On the Spectrum, Complete Trajectories and Asymptotic Stability of Linear Semi-Dynamical Systems

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Abstract. Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup in a Banach space X with generator A. We prove that if $\{T(t)\}_{t\geq 0}$ is bounded and sun-reflexive, and the sun-dual semigroup is not asymptotically stable, then there exist bounded complete trajectories under $\{T(t)\}_{t\geq 0}$, provided: (i) there is $t_0 > 0$ such that $\mathrm{ran}(T^{\odot}(t_0))$ is dense in $X^{\odot}$, or (ii) $\sigma(A) \not\supseteq i\mathbb{R}$. Questions of almost periodicity of complete trajectories are also discussed and a new proof of our earlier theorem (jointly with Yu.I. Lyubich) on asymptotic stability is given.

1. Introduction.

Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous one-parameter semigroup (C_0-semigroup) of bounded linear operators in a Banach space X, and A be its generator. For each vector x in X the *semi-trajectory* $\gamma_+(x)$ through x is defined by $\gamma_+(x) = \{T(t)x : t \geq 0\}$. A continuous function $\mathbf{x}(t) : \mathbb{R} \to X$ is called a *complete trajectory* through x if $\mathbf{x}(t) = T(t-s)\mathbf{x}(s)$ for each $t, s \in \mathbb{R}$ such that $t \geq s$, and $\mathbf{x}(0) = x$. While a semi-trajectory through any vector x in X always exists, it may happen that there is no complete trajectory, except the trivial one through 0. For instance, the semigroup $\{T(t)\}_{t\geq 0}$ in $L^p(\mathbb{R}_+), 1 \leq p < \infty$, defined by $T(t)f(s) = f(s-t)$ if $s \geq t$ and $T(t)f(s) = 0$ if $s < t$, doesn't have a non-trivial complete trajectory. Note that a function $\mathbf{x}(t)$ is a complete trajectory under the semigroup $\{T(t)\}_{t\geq 0}$ if and only if it is a mild solution of the differential equation $d\mathbf{x}(t)/dt = A\mathbf{x}(t), \ -\infty < t < \infty$ (see e.g. [19]). Thus, from the standpoint of application to the theory of differential equations and dynamical systems, a natural and important question arised, under which conditions there exists a non-trivial bounded or almost periodic complete trajectory for a given C_0-semigroup $\{T(t)\}_{t\geq 0}$? It should be noted that various conditions for almost periodicity of bounded solutions of differential equations in Banach spaces are available in the literature

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[1], [15] (cf. also [7], [11], [12]). However, to our knowledge, a general criterion for existence of a bounded complete trajectory was not known.

The primary object of the present paper is to establish such a general criterion for existence of bounded complete trajectories under C_0-semigroups. Our main result asserts that, if the semigroup $\{T(t)\}_{t\geq 0}$ is bounded and sun-reflexive, and its sun-dual semigroup $\{T^{\odot}(t)\}_{t\geq 0}$ is not asymptotically stable, then there exist (non-trivial) bounded complete trajectories, provided one of the following conditions holds: i) $\sigma(A) \not\supseteq i\mathbb{R}$, ii) $\mathrm{ran} T^{\odot}(t_0)$ is dense in $X^{\odot}$ for some $t_0 > 0$ (Corollary 2.4). The proof provides also a constructive method of obtaining a large family of bounded complete trajectories. An example is given showing that in this result sun-reflexivity is essential. Then we prove that, if the intersection of the approximate point spectrum of A and the imaginary axis is countable, then every uniformly continuous bounded complete trajectory is almost periodic, provided the space X does not contain an isomorphic copy of c_0 (the Banach space of sequences convergent to 0), or the trajectory itself is weakly compact (Theorem 3.10).

As an unexpected by-product we obtain one more new proof of a theorem on asymptotic stability of C_0-semigroups, which was established in [23] (see also [18]) and independently in [2]. The proof in [2] is, however, obscure, involving the transfinite induction method. A third different proof of this result, which is in the same spirit of [2] but does not require the transfinite induction, was subsequently given in [10]. Our new proof is rather short and elegant and has the flavour of the proof in [23], though we don't use here the Gelfand theorem and the spectral theory of isometric groups. Instead, we use the classical Wiener General Tauberian Theorem.

Of independent interest are analogous results for single operators, which we formulate in the last section. The proofs are easy modifications, with corresponding simplifications, of the proofs for C_0-semigroups.

Throughout this paper $\{T(t)\}_{t\geq 0}$ is a one-parameter semigroup in a Banach space X, with a generator A. The domain of A is denoted by $\mathcal{D}(A)$, while the spectrum, point spectrum, approximate point spectrum and the resolvent set of A will be denoted by $\sigma(A)$, $P\sigma(A)$, $A\sigma(A)$ and $\rho(A)$, respectively. Further, $C(\mathbb{R}, X)$ ($BUC(\mathbb{R}, X)$, $C_0(\mathbb{R}, X)$) is the Banach space of continuous bounded functions (respectively, uniformly continuous bounded functions, continuous functions vanishing at infinity) from $\mathbb{R}$ to the Banach space X.

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2. Bounded complete trajectories.

In this section we prove that under some general conditions there exist bounded complete trajectories; moreover, we present some constructive method of finding them.

