already proved all its components have the same dimension, and therefore it follows that $\mathcal{F}_r(\omega; \pi)$ is irreducible.

**Theorem 1.2.8** *Let $V(\omega_\alpha)$, $V(\omega_\beta)$, $V(\omega_\gamma)$ be three irreducible representations of the group $GL(E)$ with the highest weights $\omega_\alpha$, $\omega_\beta$, $\omega_\gamma$ respectively. Then the following conditions are equivalent:*

*i) Filtrations $E^\alpha$, $E^\beta$, $E^\gamma$ of the types $\omega_\alpha$, $\omega_\beta$, $\omega_\gamma$ in general position are semistable;*

*ii) Tensor product $V(\omega_\alpha) \otimes V(\omega_\beta) \otimes V(\omega_\gamma)$ contains one of the representations $(\det E)^m \bigwedge^k E$ as its component.*

Appearance of the irreducible representations $V(\omega)$ seems very surprising. It may be explained as follows. By the stability criterion we have to check inequality (19) for every subspace $V \subset E$. This inequlity depends only on the relative position of the subspace $V$ with respect to each of the filtrations $E^\alpha$, $E^\beta$, $E^\gamma$. The subspaces of a fixed relative position with respect to one of the filtrations, say $E^\alpha$, form a *Schubert cell* $\sigma_a$ in the Grassmann variety $G_p^q = \{V \subset E |\, \dim E = p; \operatorname{codim} E = q\}$. Here $a$ is the Young diagram or the partition

$$a = (a_1, a_2, \ldots, a_p),\ q \geq a_1 \geq a_2 \geq \ldots \geq a_p \geq 0.$$

If the filtrations $E^\alpha$, $E^\beta$, $E^\gamma$ are in a general position then the following conditions are equivalent:

i) There is a subspace $V \subset E$ in positions $a, b, c$ with respect to the filtrations $E^\alpha$, $E^\beta$, $E^\gamma$.

ii) Schubert cells $\sigma_a$, $\sigma_b$, $\sigma_c$ have nontrivial intersection.

iii) Schubert cycles $\overline{\sigma}_a$, $\overline{\sigma}_b$, $\overline{\sigma}_c$ have nonzero product in the cohomology or in the Chow ring of the Grassmannian $G_p^q$.

iv) Tensor product $V_a \otimes V_b \otimes V_c$ of the irreducible $SL(V)$ representations with the Young diagrams $a, b, c$ contains a nonzero $SL(V)$ invariant.

The last item is related to the previous one by *the Giambelli formula* [13]

$$\sigma_a = \begin{vmatrix} \sigma_{a_1} & \sigma_{a_1+1} & \cdots & \sigma_{a_1+p-1} \\ \sigma_{a_2-1} & \sigma_{a_2} & \cdots & \sigma_{a_2+p-2} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{a_p-p+1} & \sigma_{a_p-p+2} & \cdots & \sigma_{a_p} \end{vmatrix}.$$

This formula is similar to the *Weyl determinant* for an irreducible representation $V_a$ of the general linear group. This similarity leads to the equality

$$\begin{pmatrix} \textit{multiplicity of } \sigma_c \textit{ in the de-} \\ \textit{composition of product } \sigma_a\sigma_b \end{pmatrix} = \begin{pmatrix} \textit{multiplicity of the component} \\ V_c \textit{ in the tensor product } V_a \otimes V_b \end{pmatrix}.$$

It follows that

$$\sigma_a\sigma_b\sigma_c \neq 0 \Leftrightarrow (V_a \otimes V_b \otimes V_c)^{SL(V)} \neq 0.$$

So we heve a description of the cone of stable weights $\omega = (\omega_\alpha, \omega_\beta, \omega_\gamma)$ in purely group representational terms. Then making use of the Littlewood-Richardson rule [15] we can compare it with the cone generated by the weights satisfying condition ii) of the theorem. It turns out that they coincide.

## 1.3 Toric torsion free sheaves

We begin with a theorem that describes torsion free sheaves in terms of linear algebra in the same way as theorem 1.2.2 describes vector bundles. To state the result we need some definitions and notation. Let $X = X(\Sigma)$ be a nonsingular toric variety with the fan $\Sigma$. For a vector space $E$ we will denote by $\{E^\sigma | \sigma \in \Sigma\}$ a family of the *decreasing* multifiltrations

$$E^\sigma(I) = E^{(\alpha_1,\alpha_2,\ldots,\alpha_k)}(i_1, i_2, \ldots, i_k)$$

for $\sigma = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \Sigma$ and $I = (i_1, i_2, \ldots, i_k), i \in \mathbf{Z}$. Often it is convenient to regard a multifiltration as a function of a character:

$$E^\sigma(\chi) := E^{(\alpha_1,\alpha_2,\ldots,\alpha_k)}(\chi(\alpha_1), \chi(\alpha_2), \ldots, \chi(\alpha_k))$$

**Definition 1.3.1** *A family of multifiltrations $(E^\sigma | \sigma \in \Sigma)$ is said to be compatible if the following condition holds for every pair of cones $\tau \subset \sigma$, $\tau = (\alpha_1, \alpha_2, \ldots, \alpha_p)$, $\sigma = (\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ :*

$$E^\tau(i_1, i_2, \ldots, i_p) = E^\sigma(i_1, i_2, \ldots, i_p, -\infty, \ldots, -\infty) \tag{22}$$

It follows from the definition that a compatible family of multifiltrations is completely determined by the filtrations $E^\Delta$ for the *maximal cone* $\Delta \in \Sigma$.

