On geometry of fronts in wave propagations

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Abstract

We give a geometric descriptions of (wave) fronts in wave propagation processes. Concrete form of defining function of wave front issued from initial algebraic variety is obtained by the aid of Gauss-Manin systems associated with certain complete intersection singularities. In the case of propagations on the plane, we get restrictions on types of possible cusps that can appear on the wave front.

0. Introduction.

In this note, we intend to develop a preparatory theory toward asymptotic analysis around singular loci that appear in wave propagation process with initial data whose C^∞ singularities are located on certain smooth algebraic variety. For this purpose, we study the geometry of the (wave) front i.e. the singular support $S.S.$ of the solutions to the Cauchy problem where it loses C^∞ smoothness. We give a concrete expression of defining function of the wave front in terms of the initial data (Theorem 10). We impose conditions on the initial data so that it is defined by a quasihomogeneous polynomial with isolated singularities. The key trick is to understand the (wave) front as a discriminantal set for a deformation of complete intersection singularity. This point of view has been exposed in [21].

As it is well known, the fundamental solutions of hyperbolic operators are expressed by means of certain kinds of Gel'fand-Leray integrals. Thus the geometry of singular support (i.e. front) of the solutions is reduced to the study of singular support of these types of integrals. Homological and cohomological approach to the analysis of sharp-diffuse type of these integrals has been initiated by [1]. V.A.Vasiliev [20] uses F.Pham's approach [15] analyzing homology classes to describe the ramification character of certain kinds of Gel'fand-Leray integrals, consequently to clarify the sharp-diffuse type (see §4 for the definition) of the solutions to hyperbolic operators in the neighborhood of singular loci of the wave front, under the condition that one knows the type of singularities of the singular loci in question. Our method furnishes such existing results with a supplementary tool while we derive informations, global in space and time, uniquely from the initial data. In the sense that global behaviour of solutions to Cauchy problem is investigated, this paper is a continuation of [16], in which a similar setting of question has been studied by means of analysis of lagrangian variety.

AMS Subject Classification: Primary 35L25, 58G17, 33C75; Secondary 32S40, 78A05, 33C20.
Partially supported by Max Planck Institut für Mathematik,Bonn.

To pursue further this study, we propose to make use of the Gauss-Manin system associated with fibre bundle structure that naturally arises in integration. Our main tool is concrete expressions of the overdetermined differential systems obtained from non trivial relations between base elements of a Brieskorn lattice for complete intersection singularities (Proposition 5, Theorem 7).

Thorem 11 and the illustration given in Section 4 serve as a motivation why we propose to carry out asymptotic analysis by means of the Gauss-Manin systems. They justify our proposition in the sense that they demonstrate the possibility to interprete solutions to the Gauss-Manin systems as generalized hypergeometric functions. This procedure is supported by the general theory of the Gauss-Manin systems for isolated complete intersection singularities [6]. More systematic explanation of this situation from singularity theoretical point of view is given in [2] and [19].

Section 3 is consacrated to illustrate how one can apply the theory developed in Section 2 to situations on the plane. Our theorem 12 says that cusps that can appear on wave fronts issued from an initial front defined by an polynomial of degree k are at worst $(2k-1, 8k-2)$. We do not come into a detailed study of all possible cusps in terms of coefficients $(g_1, \cdots, g_{k-1})$ of the defining polynomial. After the establishment of an algorithm, such kind of detailed study can be achieved by anybody who tries to write down conditions like that in Remark 1.

1. Preliminaries on Cauchy problem.

In this section we prepare fundamental notations and lemmata to develop our studies in further sections. First let us regard the following Cauchy problem:

$$(\text{C.P.}) \begin{cases} [D_t^2 - \Delta_x]u(t,x) & = & 0 \\ D_t u(0,x) & = & v(x) \\ u(0,x) & = & 0 \end{cases}$$

where $t \in \mathbf{R}, x \in \mathbf{R}^n, D_t = \frac{\partial}{i\partial t}, \Delta_x = \sum_{j=1}^n (\frac{\partial}{i\partial x_j})^2, i = \sqrt{-1}$. We will study the Cauchy problem (C.P.) under the following conditions (C.1), (C.2), (C.3) imposed on the initial data.

In order to describe these conditions, we use notation $\chi_q^\epsilon(z)$ $(\epsilon = \pm 1)$ which stands for the following distributions:

$$\chi_q^\epsilon(z) = \chi_q(z + i0) + \epsilon\chi_q(z - i0),$$

where

$$\chi_q(z) = z^q/\Gamma(q) \quad q \notin \mathbf{Z} \text{ or } q \text{ negative integer}$$

$$\chi_q(z) = \frac{z^q}{q}(-\log z + C_q) \quad q \text{ positive integer}$$

where $C_0 = 0,\ C_q = C_{q-1} + 1/q$. Remark that

$$\frac{d}{dz}\chi_q(z) = \chi_{q-1}(z).$$

(C.1) *The initial data are given by a distribution of finite order with singular support (see Definition 2.2.3 of [10]) located on cotangent bundle of a smooth algebraic surface* $S := \{x \in \mathbf{R}^n : F(x) - s = 0\}$ *defined by a real polynomial* $F(x)$,

$$v(x) = g(x)\chi_q^{\epsilon}(F(x) - s)$$

with a smooth function $g(x)$.

Further we shall denote the singular support of a distribution $v(x)$ by $S.S.v(x)$.

We impose several technical conditions also. These conditions will be used so that the reasoning on the isolated complete intersection singularities can be applied to our (C.P.)

(C.2)(Quasihomogeneity) *There exists a set of integers* $(w_1, \cdots, w_n)$, *that satisfies 1)* $w_i \neq w_j$ *for certain pair* $1 \leq i \neq j \leq n$, *2)* $G.C.D.(w_1, \cdots, w_n) = 1$ *and 3) for a positive integer* $w(F)$,

$$\left(\sum_{1\leq j\leq n} w_j x_j \frac{\partial}{\partial x_j}\right)F(x) = w(F)F(x),$$

holds.

(C.3) *The following* $\mathbf{R}[x]-$ *module is of finite dimension:*

$$\mathbf{R}[x]/\mathcal{I}$$

where $\mathcal{I} = \langle F(x), \frac{\partial F}{\partial x_1}(x), \ldots, \frac{\partial F}{\partial x_n}(x)\rangle$ *(ideal generated by the entries).*

Let us introduce the following notations.

a)The phase function $\psi(x,t,z)$ is defined as follows:

$$\psi(x,t,z) = \langle x - z, grad_z F(z)\rangle^2 - t^2|grad_z F(z)|^2.$$

b) The paired oscillatory integrals introduced by [5] with phase function $\psi(x,t,z)$ of a):

$$I_p^{\epsilon}(x,t,s) = \int_{\{F(z)=s\}} H_p(z)\chi_p^{\epsilon}(\psi(x,t,z))\frac{dz}{dF},$$

with regular amplitude functions of pseudo-differential operator

$$H_p(z) \sim \sum_{r=p-2}^{-\infty} h_{p,r}(z) \in S^{p-2}(\mathbf{R}^n),$$

in which $h_{p,r}(z)$ is homogeneous of order r for large values of z. One shall understand $I_p^{\epsilon}(x,t,s)$ as the Gel'fand-Leray integral (see 1.5 [4]) defined on the real algebraic set $S = \{z \in \mathbf{R}^n; F(z) = s\}$.

c) The function $\phi(x,t,s)$ denotes the defining function of the (wave) front Σ issued from S determined by $(C.P.)$.

