On Robin's criterion for the Riemann Hypothesis

Y.-J. Choie, N. Lichiardopol, P. Moree, P. Solé

Abstract

Robin's criterion states that the Riemann Hypothesis (RH) is true if and only if Robin's inequality $\sigma(n) := \sum_{d|n} d < e^{\gamma} n \log\log n$ is satisfied for $n \geq 5041$, where γ denotes the Euler(-Mascheroni) constant. We show by elementary methods that if $n \geq 37$ does not satisfy Robin's criterion it must be even and is neither squarefree nor squarefull. Using a bound of Rosser and Schoenfeld we show, moreover, that n must be divisible by a fifth power > 1. As consequence we obtain that RH holds true iff every natural number divisible by a fifth power > 1 satisfies Robin's inequality.

1 Introduction

Let $\mathcal{R}$ be the set of integers $n \geq 1$ satisfying $\sigma(n) < e^{\gamma} n \log\log n$. This inequality we will call *Robin's inequality.* Note that it can be rewritten as

$$\sum_{d|n} \frac{1}{d} < e^{\gamma} \log\log n.$$

Ramanujan [12] (in his original version of his paper on highly composite integers, only part of which, due to paper shortage, was published, for the shortened version see [11, pp. 78-128]) proved that if RH holds then every sufficiently large integer is in $\mathcal{R}$. Robin [13] proved that if RH holds, then actually every integer $n \geq 5041$ is in $\mathcal{R}$. He also showed that if RH is false, then there are infinitely many integers that are not in $\mathcal{R}$. The numbers ≤ 5040 that are not in $\mathcal{R}$ are 2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520 and 5040. Note that none of them is divisible by a 5th power of a prime.

In this paper we are interested in establishing the inclusion of various infinite subsets of the natural numbers in $\mathcal{R}$. We will prove in this direction:

Theorem 1 *Put* $\mathcal{A} = \{2, 3, 5, 6, 10, 30\}$. *Every squarefree integer that is not in* $\mathcal{A}$ *is an element of* $\mathcal{R}$.

A similar result for the odd integers will be established:

Mathematics Subject Classification (2000). 11Y35, 11A25, 11A41

Theorem 2 *Any odd positive integer n distinct from $1, 3, 5$ and 9 is in $\mathcal{R}$.*

On combining Robin's result with the above theorems one finds:

Theorem 3 *The RH is true if and only for all even non-squarefree integers ≥ 5044 Robin's inequality is satisfied.*

It is an easy exercise to show that the even non-squarefree integers have density $\frac{1}{2} - \frac{2}{\pi^2} = 0.2973\cdots$ (cf. Tenenbaum [15, p. 46]). Thus, to wit, this paper gives at least half a proof of RH !

Somewhat remarkably perhaps these two results will be proved using only very elementary methods. The deepest input will be Lemma 1 below which only requires pre-Prime Number Theorem elementary methods for its proof (in Tenenbaum's [15] introductory book on analytic number theory it is already derived within the first 18 pages).

Using a bound of Rosser and Schoenfeld (Lemma 4 below), which ultimately relies on some explicit knowledge regarding the first so many zeros of the Riemann zeta-function, one can prove some further results:

Theorem 4 *The only squarefull integers not in $\mathcal{R}$ are $4,\ 8,\ 9,\ 16,\ 36$.*

We recall that an integer n is said to be squarefull if for every prime divisor p of n we have $p^2|n$. An integer n is called t-free if $p^t \nmid m$ for every prime number p. (Thus saying a number is squarefree is the same as saying that it is 2-free.)

Theorem 5 *All 5-free integers satisfy Robin's inequality.*

Together with the observation that all exceptions ≤ 5040 to Robin's inequality are 5-free and Robin's criterion, this result implies the following alternative variant of Robin's criterion.

Theorem 6 *The RH holds iff for all integers n divisible by the fifth power of some prime we have $\sigma(n) < e^\gamma n \log\log n$.*

2 Proof of Theorem 1 and Theorem 2

Our proof of Theorem 1 requires the following lemmata.

Lemma 1
1) *For $x \geq 2$ we have*

$$\sum_{p\leq x} \frac{1}{p} = \log\log x + B + O(\frac{1}{\log x}),$$

where the implicit constant in Landau's symbol does not exceed $2(1 + \log 4) < 5$ and

$$B = \gamma + \sum_p \left(\log(1 - \frac{1}{p}) + \frac{1}{p}\right) = 0.2614972128\cdots$$

denotes the (Meissel-)Mertens constant.
2) *For $x \geq 5$ we have*

$$\sum_{p\leq x} \frac{1}{p} \leq \log\log x + \gamma.$$

Proof. 1) This result can be proved with very elementary methods. It is derived from scratch in the book of Tenenbaum [15], p. 16. At p. 18 the constant B is determined.
2) One checks that the inequality holds true for all primes p satisfying $5 \le p \le 3673337$. On noting that

$$B + \frac{2(1+\log 4)}{\log 3673337} < \gamma,$$

the result then follows from part 1. □

`Remark 1`. More information on the (Meissel-)Mertens constant can be found e.g. in the book of Finch [6, §2.2].
`Remark 2`. Using deeper methods from (computational) prime number theory Lemma 1 can be considerably sharpened, see e.g. [14], but the point we want to make here is that the estimate given in part 2, which is the estimate we need in the sequel, is a rather elementary estimate.