PROPOSITION 2.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of isometries in X with generator A. Assume that $\sigma(A) \not\supset i\mathbb{R}$. Then every operator $T(t)$ is invertible (so that $\{T(t)\}_{t\geq 0}$ can be extended to an isometric group).*

PROOF: This proposition is already implicitely contained in [18], [22], [23]. We recall the main arguments from these papers. It is shown in [18], that if $\{T(t)\}_{t\geq 0}$ is an isometric semigroup, then

$$\|Ax - \lambda x\| \geq |\mathrm{Re}\lambda| \|x\|, \tag{2.1}$$

for every λ, $\mathrm{Re}\lambda < 0$, and every x in X. From this inequality it follows that the half-plane $\mathbb{C}_- \equiv \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\}$ lies in a regular component of the operator A. The right half-plane $\mathbb{C}_- \equiv \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ is, of course, also contained in a regular component of A (moreover, it is in the resolvent set $\rho(A)$). Since $\sigma(A) \not\supset i\mathbb{R}$, both open left and right half-planes are in the same regular component of A, which implies that

$$\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < 0\} \subset \rho(A). \tag{2.2}$$

From (2.1), (2.2) and the Hille-Yosida Theorem (see e.g. [13]) it follows that the operator $(-A)$ is the generator of a strongly continuous contraction semigroup $\{S(t)\}_{t\geq 0}$. It is easy to see that

$$\frac{d}{dt}\{T(t)S(t)x\} = T(t)AS(t)x - T(t)S(t)Ax = 0, \quad \forall x \in \mathcal{D}(A),$$

so that $T(t)S(t) = I$, for each $t \geq 0$, i.e. $T(t)$ are invertible and $T(t)^{-1} = S(t)$. ∎

The following proposition represents the method of the *limit isometric semigroup.* In this form it appears firstly in [22], though the same idea was used in [18], [20-24].

PROPOSITION 2.2. *Let* $\{T(t)\}_{t\geq 0}$ *be a bounded* C_0*-semigroup in* X, *with generator* A. *Then there exists a Banach space* E, *a continuous homomorphism* π *from* X *to* E, *with dense range, and an isometric semigroup* $\{V(t)\}_{t\geq 0}$ *in* E, *with generator* S, *such that the following properties hold:*

i) $\sigma(S) \subset \sigma(A)$, $P\sigma(S^*) \subset P\sigma(A^*)$;

ii) $V(t) \circ \pi = \pi \circ T(t)$ *for each* $t \geq 0$;

iii) $\|\pi x\|_E = \lim_{t\to\infty} \|T(t)x\|$ *for each* x *in* X.

In order to know how to find bounded complete trajectories in next theorem, let us recall the construction of E and $V(t)$ in Proposition 2.2. First we define a semi-norm l in X by

$$l(x) \stackrel{\text{def}}{=} \lim_{t\to\infty} \|T(t)x\|.$$

Then E is defined as completion of the quotient space $X/\ker(l)$ in the norm

$$\hat{l}(\pi x) \stackrel{\text{def}}{=} l(x),$$

where π is the canonical mapping from X to $X/\ker(l)$. The operator $V(t)$ is defined by $V(t)\pi x = \pi(T(t)x)$ for each x in X and then extended to the whole E by continuity. The isometric semigroup $\{V(t)\}_{t\geq 0}$ is uniquely determined (up to unitary equivalence) and is called the *limit isometric semigroup* of $\{T(t)\}_{t\geq 0}$.

If the Banach space X is non-reflexive, then the adjoint semigroup $\{T^*(t)\}_{t\geq 0}$ will not be, in general, strongly continuous. However, the subspace

$$X^{\odot} = \{\varphi \in X^* : T^*(t)\varphi \text{ is strongly continuous}\}$$

is a closed subspace in X^*, which is invariant with respect to every $T^*(t)$. Moreover, $X^{\odot} = \overline{\mathcal{D}(A^*)}$. The restriction $T^{\odot}(t) = T^*(t)|X^{\odot}$ defines a strongly continuous semigroup in $X^{\odot}$, which is called *sun-dual semigroup* of $\{T(t)\}_{t\geq 0}$. Repeating this construction, we can define the sun-dual semigroup of $\{T^{\odot}(t)\}_{t\geq 0}$, which will be denoted by $\{T^{\odot\odot}(t)\}_{t\geq 0}$. There is a natural imbedding from X into the corresponding second sun-dual space $X^{\odot\odot}$. If $X = X^{\odot\odot}$, then the semigroup $\{T(t)\}_{t\geq 0}$ is said to be *sun-reflexive*. For these and related facts we refer the reader to [6], [13]. Let us recall that the semigroup $\{T(t)\}_{t\geq 0}$ is said to be *asymptotically stable*, if $\|T(t)x\| \to 0$ as $t \to \infty$ for each x in X.

THEOREM 2.3. *Suppose that $\{T(t)\}_{t\geq 0}$ is a bounded C_0-semigroup which is not asymptotically stable. Suppose that, moreover, one of the following conditions is satisfied:*

i) There is an $t_0 > 0$ such that $T(t_0)$ has dense range;

ii) $\sigma(A) \not\supset i\mathbb{R}$.

Then there is a non-trivial bounded complete trajectory for the sun-dual semigroup $\{T^{\odot}(t)\}_{t\geq 0}$.

PROOF: Since $\{T(t)\}_{t\geq 0}$ is not asymptotically stable, the limit isometric semigroup $\{V(t)\}_{t\geq 0}$ is not identically zero. Condition (i) yields that operator $V(t_0)$ also has dense range, hence it is an invertible isometry. Therefore, $\{V(t)\}_{t\geq 0}$ can be extended to an isometric group (see e.g. [19, p. 24]). If condition (ii) holds, then by Proposition 2.2 $\sigma(S) \subset \sigma(A)$, so that $\sigma(S) \not\supset i\mathbb{R}$. By Proposition 2.1 $\{V(t)\}_{t\geq 0}$ can be extended to an isometric group. Let $\varphi \in E^{\odot}$ and $\varphi(t) = V^{\odot}(t)\varphi$, $-\infty < t < \infty$. We put $(\mathbf{f}(t))(x) \equiv (\varphi(t))(\pi x)$ and $f = \mathbf{f}(0)$. It is easy to see that $\mathbf{f}(t)$ is uniformly bounded. Moreover

$$\begin{aligned}
&\|T^*(t)f - f\| = \sup_{\|x\|\leq 1} |(T^*(t)f - f)(x)| = \\
&\sup_{\|x\|\leq 1} |f(T(t)x) - f(x)| = \sup_{\|x\|\leq 1} |\varphi(\pi \circ T(t)x) - \varphi(\pi x)| = \\
&\sup_{\|x\|\leq 1} |\varphi(V(t)\pi x) - \varphi(\pi x)| \leq \sup_{\|z\|_E\leq 1} |(V^*(t)\varphi)(z) - \varphi(z)| = \\
&\|V^*(t)\varphi - \varphi\| \to 0 \text{ as } t \to 0.
\end{aligned}$$