Proposition 1 *The following assertions hold for solutions to the Cauchy problem (C.P.) with the notations introduced in a),b) and c) as above.*

d) The solution to the Cauchy problem (C.P.) $u(x,t)$ admits an asymptotic expansion

$$u(x,t) \sim \sum_{j=0}^{\infty} I^{\epsilon}_{-n/2+q+j}(x,t,s). \tag{1.1}$$

That is to say, for every $N >> 0$ there exists such $C_N > 0$ that

$$\mid u(x,t) - \sum_{j=0}^{N} I^{\epsilon}_{-n/2+q+j}(x,t,s) \mid \leq C_N \mid \phi(x,t,s) \mid^{q+N+1}, \tag{1.2}$$

in the neighbourhood of $S.S.u(x,t)$.

Proof We give only a sketch of proof while a detailed one shall appear in [19]. First of all we show that the phase function of the integrals $I^{\epsilon}_p(x,t,s)$ coming into a) is given by c).

In solving the Hamilton-Jacobi equation associated with the Hamiltonian $\tau \pm |\xi|$ (in a symplectic coordinate with canonical symplectic form $dt \wedge d\tau + \sum_{j=1}^n dx_j \wedge d\xi_j$. For the symplectic geometry see Chapter XXI [10]),

$$\left\{ \begin{array}{ccc} \dot{t} & = & 1 \\ \dot{\tau} & = & 0 \\ \dot{x}_j & = & \pm\frac{\partial |\xi|}{\partial \xi_j} \\ \dot{\xi}_j & = & 0 \\ x_j(0) & = & z_j, \quad \{z \in \mathbf{R}^n; F(z) = s\}, 1 \leq j \leq n, \end{array} \right.$$

we get

$$x_j = \pm t\frac{\partial |\xi|}{\partial \xi_j} + z_j = \pm t \frac{\xi_j}{|\xi|} + z_j \text{ with } z \in S = \{z \in \mathbf{R}^n; F(z) = s\}. \tag{1.3}$$

This means that the singularities of the solutions to (C.P.) lies on the rays (1.3). These lines are interpreted as rays issued from the initial front $S = \{z \in \mathbf{R}^n : F(z) = s\}$ in the direction determined by the Hamiltonians $\tau^2 - |\xi|^2$. Consequently they are expressed by integrals with phase

$$\psi(x,t,z) = \langle x - z, grad_z F(z) \rangle^2 - t^2 |grad_z F(z)|^2. \tag{1.4}$$

Here we remark that for every $p \in \mathbf{Q}$ and $H(z) \in \mathcal{D}'(\mathbf{R}^n_z)$, we have

$$[D_t^2 - \Delta_x] \int_S H(z)(\psi(x,t,z))^p \frac{dz}{dF} = 0.$$

One can prove this equality with the aid of Gauss-Ostrogradskii-Stokes' theorem. Thus the question is how to find a series of integrals

$$I_p^\epsilon(x,t,s) = \int_S H_p(z)\chi_p^\epsilon(\psi(x,t,z))\frac{dz}{dF}, \quad p \in \mathbf{Q}$$

whose suitably converging sum produces a distribution $u(x,t)$ satisfying (C.1). The possibility of an asymptotic expansion (1.2) consisting of terms like $b)$ can be proven by well known estimates on the stationary phase ([10], Theorem 7.7.12). More precisely, let us remind the following lemma.

Lemma 2 *Let $(G)^0(x,t,s)$ be a residue of a smooth function $G(x,t,z)$ after division by an Jacobi ideal generated by $\frac{\partial\psi(x,t,z)}{\partial z_j}$, $\frac{\partial F(z)}{\partial z_j}$, $1 \leq j \leq n$, and $F(z)-s$, i.e.*

$$(1.6) G(x,t,z) = G^0(x,t,s) + \textstyle\sum_{j=1}^n f_j(x,t,s,z)\frac{\partial\psi(x,t,z)}{\partial z_j} +$$

$$+\sum_{j=1}^n g_j(x,t,s,z)\frac{\partial F(z)}{\partial z_j} + h(x,t,s,z)(F(z)-s),$$

with some smooth functions $h(x,t,s,z), f_j(x,t,s,z), g_j(x,t,s,z)$, $1 \leq j \leq n$.

Then for every smooth function $a(z)$ the following asymptotic estimate with some $C_N > 0$ holds in the neighbourhood of the wave front $\Sigma = \{(x,t) \in \mathbf{R}^{n+1}; (\psi)^0(x,t,s) = 0\}$:

$$(1.5) \mid \textstyle\int_S a(z)\chi_q^\epsilon(\psi(x,t,z))dz - \sum_{j=0}^N (L_{\psi,j}a)^0(x,t,s)\chi^\epsilon_{q+n/2+j}((\psi)^0(x,t,s)) \mid$$

$$< C_N \mid (\psi)^0(x,t,s) \mid^{n/2+N+1+q}$$

with differential operators $L_{\psi,j}$ of degree $2j$. Furthermore we have:

$$(L_{\psi,0}a)^0(x,t,s) = i^{\frac{n}{2}}(2\pi)^{\frac{n-1}{2}}(a)^o(x,t,s) \mid det(\frac{\psi_{zz}}{2\pi i})^0(x,t,s) \mid^{-1/2}.$$

Let us briefly sketch proof of the lemma. Malgrange's division theorem yields the decomposition (1.6) in connexion with the fact that the follwing $\mathcal{O}_{\mathbf{C}^n}-$ module is a finite dimensional vector space under the assumption (C.3):

$$\frac{\Omega^n_{\mathbf{C}^n}}{dF(z)\wedge\Omega^{n-1}_{\mathbf{C}^n} + d\psi(0,0,z)\wedge\Omega^{n-1}_{\mathbf{C}^n} + d_z\Omega^{n-1}_{\mathbf{C}^n} + F(z)\wedge\Omega^n_{\mathbf{C}^n} + \psi(0,0,z)\wedge\Omega^n_{\mathbf{C}^n}}.$$

Further it suffices to apply above mentioned stationary phase method.

After Lemma 2 the function $(\psi)^0(x,t,s)$ given by (1.6) for $G(x,t,z) = \psi(x,t,z)$ in (1.4) can be considered as a defining function $\phi(x,t,s)$ of the wave front issued from S determined by the Hamiltonian flow (1.3). The initial condition means $S = \{x : \phi(x,0,s) = 0\}$ which in turn entails $c)$.

It remains to justify asymptotic estimates in $b)$ and $d)$. This can be achieved in view of (1.5) and well known construction of an elementary solution to strictly hyperbolic Cauchy problem (see for example [8], [9]). Hence the assertion follows. **Q.E.D.**

We formulate a simple lemma before introducing necessary notations.

Lemma 3 *Under the assumptions (C.2), (C.3) imposed on $F(x)$ there exists a collection of at most quadratic polynomials $W_1(x,t), \cdots W_m(x,t)$, with m an integer smaller than $2^n \prod_{i=1}^n (\frac{1}{w_i})$, satisfying*

$$\psi(x,t,z) = \langle z, grad_z F(z)) \rangle^2 + \sum_{i=1}^m W_i(x,t) z^{\alpha^{(i)}} \tag{1.7}$$

for $\psi(x,t,z)$ of (1.4). Here $\alpha^{(i)} = (\alpha_1^{(i)}, \cdots, \alpha_n^{(i)}) \in (\mathbf{Z}_{\geq 0})^n$ stands for multi-indiex under restriction $\sum_{j=1}^n w_j \alpha_j^{(i)} < 2$.