We point out that 15 is in $\mathcal{R}$.

Lemma 2 *If r is in $\mathcal{A}$ and $q \ge 7$ is a prime, then rq is in R.*

Proof. Suppose that r is in $\mathcal{A}$. Direct computation shows that $7r$ is in $\mathcal{R}$. From this we obtain that

$$(1+\frac{1}{q})\frac{\sigma(r)}{r} \le \frac{8\sigma(r)}{7r} < e^\gamma \log\log(7r) \le e^\gamma \log\log(qr),$$

for $q \ge 7$, whence the result follows on noting that $\sigma(rq) = \sigma(r)\sigma(q)$. □

Proof of Theorem 1. By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n. Put $\omega(n) = m$. The assertion is easily provable for those integers with $m = 1$ (the primes that is). Suppose it is true for $m-1$, with $m \ge 2$ and let us consider the assertion for those squarefree n with $\omega(n) = m$. So let $n = q_1 \cdots q_m$ be a squarefree number that is not in $\mathcal{A}$ and assume w.l.o.g. that $q_1 < \cdots < q_m$. We consider two cases:
Case 1: $q_m \ge \log(q_1 \cdots q_m) = \log n$.
If $q_1 \cdots q_{m-1}$ is in $\mathcal{A}$, then if q_m is not in $\mathcal{A}$, $n = q_1 \ldots q_{m-1}q_m$ is in $\mathcal{R}$ (Lemma 2) and we are done, and if q_m is in $\mathcal{A}$, the only possibility is $n = 15$ which is in $\mathcal{R}$ and we are also done.
If $q_1 \cdots q_{m-1}$ is not in $\mathcal{A}$, by the induction hypothesis we have

$$(q_1+1)\cdots(q_{m-1}+1) < e^\gamma q_1 \cdots q_{m-1} \log\log(q_1 \cdots q_{m-1}),$$

and hence

$$(q_1+1)\cdots(q_{m-1}+1)(q_m+1) < e^\gamma q_1 \cdots q_{m-1}(q_m+1) \log\log(q_1 \cdots q_{m-1}). \tag{1}$$

We want to show that

$$\begin{aligned} &e^\gamma q_1 \cdots q_{m-1}(q_m+1) \log\log(q_1 \cdots q_{m-1}) \\ &\le e^\gamma q_1 \cdots q_{m-1} q_m \log\log(q_1 \cdots q_{m-1} q_m) = e^\gamma n \log\log n. \end{aligned} \tag{2}$$

Indeed (2) is equivalent with $q_m \log\log(q_1\cdots q_{m-1}q_m) \ge (q_m+1)\log\log(q_1\cdots q_{m-1})$, or alternatively

$$\frac{q_m(\log\log(q_1\cdots q_{m-1}q_m) - \log\log(q_1\cdots q_{m-1}))}{\log q_m} \ge \frac{\log\log(q_1\cdots q_{m-1})}{\log q_m}. \tag{3}$$

Suppose that $0 < a < b$. Note that we have

$$\frac{\log b - \log a}{b-a} = \frac{1}{b-a}\int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{4}$$

Using this inequality we infer that (3) (and thus (2)) is certainly satisfied if the next inequality is satisfied:

$$\frac{q_m}{\log(q_1\cdots q_m)} \ge \frac{\log\log(q_1\cdots q_{m-1})}{\log q_m}.$$

Note that our assumption that $q_m \ge \log(q_1\cdots q_m)$ implies that the latter inequality is indeed satisfied.
Case 2: $q_m < \log(q_1\cdots q_m) = \log n$.
It is easy to see that $\sigma(n) < e^\gamma n \log\log n$ is equivalent with

$$\log(q_1+1) - \log q_1 + \cdots + \log(q_m+1) - \log q_m < \gamma + \log\log\log(q_1\cdots q_m). \tag{5}$$

Note that

$$\log(q_1+1) - \log q_1 = \int_{q_1}^{q_1+1} \frac{dt}{t} < \frac{1}{q_1}.$$

In order to prove (5) it is thus enough to prove that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \le \sum_{p\le q_m} \frac{1}{p} \le \gamma + \log\log\log(q_1\cdots q_m). \tag{6}$$

Since $q_m \ge 7$ we have by part 2 of Lemma 1 and the assumption $q_m < \log(q_1\cdots q_m)$ that

$$\sum_{p\le q_m} \frac{1}{p} \le \gamma + \log\log q_m < \gamma + \log\log\log(q_1\cdots q_m),$$

and hence (6) is indeed satisfied. □

Theorem 2 will be derived from the following stronger result.