**Theorem 1.3.2** *Let $X = X(\Sigma)$ be a nonsingular toric variety. Then*

*i) The category of the toric torsion free sheaves $\mathcal{E}$ on $X$ is equivalent to the category of vector spaces $E$ with a compatible family of multifiltrations $(E^\sigma | \sigma \in \Sigma)$.*

*ii) For a toric torsion free sheaf $\mathcal{E}$ its bidual sheaf $\mathcal{E}^{**}$ is defined by the multifiltrations*

$$E^{**(\alpha_1,\ldots,\alpha_p)}(i_1, \ldots, i_p) = E^{\alpha_1}(i_1) \cap \ldots \cap E^{\alpha_p}(i_p)$$

*iii) Torsion free sheaf $\mathcal{E}$ is reflexive (i.e. equal to its bidual $\mathcal{E}^{**}$) if and only if* $\forall\sigma = (\alpha_1, \dots, \alpha_p) \in \Sigma$

$$E^{(\alpha_1,\dots,\alpha_p)}(i_1, \dots, i_p) = E^{\alpha_1}(i_1) \cap \dots \cap E^{\alpha_p}(i_p)$$

*The category of the toric reflexive sheaves is equivalent to the category of vector spaces $E$ with a family of filtrations $(E^\alpha(i)|\alpha \in |\Sigma|)$ and no compatibility condition.*

*iv) Reflexive sheaf $\mathcal{E}$ is a vector bundle (i.e. it is locally free) iff the corresponding filtrations $(E^\alpha|\ \alpha \in |\Sigma|)$ in iii) satisfy the compatibility condition (8) of theorem 1.2.2.*

For a toric vector bundle $\mathcal{E}$ the filtrations $(E^\alpha|\ \alpha \in |\Sigma|)$ in theorem 1.3.2 are the same as in theorem 1.2.2.

To show how the multifiltrations appear let us consider an open affine $T$-equivariant subset $U_\Delta \subset X$ corresponding to a maximal cone $\Delta \in \Sigma$; $U_\Delta = \mathrm{Spec}\mathbf{C}[\omega_1, \dots, \omega_n]$ where $(\omega_1, \dots, \omega_n)$ is a basis of the character lattice $\hat{T}$ dual to the basis $|\Delta|$ of $\hat{T}^\circ$. Then the $T$-equvariant sheaf $\mathcal{E}^\Delta = \mathcal{E}|U_\Delta$ is nothing else but the multigraded module

$$\mathcal{E}^\Delta = \bigoplus_{i_1,\dots,i_n} E^\Delta(i_1, \dots, i_n) \tag{23}$$

over the polinomial ring $\mathbf{C}[\omega_1, \dots, \omega_n]$. Multiplication by a character $\omega_1^{a_1} \dots \omega_n^{a_n}$ defines a morphism

$$\varphi : E^\Delta(i_1, \dots, i_n) \hookrightarrow E^\Delta(i_1 + a_1, \dots, i_n + a_n)$$

which is a monomorphism because the module is torsion free. So the multigrading (23) is completely determined by the natural multifiltration of a limit space

$$E = \lim_{i_1,\dots,i_n \to \infty} E^\Delta(i_1, \dots, i_n).$$

This space is easily identified with a general fiber of $\mathcal{E}$. The compatibility condition (22) allows us to glue together all the modules $\mathcal{E}^\Delta, \Delta \in \Sigma$ in a single sheaf $\mathcal{E}$.

The main difference between the theory of toric vector bundles described in the previous section and the theory of torsion free sheaves is that the latter one has been never published. In all others respects they are quite similar.

**Metatheorem 1.3.3** *The Glossary of the previous section is valid for torsion free toric sheaves.*

The only way to prove this theorem is to write a sheaf-theoretic version of the paper [19]. The author will do it in an appropriate place and at the proper time. But right now I should like to explain how one can use the Glossary for torsion free sheaves. Most of the entries of the Glossary are expressed in terms of the multifiltrations $E^\sigma(\chi)$. In this case no changes are necessary. In some places (e.g. in the formulae for the characteristic classes) a composition factors $E^{[\sigma]}(I)$ of multifiltrations $E^\sigma(I)$ appears. It must be interpreted as a formal linear combination of vector spaces

$$E^{[\sigma]}(i_1,\ldots,i_p) = \Delta_1\ldots\Delta_p E^\sigma(i_1,\ldots,i_p),$$

where $\Delta_k$ is a difference operator

$$\Delta_k E^\sigma(i_1,\ldots,i_k,\ldots,i_p) = E^\sigma(i_1,\ldots,i_k,\ldots,i_p) - E^\sigma(i_1,\ldots,i_k+1,\ldots,i_p)$$

The dimension of $E^{[\sigma]}(I)$ may be negative.

It seems worthwhile to say a little bit more about the *trace formula.* As Serre [29] and Grothendieck taught one has to deal with the Euler characteristic

$$\mathcal{E}(x) := Tor_*(\mathcal{E}, k(x)) = \sum_i (-1)^i Tor_i(\mathcal{E}, k(x)) \tag{24}$$

instead of the fiber $\mathcal{E}(x)$ of a vector bundle $\mathcal{E}$ in the case of an arbitrary sheaf $\mathcal{E}$ (here $k(x)$ is the residue field of a point $x$). We have the same formula for the multiplicity of the character for this fiber as in (11):

$$m(\chi, Tor_*(\mathcal{E}, k(x_\sigma))) = \dim E^{[\sigma]}(\chi). \tag{25}$$

The difference operators $\Delta_i$, which enter in the definition on right hand side of this formula correspond to the Koszul complex for the calculation of the $Tor$'s in the left hand side. This "fiber" satisfies the same trace formula *trace formula* (15):

$$\sum_p (-1)^p H^p(X, \mathcal{E}) = \sum_\Delta \frac{\mathcal{E}(x_\Delta)}{\prod_{\omega\in\Delta^*}(1-\omega^{-1})}. \tag{26}$$

This formula gives us an equality in the character ring of the torus $T$. The multiplicity formula (25) implies that the "fiber" $\mathcal{E}(x_\Delta)$ has the following character:

$$\mathcal{E}(x_\Delta) = \sum_{\chi \in \hat{T}} \chi \dim E^{[\Delta]}(\chi) \tag{27}$$

For our applications we need not a trace formula for cohomology but rather a trace formula for $Ext^i(\mathcal{E}, \mathcal{E})$. The latter groups have a natural interpretation in terms of of the moduli space structure in a neighbourhood of the "point" $\mathcal{E}$[23, 1]:

$$\begin{aligned} Ext^0(\mathcal{E}, \mathcal{E}) &= \mathbf{C}, \textit{for stable sheaves;} \\ Ext^1(\mathcal{E}, \mathcal{E}) &= \textit{(tangent space to the moduli);} \\ Ext^2(\mathcal{E}, \mathcal{E}) &= 0, \textit{if } \mathcal{E} \textit{ is nonsingular point of the moduli space.} \end{aligned} \tag{28}$$

The trace formula for $Ext$ is easily deduced from the above one.