Therefore $f \in X^{\odot}$, and hence $\mathbf{f}(t) \in X^{\odot}$ and $\mathbf{f}(t)$ is continuous. Analogously we have, for $t \geq s$

$$\begin{aligned}
&(T^{\odot}(t-s)\mathbf{f}(s))(x) = \mathbf{f}(s)(T(t-s)x) = \varphi(s)(\pi \circ T(t-s)x) \\
&= \varphi(s)(V(t-s) \circ \pi x) = (V^*(t-s)\varphi(s))(\pi x) = (\varphi(t))(\pi x) = \mathbf{f}(t)(x),
\end{aligned}$$

i.e. $\mathbf{f}(t)$ is a bounded complete trajectory under $\{T^{\odot}(t)\}_{t\geq 0}$. ∎

Applying Theorem 2.3 to the sun-dual semigroup, we obtain the following corollary (note that $\sigma(A^{\odot}) = \sigma(A)$, see [13]).

COROLLARY 2.4. *Suppose that* $\{T(t)\}_{t\geq 0}$ *is a bounded sun-reflexive semigroup, such that* $\{T^{\odot}(t)\}_{t\geq 0}$ *is not asymptotically stable and one of the following conditions holds:*

i) There is $t_0 > 0$ *such that* $\operatorname{ran}(T^{\odot}(t_0))$ *is dense in* $X^{\odot}$*;*

ii) $\sigma(A) \not\supset i\mathbb{R}$.

Then there exists a non-trivial bounded complete trajectory under $\{T(t)\}_{t\geq 0}$.

The following is a consequence of Corollary 2.4 because a semigroup in X, with bounded generator, is sun-reflexive iff X is reflexive, and always satisfies condition (ii).

COROLLARY 2.5. *Suppose that* X *is reflexive and* $\{T(t)\}_{t\geq 0}$ *is a bounded semigroup in* X *with bounded generator* A, *such that* $\{T^{*}(t)\}_{t\geq 0}$ *is not asymptotically stable. Then there exists a non-trivial bounded complete trajectory under* $\{T(t)\}_{t\geq 0}$.

If X is reflexive, then every C_0-semigroup in X is sun-reflexive. Moreover, condition (i) is equivalent to

(i') There is $t_0 > 0$ such that $\ker(T(t_0)) = \{0\}$.

Therefore, the following holds.

COROLLARY 2.6. *If* $\{T(t)\}_{t\geq 0}$ *is a bounded* C_0*-semigroup in a reflexive Banach space, such that* $T^{*}(t)$ *does not converge strongly to zero and one of the conditions (i') or (ii) holds, then there exists a non-trivial bounded complete trajectory under* $\{T(t)\}_{t\geq 0}$.

The proof in Theorem 2.3 (and thus in Corollaries 2.4 - 2.6) also gives a way of obtaining bounded complete trajectories: for this it is enough to pick up any functional φ from $E^{\odot}$ and to lift each functional $\varphi(t) = V^{\odot}(t)\varphi$, $t \in \mathbb{R}$, to a functional $\mathbf{f}(t)$ on X using the exact sequence of natural homomorphisms

$$X \to X/\ker(l) \to E.$$

Note that the sun-refexivity condition in Corollary 2.4 is essential, as is shown by the following example.

EXAMPLE 2.6. Consider the diffusion semigroup $\{T(t)\}_{t\geq 0}$ in $C_0(\mathbb{R})$ defined by

$$T(t)f(u) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} \exp(-|u-v|^2/4t) f(v) dv = \mu_t * f(u),$$

where $\mu_t(u) = (4\pi t)^{-1/2} \exp(-|u|^2/4t)$ is the Gaussian probability density. The generator of this semigroup is the differential operator $A = d^2/du^2$ in $C_0(\mathbb{R})$, with the domain $\mathcal{D}(A) = \{f \in C_0(\mathbb{R}) : f$ is twice differentiable and $f'' \in C_0(\mathbb{R})\}$.

It is easy to see that the semigroup $\{T(t)\}_{t\geq 0}$ is not sun-reflexive and its sun-dual semigroup $\{T^{\odot}(t)\}_{t\geq 0}$ is not asymptotically stable. Indeed, the dual space of $C_0(\mathbb{R})$ can be identified with the space $M_b(\mathbb{R})$ of all bounded (complex) Borel measures on $\mathbb{R}$. Since A is the square of the differentiation operator on $C_0(\mathbb{R})$, the adjoint of A is given by

$$\mathcal{D}(A^*) = \{\mu \in M_b(\mathbb{R}) : D^2\mu \in M_b(\mathbb{R})\}, \quad A^*\mu = D^2\mu,$$

(here $D\mu$ denotes the distributional derivative of μ). Thus, by the same reasoning as in [9], one can show that $X^{\odot} = L^1(\mathbb{R})$, $T^{\odot}(t)$ and $A^{\odot}$ have the same form, but are defined on the space $L^1(\mathbb{R})$. Moreover, $X^{\odot\odot} = BUC(\mathbb{R})$, hence the semigroup $\{T(t)\}_{t\geq 0}$ is not sun-reflexive. Note that the second sun-dual semigroup $\{T^{\odot\odot}(t)\}_{t\geq 0}$ is again the diffusion semigroup on $BUC(\mathbb{R})$. Note also that $\sigma(A) \not\supseteq i\mathbb{R}$ (indeed $\sigma(A) \cap i\mathbb{R} = \{0\}$).