Theorem 7 *For all odd integers except* $1, 3, 5, 9$ *and* 15 *we have*

$$\frac{n}{\varphi(n)} < e^\gamma \log\log n. \tag{7}$$

To see that this is a stronger result, let $n = \prod_{i=1}^k p_i^{e_i}$ be the prime factorisation of n and note that for $n \ge 2$ we have

$$\frac{\sigma(n)}{n} = \prod_{i=1}^k \frac{1-p_i^{-e_i-1}}{1-p_i^{-1}} < \prod_{i=1}^k \frac{1}{1-p_i^{-1}} = \frac{n}{\varphi(n)}, \tag{8}$$

where $\varphi(n)$ denotes Euler's totient function.

We let $\mathcal{N}$ ($\mathcal{N}$ in acknowledgement of the contributions of J.-L. Nicolas to this subject) denote the set of integers $n \geq 1$ satisfying (7). Our proofs of Theorems 2 and 7 use the next lemma, the proof of which rests on some numerical estimates in combination with very straightforward manipulations and is left to the interested reader.

Lemma 3 *Put* $S = \{3^a \cdot 5^b \cdot q^c : q \geq 7 \text{ is prime, } a, b, c \geq 0 \text{ and } \omega(3^a \cdot 5^b \cdot q^c) \geq 2\}$. *Then* $\mathcal{S} \subset \mathcal{R}$. *Moreover, all elements from* $\mathcal{S}$ *except for* 15 *are in* $\mathcal{N}$.

`Remark`. Let t be any integer. Suppose that we have an infinite set of integers all having no prime factors $> t$. Then $\sigma(n)/n$ and $n/\varphi(n)$ are bounded above on this set, whereas $\log\log n$ tends to infinity. Thus only finitely many of those integers will not be in $\mathcal{R}$, respectively $\mathcal{N}$. It is a finite computation to find them all.

Proof of Theorem 7. As before we let $m = \omega(n)$. If $m \leq 1$ it is easy to check that n is in $\mathcal{N}$, except when $n = 1, 3, 5$ or 9. So we may assume $m \geq 2$. Let $\kappa(n) = \prod_{p|n} p$ denote the squarefree kernel of n. Since $n/\varphi(n) = \kappa(n)/\varphi(\kappa(n))$ it follows that if r is a squarefree number satisfying (7), then all integers n with $\kappa(n) = r$ satisfy (7) as well. Thus we consider first the case where $n = q_1 \cdots q_m$ is an odd squarefree integer with $q_1 < \cdots < q_m$. In this case n is in $\mathcal{N}$ iff

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \log\log n.$$

Note that

$$\frac{q_i}{q_i - 1} \leq \frac{3}{2} \text{ and } \frac{q_i}{q_i - 1} < \frac{q_{i-1} + 1}{q_{i-1}},$$

and hence

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < \frac{3}{2} \prod_{i=1}^{m-1} \frac{q_i + 1}{q_i} = \frac{\sigma(n_1)}{n_1},$$

where $n_1 = 2n/q_m < n$. Thus, $n/\varphi(n) < \sigma(n_1)/n_1$. If n_1 is in $\mathcal{R}$, then invoking Theorem 1 we find

$$\frac{n}{\varphi(n)} < \frac{\sigma(n_1)}{n_1} < e^{\gamma} \log\log n_1 < e^{\gamma} \log\log n,$$

and we are done.

If n_1 is not in $\mathcal{R}$, then by Theorem 1 it follows that n must be in $\mathcal{S}$. The proof is now completed on invoking Lemma 3. □

Proof of Theorem 2. One checks that $1, 3, 5$ and 9 are not in $\mathcal{R}$, but 15 is in $\mathcal{R}$. The result now follows by Theorem 7 and inequality (8). □

`Remark`. Note that the proofs of our theorems could have been nicer, if instead of Robin's criterion we had a criterion involving every integer $n \geq 1$. Such a criterion was found in 2002 by Lagarias [7] who, using Robin's work, showed that the RH is equivalent with the inequality

$$\sigma(n) \leq h(n) + e^{h(n)} \log(h(n)),$$

where $h(n) = \sum_{k=1}^{n} 1/k$ is the harmonic sum. Unfortunately our methods, which rest on the multiplicativity of $\sigma(n)/n$, break down for this inequality.

2.1 Theorem 7 put into perspective

Since the proof of Theorem 7 can be carried out with such simple means, one might expect it can be extended to quite a large class of even integers. However, even a superficial inspection of the literature on $n/\varphi(n)$ shows this expectation to be wrong.