**Proposition 1.3.4** *For any equivariant sheaves $\mathcal{E}$ and $\mathcal{F}$ on a toric variety $X = X(\Sigma)$ the following equality in the character ring of the torus holds:*

$$\sum_p (-1)^p Ext^p(\mathcal{E}, \mathcal{F}) = \sum_\Delta \frac{\mathcal{E}(x_\Delta)\overline{\mathcal{F}(x_\Delta)}}{\prod_{\omega \in \Delta^*}(1 - \omega^{-1})}, \tag{29}$$

*where the characters of $\mathcal{E}(x_\Delta)$ and $\mathcal{F}(x_\Delta)$ are given by the formula (27), the bar denotes the automorphism $\chi \mapsto \chi^{-1}$ of the character ring of the torus and the rest of the notations are the same as in the trace formula (15).*

For the ptoof it is sufficient to note that both parts of the formula are bilinear in $\mathcal{E}$and $\mathcal{F}$ in the Grothendieck ring of coherent sheaves on $X$. In the case of the locally free sheves it is reduced to the ordinary trace formula (15). Therefore it is valid in general.

**Example 1.3.5** *Torsion free sheaves on the projective plane.*

For the projective plane we have an equivalence of categories:

$$\left(\textit{Torsion free sheaves on } \mathbf{P}^2\right) \leftrightarrow \begin{pmatrix} \textit{Triplets of compatible bifiltrations} \\ E^{\alpha\beta}(i,j),\ E^{\beta\gamma}(j,k),\ E^{\gamma\alpha}(k,i) \end{pmatrix},$$

where the compatibility means that the following conditions hold

$$\begin{aligned} \lim_{i\to-\infty} E^{\alpha\beta}(i,j) &= \lim_{k\to-\infty} E^{\beta\gamma}(j,k) := E^{\beta}(j); \\ \lim_{j\to-\infty} E^{\beta\gamma}(j,k) &= \lim_{i\to-\infty} E^{\gamma\alpha}(k,i) := E^{\gamma}(k); \\ \lim_{k\to-\infty} E^{\gamma\alpha}(k,i) &= \lim_{j\to-\infty} E^{\alpha\beta}(i,j) := E^{\alpha}(i). \end{aligned} \tag{30}$$

We call the filtrations $E^{\alpha}$, $E^{\beta}$, $E^{\gamma}$ from (31) the *limit filtrations* of the triplet $E^{\alpha\beta}, E^{\beta\gamma}, E^{\gamma\alpha}$. This relations give us a natural map from the moduli space of the compatible *bifiltrations* to the moduli space of *filtrations*:

$$(E^{\alpha\beta}, E^{\beta\gamma}, E^{\gamma\alpha}) \mapsto (E^{\alpha}, E^{\beta}, E^{\gamma}).$$

On the level of sheaves $\mathcal{E}$ this map corresponds to the minimal desingularisation

$$\mathcal{E} \mapsto \mathcal{E}^{**},$$

where $\mathcal{E}^{**}$ is the bidual sheaf for $\mathcal{E}$ which is in fact a vector bundle in the two dimentional case. This desingularisation map play an essential role in what follows. We shall normally try to separate the contribution to from the vector bundle $\mathcal{E}^{**}$ and from the singularity:

$$S = S^{\alpha\beta} + S^{\beta\gamma} + S^{\gamma\alpha} := \mathcal{E}^{**}/\mathcal{E},$$

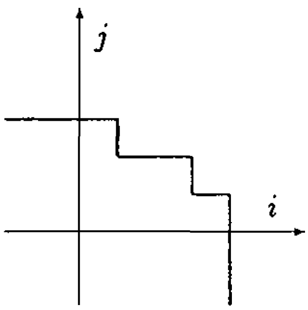
which is a skyscraper sheaf with the stalks $S^{\alpha\beta}$, $S^{\beta\gamma}$, $S^{\gamma\alpha}$ in the fix points $x_{\alpha\beta}, x_{\beta\gamma}, x_{\gamma\alpha}$ of the torus $T$ on $\mathbf{P}^2$.

For example the singularity $S$ does not influence the first Chern class but increases the second class and the discriminant:

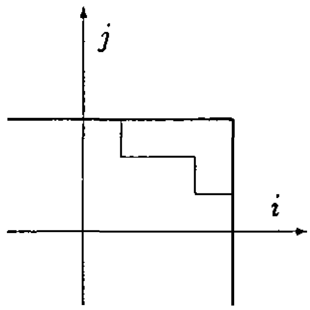
$$c_1(\mathcal{E}) = c_1(\mathcal{E}^{**}),\ c_2(\mathcal{E}) = c_2(\mathcal{E}^{**}) + \dim S;$$

$$D(\mathcal{E}) = D(\mathcal{E}^{**}) + 2r \dim S. \tag{31}$$

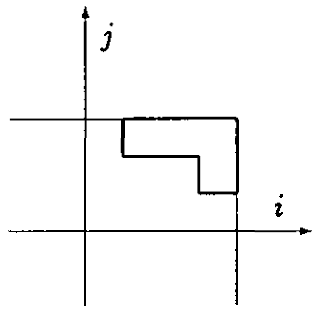
*Subexample.* Let us consider a toric torsion free sheaf $\mathcal{L}$ of rank 1. The corresponding bifiltrations depend only on the discrete parameters $\dim L^{\alpha\beta}(i,j)$ and may be represented by the following picture on the $(i,j)$-plane:




*The diagram* $\mathcal{D}^{\alpha\beta}(\mathcal{L})$




*The diagram* $\mathcal{D}^{\alpha\beta}(\mathcal{L}^{**})$




*The singularity* $S^{\alpha\beta}$

The polygonal line on the picture divides the plane on two parts, in one of which $\dim L^{\alpha\beta}(i,j) = 1$ and while in the other one the dimension is zero. The first part contains the line and all points to the south-west of it. We will refer to this line or to the whole picture as a *diagram* $\mathcal{D}^{\alpha\beta}$ of the rank one sheaf $\mathcal{L}$. The diagram of the bidual line bundle $\mathcal{L}^{**}$ is the whole south-west angle with the same asymptotic lines. The integral points between the diagrams $\mathcal{D}^{\alpha\beta}(\mathcal{L})$ and $\mathcal{D}^{\alpha\beta}(\mathcal{L}^{**})$ represent the spectrum of the torus representation in the singularity stalk $S^{\alpha\beta}$.