We show that there are no complete bounded trajectories under $\{T(t)\}_{t\geq 0}$. In fact, suppose that $\mathbf{f}(t,u)$ is a bounded complete trajectory under $\{T(t)\}_{t\geq 0}$. Then from Propositions 3.7 and 3.8 (see section 3) it follows that $\mathbf{f}(t,u)$ is an almost periodic function from $\mathbb{R}$ to $C_0(\mathbb{R})$. This implies that the operator $A = d^2/du^2$ on $C_0(\mathbb{R})$ has an eigenvalue on $i\mathbb{R}$ (namely $\lambda = 0$), which is impossible.

3. Almost periodicity .

In this section we prove that, under suitable conditions on the spectrum of the generator A, a bounded complete trajectory is almost periodic. For this purpose we carry out a comparative analysis of spectra of functions, trajectories and associated generators, which may be of independent interest.

Let $\mathbf{x}(t)$, $-\infty < t < \infty$, be a bounded measurable function with values in X. We recall, that the *Carleman transform* of the function $\mathbf{x}(t)$ is defined as follows (see [5], [14])

$$\tilde{\mathbf{x}}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t}\mathbf{x}(t)dt, & \text{Re}\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t}\mathbf{x}(t)dt, & \text{Re}\lambda < 0, \end{cases}$$

and is a function holomorphic in $\mathbb{C} \setminus i\mathbb{R}$. A point λ_0 on $i\mathbb{R}$ is called *regular point* of $\mathbf{x}(t)$ if $\tilde{\mathbf{x}}(\lambda)$ can be continued analytically into a neighborhood of λ_0. The complement in $i\mathbb{R}$ of the set of regular points is called *(Carleman) spectrum* of $\mathbf{x}(t)$ and is denoted by $\text{Sp}(\mathbf{x})$.

The following proposition is a vector-valued version of the Carleman Lemma on analytic continuation (for the scalar case, see e.g. [5], [14]).

PROPOSITION 3.1. *Let* $\mathbf{x}(t) \in L^\infty(\mathbb{R}, X)$ *and* f *be a (scalar) function in* $L^1(\mathbb{R})$ *such that*

$$f * \mathbf{x} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(t-\tau)\mathbf{x}(\tau)d\tau \equiv 0. \tag{3.1}$$

Suppose that the Fourier transform $\hat{f}$ *is different from zero for all* ξ *from an interval* (a, b). *Then the Carleman transform* $\tilde{\mathbf{x}}(\lambda)$ *has analytic continuation through* $i(a, b)$.

PROOF: We note, first of all, that if $h(t)$ and $f(t)$ are functions in $L^1(\mathbb{R})$ such that $\hat{h}$ has a compact support and $\hat{f}(\xi) \neq 0$ for each $\xi \in \text{supp}\,\hat{h}$, then the convolution equation $f * u = h$ has a solution $u \in L^1(\mathbb{R})$ which depends continuously on f and h, and $\hat{u} = \hat{h}/\hat{f}$. Let $\varepsilon > 0$ be a positive number less than $(b-a)/2$ and λ_0 be any point in the interval $\Delta_\varepsilon \equiv i(a+\varepsilon, b-\varepsilon)$. Then the convolution equation

$$(f * u)(t) = K_\varepsilon(t)e^{\lambda_0 t}$$

has a solution which depends continuously on $\lambda_0 \in \Delta_\varepsilon$ in the norm of $L^1(\mathbb{R})$, where

$$K_\varepsilon(t) = \frac{\sin^2(\frac{\varepsilon}{2}t)}{(\frac{\varepsilon}{2}t)^2}.$$

From (3.1) it follows that

$$\int_{-\infty}^{\infty} \mathbf{x}(t)K_\varepsilon(t)e^{\lambda_0 t}dt = 0, \quad \forall\lambda_0 \in \Delta_\varepsilon.$$

Consequently, the Carleman transform $\widetilde{(\mathbf{x}\cdot K_\varepsilon)}(\lambda)$ has analytic continuation into Δ_ε. Since $\widetilde{(\mathbf{x}\cdot K_\varepsilon)}(\lambda)$ converges to $\hat{\mathbf{x}}(\lambda)$ as $\varepsilon \to 0$ uniformly on every compact set inside $\mathbb{C}_-$ and $\mathbb{C}_+$, it is enough to show that $\widetilde{(\mathbf{x}\cdot K_\varepsilon)}(\lambda)$ is uniformly bounded in some neighborhood of Δ_ε.

Consider the following function

$$y(\lambda) = (\lambda - a - \varepsilon)(\lambda - b + \varepsilon)\widetilde{(\mathbf{x}\cdot K_\varepsilon)}(\lambda), \;\; \lambda \in \mathbb{C}\setminus i\mathbb{R}.$$

It is easy to see that the following estimate holds

$$\|\widetilde{(\mathbf{x}\cdot K_\varepsilon)}(\lambda)\| \le \frac{\|\mathbf{x}\|}{|\mathrm{Re}\lambda|}, \;\; \forall\lambda \in \mathbb{C}\setminus i\mathbb{R}. \tag{3.2}$$

Let Q be the square with vertices at $i(a+\varepsilon)$, $i(b-\varepsilon)$, $,(b-a)/2-\varepsilon+i(b+a)/2$, and $-(b-a)/2+\varepsilon+i(b+a)/2$. From (3.2) it follows that $\|y(\lambda)\| \le C$ for each λ on the boundary of Q, thus for each λ in Q by virtue of the Maximum Modulus Principle; moreover, the constant C does not depend on ε. Therefore, the family $\widetilde{(\mathbf{x}\cdot K_\varepsilon)}(\lambda)$ is uniformly bounded in Q. ∎

The following proposition is in fact the theorem on equality between the Carleman spectrum and the Beurling spectrum. For scalar functions the proof can be found in [14, Chapter VI, Theorem 8.2].