The proof follows direct calculation of (1.4). Quasihomogeneous type of $F(z)$ yields the estimate on term number $''m''$. Let us denote by

$$w(z^{\alpha^{(i)}}) = \sum_{j=1}^n w_j \alpha_j^{(i)},$$

quasihomogeneous weight of monomial $z^{\alpha^{(i)}}$ for $\alpha^{(i)} \in \mathbf{N}^n$. In terms of the quasihomogeneous weight we distinguish two cases.

Case 1 If there is a term with $w(z^{\alpha^{(i)}}) = 0$, let us mark it as $\alpha^{(1)}$ and define a polynomial

$$f_1(y(x,t), z) = \langle z, grad_z F(z)) \rangle^2 + \sum_{i=1}^m y_i(x,t) z^{\alpha^{(i)}}.$$

Here $y_i(x,t) = W_i(x,t), 1 \leq i \leq m$, for polynomials introduced in Lemma 3, (1.7).

Case 2 If all terms of (1.7) has positive weight, we shall define

$$f_1(y(x,t), z) = \langle z, grad_z F(z)) \rangle^2 + \sum_{i=2}^{m+1} y_i(x,t) z^{\alpha^{(i-1)}} + y_1.$$

with $y_{i+1}(x,t) = W_i(x,t), 1 \leq i \leq m.$

For the sake of simplicity we adopt notation $\mu = m$ for Case 1 and $\mu = m+1$ for Case 2.

Further we define integrals

$$I_p(y(x,t), s) = \int_S H_p(z) \chi_p(f_1(y(x,t), z)) \frac{dz}{dF} \tag{1.8}$$

Hence if one denotes $y' = (y_2(x,t), \cdots, y_\mu(x,t))$,

$$I_p^\epsilon(x,t,s) = I_p(y_1 + i0, y'(x,t), s) + \epsilon I_p(y_1 - i0, y'(x,t), s)$$

on understanding that the boundary value shall be taken at $y_1 = 0$ in the above mentioned Case 2. Thus it is essential to study $I_p(y(x,t), s)$ of (1.8) to estimate asymptotic behaviour of $I_p^\epsilon(x,t,s)$. From now on we shall regard the integral (1.8) as a function in variables $y(x,t) = (y_1, y_2(x,t), \cdots, y_\mu(x,t))$. Therefore our main concern will be to investigate the differential equations that satisfy $I_p(y,s)$ corresponding to various amplitudes $H_p(z)$ with the aid of Gauss-Manin connexions associated to complete intersection singularities.

2. Gauss-Manin connexions for quasihomogeneous complete intersections.

Here we propose to study the integrals $I_p(y,s)$ defined in (1.8) by means of the Gauss-Manin system associated with complete intersection singularities. In effect, it is well known that the Gauss-Manin connexion can be defined on the relative de Rham cohomology groups. Instead of that here we propose to calculate it on spaces so called Brieskorn lattices (see [4], [6]).

The formulation of this section is a modification of [17], §1 adapted to our situation. Let us observe a mapping between complex manifolds $X = (\mathbf{C}^{N+K}, 0), Y = (\mathbf{C}^K, 0)$,

$$f : X \to Y$$

that defines an isolated quasihomogeneous complete intersection singularity at the origin. That is to say, if we denote

$$X_y := \{u \in X; f_0(u) = y_0, \ldots, f_{K-1}(u) = y_{K-1}\}, \tag{2.1}$$

then $dim X_y = N \geq 0$ and the critical set of mapping $f : X_0 \to Y$ is isolated in X_0. Further we assume that polynomials $f_0(u), \cdots, f_{K-1}(u)$ are quasihomogeneous i.e. there exists a collection of integers $v_1, \cdots, v_{N+K}$ whose greatest common divisor equals 1 and

$$(v_1 u_1 \frac{\partial}{\partial u_1} + \cdots + v_{N+K} u_{N+K} \frac{\partial}{\partial u_{N+K}}) f_\ell(u) = p_\ell f_\ell(u), \quad \ell = 0, 1, \cdots, K-1,$$

for certain integers $p_0, \cdots, p_{K-1}$. We shall call the vector field

$$E = \sum_{i=1}^{N+K} v_i u_i \frac{\partial}{\partial u_i}, \tag{2.2}$$

Euler vector field and $v_1, \cdots, v_{N+K}$ (resp. $p_0, \cdots, p_{K-1}$) weights of variables $u_1, \cdots, u_{N+K}$ (resp. polynomials $f_0, \cdots, f_{K-1}$) i.e. $v_1 = w(u_1)$, $p_0 = w(f_0)$ etc.

In order to calculate the Gauss-Manin connexion for isolated complete intersection singularity X_0, we introduce two vector spaces V and F. After Greuel-Hamm [7], we look at a space whose dimension as a vector space over $\mathbf{C}$ is known to be the Minor number $\mu(X_0)$ of singularity X_0,

$$V := \frac{\Omega_X^N}{df_0 \wedge \Omega_X^{N-1} + \cdots + df_{K-1} \wedge \Omega_X^{N-1} + d\Omega_X^{N-1} + f_0\Omega_X^N + \cdots + f_{K-1}\Omega_X^N}. \tag{2.3}$$

The second one will later turn out to be isomorphic to V (see Proposition 6),

$$F := \frac{\Omega_X^{N+1}}{df_0 \wedge \Omega_X^N + \cdots + df_{K-1} \wedge \Omega_X^N + i_E(\Omega_X^{N+2})}. \tag{2.4}$$

Here i_E means the inner contraction with Euler field E defined by (2.2). The third vector space associated with the singularity X_0 is defined as follows

$$\Phi := \frac{\Omega_X^{N+K}}{df_0 \wedge \cdots \wedge df_{K-1} \wedge \Omega_X^N + f_0\Omega_X^{N+K} + \cdots + f_{K-1}\Omega_X^{N+K}}. \tag{2.5}$$

Later we define period integrals as coupling of forms of V or of Φ with base element of homology groups $H_N(X_y)$. We remember also the definition of the Brieskorn lattice $''H$ from [6],

$$''H = \frac{\Omega_X^{N+K}}{df_0 \wedge \cdots \wedge df_{K-1} \wedge d\Omega_X^{N-1}},$$

whose rank as $\mathbf{O}_S-$ module equals the Minor number $\mu(X_0)$ of the singularity X_0.

It is easy to show

Lemma 4 *For $f_0, \cdots, f_{K-1}$ quasihomogeneous polynomials defining an isolated complete intersection singularity,*

$$\Phi \cong {''H}/(f_0, \cdots, f_{K-1}).$$

Thus $dim_{\mathbf{C}}\Phi = \mu(X_0)$.

From definitions (2.4) and (2.5) we easily deduce the following.

Proposition 5 *For every form $\tilde{\omega}_i \in F$, one has the following decomposition,*

$$\tilde{\omega}_i \wedge df_0 \wedge \cdots^{\vee \ell} \wedge df_{K-1} = \sum_{j=1}^{\mu(X_0)} P_{ij}^{(\ell)} \phi_j(u)du \tag{2.6}$$

with $P_{ij}^{(\ell)} \in \mathbf{C}[f_0, \cdots, f_{K-1}]$ and $\phi_j(u)du \in \Phi$, for $1 \leq i, j \leq \mu(X_0), 0 \leq \ell \leq K-1$ and $df_1 \wedge \cdots^{\vee \ell} \wedge df_{K-1} = \bigwedge_{i \neq \ell} df_i$.