Let us return to a general torsion free toric sheaf $\mathcal{E}$ of an arbitrary rank $r$ on $\mathbf{P}^2$. The restriction $\mathcal{E}^{\alpha\beta}$ of $\mathcal{E}$ to the affine equivariant neighbourhood $U_{\alpha\beta} = \mathbf{P}^2 \backslash X_\gamma$ of the fixed point $x_{\alpha\beta}$ has an equivariant filtration

$$\mathcal{E}^{\alpha\beta} \supset \mathcal{E}_1^{\alpha\beta} \supset \ldots \supset \mathcal{E}_{r-1}^{\alpha\beta} \supset 0 \tag{32}$$

with torsion free modules of rank one $\mathcal{E}_i^{\alpha\beta}/\mathcal{E}_{i+1}^{\alpha\beta}$ as composition factors. The standard way to construct the filtration (32) of $\mathcal{E}$ is to decompose the restriction of the bidual bundle $\mathcal{E}^{**}$ to the affine space $U_{\alpha\beta}$ in a direct sum of the rank one modules and then to take the filtration of $\mathcal{E}^{\alpha\beta}$ induced by this decomposition.

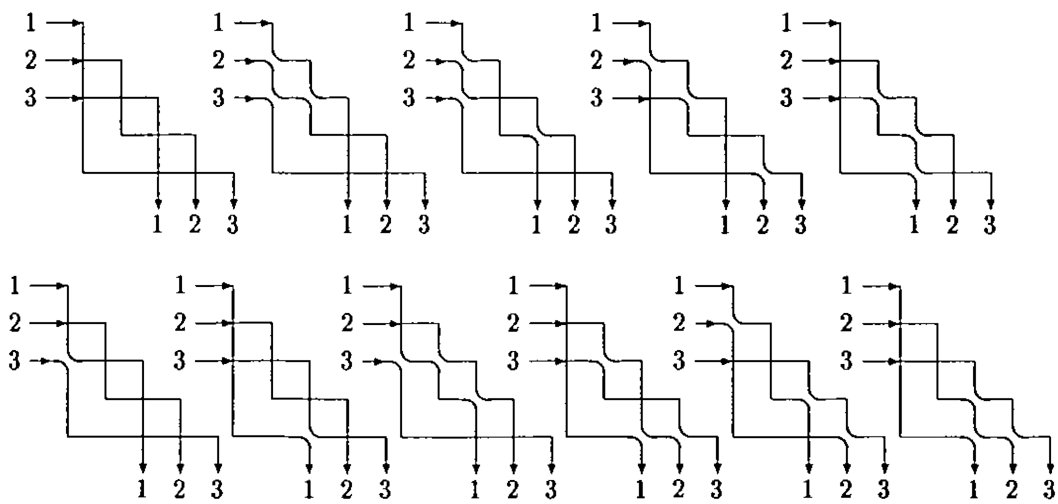
The set of this composition factors *depends* of the filtration (32). It follows from the above construction that this filtration may be chosen uniformly for all the sheaves $\mathcal{E}$ with a given desingularisation $\mathcal{E}^{**}$, i.e. with given *limit filtrations* (31).

**Definition 1.3.6** *The diagram of a bifiltration* $E^{\alpha\beta}(i,j)$ *is the set of dimensions* $\dim E^{\alpha\beta}(i,j)$ *written down on (i,j)-plane. The display of a bifiltration* $E^{\alpha\beta}$ *is the set of diagrams of the rank one composition factors (32).*

It follows from this definition that the diagram of a bifiltration is a sum of the rank one diagrams of its display. The display of a bifiltration $E^{\alpha\beta}$ allows also to reconstruct the relative position of its *limit filtrations*:

$$E^{\alpha}(i) = \lim_{j\to-\infty} E^{\alpha\beta}(i,j); \; E^{\beta}(j) = \lim_{i\to-\infty} E^{\alpha\beta}(i,j). \tag{33}$$

It is a good idea to picture the diagram of a bifiltration $E^{\alpha\beta}$ with the fixed relative position of the "initial" and the "final" limit filtrations $E^{\alpha}$ and $E^{\beta}$ as an oriented network on the (i,j)-plane with an indication which of the horizontal input rays correspond to an output vertical ray. All the horizontal edges of the network are oriented to the east and all the vertical ones are oriented to the south with the balance of the input and output edges at each vertex. The edges of the network divide the plane into regions with a constant value of $\dim E^{\alpha\beta}(i,j)$. The multiplicity of an edge is equal to the difference of these dimensions on the right hand side and on the left hand side of it. The display of a bifiltration is one of the way to decipher the diagram i.e. to point out for each of the horisontal rays a specific path connecting it with the corresponding vertical ray. May be the following comics may clarifiy the situation and compensate for the poorness of my English.



*This is the diagram of a bifiltration and all of its displays*

The first picture in this series is the diagram of a bifiltration of 3-dimensional space with the initial and the final filtrations in general position. Further follow all displays which are compatible with this configuration of the limit filtrations.