Rosser and Schoenfeld [14] showed in 1962 that

$$\frac{n}{\varphi(n)} \le e^{\gamma} \log\log n + \frac{5}{2\log\log n},$$

with one exception: $n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. They raised the question of whether there are infinitely many n for which

$$\frac{n}{\varphi(n)} > e^{\gamma} \log\log n, \tag{9}$$

which was answered in the affirmative by J.-L. Nicolas [9]. More precisely, let $N_k = 2 \cdot 3 \cdot \cdots \cdot p_k$ be the product of the first k primes, then if the RH holds true (9) is satisfied with $n = N_k$ for every $k \ge 1$. On the other hand, if RH is false, then there are infinitely many k for which (9) is satisfied with $n = N_k$ and there are infinitely many k for which (9) is not satisfied with $n = N_k$. Thus the approach we have taken to prove Theorem 2, namely to derive it from the stronger result Theorem 7, is not going to work for even integers.

3 Proof of Theorem 4

The proof of Theorem 4 is an immediate consequence of the following stronger result.

Theorem 8 *The only squarefull integers $n \ge 2$ not in $\mathcal{N}$ are* 4, 8, 9, 16, 36, 72, 108, 144, 216, 900, 1800, 2700, 3600, 44100 *and* 88200.

Its proof requires the following two lemmas.

Lemma 4 [14]. *For $x > 0$ we have*

$$\prod_{p \le x} \frac{p}{p-1} \le e^{\gamma}(\log x + \frac{1}{\log x}).$$

Lemma 5 *Let $p_1 = 2,\ p_2 = 3, \ldots$ denote the consecutive primes. If*

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \ge e^{\gamma} \log(2\log(p_1 \cdots p_m)),$$

then $m \le 4$.

Proof. Suppose that $m \geq 26$ (i.e. $p_m \geq 101$). It then follows by Theorem 10 of [14] which states that $\theta(x) := \sum_{p \leq x} \log p > 0.84x$ for $x \geq 101$, that $\log(p_1 \cdots p_m) = \theta(p_m) > 0.84 p_m$. We find that

$$\log(2 \log(p_1 \cdots p_m)) > \log p_m + \log 1.64 \geq \log p_m + \frac{1}{\log p_m},$$

and so, by Lemma 4, that

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq e^\gamma \left(\log p_m + \frac{1}{\log p_m} \right) < e^\gamma \log(2 \log(p_1 \cdots p_m)).$$

The proof is then completed on checking the inequality directly for the remaining values of m. □

Proof of Theorem 8. Suppose that

$$\frac{n}{\varphi(n)} \geq e^\gamma \log \log n.$$

Put $\omega(n) = m$. Then

$$\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \geq \frac{n}{\varphi(n)} \geq e^\gamma \log \log n \geq e^\gamma \log(2 \log(p_1 \ \cdots \ p_n)).$$

By Lemma 5 it follows that $m \leq 4$. In particular we must have

$$2 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} = \frac{35}{8} \geq e^\gamma \log \log n,$$

whence $n \leq \exp(\exp(e^{-\gamma} 35/8)) \leq 116144$. On numerically cheeking the inequality for the squarefull integers ≤ 116144, the proof is then completed. □

`Remark`. The squarefull integers ≤ 116144 are easily produced on noting that they can be unqiuely written as $a^2 b^3$, with a a positive integer and b squarefree.

4 On the ratio $\sigma(n)/(n \log \log n)$ as n ranges over various sets of integers

We have proved that Robin's inequality holds for large enough odd numbers, squarefree and squarefull numbers. A natural question to ask is how large the ratio $f_1(n) := \sigma(n)/(n \log \log n)$ can be when we restrict n to these sets of integers. We will consider the same question for the ratio $f_2(n) := n/(\varphi(n) \log \log n)$. Our results in this direction are summarized in the following result:

Theorem 9 *We have*

$$(1)\ \limsup_{n \to \infty} f_1(n) = e^\gamma,\ (2)\ \limsup_{\substack{n \to \infty \\ n \text{ is squarefree}}} f_1(n) = \frac{6e^\gamma}{\pi^2},\ (3)\ \limsup_{\substack{n \to \infty \\ n \text{ is odd}}} f_1(n) = \frac{e^\gamma}{2},$$

and, moreover,

$$(4)\ \limsup_{n\to\infty} f_2(n) = e^\gamma,\ (5)\ \limsup_{\substack{n\to\infty \\ n \text{ is squarefree}}} f_2(n) = e^\gamma,\ (6)\ \limsup_{\substack{n\to\infty \\ n \text{ is odd}}} f_2(n) = \frac{e^\gamma}{2}.$$

Furthermore,

$$(7)\ \limsup_{\substack{n\to\infty \\ n \text{ is squarefull}}} f_1(n) = e^\gamma,\ (8)\ \limsup_{\substack{n\to\infty \\ n \text{ is squarefull}}} f_2(n) = e^\gamma.$$

(The fact that the corresponding lim infs are all zero is immediate on letting n run over the primes.)