From [17] we remember the following

Proposition 6 *Under the situation and definitions as above, the mapping,*

$$i_E : F \to V$$

induces an isomorphism. Consequently $dim_{\mathbf{C}} F = dim_{\mathbf{C}} V = \mu(X_0)$.

In view of Proposition 6, let us denote the base of V by ω_i such that $\omega_i = i_E(\tilde{\omega}_i), \tilde{\omega}_i \in F,\ \ 1 \leq i \leq \mu(X_0)$.

To make a transition from $(N+K)-$ forms to period integrals, we introduce meromorphic $N-$forms ψ_i satisfying

$$df_0 \wedge \cdots \wedge df_{K-1} \wedge \psi_i = \phi_i(u)du, \quad 1 \leq i \leq \mu(X_0).$$

Then we derive the following relation from Proposition 6,

(2.7) $d\omega_j = (di_E + i_E d)(\tilde{\omega}_j) \equiv \ell_j \tilde{\omega}_j \equiv \ell_j (\sum_{q=1}^{\mu(X_0)} P_{jq}^{(0)} df_0 \wedge \psi_q + \cdots$
$+(-1)^{K-1} \sum_{q=1}^{\mu(X_0)} P_{jq}^{(K-1)} df_{K-1} \wedge \psi_q)\ mod((df_0, \cdots, df_{K-1})\Omega_X^N, d\Omega_X^N)$,

where ℓ_j denotes weight of the form ω_j. Hence,

(2.8) $$\omega_j = i_E(\tilde{\omega}_j) \equiv$$

$$\sum_{i=0}^{K-1} (-1)^i [\sum_{q=1}^{\mu(X_0)} P_{jq}^{(i)} p_i f_i \psi_q - \sum_{q=1}^{\mu(X_0)} P_{jq}^{(i)} df_i \wedge i_E(\psi_q)] mod((df_0, \cdots, df_{K-1})\Omega_X^{N-1}, d\Omega_X^{N-1}).$$

As a consequence

(2.9) $d\omega_j = \sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i (d(p_i P_{jq}^{(i)} f_i) - w(\psi_q) P_{jq}^{(i)} df_i)] \wedge \psi_q +$
$\sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i p_i P_{jq}^{(i)} f_i] \wedge d\psi_q,\ mod((df_0, \cdots, df_{K-1})\Omega_X^N, d\Omega_X^N)$,

where $w(\psi_q)$, quasihomogeneous weight of form ψ_q. The expression (2.9) can be simplified if one lets them couple with a vanishing $N-$cycle, say $\gamma(y)$ and attains non trivail relations between integrals $\int_{\gamma(y)} \psi_q$, instead of those between forms. One defines so called period integral $I_{\phi_q, \gamma(y)}(y)$ taken along a vanishing cycle $\gamma(y)$ whose ambiguity in homology class $H_N(X_y, \mathbf{Z})$ we do not care for the moment,

(2.10) $I_{\phi_q, \gamma(y)}(y) := \int_{\gamma(y)} \psi_q = (\frac{1}{2\pi i})^K \int_{\partial\gamma(y)} \frac{df_0 \wedge \cdots \wedge df_{K-1} \wedge \psi_q}{(f_0 - y_0) \cdots (f_{K-1} - y_{K-1})}$

$$= (\frac{1}{2\pi i})^K \int_{\partial\gamma(y)} \frac{\phi_q(u)du}{(f_0 - y_0) \cdots (f_{K-1} - y_{K-1})},$$

where $\partial\gamma(y)$ is a cycle obtained by the aid of Leray's coboundary operator ∂. One may consult a booklet by F.Pham [15], or a book by V.A.Vasiliev [20] on the coboundary operator.

From (2.8) we can deduce

$$(2.11) \qquad \int_{\gamma(y)} \omega_j = \sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i p_i y_i P_{jq}^{(i)}(y)] I_{\phi_q,\gamma(y)}(y).$$

It is easily seen from the following evident equality in view of definition (2.10),

$$\int_{\partial\gamma(y)} \frac{df_0 \wedge \cdots \wedge df_{K-1}}{(f_0 - y_0)\cdots(f_{K-1} - y_{K-1})} \wedge df_i \wedge i_E(\psi_q) = 0,\ 0 \leq i \leq K-1.$$

Let us compare the relation

$$(2.12) \quad d\int_{\gamma(y)} \omega_j = \ell_j \sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i P_{jq}^{(i)}(y) dy_i] I_{\phi_q,\gamma(y)}(y),$$

obtained from (2.7) and (2.11). As a result we get equations between $I_{\phi_q}(y)$ and $\frac{\partial}{\partial y_\ell} I_{\phi_q}$, $0 \leq \ell \leq K-1$, (we omit to specify $\gamma(y)$ except necessary cases),

$$\frac{\partial}{\partial y_\ell}[\sum_{q=1}^{\mu(X_0)} \sum_{i=0}^{K-1} (-1)^i p_i y_i P_{jq}^{(i)} I_{\phi_q}]$$
$$= \ell_j \sum_{q=1}^{\mu(X_0)} [\sum_{i=0}^{K-1} (-1)^i P_{jq}^{(i)}(y) dy_i] I_{\phi_q}(y), 1 \leq j \leq \mu(X_0).$$

Thus we have obtained a system of differential equations to be understood as the Gauss-Manin connexion of the singularity X_0. To state the theorem in a simple form, we introduce the following notations: $\mathbf{I}_V = (\int_{\gamma(y)} \omega_1, \cdots, \int_{\gamma(y)} \omega_{\mu(X_0)})$, $\mathbf{I}_\Phi = (I_{\phi_1,\gamma(y)}(y), \cdots, I_{\phi_{\mu(X_0)},\gamma(y)}(y))$, i.e. vectors of integrals taken along a certain vanishing cycle $\gamma(y)$. We define several other $\mu(X_0) \times \mu(X_0)$ matrices as follows,

$$L_V = diag(\ell_1, \cdots, \ell_{\mu(X_0)})$$

with $\ell_i = w(\omega_i)$, $P^{(0)}(y) = (P_{jq}^{(0)}(y)), \cdots, P^{(K-1)}(y) = (P_{jq}^{(K-1)}(y)), 1 \leq j,q \leq \mu(X_0)$, matrices consisting of elements defined in (2.6).

In summing up the above arguments and the theory due to Greuel [6], we obtain the following.

Theorem 7 *1). For a quasihomogeneous mapping*

$$f : X \to Y$$

with isolated complete intersection singularities of dimension N like (2.1), the Gauss-Manin system satisfied by $\mathbf{I}_\Phi$ is described as follows:

$$(2.13) \quad d[\sum_{i=0}^{K-1} (-1)^i p_i y_i P^{(i)}(y) \mathbf{I}_\Phi] = L_V [\sum_{i=0}^{K-1} (-1)^i P^{(i)}(y) dy_i] \mathbf{I}_\Phi.$$

2). The critical value D (singular locus of the system (2.13)) of deformation X_y is given by $D = \{y \in Y : \Delta(y) = 0\}$ where

$$(2.14) \qquad \Delta(y) = det(\sum_{i=0}^{K-1} (-1)^i p_i y_i P^{(i)}(y)).$$

Let us return to the problem (C.P.) of §1. Our main concern is to understand the integral (1.8) as a sum of integrals like (2.10) for certain mapping f. To adapt our (C.P.) to the scheme explained before Theorem 7, we treat the following mapping $f : X \rightarrow Y$ for $X = (\mathbf{C}^{n+\mu}_u, 0), Y = (\mathbf{C}^{\mu+1}_y, 0)$. Concretely, it is defined as follows,

$$
(2.15)\left\{\begin{array}{ccccc}
f_0(u) & = & F(z) & = & y_0 \\
f_1(u) & = & z^P_{n+\mu} + \langle z, grad_z F(z)\rangle^2 + \sum_{i=1}^{\mu-1} z_{n+i} z^{\alpha(i)} & = & y_1 \\
f_2(u) & = & z_{n+1} & = & y_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
f_{i+1}(u) & = & z_{n+i} & = & y_{i+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
f_\mu(u) & = & z_{n+\mu-1} & = & y_\mu
\end{array}\right.
$$

with notation $z = (z_1, \cdots, z_n), z' = (z_{n+1}, \cdots, z_{n+\mu-1}), u = (z, z', z_{n+\mu})$. Here the power P is an integer that corresponds to the denominator of $q \in \mathbf{Q}$.