Now we can state the main technical result on the structure of the space of bifiltrations with given limit filtrations. Let $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ (respectively $\mathcal{B}(\Delta, E^\alpha, E^\beta)$) be the space of all the bifiltrations $E^{\alpha\beta}(i,j)$ with fixed limit filtrations $E^\alpha$ and $E^\beta$ and a given diagram $\mathcal{D}$ (resppectively display $\Delta$). Most of the geometrical and topological properties of the variety $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ may be deduced from the following stratification

$$\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta) = \bigsqcup_{\Delta} \mathcal{B}(\Delta, E^\alpha, E^\beta), \tag{34}$$

where the union is taken over all the displays $\Delta$ of the diagram $\mathcal{D}$ compatible with the relative position of the limit filtrations.

**Theorem 1.3.7** *All the stratas $\mathcal{B}(\Delta, E^\alpha, E^\beta)$ are affine spaces. Hence (34) gives us a cell decomposition of the space $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$.*

**Corollary 1.3.8** *The Euler characteristic of the space $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ is equal to the number of displays of the diagram $\mathcal{D}$ compatible with the relative positions of the limit filtrations $E^\alpha$ and $E^\beta$.*

Thus the displays play in our theory the same role as the Young diagrams in the Schubert calculus. To get an explicit formula for the stratas $\mathcal{B}(\Delta)$ and their dimentions one has to fix a reference filtration (32) on which the definition of the stratas depends. It is a sheaf filtration, but the Glossary explains that in fact it is determined by a complete filtration $E^\delta$ of the space $E$ which forms a *split triplet* with the limit filtrations $E^\alpha$ and $E^\beta$. There is no canonical choice of this filtration. Nevertheless it is convenient to use as a reference one of the limit filtrations if it is complete. Let it be the final filtration $E^\alpha$. Then the decomposition (34) may be explained as follows. Let us say that the factors

$$E^{[\alpha]\beta}(i,j) := E^{\alpha\beta}(i,j)/E^{\alpha\beta}(i+1,j)$$

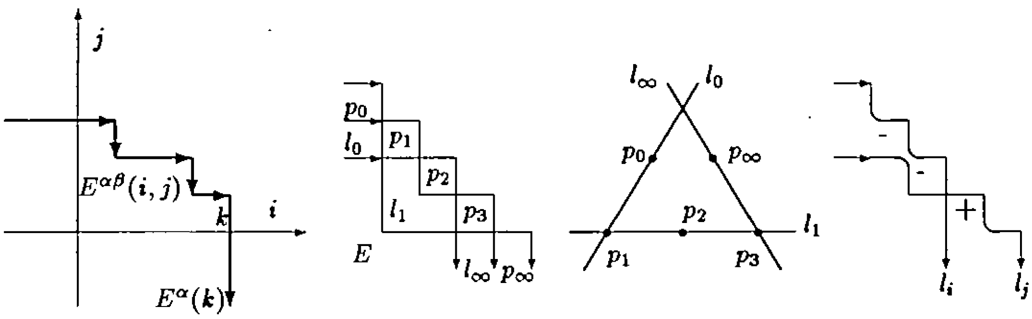
$$E^{[\alpha]}(k) := E^\alpha(k)/E^\alpha(k+1)$$

of a bifiltration $E^{\alpha\beta}$ and its limit filtration $E^\alpha(k) = E^{\alpha\beta}(k, -\infty)$ are *connected in display* $\Delta$ if a display's line enteres vertically at the point $(i, j)$ and proceeds to $(k, -\infty)$ (see the first picture below).

Now the cell $\mathcal{B}(\Delta, E^\alpha, E^\beta) \subset \mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ may be described as follows:

$$\mathcal{B}(\Delta, E^\alpha, E^\beta) = \begin{pmatrix} \textit{bifiltrations } E^{\alpha\beta} \textit{ such that the factor } E^{[\alpha]\beta}(i,j) \\ \textit{has a nonzero projection in } E^{[\alpha]}(k) \textit{ if and only} \\ \textit{if these factors are connected in } \Delta \end{pmatrix} \tag{35}$$

( the projection of $A/B$ in $C/D$ is $C \cap (A + D)/C \cap (B + D)$). Let us look once again on the previous picture, which begins with the diagram of a bifiltration $E^{\alpha\beta}$ of the three-dimentional space $E$. The diagram divides $(i, j)$-plane into the pieces with equal spaces $E^{\alpha\beta}(i, j)$. This spaces form the following configuration on the projective plane $\mathbf{P}(E)$ with the limit filtrations $p_0 \in l_0$ and $p_\infty \in l_\infty$ in general position.




The design gives us a view of a general configuration with the above diagram $\mathcal{D}$. The cell (35) consists of configurations with a specific degeneration. They are given (in the same order as the corresponding displays on page 23) by the following table.

| No | $\mathcal{B}(\Delta)$ | dim | No | $\mathcal{B}(\Delta)$ | dim |
|---|---|---|---|---|---|
| 1 | $p_\infty \notin l_1; p_1 \notin l_\infty; p_2 \notin l_\infty$ | 3 | 6 | $l_1 = l_\infty; p_2 = p_\infty \neq p_3$ | 1 |
| 2 | $p_\infty \notin l_1; p_1 \notin l_\infty; p_2 \in l_\infty$ | 2 | 7 | $l_1 \neq l_\infty; p_1 = p_2 = p_3$ | 1 |
| 3 | $p_3 = p_\infty \neq p_2; p_1 \notin l_\infty$ | 2 | 8 | $l_1 = l_\infty; p_3 = p_\infty \neq p_2$ | 1 |
| 4 | $l_1 = l_\infty; p_2 \neq p_\infty \neq p_3$ | 2 | 9 | $p_2 = p_3 = p_\infty; p_1 \notin l_\infty$ | 1 |
| 5 | $p_\infty \notin l_1; p_2 \notin l_\infty; p_1 \in l_\infty$ | 2 | 10 | $l_1 = l_\infty; p_2 = p_3 = p_\infty$ | 0 |

One can check straightforwardly that the stratas are indeed the affine spaces of dimension given in the table. Thus we obtain a Poincare polinomial

$$P(\mathcal{B}(\mathcal{D})) = 1 + 4t^2 + 4t^4 + t^6.$$

In general there is a simple combinatorial formula for the dimentions of the stratas (35). To explain this formula we need a notion of the *intersection index* of two lines of display. It is illustrated by the last figure in the above picture, where two lines $l_i$ and $l_j$ of the display are shown. Let us look at a domain between the two lines. Such a domain has an orientation depending of the side with respect to the line $l_i$ where it is situated. There are two unbounded domains between lines. One of them unbounded on the left will be called the *initial* domain , and the other one unbounded on the bottom will be called the *final* one. We define the *intersection index* as follows:

$$\mathrm{Ind}(l_i, l_j) = \begin{pmatrix} \textit{a number of bounded domains between lines } l_i \textit{ and } l_j \\ \textit{whose orientation is opposite to that of final domain} \end{pmatrix}.$$

This index is used in the following theorem.