PROPOSITION 3.2. *A point* λ_0 *is in the spectrum of a function* $\mathbf{x}(t) \in L^\infty(\mathbb{R}, X)$ *if and only if, for each neighborhood* $\mathcal{U}$ *of* λ_0*, there exists a scalar function* $u \in L^1(\mathbb{R})$ *with* $\operatorname{supp}\hat{u} \subset i\mathcal{U}$ *such that* $\mathbf{x} * u \ne 0$.

PROOF: We show that λ_0 is a regular point of $\tilde{\mathbf{x}}$ if and only if there exists a neighborhood $\mathcal{U}$ of λ_0 such that $\mathbf{x}*u \equiv 0$ for each function $u \in L^1(\mathbb{R})$ with $\operatorname{supp}\hat{u} \in i\mathcal{U}$. Let λ_0 be a regular point of $\tilde{\mathbf{x}}(\lambda)$, that is, $\tilde{\mathbf{x}}(\lambda)$ has analytic continuation into a neighborhood $\mathcal{U}$ of λ_0. Let $u \in L^1(\mathbb{R})$ such that $\operatorname{supp}\hat{u} \subset i\mathcal{U}$. For each $\varepsilon > 0$ we have $\mathbf{x} * K_\varepsilon \in L^1(\mathbb{R})$ and $\widetilde{(\mathbf{x} * K_\varepsilon)}|_{i\mathcal{U}} = 0$. Therefore, $u * \mathbf{x} * K_\varepsilon = 0$. This implies $u * \mathbf{x} = 0$.

Conversely, assume that $\lambda_0 \in i\mathbb{R}$ and $\mathcal{U}$ is a neighborhood of λ_0 such that $u \in L^1(\mathbb{R})$ with $\operatorname{supp} \hat{u} \subset i\mathcal{U}$ implies $u * \mathbf{x} = 0$. By Proposition 3.1, $\tilde{\mathbf{x}}(\lambda)$ has analytic continuation into $\mathcal{U}$. ∎

REMARK 3.3. The proof of Proposition 3.2 also shows that if $\{\mathbf{x}_n\}_{n=1}^{\infty} \subset L^{\infty}(\mathbb{R}, X)$, $\mathbf{x}_n \to \mathbf{x}$ as $n \to \infty$ (in L^{∞}-norm), $\lambda_0 \in i\mathbb{R}$, and $\mathcal{U}$ is a neighborhood of λ_0 such that $\tilde{\mathbf{x}}_n(\lambda)$ has analytic continuation into $\mathcal{U}$ for each n, then $\tilde{\mathbf{x}}(\lambda)$ also has analytic continuation into $\mathcal{U}$.

Two well-known central results of the classical harmonic analysis which will be used in section 4 can be formulated in terms of a spectrum of a function. The first is the Wiener General Tauberian Theorem: *the translates of a function $f \in L^1(\mathbb{R})$ span a dense subspace of $L^1(\mathbb{R})$ if its Fourier transform never vanishes.* It is not hard to see (having in mind the equivalent definition of a spectrum given by Proposition 3.2) that the following is an equivalent formulation of this statement: *a function $\mathbf{x} \in L^{\infty}(\mathbb{R}, X)$ has empty spectrum only if $\mathbf{x}$ is identically zero.*

The second result is the Primary Ideal Theorem due to V. Ditkin: *a closed ideal in $L^1(\mathbb{R})$ whose Fourier transforms have only one common zero necessarily contains all functions whose Fourier transforms vanish at that point.* In other words, every closed primary ideal is maximal. In terms of a spectrum this theorem reads as follows: *a function $\mathbf{x} \in L^{\infty}(\mathbb{R}, X)$ whose spectrum is one point $\lambda \in i\mathbb{R}$ has the form $\mathbf{x}(t) = e^{\lambda t} x_0$, where $x_0 \in X$.*

We note that both theorems follow from the Carleman Lemma on analytic continuation, the estimate

$$\|\tilde{\mathbf{x}}(\lambda)\| \leq \frac{1}{|\mathrm{Re}\lambda|} \|\mathbf{x}\|, \quad \lambda \notin i\mathbb{R},$$

and a standard argument of the complex analysis involving the Phragmén-Lindelöf Theorem (see e.g. [14, p. 181]).

Now let $\{T(t)\}_{t \in \mathbb{R}}$ be a bounded C_0-group of linear operators in a Banach space X. Then for each vector x in X there is a complete trajectory through x, defined by $\gamma(x) = \{\mathbf{x}(t) = T(t)x : t \in \mathbb{R}\}$. Let M_x denote the closed subspace of X spanned by $T(t)x$, $t \in \mathbb{R}$. Clearly, M_x is invariant for each $T(t)$, thus also for A. We have

PROPOSITION 3.4. $\mathrm{Sp}(\mathbf{x}) = \sigma(A|M_x)$.

