Integrable structure of $\mathcal{W}_3$ Conformal Field Theory, Quantum Boussinesq Theory and Boundary Affine Toda Theory.

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Abstract

In this paper we study the Yang-Baxter integrable structure of Conformal Field Theories with extended conformal symmetry generated by the $\mathcal{W}_3$ algebra. We explicitly construct various $\mathbf{T}$ and $\mathbf{Q}$-operators which act in the irreducible highest weight modules of the $\mathcal{W}_3$ algebra. These operators can be viewed as continuous field theory analogues of the commuting transfer matrices and Q-matrices of the integrable lattice systems associated with the quantum algebra $U_q(\widehat{sl}(3))$. We formulate several conjectures detailing certain analytic characteristics of the $\mathbf{Q}$-operators and propose exact asymptotic expansions of the $\mathbf{T}$ and $\mathbf{Q}$-operators at large values of the spectral parameter. We show, in particular, that the asymptotic expansion of the $\mathbf{T}$-operators generates an infinite set of local integrals of motion of the $\mathcal{W}_3$ CFT which in the classical limit reproduces an infinite set of conserved Hamiltonians associated with the classical Boussinesq equation. We further study the vacuum eigenvalues of the $\mathbf{Q}$-operators (corresponding to the highest weight vector of the $\mathcal{W}_3$ module) and show that they are simply related to the expectation values of the boundary exponential fields in the non-equilibrium boundary affine Toda field theory with zero bulk mass.

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1 Introduction

The Yang-Baxter equation and its associated integrability structures play an important role in the understanding of Integrable Quantum Field Theory (IQFT) [1]. Recently, the authors of [2–5] developed a continuous version of the Quantum Inverse Problem Method (QIPM) [6] or Baxter's commuting transfer matrix method [7] to study IQFT, especially the finite volume IQFT. This new method inherits all the analytic power of the theory of integrable lattice models but does not require intermediate lattice regularization and allows from the very beginning to work in terms of the continuous IQFT. The method has been most successfully applied [2–4] to the $c < 1$ Conformal Field Theory (CFT) which is a simplest nontrivial example of IQFT. The fundamental problem addressed in these papers is the simultaneous diagonalization of an infinite set of mutually commuting local integrals of motion (IM) in CFT.

In this paper we extend the results of [2–4] to CFT's with extended conformal symmetry generated by the simplest W-algebra $\mathcal{W}_3$ [8]. Such a CFT involves additional local spin-3 fields $W(u)$ and $\overline{W}(\overline{u})$ besides the usual left and right components $T(u)$ and $\overline{T}(\overline{u})$ of the stress-energy tensor. In what follows we will concentrate on the left chiral component $\mathcal{W}_3$ of the full symmetry algebra $\mathcal{W}_3 \times \overline{\mathcal{W}}_3$. The algebra $\mathcal{W}_3$ is generated by the fields $T(u)$ and $W(u)$ along with all composite fields built from products of $T(u)$ and $W(u)$ and their derivatives. We will interpret the variable u as a complex coordinate on a 2D cylinder of circumference 2π and impose the periodic boundary conditions

$$T(u+2\pi) = T(u), \qquad W(u+2\pi) = W(u). \tag{1.1}$$

The algebra $\mathcal{W}_3$ possesses an infinite-dimensional Abelian subalgebra [9,10] generated by the local integrals of motion (IM), $\mathbf{I}_k \in \mathcal{W}_3$

$$\mathbf{I}_k = \int_0^{2\pi} \frac{du}{2\pi}\, T_{k+1}(u), \qquad k \in Z_+, \qquad k \neq 0 \pmod 3) \tag{1.2}$$

where the densities $T_k(u)$ are appropriately regularized polynomials in $T(u)$ and $W(u)$ and their derivatives. The first two densities read

$$T_2(u) = T(u), \qquad T_3(u) = W(u), \tag{1.3}$$

while the higher densities are determined by the requirement of commutativity

$$[\mathbf{I}_k, \mathbf{I}_l] = 0, \tag{1.4}$$

and by the "spin" assignment [9] (see Section 2 for the details).

We explicitly construct operator valued functions $\mathbb{T}(t)$ and $\overline{\mathbb{T}}(t)$ where t is a complex parameter. These operators act invariantly in each irreducible highest weight module of the $\mathcal{W}_3$ algebra

$$\mathbb{T}(t), \overline{\mathbb{T}}(t): \qquad \mathcal{V}_{\Delta_2,\Delta_3} \to \mathcal{V}_{\Delta_2,\Delta_3} \tag{1.5}$$

where Δ_2, Δ_3 are the highest weights and they commute among themselves for any values of t, i.e.,

$$[\mathbb{T}(t), \mathbb{T}(t')] = [\mathbb{T}(t), \overline{\mathbb{T}}(t')] = [\overline{\mathbb{T}}(t), \overline{\mathbb{T}}(t')] = 0. \tag{1.6}$$

These operators are defined in terms of certain monodromy matrices associated with the three-dimensional representation of the quantum algebra $U_q(\widehat{sl}(3))$ where

$$q = e^{i\pi g}, \tag{1.7}$$

and g is related to the central charge c of the $\mathcal{W}_3$ algebra as

$$c = 50 - 24(g + g^{-1}), \tag{1.8}$$

Here we mostly restrict our considerations to the "quasiclassical domain" [2]

$$-\infty < c < -2, \tag{1.9}$$

however, the results are valid in a wider region $-\infty < c < 2$, corresponding to $0 < g < 1$.