**Theorem 1.3.9** *A dimension of the cell (35) is given by the formula*

$$\dim \mathcal{B}(\Delta, E^\alpha, E^\beta) = \sum_{i<j} Ind(l_i, l_j), \qquad (36)$$

*(the summation is over all pairs of display's line).*

One may proceed as above to obtain formulae of the same type even if the limit filtration $E^\alpha$ is not complete.

The space of bifiltrations $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ may be reducible and may have a singularity. For example if we use the diagram on the page 25 with the initial and the final filtrations not in a general position, say $l_0 = l_\infty$, then the space $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ will consist of two components intersected by line. Nevertheless

**Theorem 1.3.10** *If the initial and the final filtrations $E^\alpha$, $E^\beta$ are in a general position then for any diagram $\mathcal{D}$ the space of bifiltrations $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ is complete nonsingular irreducible variety (or empty).*

It is very interesting to understand what happens when a configuration of the initial and the final filtrations degenerates. The most surprising thing which may happen is that nothing happens at all. For example let us consider the same diagram $\mathcal{D}$ as above but under *stable* degenerations of filtrations represented on page 13. The first two of them do not change the pairwise relative positions of the limit filtrations. Hence the space of bifiltrations is not changes as well. But for degeneration of the third type a pair of filtrations is not in a general position. Nevertheless if one writes down the list of displays, similar to the one on the page 23, then it turns out that the Poincare polinomial is the same as before. It seems that this is a special case of a general phenomenon but I can prove it for rank three only.

**Proposition 1.3.11** *If* $\dim E = 3$ *then the natural map from the moduli space of stable compatible triplets of bifiltrations* $E^{\alpha\beta}$, $E^{\beta\gamma}$, $E^{\gamma\alpha}$ *to the moduli space of their limit filtrations is a fibre bundle.*

To sum up we may say that the fix points $\overline{\mathcal{M}}_r(D)^T$ of the moduli space of torsion free sheaves essentially coincide with the moduli space of compatible triplets of bifiltrations. The desingularisation

$$\mathcal{E} \mapsto \mathcal{E}^{**}$$

corresponds on the level of filtrations to the map

$$\begin{pmatrix} \text{moduli space of compatible} \\ \text{bifiltrations } E^{\alpha\beta},\ E^{\beta\gamma},\ E^{\gamma\alpha} \end{pmatrix} \mapsto \begin{pmatrix} \text{moduli space of theirs limit} \\ \text{filtrations } E^{\alpha},\ E^{\beta},\ E^{\gamma} \end{pmatrix} \tag{37}$$

We have a complete information on the fibres of this map. For fixed *diagrams* $\mathcal{D}^{\alpha\beta}$, $\mathcal{D}^{\beta\gamma}$, $\mathcal{D}^{\gamma\alpha}$ of the bifiltrations $E^{\alpha\beta}$, $E^{\beta\gamma}$, $E^{\gamma\alpha}$ the fibre splits in the product

$$\mathcal{B}(\mathcal{D}^{\alpha\beta}, E^{\alpha}, E^{\beta}) \times \mathcal{B}(\mathcal{D}^{\beta\gamma}, E^{\beta}, E^{\gamma}) \times \mathcal{B}(\mathcal{D}^{\gamma\alpha}, E^{\gamma}, E^{\alpha}).$$

Theorem 1.3.7 gives a cell decomposition of the factors. They depend only on the diagrams and paiwise relative positions of the filtrations. Moreover the map (37) is a fibre bundle over the space of triplets of filtration with fixed pairwise relative positions, that is over the stratas of the decomposition (21). Therefore the investigation of the invariant space $\overline{\mathcal{M}}_r(D)^T$ to a great extent may be reduced to the problem 1.2.6 of the description the stable triplets of filtrations. In low dimentions no problem arises: for rank two all components are points and for rank three all the components are either points or projective lines (example 1.2.5).

## 1.4 Moduli space of torsion free sheaves on $\mathbf{P}^2$

We are reaby now to explain the results mentioned in the beginning of the paper. Let us start with the Euler characteristic of the moduli space $\overline{\mathcal{M}}_r(D)$.

**Theorem 1.4.1** *Let $\mathcal{M}_r(D)$ and $\overline{\mathcal{M}}_r(D)$ be the moduli space of the stable vector bundles (respectively torsion free sheaves) of rank $r$ and of discriminant $-D = (r-1)c_1^2 - 2rc_2$ on the projective plane $\mathbf{P}^2$. Then their Euler characteristic for any fixed rank $r$ satisfies the following identity*

$$\sum_{(D,r)=1} \chi(\overline{\mathcal{M}}_r(D))q^D = \frac{1}{\prod_{n>0}(1-q^{2rn})^{3r}} \sum_{(D,r)=1} \chi(\mathcal{M}_r(D))q^D$$

The main part of the proof was outlined above. One start with the relations

$$\chi(\overline{\mathcal{M}}_r(D)) = \chi(\overline{\mathcal{M}}_r(D)^T),$$
$$\chi(\mathcal{M}_r(D)) = \chi(\mathcal{M}_r(D)^T).$$