Part 4 of Theorem 9 was proved by Landau in 1909, see e.g. [2, Theorem 13.14], and the remaining parts can be proved in a similar way. Gronwall in 1913 established part 1. Our proof makes use of a lemma involving t-free integers (Lemma 6), which is easily proved on invoking a celebrated result due to Mertens (1874) asserting that

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} \sim e^\gamma \log x, \quad x\to\infty.$$

Lemma 6 *Let $t \ge 2$ be a fixed integer. We have*

$$(1)\ \limsup_{\substack{n\to\infty \\ t-\text{free integers}}} f_1(n) = \frac{e^\gamma}{\zeta(t)},\ (2)\ \limsup_{\substack{n\to\infty \\ \text{odd } t-\text{free integers}}} f_1(n) = \frac{e^\gamma}{2\zeta(t)(1-2^{-t})}.$$

Proof. 1) Let us consider separately the prime divisors of n that are larger than $\log n$. Let us say there are r of them. Then $(\log n)^r < n$ and thus $r < \log n/\log\log n$. Moreover, for $p > \log n$ we have

$$\frac{1-p^{-t}}{1-p^{-1}} < \frac{1-(\log n)^{-t}}{1-(\log n)^{-1}}.$$

Thus,

$$\prod_{\substack{p|n \\ p>\log n}} \frac{1-p^{-t}}{1-p^{-1}} < \left(\frac{1-(\log n)^{-t}}{1-(\log n)^{-1}}\right)^{\frac{\log n}{\log\log n}}.$$

Let p_k denote the largest prime factor of n. We obtain

$$\begin{aligned}\frac{\sigma(n)}{n} &= \prod_{i=1}^{k} \frac{1-p_i^{-e_i-1}}{1-p_i^{-1}} \le \prod_{i=1}^{k} \frac{1-p_i^{-t}}{1-p_i^{-1}} \\ &< \left(\frac{1-(\log n)^{-t}}{1-(\log n)^{-1}}\right)^{\frac{\log n}{\log\log n}} \prod_{p\le\log n} \frac{1-p^{-t}}{1-p^{-1}}, \qquad (10)\end{aligned}$$

where in the derivation of the first inequality we used that $e_i < t$ by assumption. Note that the factor before the final product satisfies $1 + O((\log\log n)^{-1})$ and thus tends to 1 as n tends to infinity. On invoking Mertens' theorem and noting that $\prod_{p\le\log n}(1-p^{-t}) \sim \zeta(t)^{-1}$, it follows that the $\limsup \le e^\gamma/\zeta(t)$.

In order to prove the $\geq$ part of the assertion, take $n = \prod_{p \leq x} p^{t-1}$. Note that n is t-free. On invoking Mertens' theorem we infer that

$$\frac{\sigma(n)}{n} = \prod_{p \leq x} \frac{1 - p^{-t}}{1 - p^{-1}} \sim \frac{e^\gamma}{\zeta(t)} \log x.$$

Note that $\log n = t \sum_{p \leq x} p = t\theta(x)$, where $\theta(x)$ denotes the Chebyshev theta function. By an equivalent form of the Prime Number Theorem we have $\theta(x) \sim x$ and hence $\log\log n = (1 + o_t(1)) \log x$. It follows that for the particular sequence of infinitely many n values under consideration we have

$$\frac{\sigma(n)}{n \log\log n} = \frac{e^\gamma}{\zeta(t)} \Big(1 + o_t(1)\Big).$$

Thus, in particular, for a given $\epsilon > 0$ there are infinitely many n such that

$$\frac{\sigma(n)}{n \log\log n} > \frac{e^\gamma}{\zeta(t)} (1 - \epsilon).$$

2) Can be proved very similarly to part 1. Namely, the third product in (10) will extend over the primes $2 < p \leq \log n$ and for the $\geq$ part we consider the integers n of the form $n = \prod_{2 < p \leq x} p^{t-1}$. $\square$

`Remark`. Robin [13] has shown that if RH is false, then there are infinitely many integers n not in $\mathcal{R}$. As n ranges over these numbers, then by part 1 of Lemma 6 we must have $\max\{e_i\} \to \infty$, where $n = \prod_{i=1}^{k} p_i^{e_i}$.