PROOF: We have, for each $\mathrm{Re}\lambda > 0$,

$$\tilde{\mathbf{x}}(\lambda) = \int_0^\infty e^{-\lambda t} T(t) x dt = (\lambda - A|M_x)^{-1} x,$$

and, since $(-A)$ is the generator of the semigroup $\{T(-s)\}_{s \geq 0}$, for each $\mathrm{Re}\lambda < 0$,

$$\begin{aligned} \tilde{\mathbf{x}}(\lambda) &= -\int_{-\infty}^0 e^{-\lambda t} T(t) x dt = -\int_0^\infty e^{\lambda s} T(-s) x ds \\ &= -\int_0^\infty e^{-(-\lambda)s} T(-s) x ds = -(-\lambda - (-A)|M_x)^{-1} x = (\lambda - A|M_x)^{-1} x. \end{aligned}$$

From this it follows that every point λ in $\rho(A|M_x)$ is a regular point of $\tilde{\mathbf{x}}$, hence $\mathrm{Sp}(\mathbf{x}) \subset \sigma(A|M_x)$. Conversely, let λ be a regular point of $\tilde{\mathbf{x}}$ and $\mathcal{U}$ be a corresponding neighborhood of λ into which $\tilde{\mathbf{x}}$ has analytic continuation. Let $y \in \mathrm{span}\{T(t)x : t \in \mathbb{R}\}$ and $\mathbf{y}(t) \equiv T(t)y$. It is easy to see that $\tilde{\mathbf{y}}$ also has analytic continuation into $\mathcal{U}$. By Remark 3.3, for every $y \in M_x$ the function $\tilde{\mathbf{y}}$ has analytic continuation into $\mathcal{U}$. From this it is easy to see that the operator B in M_x defined by $By = \tilde{\mathbf{y}}(\lambda)$ is a bounded inverse of $(\lambda - A|M_x)$. ∎

Below we denote the differentiation operator d/dt by D for simplicity of notations.

PROPOSITION 3.5. *Let* $\mathbf{x}(t)$ *be a function in* $BUC(\mathbb{R}, X)$ *and* $\mathcal{M}_{\mathbf{x}}$ *be a closed subspace in* $BUC(\mathbb{R}, X)$ *spanned by all translations of* $\mathbf{x}$. *Then* $\mathrm{Sp}(\mathbf{x}) = \sigma(D|\mathcal{M}_{\mathbf{x}})$.

PROOF: Apply Proposition 3.4 to the translation group $\{S(t)\}_{t \in \mathbb{R}}$ in $BUC(\mathbb{R}, X)$. ∎

REMARK 3.6. Using the fact that isometric groups have *separable spectrum* [3], [17] (see also [8]), one can show that, if $\mathbf{x}(t) \in BUC(\mathbb{R}, X)$ is a complete trajectory, then for every compact subset $\Delta \subset \mathrm{Sp}(\mathbf{x})$, such that $\mathrm{Int}(\Delta)$ is non-empty (where Int means interior with regard to $\mathrm{Sp}(\mathbf{x})$), there is a non-trivial complete trajectory $\mathbf{y}(t) \in BUC(\mathbb{R}, X)$ under $\{T(t)\}_{t \geq 0}$ such that $\mathrm{Sp}(\mathbf{y}) \subset \Delta$. This fact will not be used in the sequel.

PROPOSITION 3.7. *If* $\mathbf{x}(t)$ *is a uniformly continuous bounded complete trajectory under the semigroup* $\{T(t)\}_{t\geq 0}$, *then* $\mathrm{Sp}(\mathbf{x}) \subset A\sigma(A) \cap i\mathbb{R}$.

PROOF: Suppose that $\lambda \in \mathrm{Sp}(\mathbf{x})$. By Proposition 3.5, $\lambda \in \sigma(D|\mathcal{M}_{\mathbf{x}})$. Since $D|\mathcal{M}_{\mathbf{x}}$ is a generator of an isometric group, its spectrum coincides with the approximate point spectrum, so that there exists $\{\mathbf{y}_n\} \subset \mathcal{M}_{\mathbf{x}}$, $\|\mathbf{y}_n\| = 1$, such that

$$\lim_{n\to\infty} \|S(t)\mathbf{y}_n - e^{\lambda t}\mathbf{y}_n\|_\infty = 0, \quad \forall t \in \mathbb{R}.$$

Without lost of generality we can assume that all $\mathbf{y}_n$ are from the dense linear manifold spanned by translations of $\mathbf{x}$, that is,

$$\mathbf{y}_n(s) = \sum_{j=1}^{k_n} \xi_j^{(n)} \mathbf{x}_{\tau_j^{(n)}}(s),$$

for some $\xi_j^{(n)} \in \mathbb{C}$, $\tau_j^{(n)} \in \mathbb{R}$. Thus we have

$$\lim_{n\to\infty} \|\sum_{j=1}^{k_n} \xi_j^{(n)} \mathbf{x}_{\tau_j^{(n)}+t} - e^{\lambda t}\sum_{j=1}^{k_n} \xi_j^{(n)} \mathbf{x}_{\tau_j^{(n)}}\|_\infty = 0, \quad \forall t \in \mathbb{R}. \tag{3.3}$$

Since $\|\mathbf{y}_n\| = 1$, there exists, for each n, a real number s_n such that

$$\|\sum_{j=1}^{k_n} \xi_j^{(n)} \mathbf{x}(\tau_j^{(n)} + s_n)\| \geq 1 - \varepsilon.$$

We put $z_n = \sum_{j=1}^{k_n} \xi_j^{(n)} \mathbf{x}(\tau_j^{(n)} + s_n)$. Then $\|z_n\| \geq 1 - \varepsilon$ and, since $\mathbf{x}(t)$ is a complete trajectory under $T(t)$, we have, for each positive t, $\mathbf{x}(\tau_j^{(n)} + s_n + t) = T(t)\mathbf{x}(\tau_j^{(n)} + s_n)$. Therefore, from (3.3) it follows

$$\lim_{n\to\infty} \|T(t)z_n - e^{\lambda t}z_n\| = 0, \quad \forall t \geq 0,$$

i.e. $\lambda \in A\sigma(A)$. ∎

Recall that a function $\mathbf{x}(t) \in BUC(\mathbb{R}, X)$ is said to be *almost periodic* if the family of translations $\mathcal{H}_{\mathbf{x}} = \{S(t)\mathbf{x} : t \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R}, X)$.