Then we consider the minimal desingularisation $\mathcal{E}^{**}$ of the torsion free sheaf $\mathcal{E}$ and its skyscraper singularity sheaf

$$S = \mathcal{E}^{**}/\mathcal{E}.$$

The discriminants of $\mathcal{E}$ and $\mathcal{E}^{**}$ satisfy the relation (31)

$$D(\mathcal{E}) = D(\mathcal{E}^{**}) + 2r \dim S$$

Now we make use of the interpretation of the invariants $\overline{\mathcal{M}}_r(D)^T$ as a moduli space of compatible bifiltrations and of the corollary of theorem 1.3.7 to get an equality

$$\begin{pmatrix}\textit{Euler characteristic of the space of toric} \\ \textit{sheaves } \mathcal{E} \textit{ with a fixed desingularisation } \mathcal{E}^{**} \\ \textit{and singularity } \mathcal{E}^{**}/\mathcal{E} \textit{ of dimension } d\end{pmatrix} = \begin{pmatrix}\textit{a number of 3r-tuples of} \\ \textit{Young diagrams contain-} \\ \textit{ing d cells all together}\end{pmatrix}$$

To comment on this formula let us note that the sheaf $\mathcal{E}$ corresponds to a triplet of bifiltrations $E^{\alpha\beta}$, $E^{\beta\gamma}$, $E^{\gamma\alpha}$ with fixed limit filtrations $E^{\alpha}$, $E^{\beta}$, $E^{\gamma}$. A dimension of the singularity stalk $S^{\alpha\beta}$ at a fixed point $x_{\alpha\beta} \in \mathbf{P}^2$ is equal to the sum of singularity dimension of rank one factors of the reference filtration (32). In turn the singularity dimention $\dim S^{\alpha\beta}$ of rank one sheaf $\mathcal{L}$ is equal to a number of cells in the corresponding Young diagram (it is shown on the right picture on page 22). Thus the number of displays $\Delta^{\alpha\beta}$ that corresponds to a singularity of dimension $d^{\alpha\beta}$ is equal to the number of $r$-tuples of Young diagrams with the sum of the areas equal to $d^{\alpha\beta}$. On the other hand by the corollary of theorem 1.3.7 this number is equal to the Euler characteristic of the space of bifiltrations $\bigsqcup_{\mathcal{D}} \mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ with a singularity of given dimension $d^{\alpha\beta}$. If we consider all the three fix points $x_{\alpha\beta}$, $x_{\beta\gamma}$, $x_{\gamma\alpha}$ we obtain the above equality.

Its left hand side gives us the Euler characteristic of a fibre of the map

$$\overline{\mathcal{M}}_r(D,d)^T \to \mathcal{M}_r(D-2rd)^T; \mathcal{E} \mapsto \mathcal{E}^{**}, \tag{38}$$

where $\overline{\mathcal{M}}_r(D,d)^T \subset \overline{\mathcal{M}}_r(D)^T$ denotes the subspace of sheaves with a singularity of dimension $d$. The map (38) is in fact a fibre bundle. Therefore from the multiplicativity of the Euler characteristic it follows

$$\chi(\overline{\mathcal{M}}_r(D) = \sum_d \chi(\mathcal{M}_r(D-2rd)) \begin{pmatrix} \textit{a number of 3r-tuples of Young dia-} \\ \textit{grams containing d cells all together} \end{pmatrix}.$$

This formula is equivalent to the assertion of the theorem. For vector bundles of rank two the Euler characteristic of the moduli space is known [20]. Thus we get

**Corollary 1.4.2** *The Euler characteristic of the moduli space of torsion free sheaves of rank two on* $\mathbf{P}^2$ *satisfies the identity*

$$\frac{1}{3}\sum_{n=1}^{\infty}\chi(\overline{\mathcal{M}}_2(4n-1))q^{n-\frac{1}{4}} = \frac{1}{\eta(q)^6}\sum_{n=1}^{\infty} H(4n-1)q^n,$$

*where* $\eta(q) = q^{\frac{1}{24}}\prod_1^\infty(1-q^n)$ *is the Dedekind* $\eta$*-function and* $H(D)$ *is the Hurwitz class number (4).*

As explained in section 1.1 to get further information on the moduli space $\overline{\mathcal{M}}_r(D)$ we need to know

- the structure of the invariant components $\overline{\mathcal{M}}_r(D)^T$;
- the torus representation in the tangent space of each of the components.

As far as the first item is concerned the complete structure of invariants $\overline{\mathcal{M}}_r(D)^T$ is known only for the small rank $r$.

**Proposition 1.4.3** *In the case of rank two each of the components of* $\overline{\mathcal{M}}_2(D)^T$ *is a product of lines* $\mathbf{P}^1 \times \ldots \times \mathbf{P}^1$.

Really in this case the range of the desingularisation map $\mathcal{E} \mapsto \mathcal{E}^{**}$ from the moduli space of toric sheaves to the moduli of vector bundles is of dimension zero (c.f. example 1.2.5). As we have seen in the previous section the components of the fibres of this map for any rank are of the form

$$\mathcal{B}(\mathcal{D}^{\alpha\beta}, E^\alpha, E^\beta) \times \mathcal{B}(\mathcal{D}^{\beta\gamma}, E^\beta, E^\gamma) \times \mathcal{B}(\mathcal{D}^{\gamma\alpha}, E^\gamma, E^\alpha). \quad (39)$$

It is easy to check that for a two dimensional space $E$ the space of bifiltrations $\mathcal{B}(\mathcal{D}, E^\alpha, E^\beta)$ is a product of projective lines $\mathbf{P}(E)$ (a number of the factors is equal to a number of restricted domains in the plane partition determined by the diagram of a bifiltration ).

For rank three we also have an almost complete information.

**Proposition 1.4.4** *The components of* $\overline{\mathcal{M}}_3(D)^T$ *are either of the form (39) or form a bundle over* $\mathbf{P}^1$ *with fibre(39).*

The proof follows along the same lines as in rank two case except that the range of the desingularisation map $\mathcal{E} \mapsto \mathcal{E}^{**}$ consists of points and lines (see example 1.2.5). In the case of lines we have to use propsition 1.3.11.