Proof of Theorem 9.
1) Follows from part 1 of Lemma 6 on letting t tend to infinity. A direct proof (similar to that of Lemma 6) can also be given, see e.g. [4]. This result was proved first by Gronwall in 1913.
2) Follows from part 1 of Lemma 6 with $t = 2$.
3) Follows on letting t tend to infinity in part 2 of Lemma 6.
4) Landau (1909).
5) Since $f_2(n) \leq f_2(\kappa(n))$, part 5 is a consequence of part 4.
6) A consequence of part 4 and the fact that for odd integers n and $a \geq 1$ we have $f_2(2^a n) = 2f_2(n)(1 + O((\log n \log\log n)^{-1}))$.
7) Consider numbers of the form $n = \prod_{p \leq x} p^{t-1}$ and let t tend to infinity. These are squarefull for $t \geq 3$ and using them the $\geq$ part of the assertion follows. The $\leq$ part follows of course from part 3.
8) It is enough here to consider the squarefull numbers of the form $n = \prod_{p \leq x} p^2$. $\square$

5 Reduction to Hardy-Ramanujan integers

Recall that $p_1, p_2, \ldots$ denote the consecutive primes. An integer of the form $\prod_{i=1}^{s} p_i^{e_i}$ with $e_1 \geq e_2 \geq \cdots \geq e_s \geq 0$ we will call an *Hardy-Ramanujan integer*. We name them after Hardy and Ramanujan who in a paper entitled 'A problem in the analytic theory of numbers' (Proc. London Math. Soc. **16** (1917), 112-132) investigated them. See also [11, pp. 241-261], where this paper is retitled 'Asymptotic formulae for the distribution of integers of various types'.

Proposition 1 *If Robin's inequality holds for all Hardy-Ramanujan integers* $5041 \leq n \leq x$*, then it holds for all integers* $5041 \leq n \leq x$*. Asymptotically there are*

$$\exp((1+o(1))2\pi\sqrt{\log x/3\log\log x})$$

Hardy-Ramanujan numbers $\leq x$*.*

Hardy and Ramanujan proved the asymptotic assertion above. The proof of the first part requires a few lemmas.

Lemma 7 *For* $e > f > 0$*, the function*

$$g_{e,f} : x \to \frac{1-x^{-e}}{1-x^{-f}}$$

is strictly decreasing on $(1, +\infty]$*.*

Proof. For $x > 1$, we have

$$g'_{e,f}(x) = \frac{ex^f - fx^e + f - e}{x^{e+f+1}(1-x^{-f})^2}.$$

Let us consider the function $h_{e,f} : x \to ex^f - fx^e + f - e$. For $x > 1$, we have $h'_{e,f}(x) = efx^f\left(1 - x^{e-f}\right) < 0$. Consequently $h_{e,f}$ is decreasing on $(1, +\infty]$ and since $h_{e,f}(1) = 0$, we deduce that $h_{e,f}(x) < 0$ for $x > 1$ and so $g_{e,f}(x)$ is strictly decreasing on $(1, +\infty]$. □

`Remark`. In case f divides e, then

$$\frac{1-x^{-e}}{1-x^{-f}} = 1 + \frac{1}{x^f} + \frac{1}{x^{2f}} + \cdots + \frac{1}{x^e},$$

and the result is obvious.

Lemma 8 *If* $q > p$ *are primes and* $f > e$*, then*

$$\frac{\sigma\left(p^f q^e\right)}{p^f q^e} > \frac{\sigma\left(p^e q^f\right)}{p^e q^f}. \tag{11}$$

Proof. Note that the inequality (11) is equivalent with

$$(1-p^{-1-f})(1-p^{-1-e})^{-1} > (1-q^{-1-f})(1-q^{-1-e})^{-1}.$$

It follows by Lemma 7 that the latter inequality is satisfied. □

Let $n = \prod_{i=1}^{s} {q_i}^{e_i}$ be a factorisation of n, where we ordered the primes q_i in such a way that $e_1 \geq e_2 \geq e_3 \geq \cdots$ We say that $\bar{e} = (e_1, \ldots, e_s)$ is the exponent pattern of the integer n. Note that $\Omega(n) = e_1 + \ldots + e_s$, where $\Omega(n)$ denotes the total number of prime divisors of n. Note that $\prod_{i=1}^{s} {p_i}^{e_i}$ is the minimal number having exponent pattern $\bar{e}$. We denote this (Hardy-Ramanujan) number by $m(\bar{e})$.

Lemma 9 *We have*

$$\max \left\{ \frac{\sigma(n)}{n} \mid n \text{ has factorisation pattern } \bar{e} \right\} = \frac{\sigma(m(\bar{e}))}{m(\bar{e})}.$$

Proof. Since clearly

$$\frac{\sigma\left(p^{e}\right)}{p^{e}} > \frac{\sigma\left(q^{e}\right)}{q^{e}}$$

if $p < q$, the maximum is assured on integers $n = \prod_{i=1}^{s} p_i{}^{f_i}$ having factorisation pattern $\bar{e}$. Suppose that n is any number of this form for which the maximum is assumed, then by Lemma 8 it follows that $f_1 \geq f_2 \geq \cdots \geq f_s$ and so $n = m(\bar{e})$. □

Lemma 10 *Let $\bar{e}$ denote the factorisation pattern of n.*
1) *If $\sigma(n)/n \geq e^{\gamma} \log\log n$, then $\sigma(m(\bar{e}))/m(\bar{e}) > e^{\gamma} \log\log m(\bar{e})$.*
2) *If $\sigma(m(\bar{e}))/m(\bar{e}) < e^{\gamma} \log\log m(\bar{e})$, then $\sigma(n)/n < e^{\gamma} \log\log n$ for every integer n having exponent pattern $\bar{e}$.*

Proof. A direct consequence of the fact that $m(\bar{e})$ is the smallest number having exponent pattern $\bar{e}$ and Lemma 9. □

On invoking the second part of the latter lemma, the proof of Proposition 1 is completed.