PROPOSITION 3.8. *Suppose that* $\mathbf{x} \in BUC(\mathbb{R}, X)$ *and* $\mathrm{Sp}(\mathbf{x})$ *is countable. Then* $\mathbf{x}$ *is almost periodic, provided one of the following conditions is satisfied:*

i) X *does not contain* c_0*;*

ii) The set $\{\mathbf{x}(t) : -\infty < t < \infty\}$ *is weakly relatively compact in* X.

This is the vector-valued version of the well known Loomis' Theorem (see [15], [16]). Here "X does not contain c_0" means that there is no subspace of X, which is isomorphic to c_0, the Banach space of (numerical) sequences which converge to zero. For example, reflexive or weakly complete Banach spaces do not contain c_0.

THEOREM 3.9. *Let* $\mathcal{T}$ *be a* C_0*-semigroup with generator* A *such that* $A\sigma(A) \cap i\mathbb{R}$ *is countable and* $P\sigma(A) \cap i\mathbb{R}$ *is empty. Then there exists no non-trivial uniformly continuous weakly compact complete trajectory. If, in addition,* X *does not contain* c_0*, then there is no non-trivial uniformly continuous bounded complete trajectory.*

PROOF: Indeed, if $\mathbf{x}(t)$ is a non-trivial uniformly continuous bounded complete trajectory, then from Proposition 3.7 it follows that $\mathrm{Sp}(\mathbf{x})$ is countable. Thus, by Proposition 3.8, $\mathbf{x}(t)$ is an almost periodic function. Consider the subspace $\mathcal{M}_{\mathbf{x}}$ in $BUC(\mathbb{R}, X)$, which is spanned by translations $\{S(\tau)\mathbf{x}\}_{\tau \in \mathbb{R}}$. It is easy to see that the restriction of $\{S(\tau)\}_{\tau \in \mathbb{R}}$ onto $\mathcal{M}_{\mathbf{x}}$ is an almost periodic group. Therefore, there exists a function $\mathbf{y}(t)$ in $\mathcal{M}_{\mathbf{x}}$ and a real number λ, such that $S(\tau)\mathbf{y}(t) = e^{i\lambda\tau}\mathbf{y}(t), \forall \tau, t \in \mathbb{R}$, (see e.g. [4]). As indicated in the proof of Proposition 3.6, $\mathbf{y}(t)$ is a complete trajectory under $\{T(t)\}_{t \geq 0}$. Consequently, we have

$$(T(\tau)\mathbf{y})(t) = \mathbf{y}(t+\tau) = (S(\tau)\mathbf{y})(t) = e^{i\lambda\tau}\mathbf{y}(t), \forall \tau \geq 0, t \in \mathbb{R}.$$

This implies that $\lambda \in P\sigma(A)$, which is a contradiction. ∎

In [15] it was proved, that if $\sigma(A) \cap i\mathbb{R}$ is countable and $\{\mathbf{x}(t)\}_{t \in \mathbb{R}}$ is a uniformly continuous bounded complete trajectory, then $\mathbf{x}(t)$ is almost periodic, provided that condition (i) or (ii) in Theorem 3.10 holds. Propositions 3.7 and 3.8 give an alternative proof of the following slightly improved version of this result.

THEOREM 3.10. *Let* $\{T(t)\}_{t\geq 0}$ *be a* C_0*-semigroup with generator* A*, such that* $A\sigma(A)\cap i\mathbb{R}$ *is countable. Then every uniformly continuous bounded complete trajectory* $\{\mathbf{x}(t)\}_{t\in\mathbb{R}}$ *is almost periodic, provided one of the following conditions is satisfied:*

i) X *does not contain* c_0*;*

ii) The set $\{\mathbf{x}(t) : t \in \mathbb{R}\}$ *is weakly relatively compact in* X.

4. Asymptotic stability.

COROLLARY 4.1. *Suppose that* $\{T(t)\}_{t\geq 0}$ *is a bounded* C_0*-semigroup with generator* A*. Assume that:*

i) $\sigma(A)\cap i\mathbb{R}$ *is countable;*

ii) $P\sigma(A^*)\cap i\mathbb{R}$ *is empty.*

Then the semigroup $\{T(t)\}_{t\geq 0}$ *is asymptotically stable.*

PROOF: Assume, on contrary, that $\{T(t)\}_{t\geq 0}$ is not asymptotically stable. By Theorem 2.3, there is a non-trivial complete trajectory $\mathbf{f}(t) \in BUC(\mathbb{R}, X^{\odot})$. By Proposition 3.7, $\mathrm{Sp}(\mathbf{f}) \subset \sigma(A^{\odot})\cap i\mathbb{R}$, so that $\mathrm{Sp}(\mathbf{f})$ also is countable. By the Wiener General Tauberian Theorem, $\mathrm{Sp}(\mathbf{f})$ is non-empty, so that $\sigma(D|\mathcal{M}_{\mathbf{f}})$ also is non-empty, by Proposition 3.5. Let λ_0 be an isolated point in $\sigma(D|M_{\mathbf{f}})$ and P be the corresponding Riesz projection in $\mathcal{M}_{\mathbf{f}}$. It is not hard to see that every function $\mathbf{g}(t)$ in $\mathrm{ran}(P) \subset BUC(\mathbb{R}, X^{\odot})$ is a complete trajectory under $\{T^{\odot}(t)\}_{t\geq 0}$ with a single spectrum at $\lambda_0 \in i\mathbb{R}$. Therefore, $\mathbf{g}(t) = e^{\lambda_0 t}g_0$ for some $g_0 \in X^*$, i.e. $T^*(t)g_0 = e^{\lambda_0 t}g_0$, $\forall t \geq 0$, or $A^*g_0 = \lambda_0 g_0$, which is a contradiction. ∎

We remark that Theorem 2.3 itself can be also reformulated as a criterion of asymptotic stability.