As an easy corollary we obtain that all the components of the space $\overline{\mathcal{M}}_3(D)^T$, $g.c.d.(D,3) = 1$ are rational. Makeing use of the Bialynicki-Birula stratification, described in section 1.1 we get

**Corollary 1.4.5** *The moduli space* $\acute{\mathcal{M}}_3(D)$, $g.c.d.(D,3) = 1$ *is rational.*

In both this cases it is easy to find the Betti number of a component of $\overline{\mathcal{M}}_3(D)^T$, $(D,3) = 1$. For rank two case a Poincare polinomial is of the form $(1+t^2)^m$. In the case of rank three the Poincare polinomial of a component is the product that of the base and of the fibre, because one can not find a place for nonzero differential in the spectral sequence for the bundle of proposition 1.4.4. An odd Betti number of the fibre is zero and an even one may be find from the cell decomposition of theorem 1.3.7 in conjunction with the dimension formula of theorem 36. It seems that no explicit formula for this Betti numbers exists.

Let us turn to the second item, that is the torus representation in the tangent space to the moduli $\overline{\mathcal{M}}_r(D)$ in $T$-invariant point. This represntation gives us the degree shift in the Ginsbourg's formula (7) relating cohomology of $\overline{\mathcal{M}}_r(D)$ to the cohomology of $\overline{\mathcal{M}}_r(D)^T$.

The tangent space of the moduli space $\overline{\mathcal{M}}_r(D)$ in a point corresponding to a toric torsion free sheaf $\mathcal{E}$ is $Ext^1(\mathcal{E},\mathcal{E})$. The character of this module may be find from the trace formula of proposition 1.3.4 in conjunction with the equalities (28).

**Proposition 1.4.6** *Let $\mathcal{E}$ be a stable toric torsion free sheaf on $\mathbf{P}^2$ and $E^{\alpha\beta}$, $E^{\beta\gamma}$, $E^{\gamma\alpha}$ are the corresponding bifiltrations. Then a character of the torus representation in the tangent space to the moduli $\overline{\mathcal{M}}_r(D)$ at the point $\mathcal{E}$ is given by the formula*

$$1-\frac{\mathcal{E}(x_{\alpha\beta})\overline{\mathcal{E}(x_{\alpha\beta})}}{(1-\frac{\omega_\gamma}{\omega_\alpha})(1-\frac{\omega_\gamma}{\omega_\beta})}-\frac{\mathcal{E}(x_{\beta\gamma})\overline{\mathcal{E}(x_{\beta\gamma})}}{(1-\frac{\omega_\alpha}{\omega_\beta})(1-\frac{\omega_\alpha}{\omega_\gamma})}-\frac{\mathcal{E}(x_{\gamma\alpha})\overline{\mathcal{E}(x_{\gamma\alpha})}}{(1-\frac{\omega_\beta}{\omega_\gamma})(1-\frac{\omega_\beta}{\omega_\alpha})}, \tag{40}$$

*where we suppose that the torus $T$ consists of diagonal matrices $diag(\omega_\alpha,\omega_\beta,\omega_\gamma)$, the bar denote an automorphism of character ring induced by $\chi\mapsto\chi^{-1}$, $\chi\in\hat{T}$ and the representation in the "fibre" $\mathcal{E}(x)$ of a fixed point $x$ is given by the formula (27):*

$$\mathcal{E}(x_{\alpha\beta})=\sum_{i,j}\left(\frac{\omega_\alpha}{\omega_\gamma}\right)^i\left(\frac{\omega_\beta}{\omega_\gamma}\right)^j\dim E^{[\alpha\beta]}(i,j).$$

For practical purposes it is conveniant to share out the contribution in the spectre the sheaf singularity at each fix point. This may be done by make use of the proposition both to a sheaf $\mathcal{E}$and to its desingularisation $\mathcal{E}^{**}$:

$$\begin{aligned}Ext^1(\mathcal{E},\mathcal{E}) &= Ext^1(\mathcal{E}^{**},\mathcal{E}^{**})+ \\ &\overline{S}_{\alpha\beta}\mathcal{E}(x_{\alpha\beta})+\frac{\omega_\alpha}{\omega_\gamma}\frac{\omega_\beta}{\omega_\gamma}S_{\alpha\beta}\overline{\mathcal{E}}(x_{\alpha\beta})-\left(1-\frac{\omega_\alpha}{\omega_\gamma}\right)\left(1-\frac{\omega_\beta}{\omega_\gamma}\right)S_{\alpha\beta}\overline{S}_{\alpha\beta}+ \\ &\overline{S}_{\beta\gamma}\mathcal{E}(x_{\beta\gamma})+\frac{\omega_\beta}{\omega_\alpha}\frac{\omega_\gamma}{\omega_\alpha}S_{\alpha\beta}\overline{\mathcal{E}}(x_{\beta\gamma})-\left(1-\frac{\omega_\beta}{\omega_\alpha}\right)\left(1-\frac{\omega_\gamma}{\omega_\alpha}\right)S_{\beta\gamma}\overline{S}_{\beta\gamma}+ \\ &\overline{S}_{\gamma\alpha}\mathcal{E}(x_{\gamma\alpha})+\frac{\omega_\gamma}{\omega_\beta}\frac{\omega_\alpha}{\omega_\beta}S_{\gamma\alpha}\overline{\mathcal{E}}(x_{\gamma\alpha})-\left(1-\frac{\omega_\gamma}{\omega_\beta}\right)\left(1-\frac{\omega_\alpha}{\omega_\beta}\right)S_{\gamma\alpha}\overline{S}_{\gamma\alpha},\end{aligned} \tag{41}$$

where $S_{\alpha\beta}$ is the stalk of a singularity sheaf $S = \mathcal{E}^{**}/\mathcal{E}$ at the fixed point $x_{\alpha\beta}$. Thus we have an explicit formula for a spectrum of the torus representation in tangent space of the moduli $\overline{\mathcal{M}}_r(D)$ at a fix point of the torus $T$. Then we may combine it with the above information on invariant components $\mathcal{M}_r(D)^T$ and Ginsburg's formula (7) to find the cohomology of $\overline{\mathcal{M}}_r(D)$.

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