Lemma 8 *For $F(z)$ under conditions (C.2), (C.3), the mapping (2.15) $(f_0, \cdots, f_\mu)$ defines an isolated quasihomogeneous complete intersection singularity,*

$$X_0 = \{u \in X : f_0(u) = \ldots = f_\mu(u) = 0\}.$$

Namely $\mathbf{O}_X-$ module

$$V = \frac{\Omega^{n+\mu}_X}{f_0\Omega^{n+\mu}_X + f_1\Omega^{n+\mu}_X + \sum_{i=1}^{\mu} z_{i+n}\Omega^{n+\mu}_X + dF\wedge\Omega^{n+\mu-1}_X + df_1\wedge\Omega^{n+\mu-1}_X + \sum_{i=1}^{\mu} dz_{i+n}\wedge\Omega^{n+\mu-1}_X},$$

is a finite dimensional vector space.

Proof The complete intersection property follows from the fact that two polynomials $F(z)$ and $\langle z, grad_z F(z)\rangle$ are of the same quasihomogeneous weight but with different coefficients. This is a consequence of (C.2), 1) which supposes that $F(z)$ is not a homogeneous polynomial. The condition (C.3) entails immediately the finite dimensionality of V. **Q.E.D.**

To see that the components of $\mathbf{I}_\Phi$ defined for the mapping (2.15) give rise to integrals of type (1.8), we prepare the following.

Lemma 9 *1). Let us denote*

$$\Phi(z, z') = \frac{\Omega^{n+\mu+1}_X}{df_0 \wedge \ldots df_\mu \wedge \Omega^n_X + \sum_{i=0}^{\mu} f_i\Omega^{n+\mu+1}_X}|_{z_{n+\mu}=0}.$$

Then the following natural isomorphism holds

$$\Phi \cong \Phi(z, z') \otimes (\mathbf{C}[z_{n+\mu}]/ < z^P_{n+\mu} >).$$

2). For $\partial\gamma_{n-2}$ a Leray coboundary of a vanishing cycle $\gamma_{n-2} \in H_{n-2}(X_y \mid_{z_{n+\mu}=0}, \mathbf{Z})$ one can choose a corresponding vanishing cycle $\tilde{\gamma}_{n-1} \in H_{n-1}(X_y, \mathbf{Z})$ such that an equality

(2.16) $\int_{\partial\gamma_{n-2}} \phi(z)(f_1(z, y_2, \cdots, y_\mu, 0) - y_1)^{\frac{r+1}{P}-1} \frac{dz}{dF} =$

$$= \epsilon(\frac{1}{2\pi i})^\mu \int_{\partial\tilde{\gamma}_{n-1}} \phi(z) z_{n+\mu}^r \frac{du}{df_0 \wedge \cdots \wedge df_\mu}$$

holds, where $\epsilon \in \mathbf{C}^\times$ such that $\epsilon^P = 1$. Furthermore, the cycle $\partial\tilde{\gamma}_{n-1}$ is topologically equivalent to a product of a small circle on complex $z_{n+\mu}-$ plane and $\partial\gamma_{n-2}$.

Proof The statement 1) is evident. The statement 2) is an integral version of statement 1), which can be shown by means of equality (2.10). **Q.E.D.**

Thus the singular locus of the integral (1.8) can be given by that of

$$\int_{\partial\tilde{\gamma}_{n-1}} \phi(z) z_{n+\mu}^r \frac{du}{df_0 \wedge \cdots \wedge df_\mu}$$

with $\tilde{\gamma}_{n-1} \in H_{n-1}(X_y, \mathbf{Z})$ after substitution $y_1 = -W_1(x,t)$ (Case 1 after Lemma 3), or $y_1 = 0$ (Case 2 after Lemma 3), $y_i = W_i(x,t), 2 \leq i \leq \mu$. Let us remind that we denoted the quasihomogeneous weight of function f_i by p_i, $0 \leq i \leq \mu$. We define matrices $P^{(i)}(y), 2 \leq i \leq \mu$ for the mapping (2.15) after the master (2.6) and Theorem 7. Combining Theorem 7 with Lemma 9, we obtain the following.

Theorem 10 *The defining equation of wave front (Proposition 1, c)) is given by the following polynomial:*

$$\phi(x,t,s) = det(\sum_{i=0}^{\mu}(-1)^i p_i y_i P^{(i)}(y)) \mid_{y_0=s, y_i=W_i(x,t), 2\leq i\leq\mu} . \tag{2.17}$$

Here the restriction shall be imposed in accordance with two cases treated just after Lemma 3 i.e. $y_1 = -W_1(x,t)$ in Case 1 and $y_1 = 0$ in Case 2.

3.Fronts in the wave propagation on the plane.

In this section we illustrate how effectively our machinery explained in §2 works in studying wave propagation process on the plane (i.e. $n = 2$). Our main argument consists in the fact that the integrals (1.8) can be expressed as a sum of solutions to A_μ type the Gauss-Manin system. We shall use further notations from the former sections if they can be understood without special modifications.

Let us suppose

$$F(z) = z_1 - G(z_2), \tag{3.1}$$

for a polynomial of degree k with real coefficients,

$$G(z_2) = z_2^k + g_1 z_2^{k-1} + g_2 z_2^{k-2} + \ldots + g_{k-1} z_2. \tag{3.2}$$

Without loss of generality, we can choose as the initial wave front $S = \{x \in \mathbf{R}^2 : F(x) = 0\}$, i.e. $s = 0$. As the phase function described in Proposition 1, c) restrictricted on S we get,

$$(3.3)\psi(x,t,z)|_S = \langle (x_1 - G(z_2), x_2 - z_2), (-1, G'(z_2)) \rangle^2 - t^2(1 + G'(z_2)^2)$$
$$= (k-1)^2 [z_2^{2k} + y_1(x,t) z_2^{2k-1} + y_2(x,t) z_2^{2k-2} + \ldots + y_{2k-1}(x,t) z_2 + y_{2k}(x,t)],$$

where $y_i(x,t), 1 \leq i \leq 2k$ are polynomials at most quadratic in (x,t) variables defined as those which make (3.3) valid. More precisely, $y_1(x,t), \ldots, y_k(x,t)$ are linear in x_1, x_2, t^2. The remaining coefficients $y_{k+1}(x,t), \ldots, y_{2k}(x,t)$ are linear in $x_1, x_2, t^2, x_1x_2, x_1^2, x_2^2$. Especially,

$$y_1(x,t) = (k-1)^{-2}(2x_2 + k - (2k-1)g_1),$$
$$y_{2k}(x,t) = (k-1)^{-2}((x_1 + g_{k-1}x_2)^2 - t^2(1 + g_{k-1}^2)).$$