6 Superabundant numbers

For an arithmetic function f, an integer n is said to be a champion number if $f(m) < f(n)$ for all $m < n$. The most well-known champion numbers are the highly composite numbers, which are the champion numbers for the divisor function $\sum_{d|n} 1$. They were studied in depth by Ramanujan in a celebrated paper [11, pp. 78-128].

The champion numbers for σ are called highly abundant numbers. The champion numbers for $\sigma(n)/n$ are called superabundant numbers. An integer N for which there exists $\epsilon > 0$ such that

$$\frac{\sigma(m)}{m^{1+\epsilon}} \leq \frac{\sigma(N)}{N^{1+\epsilon}}$$

for all natural numbers m, is called a colossally abundant number. The first 30 superabundant numbers are 1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, 10080, 15120, 25200, 27720, 55440, 110880, 166320.

It is easy to show that if N is colossaly abundant, then it is superabundant and if N is superabundant, then it is highly abundant. These implications can not be reversed. For example, Nicolas [8] showed that there is a constant $c > 0$ such that the number of integers $n \leq x$ that are highly abundant, but not superabundant is $\geq c(\log x)^{3/2}$. Erdős and Nicolas [5] proved that if $c_1 < 5/48$, then the number of integers $n \leq x$ that are superabundant is at least $(\log x)^{1+c_1}$ for x sufficiently large. It is not known (see [10]) whether this number can be bounded above by $(\log x)^{\Delta}$ for some Δ.

Alaoglu and Erdős proved the following result concerning superabundant numbers.

Theorem 10 [1]. *Let $n = \prod_{i=1}^{s} p_i^{e_i}$ denote the factorisation of a superabundant number n, with p_s its largest prime factor. Then n is a Hardy-Ramanujan number and $e_s = 1$ except if $n = 4$ or $n = 36$. Furthermore if i and hence n tends to infinity, then $p_i^{e_i} \sim p_s \log p_s / \log q_i$ and $p_s \sim \log n$. Moreover, $p_i^{e_i} < 2^{e_1+2}$.*

Superabundant numbers seem to have been first studied by Ramanujan [12]. For some recent computational results on superabundant and colossally abundant numbers, see Briggs [3].

7 The proof of Theorem 5

It is easy to see that the smallest integer ≥ 5041 not satisfying Robin's inequality, provided it exists, must be a superabundant number. We will prove Theorem 5 by using the latter observation, Theorem 10 and the lemma below.

By $P(n)$ we denote the largest prime factor of n.

Lemma 11 *Suppose that there exists an integer exceeding* 5040 *that does not satisfy Robin's inequality. Let n be the smallest such integer. Then $P(n) < \log n$.*

Proof. One numerically checks that we must have $n \geq 10081$. Suppose that n is not superabundant. Since 10080 is a superabundant number, it follows that there is an integer $5041 \leq n_0 < n$ that is superabundant and for which $\sigma(n_0)/n_0 > \sigma(n)/n \geq e^{\gamma} \log\log n > e^{\gamma} \log\log n_0$. This contradicts the minimality assumption on n and shows that n must be superabundant. By Theorem 10 it then follows that we can write $n = r \cdot q_m$ with $P(n) = q_m$ and $q_m \nmid r$. The minimality assumption on n implies that either r is a Hardy-Ramanujan number ≤ 5040 not satisfying Robin's inequality or that r is in $\mathcal{R}$. It is not difficult to exclude the former case, and so we infer that r is in $\mathcal{R}$. We will now show that this together with the assumption $q_m \geq \log n$ leads to a contradiction, whence the result follows.