COROLLARY 4.2. . *Let* $\{T(t)\}_{t\geq 0}$ *be a bounded* C_0*-semigroup. Assume, that the sun-dual semigroup* $\{T^{\odot}(t)\}_{t\geq 0}$ *has no non-trivial bounded complete trajectory. Assume, moreover, that one of the following conditions is satisfied:*

i) There is $t_0 > 0$ *such that* $T(t_0)$ *has dense range;*

ii) $\sigma(A) \not\supset i\mathbb{R}$.

Then the semigroup $\{T(t)\}_{t\geq 0}$ *is asymptotically stable.*

In particular, if the semigroup $\{T(t)\}_{t\geq 0}$ has bounded generator and the adjoint semigroup does not have a non-trivial bounded complete trajectory, then $\{T(t)\}_{t\geq 0}$ is asymptotically stable.

5. Discrete case.

The results of sections 2-4 remain true for discrete semigroup $\{T^n\}_{n\in\mathbb{Z}_+}$, where $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. Since the proofs are obvious modifications of the proofs for the case of C_0-semigroups, we restrict ourselves to formulations of the results. Let T be a linear bounded operator in a Banach space X. A two-sided sequence $\mathbf{x} \equiv \{x_n\}_{n\in\mathbb{Z}}$ is called a *complete trajectory* under T if $x_{n+k} = T^k x_n$ for every $n \in \mathbb{Z}$ and every $k \in \mathbb{Z}_+$. The operator T is said to be *power-bounded*, if $\sup_{n\in\mathbb{Z}_+} \|T^n\| < \infty$. If $\|T^n x\| \to 0$ as $n \to \infty$ for all x in X, then we say that T *is of class* $C_{0\cdot}$. Below we denote by Γ the set $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

THEOREM 5.1. *Suppose that* T *is a power-bounded operator in a Banach space* X*, such that* $T \notin C_{0\cdot}$. *Suppose, moreover, that one of the following conditions is satisfied:*

i) $\mathrm{ran}(T)$ *is dense in* X*;*

ii) $\sigma(T) \not\supset \Gamma$.

Then there exists a non-trivial bounded complete trajectory for the adjoint operator T^*.

Note that for discrete semigroups (as well as for uniformly continuous one-parameter semigroups) the sun-dual semigroup coincides with the usual dual semigroup. Consequently, such semigroups are sun-reflexive iff X is reflexive.

COROLLARY 5.2. *Suppose that* X *is reflexive and* T *is a power-bounded operator in* X *such that* $T^* \notin C_{0\cdot}$ *and one of the following conditions holds:*

i) $\ker(T) = \{0\}$*;*

ii) $\sigma(T) \not\supset \Gamma$.

Then there exists a non-trivial bounded complete trajectory under T.

Let $\mathbf{x} \equiv \{x_n\}_{n\in\mathbb{Z}}$ be a bounded two-sided sequence in X. The *Carleman transform* $\tilde{\mathbf{x}}(z)$ of $\mathbf{x}$ is a function analytic on $\mathbb{C} \setminus \Gamma$ defined by

$$\tilde{\mathbf{x}}(z) = \begin{cases} \sum_{n=1}^{\infty} x_n z^{n-1}, & |z| < 1 \\ -\sum_{-\infty}^{0} x_n z^{n-1}, & |z| > 1 \end{cases}$$

A point z on Γ is called *regular point* of $\mathbf{x}$, if $\tilde{\mathbf{x}}(z)$ can be continued analytically into a neighborhood of z. The complement in Γ of the set of regular points is called *spectrum* of $\mathbf{x}$ and denoted by $\mathrm{Sp}(\mathbf{x})$.

PROPOSITION 5.3. *Let* T *be a linear operator in* X *and* $\mathbf{x} \equiv \{x_n\}_{n\in\mathbb{Z}}$ *be a bounded complete trajectory under* T. *Then* $Sp(\mathbf{x}) \subset A\sigma(T) \cap \Gamma$.

THEOREM 5.4. *Suppose that* $A\sigma(T) \cap \Gamma$ *is countable and* $P\sigma(T) \cap \Gamma$ *is empty. Then there is no non-trivial weakly compact trajectory under* T. *If, in addition,* X *does not contain* c_0, *then there is no non-trivial bounded complete trajectory under* T.

A two-sided sequence $\mathbf{x} \equiv \{x_n\}_{n\in\mathbb{Z}}$ is said to be *almost periodic* if the sequence of shifts $\mathbf{x}_k \equiv \{x_{k+n}\}_{n\in\mathbb{Z}}$, $k \in \mathbb{Z}$, is relatively compact in $l^\infty(X)$, the Banach space of bounded two-sided sequences with values in X. Equivalently, $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}}$ is almost periodic if and only if, for every $\varepsilon > 0$, there exists a natural number k_ε, such that, every interval $(m, m + k_\varepsilon)$, $m \in \mathbb{Z}$, contains a *ε-almost-period* N , i.e. $\|x_{n+N} - x_n\| < \varepsilon$ for each $n \in \mathbb{Z}$.

THEOREM 5.5. *Assume that* $A\sigma(T) \cap \Gamma$ *is countable. Then every bounded complete trajectory* $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}}$ *under* T *is almost periodic, provided one of the following conditions is satisfied:*

i) X *does not contain* c_0;

ii) $\{x_n\}_{n\in\mathbb{Z}}$ *is weakly relatively compact.*

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