Thus, in our situation the integral (1.8) is reduced to

$$I_\lambda(y(x,t)) = \int_\gamma H(G(z_2), z_2)[z_2^{2k} + y_1(x,t) z_2^{2k-1} + \ldots + y_{2k}(x,t)]^\lambda dz_2, \tag{3.4}$$

for certain vanishing cycle γ of $\psi(x,t,z)|_S = 0$. Here we took into account the equivalence relation,

$$\frac{dz_1 \wedge dz_2}{dF} \equiv dz_2.$$

Let us consider a series of integrals

$$K_i^\lambda(y) = \int_\gamma z^i f_1(y,z)^\lambda dz, \qquad i = 0, \ldots, 2k+1, \tag{3.5}$$

where we used notation

$$f_1(y,z) := (k-1)^{-2}\psi(x,t,z)|_S = z_2^{2k} + y_1 z_2^{2k-1} + y_2 z_2^{2k-2} + \ldots + y_{2k}.$$

It is evident that our object of investigation (3.4) can be interpreted as a sum of these integrals (3.5) with $\mathbf{C}[y]$ coefficients. That is to say, singular locus $D = \{y \in Y : \Delta(y) = 0\}$ with $\Delta(y)$ the discriminant of $f_1(y,z)$ coincides with singular support of (3.4) if we consider the latter a function in y variables. Thus the analysis of integrals (3.5) entails not only geometric but also analytic properties of C^∞ singularities of (3.4).

We remind the following statement that gives complete description of the overdetermined system (the Gauss-Manin connexion with A_{2k-1} singularities) satisfied by the integrals $K_i^\lambda(y)$.

Theorem 11 ([12], [13]) *1)The integrals $K_0^\lambda(y), \ldots, K_{4k-2}^\lambda(y)$ satisfy the following over-determined system of differential equations:*

$$\sum_{\ell=0}^{2k-1} y_{2k-\ell}\frac{\partial}{\partial y_{2k-\ell}}K_i^\lambda + \frac{\partial}{\partial y_{2k}}K_{2k+i}^\lambda = \lambda K_i^\lambda, \quad 0 \leq i \leq 2k-2, \tag{3.6}$$

$$\sum_{\ell=1}^{2k-1} \ell y_{2k-\ell}\frac{\partial}{\partial y_{2k-\ell}}K_j^\lambda + 2k\frac{\partial}{\partial y_{2k}}K_{2k+j}^\lambda = -(j+1)K_j^\lambda, \quad -1 \leq j \leq 2k-2. \tag{3.7}$$

2) In the form of matrix, the above systems (3.6), (3.7) admit an expression,

$$\mathit{\Pi}(y)\cdot\vec{b} = \vec{a}, \tag{3.8}$$

where

$$\mathit{\Pi}(y) = \begin{bmatrix} y_{2k} & y_{2k-1} & \cdots & y_2 & y_1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & y_{2k} & \cdots & y_3 & y_2 & y_1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & y_4 & y_3 & y_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & y_{2k-1} & y_{2k-2} & y_{2k-3} & \cdots & y_1 & 1 & 0 \\ 0 & 0 & \cdots & y_{2k} & y_{2k-1} & y_{2k-2} & \cdots & y_2 & y_1 & 1 \\ y_{2k-1} & 2y_{2k-2} & \cdots & (2k-1)y_1 & 2k & 0 & \cdots & 0 & 0 & 0 \\ 0 & y_{2k-1} & \cdots & (2k-2)y_2 & (2k-1)y_1 & 2k & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2y_{2k-2} & 3y_{2k-3} & 4y_{2k-4} & \cdots & 2k & 0 & 0 \\ 0 & 0 & \cdots & y_{2k-1} & 2y_{2k-2} & 3y_{2k-3} & \cdots & (2k-1)y_1 & 2k & 0 \\ 0 & 0 & \cdots & 0 & y_{2k-1} & 2y_{2k-2} & \cdots & (2k-2)y_2 & (2k-1)y_1 & 2k \end{bmatrix}$$

$$\vec{a} =^t (\lambda K_0^\lambda, \lambda K_1^\lambda, \cdots, \lambda K_{2k-2}^\lambda, 0, -K_0^\lambda, -2K_1^\lambda, \cdots, -(2k-1)K_{2k-2}^\lambda),$$

$$\vec{b} =^t (\frac{\partial}{\partial y_{2k}}K_0^\lambda, \frac{\partial}{\partial y_{2k}}K_1^\lambda, \cdots, \frac{\partial}{\partial y_{2k}}K_{4k-2}^\lambda).$$

To be remarked, $\frac{\partial}{\partial y_{2k-i}}K_j^\lambda(y) = \frac{\partial}{\partial y_{2k-j}}K_i^\lambda(y)$.

In [18] we derived a partial differential operator of Fuchsian type annihilating $K_j^\lambda(y)$.

By the aid of the above theorem, we get our main result on 2 dimensional wave propagation.

Let us denote by $\Delta(y)$ the discriminant of $f_1(z,y)$ concretely calculated by

$$\Delta(y) = det\mathit{\Pi}(y), \tag{3.9}$$

for the matrix $\mathit{\Pi}(y)$ of (3.8). In literature one often calls $\mathit{\Pi}(y)$ the resultant of Sylvester.

Theorem 12 *1). The wave front Σ of (C.P.) with initial front $S = \{x \in \mathbf{R}^2 : F(x) = 0\}$ given by (3.1), (3.2), admits the following expression.*

$$\Sigma = \{(x,t) \in \mathbf{R}^3 : \Delta(y(x,t)) = 0\},$$

where $y(x,t) = (y_1(x,t), \cdots, y_{2k}(x,t))$ defined in (3.3).

2). Except ordinary double points, Σ has cusps of type at most $(2k-1, 8k-2)$. In other words if under a suitable local diffeomorphism $(x_1, x_2) \rightarrow (X_1, X_2)$ the restriction of wave front at certain moment $\Sigma|_{t=const.}$ admits as its local equation

$$\{X_1(x)^{p_1} - X_2(x)^{p_2} = 0\}, p_1 \leq p_2, \tag{3.10}$$

then these powers satisfy $p_1 \leq 2k-1, p_2 \leq 8k-2$.

Proof The statement 1) follows if one applies Theorem 11 to our integral (3.5) then replace variables y by $y(x,t)$ defined in (3.3).

To see (3.10), one shall examine the matrix $\Pi(y)$. After (3.8), $\Delta(y) = det\Pi(y)$ contains the highest in variable y_{2k} term y_{2k}^{2k-1}, while $\Delta(y)$ is quasihomogeneous with respect to the weight $w(y_i) = i, 1 \leq i \leq 2k$. According to this weight, every term of $\Delta(y)$ must be of weight $2k(2k-1)$. Remark that

$$y_{2k}(x,t) = L(x,t)L(x,-t)$$

for

$$L(x,t) = (k-1)^{-1}(x_1 + g_{k-1}x_2 - (1+g_{k-1}^2)^{1/2}t).$$

Hence $L(x,t) \neq L(x,-t)$ out of $t=0$ and $y_{2k}(x,t)$ can never be a constantly zero function for any choice of $(g_1, \cdots, g_{k-1})$.