So assume that $q_m \geq \log n$. This implies that

$$\frac{q_m}{\log n} > \frac{\log\log r}{\log q_m}.$$

This (cf. the proof of case 1 of Theorem 1) implies that

$$\frac{q_m(\log\log n - \log\log r)}{\log q_m} > \frac{\log\log r}{\log q_m}.$$

The latter inequality is equivalent with $(1 + 1/q_m)\log\log r < \log\log n$. Now we infer that

$$\frac{\sigma(n)}{n} = \frac{\sigma(q_m r)}{q_m r} = (1 + \frac{1}{q_m})\frac{\sigma(r)}{r} < (1 + \frac{1}{q_m})e^{\gamma}\log\log r < e^{\gamma}\log\log n.$$

This contradicts our assumption that $n \notin \mathcal{R}$. $\square$

Proof of Theorem 5. By contradiction. So we assume that there exists at least one 5-free integer ≥ 5041 not satisfying Robin's inequality. We let n be the smallest of these. As we have observed before, the number n must be superabundant. Since $e_1 \leq 4$ we have, by Theorem 10, that $p_i^{e_i} < 64$. Put $f_1 = 4$ and for $i \geq 2$ let f_i be the largest integer such that $p_i^{f_i} < 64$. Note that $f_2 = 3$, $f_3 = f_4 = 2$ and $f_5 = \cdots = f_{18} = 1$ and $f_i = 0$ for $i \geq 19$. Put $P(n) = p_m$. Then $p_m < \log n$ by Lemma 11 and we infer that

$$\prod_{i=1}^{m} \frac{\sigma(p_i^{f_i})}{p_i^{f_i}} \geq \frac{\sigma(n)}{n} \geq e^{\gamma} \log\log n \geq e^{\gamma} \log p_m$$

and thus

$$\prod_{i=1}^{m} \frac{\sigma(p_i^{f_i})}{p_i^{f_i}} \geq e^{\gamma} \log p_m.$$

On numerically checking this for the range $m \leq 19$ we see that $m \leq 9$. Now we are left with 142 candidates for the number n: namely those numbers of the form $\prod_{i=1}^{9} p_i^{e_i}$ with $e_i \leq f_i$ and $e_1 \geq e_2 \geq \cdots \geq e_9 \geq 0$. However, those of the corresponding 142 integers that furthermore exceed 5040 all turn out (by computer calculation) to satisfy Robin's inequality, and thus we have arrived at a contradiction. □

It might be a project of some interest to replace 5-free in Theorem 5 with t-free, with t as large as possible. In this direction it should be mentioned that Briggs [3] did some floating point calculations (which as he himself writes cannot be completely regarded as rigorous) in which he verified that the superabundant numbers > 5040 with maximal two exponent 12 satisfy Robin's inequality. If this can be established rigorously, it would follow that in Theorem 5, 5-free can be replaced by 13-free.

8 Acknowledgement

We thank J.-C. Lagarias for pointing out reference [3]. Furthermore, E. Bach, P. Dusart and O. Ramaré for their remarks. Keith Briggs we thank for his willingness to do large scale computations on our behalf. In the end, however, it turned out that only modest computations are needed in order to establish Theorem 6 (the deepest result of this paper).

References

[1] L. Alaoglu and P. Erdős, On highly composite and similar numbers, *Trans. Amer. Math. Soc.* **56** (1944) 448–469.

[2] T.M. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics. (Springer-Verlag, New York-Heidelberg, 1976).

[3] K. Briggs, Abundant numbers and the Riemann hypothesis, to appear in Experimental Mathematics.

[4] J.H. Bruinier, Primzahlen, Teilersummen und die Riemannsche Vermutung, *Math. Semesterber.* **48** (2001) 79–92.

[5] P. Erdős and J.-L. Nicolas, Répartition des nombres superabondants, *Bull. Soc. Math. France* **103** (1975) 65–90.

[6] S.R. Finch, *Mathematical constants*, Encyclopedia of Mathematics and its Applications **94**, (Cambridge University Press, Cambridge, 2003).

[7] J.C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, *Amer. Math. Monthly* **109** (2002) 534–543.

[8] J.-L. Nicolas, Ordre maximal d'un élément du groupe S_n des permutations et "highly composite numbers", *Bull. Soc. Math. France* **97** (1969) 129–191.

[9] J.-L. Nicolas, Petites valeurs de la fonction d'Euler, *J. Number Theory* **17** (1983) 375–388.

[10] J.-L. Nicolas, Some open questions, *Ramanujan J.* **9** (2005) 251–264.

[11] S. Ramanujan, *Collected Papers*, (Chelsea, New York, 1962).

[12] S. Ramanujan, Highly composite numbers. Annotated and with a foreword by J.-L. Nicolas and G. Robin, *Ramanujan J.* **1** (1997) 119–153.

[13] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, *J. Math. Pures Appl.* (9) **63** (1984) 187–213.

[14] J.B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962) 64–94.

[15] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Studies in Advanced Mathematics **46**, (Cambridge University Press, Cambridge, 1995).

Y.-J. Choie, Dept of Mathematics, POSTECH, Pohang, Korea 790-784, e-mail: yjc@postech.ac.kr

P. Moree, Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany.
e-mail: moree@mpim-bonn.mpg.de

N. Lichiardopol, ESSI, Route des Colles, 06 903 Sophia Antipolis, France, e-mail: lichiard@essi.fr

P. Solé, CNRS-I3S, ESSI, Route des Colles, 06 903 Sophia Antipolis, France, e-mail: ps@essi.fr