The terms that may appear in the expression $\Delta(y)$ must be of degree less than $4k-1$ after concrete form of $\Pi(y)$. According to the remark after (3.3), $y_i(x,t)$ are at most quadratic polynomials. Suppose that there exists a set of indices $1 \leq \alpha_j \leq 2k-1$, $\{\alpha_1, \cdots, \alpha_\ell\} \subset \{1, \ldots, 2k-1\}$

$$y_{\alpha_j}(x,t) = c_{\alpha_j} X^2(x,t),$$

for a linear function $X(x,t)$. Such cases do really happen if the coefficients of $G(z_2)$, $(g_1, \ldots, g_{k-1})$ satisfy certain algebraic equations of degree 4 (see Remark 1 below). Suppose that for $X(x,t)$ in question such a set $\{\alpha_1, \cdots, \alpha_M\}$ is maximal i.e. for complementary indices $\{\beta_1, \cdots, \beta_{2k-M-1}\} = \{1, \ldots, 2k-1\} \setminus \{\alpha_1, \cdots, \alpha_M\}$, $y_{\beta_i}(x,t), 1 \leq i \leq 2k-M-1$ are not divided by $X^2(x,t)$. Then evidently,

$$\Delta(y) = y_{2k}^{2k-1} - const.X^{2(4k-1)}(x,t) + R(X, y_{\beta_1}, \cdots, y_{\beta_{2k-M-1}}, y_{2k}),$$

where $R(X, y_{\beta_1}, \cdots, y_{\beta_{2k-M-1}}, y_{2k})$ is a polynomial of degree $\leq 2(4k-2)$ in X and $\leq (4k-2)$ in $y_{\beta_1}, \cdots, y_{\beta_{2k-M-1}}, y_{2k}$. The coefficients of the polynomial R can vary in dependence of $(g_1, \cdots, g_{k-1})$. From the above expression of $\Delta(y)$ the statement 2) follows. **Q.E.D.**

Remark 1 On the divisibility of certain $y_{\alpha_j}(x,t)$ by a square of another function, one can calculate, in principle, the condition to be satisfied by $(g_1,\cdots,g_{k-1})$. For example a sufficient condition so that $y_\ell(x,t_0), \ell < k$ be expressed as a square of a linear function in x_2 at $t=t_0\neq 0$ sounds,

$(\sum_{i+j=\ell-1}(2k-\ell-2(k-i)(k-j))g_ig_j)^2 = 4[\sum_{i+j=\ell-2}(k-i)(k-j)g_ig_j][\sum_{i+j=\ell}(2(k-i)(k-j)-2k+\ell+1)g_ig_j - t_0^2\sum_{i+j=\ell-2}(k-i)(k-j)g_ig_j]$.

Conditions corresponding to other cases have similar form. We omit them in view of purely technical character of such conditions.

4. An illustration by Gauss' hypergeometric function.

In this section, we mention a little about analytic aspects of our theory. Namely we illustrate how to determine sharp-(diffuse) type at singular locus of fronts ((2,3) cusp) in an adequate way by means of the Gauss-Manin system of A_2 type that admits solutions expressed through Gauss hypergeometric functions.

We say that a distribution $u(x)$ with C^∞ singular locus on $\Sigma=\{x:\phi(x)=0\}$ has sharp (resp. diffuse) front from the side $\Sigma_+=\{x:\phi(x)>0\}$ in approaching to a point of Σ if and only if there exists (resp. does not exist) a C^∞ extension of $u(x)$ across Σ. The distribution $u(x)$ has lacuna inside Σ_+ iff $u(x)\equiv 0$ on Σ_+, see [14]. This notion has been introduced in connexion with the situation that a diffuse front can never be a front of a lacuna component. On the other hand, sharpness of a front is a necessary condition for presence of lacuna. As for details and typical examples, see [5].

Though the degree of a phase function $f_1(y(x,t),z)$ defined in (3.3) cannot be odd, an observation in terms of A_2 type singularity is useful because of the adjacency property of A_2 singularity to almost all more complicated singularities. In other words, one can expect that the solutions to (C.P.) are expressed by means of period integrals of A_2 type around the generic points on the singular loci of the wave front. Let us look at the integrals

$$K_i^\lambda(y)=\int_\Gamma z^i(z^3+y_1z+y_0)^\lambda dz, \qquad i=0,1,$$

where Γ a double loop of Pochhammer (cf. [3]) i.e. one dimensional regularized cycle. Then from Theorem 11 (or from [18]) one can deduce the following differential equations for $K_0^\lambda(y), K_1^\lambda(y)$.

$$(4.1)_0 \qquad [(27y_0^2+4y_1^3)(\frac{\partial}{\partial y_0})^2+27(1-2\lambda)y_0\frac{\partial}{\partial y_0}+27(\lambda^2-1/9)]K_0^\lambda(y)=0$$

$$(4.1)_1 \qquad [(27y_0^2+4y_1^3)(\frac{\partial}{\partial y_0})^2+27(1-2\lambda)y_0\frac{\partial}{\partial y_0}+27(\lambda^2-4/9)]K_1^{\lambda}(y)=0$$

The equation $(4.1)_0$ has two independent solutions $U_{1,\lambda}(y), U_{2,\lambda}(y)$ expressed by Gauss' hypergeometric function:

$$\begin{aligned} U_{1,\lambda}(y) &= y_1 y_0^{-\lambda+1/3} \cdot_2 F_1(\tfrac{-\lambda}{2}-\tfrac{1}{6}, \tfrac{-\lambda}{2}+\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{-4y_1^3}{27y_0^2}) \\ U_{2,\lambda}(y) &= y_1^2 y_0^{-\lambda-1/3} \cdot_2 F_1(\tfrac{-\lambda}{2}+\tfrac{1}{6}, \tfrac{-\lambda}{2}+\tfrac{2}{3}, \tfrac{4}{3}, \tfrac{-4y_1^3}{27y_0^2}) \end{aligned}$$

The solutions to $(4.1)_1$ are written down in a similar way. As these two solutions $U_{1,\lambda}(y), U_{2,\lambda}(y)$ correspond to two independent basis γ_1, γ_2 of the twisted homology group, we can write

$$U^{\epsilon}_{j,\lambda}(y)=\int_{\gamma_j}\chi^{\epsilon}_{\lambda}(z^3+y_1z+y_0)dz. \qquad (j=1,2)$$

These solutions ramify around the singular locus $\Sigma=\{y\in\mathbf{R}^2 : \Delta_2(y)=0\}$. Here $\Delta_2(y)=27y_0^2+4y_1^3$. We shall remark that $\int_{\gamma_i} z^j\chi^{\epsilon}_{\lambda}(z^3+y_1z+y_0)dz$ and $U^{\epsilon}_{i,\lambda}(y)$ have same sharp diffuse-type because of a relation

$$(\frac{\partial}{\partial y_1})^j U^{\epsilon}_{i,\lambda}(y)=(\frac{\partial}{\partial y_0})^j\int_{\gamma_i} z^j\chi^{\epsilon}_{\lambda}(z^3+y_1z+y_0)dz.$$

It shall be noted that sharp-diffuse type does not change by differentiation or integration of a distribution. To study the asymptotic behavior of $U^{\epsilon}_{i,\lambda}$ around Σ, we remind here the connexion formula for Gauss' hypergeometric function:

$$\begin{aligned} &{}_2F_1(\alpha,\beta,\gamma,x)=\tfrac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\cdot_2 F_1(\alpha,\beta,\alpha+\beta-\gamma+1,1-x) \\ &+\tfrac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-x)^{\gamma-\alpha-\beta}\cdot_2 F_1(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-x). \end{aligned}$$

These are based on a relation between 4 expressions from Kummer's 24 integrals, that goes back to Gauß(cf. [3],Chapitre I (10)). In applying this connexion formula to our $U_{1,\lambda}(y)$ and $U_{2,\lambda}(y)$, we get

$$\begin{aligned} &{}_2F_1(-\tfrac{\lambda}{2}-\tfrac{1}{6},-\tfrac{\lambda}{2}+\tfrac{1}{3},\tfrac{2}{3},\tfrac{-4y_1^3}{27y_0^2})= \\ &=\tfrac{\Gamma(\frac{2}{3})\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{\lambda}{2}+\frac{5}{6})\Gamma(\frac{\lambda}{2}+\frac{1}{3})}\cdot_2 F_1(-\tfrac{\lambda}{2}-\tfrac{1}{6},-\tfrac{\lambda}{2}+\tfrac{1}{3},-\lambda+\tfrac{1}{2},1+\tfrac{4y_1^3}{27y_0^2}) \\ &+\tfrac{\Gamma(\frac{2}{3})\Gamma(-\lambda-\frac{1}{2})}{\Gamma(-\frac{\lambda}{2}-\frac{1}{6})\Gamma(-\frac{\lambda}{2}+\frac{1}{3})}(1+\tfrac{4y_1^3}{27y_0^2})^{\lambda+\frac{1}{2}}\cdot_2 F_1(\tfrac{\lambda}{2}+\tfrac{5}{6},\tfrac{\lambda}{2}+\tfrac{1}{3},\lambda+\tfrac{3}{2},1+\tfrac{4y_1^3}{27y_0^2}), \\ &{}_2F_1(-\tfrac{\lambda}{2}+\tfrac{1}{6},-\tfrac{\lambda}{2}+\tfrac{2}{3},\tfrac{4}{3},\tfrac{-4y_1^3}{27y_0^2})= \\ &=\tfrac{\Gamma(\frac{4}{3})\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{\lambda}{2}+\frac{3}{2})\Gamma(\frac{\lambda}{2}+\frac{2}{3})}\cdot_2 F_1(-\tfrac{\lambda}{2}+\tfrac{1}{6},-\tfrac{\lambda}{2}+\tfrac{2}{3},-\lambda+\tfrac{1}{2},1+\tfrac{4y_1^3}{27y_0^2}) \\ &+\tfrac{\Gamma(\frac{4}{3})\Gamma(-\lambda-12)}{\Gamma(-\frac{\lambda}{2}+\frac{1}{6})\Gamma(-\frac{\lambda}{2}+\frac{2}{3})}(1+\tfrac{4y_1^3}{27y_0^2})^{\lambda+\frac{1}{2}}\cdot_2 F_1(\tfrac{\lambda}{2}+\tfrac{3}{2},\tfrac{\lambda}{2}+\tfrac{2}{3},\lambda+\tfrac{3}{2},1+\tfrac{4y_1^3}{27y_0^2}). \end{aligned}$$

It is evident that the second term in the above expressions ramifies around the singular locus Σ, if λ is not a half integer. Hence we obtain the following four classifications of sharp-diffuse type. We write down here the classification only for $U^{\epsilon}_{1,\lambda}$ while $U^{\epsilon}_{2,\lambda}$ behaves in a parallel manner.

Case 1. $\lambda = -1, \epsilon = -1$.

$U^{-}_{1,-1}(y) = 0$ in $y_0 > 0, \Delta_2(y) > 0$

$U^{-}_{1,-1}(y) = const.y_1|y_0|^{4/3} \cdot {}_2F_1(\frac{1}{3}, \frac{5}{6}, \frac{2}{3}; -\frac{4y_1^3}{27y_0^2})$ in $y_0 < 0, \Delta_2(y) > 0$

$U^{-}_{1,-1}(y) = const.y_1|y_0|^{7/3}\Delta_2(y)^{-1/2} \cdot {}_2F_1(1, \frac{1}{6}, \frac{1}{2}; 1 + \frac{4y_1^3}{27y_0^2})$ in $y_0 \neq, \Delta_2(y) < 0$

Hence $U^{-}_{1,-1}(y)$ has sharp front in approaching to a generic point on Σ from the side $\Delta_2(y) > 0$, while it has diffuse front in approaching to the singular point of Σ (i.e. the origin). It has diffuse front from the side $\Delta_2(y) < 0$.

Case 2. $\lambda = -1, \epsilon = +1$.

$U^{-}_{1,+1}(y) =$

$$= const.y_1|y_0|^{7/3}\Delta_2(y)^{-1/2} \cdot {}_2F_1(1, \frac{1}{6}, 1; 1 + \frac{4y_1^3}{27y_0^2}) + hol(1 + \frac{4y_1^3}{27y_0^2})$$

$$\text{in } \; y_0 \neq 0, \Delta_2(y) > 0$$

$U^{-}_{1,+1}(y) = 0$ in $\Delta_2(y) < 0$

Hence $U^{-}_{1,+1}(y)$ has diffuse front in approaching to Σ from the side $\Delta_2(y) > 0$. It has always lacuna (*a fortiori* sharp front) from the side $\Delta_2(y) < 0$.

Case 3. $\lambda = -\frac{3}{2}, \epsilon = -1$.

$U^{-}_{1,-3/2}(y) = 0$ in $y_0 > 0$

$U^{-}_{1,-3/2}(y) = const.y_1|y_0|^{11/6}\Delta_2(y)^{-1} \cdot {}_2F_1(\frac{3}{4}, \frac{-1}{12}, 0; 1 + \frac{4y_1^3}{27y_0^2})$ in $y_0 < 0$

Hence $U^{-}_{1,-3/2}(y)$ has sharp front in approaching to a generic point of Σ from any side if $y_0 > 0$. It has diffuse front from any side of Σ if $y_0 < 0$. In approaching to a singular point of $\Sigma, U^{-}_{1,-3/2}(y)$ has always diffuse front.

Case 4. $\lambda = -\frac{3}{2}, \epsilon = +1$.

$U^{+}_{1,-3/2}(y) =$
$= c_1 y_1 y_0^{11/6} p.v.(1 + \frac{4y_1^3}{27y_0^2})^{-1} \cdot {}_2F_1(\frac{3}{4}, \frac{-1}{12}, 0; 1 + \frac{4y_1^3}{27y_0^2}) +$
$+ c_2 y_1 y_0^{11/6} \cdot {}_2F_1(\frac{7}{12}, \frac{13}{12}, 2; 1 + \frac{4y_1^3}{27y_0^2})$ in $y_0 > 0$.
$U^{+}_{1,-3/2}(y) =$
$= c'_1 y_1 y_0^{11/6} \delta(1 + \frac{4y_1^3}{27y_0^2}) \cdot {}_2F_1(\frac{3}{4}, \frac{-1}{12}, 0; 1 + \frac{4y_1^3}{27y_0^2})$
$+ c'_2 y_1 y_0^{11/6} \cdot {}_2F_1(\frac{7}{12}, \frac{13}{12}, 2; 1 + \frac{4y_1^3}{27y_0^2})$ in $y_0 < 0$.

Hence we conclude that $U^{+}_{1,-3/2}(y)$ has diffuse front in approaching to a generic point of Σ from any side if $y_0 > 0$. It has sharp front from any side of Σ if $y_0 < 0$. In approaching to a singular point of Σ, $U^{-}_{1,-3/2}(y)$ has always diffuse front.

Remark 2 The above results yield not only known pictures but also add supplementary details to the figures drawn in [5]. For example we can add an arrow "d" (indicating diffuse front) directed to the singular point of cusp from the right side to figure 3. $\lambda = +1, \epsilon = -1$. Furthermore, we can add two arrows "s" directed to the singular points appearing in swallow's tail ("back bone") from the inside to figure 5. integer,+.

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