Obviously, the operators $\mathbb{T}(t)$ and $\overline{\mathbb{T}}(t)$ are $\mathcal{W}_3$ analogues of Baxter's commuting transfer matrices from lattice theory. They enjoy remarkable analyticity properties, namely, they are entire functions of t with an essential singularity at $t = \infty$ and their asymptotic expansion near this point is described in terms of the local IM (1.2). For instance,

$$\log \mathbb{T}(t) \simeq m\, t^{\frac{1}{3(1-g)}} + \sum_{k=1}^{\infty} C'_k\, t^{-\frac{k}{3(1-g)}}\, \mathbf{I}_k, \tag{1.10}$$

where m and C'_k are some explicitly known coefficients which depend on the central charge c only and $C'_k \equiv 0$ for $k = 0 \pmod 3$. A similar expansion holds for $\overline{\mathbb{T}}(t)$. Therefore the operators $\mathbb{T}(t)$ and $\overline{\mathbb{T}}(t)$ can be thought of as generating functions of the local IM since their eigenvalues contain all the information about the eigenvalues of the local IM.

Another powerful method known in the lattice theory is based on the so-called **Q**-operators. This method was introduced by Baxter in his original solution of the eight-vertex model of statistical mechanics [11]. Here we construct appropriate $\mathcal{W}_3$ versions of Baxter's **Q**-operators (actually, there are six such operators $\mathbf{Q}_i(t)$ and $\overline{\mathbf{Q}}_i(t)$, $i = 1, 2, 3$). They are defined again as traces of certain monodromy matrices this time associated with infinite-dimensional representations of the so-called q-oscillator algebra. The **Q**-operators satisfy the following relations with the operators $\mathbb{T}(t)$ and $\overline{\mathbb{T}}(t)$ discussed above

$$
\begin{aligned}
\mathbf{Q}_i(tq^3) - \mathbb{T}(tq^{\frac{1}{2}})\,\mathbf{Q}_i(tq) + \overline{\mathbb{T}}(tq^{-\frac{1}{2}})\,\mathbf{Q}_i(tq^{-1}) - \mathbf{Q}_i(tq^{-3}) &= 0, \\
\overline{\mathbf{Q}}_i(tq^3) - \overline{\mathbb{T}}(tq^{\frac{1}{2}})\,\overline{\mathbf{Q}}_i(tq) + \mathbb{T}(tq^{-\frac{1}{2}})\,\overline{\mathbf{Q}}_i(tq^{-1}) - \overline{\mathbf{Q}}_i(tq^{-3}) &= 0,
\end{aligned}
\tag{1.11}
$$

where $i = 1, 2, 3$. These relations are obviously the $U_q(\widehat{sl}(3))$-analogues [12,13] of Baxter's famous T-Q relation. The relations (1.11) are just the first representatives of a rich family of the functional relations obeyed by the $\mathbf{Q}$-operators in the $U_q(\widehat{sl}(3))$ related integrable systems. In particular, in Section 4 we define more general $\mathbf{T}$-operators $\mathbf{T}_\mu(t)$ and $\overline{\mathbf{T}}_\mu(t)$ corresponding to finite-dimensional irreducible representations π_μ of $U_q(gl(3))$. Here $\mu = (\mu_1, \mu_2, \mu_3)$ is the highest weight of the representation π_μ such that $\mu_1 - \mu_2$ and $\mu_2 - \mu_3$ are non-negative integers[1]. All these $\mathbf{T}$-operators can be expressed through $\mathbf{Q}_i(t)$ and $\overline{\mathbf{Q}}_i(t)$ by the means of only two elegant relations

$$
\begin{aligned}
\mathbf{T}_{(\mu_1,\mu_2,\mu_3)}(t) &= z_0^{-1} \det \big\| \, \mathbf{Q}_i(t\, q^{+2\mu'_j}) \big\|_{i,j=1,2,3} \\
-\overline{\mathbf{T}}_{(\mu_1,\mu_2,\mu_3)}(t) &= z_0^{-1} \det \big\| \, \overline{\mathbf{Q}}_i(t\, q^{-2\mu'_j}) \big\|_{i,j=1,2,3}
\end{aligned}
\tag{1.12}
$$

where ${\mu'}_1 = \mu_1 + 1, \quad {\mu'}_2 = \mu_2, \quad {\mu'}_3 = \mu_3 - 1$, and where z_0 is a simple operator acting diagonally in each $\mathcal{W}_3$ module which can be interpreted as the "quantum Wronskian" [3] of the third order difference equations (1.11)

$$
z_0 = \det \big\| \mathbf{Q}_i(tq^{-2j}) \big\|_{i,j=1,2,3} = -\det \big\| \overline{\mathbf{Q}}_i(tq^{2j}) \big\|_{i,j=1,2,3} \tag{1.13}
$$

We prove the relations (1.12) algebraically by using decomposition properties of direct products of certain "q-oscillator" representations of the Borel subalgebra of $U_q(\widehat{sl}(3))$[2]. These relations provide a very simply tool for derivation of various "fusion relations" for the $\mathbf{T}$-operators [16, 17]. In particular, the so-called RSOS-reductions [16] of the $\mathbf{T}$-operators simply follow from (1.12) when two columns of the three-by-three matrices therein become proportional.

The eigenvalues of the $\mathbf{Q}$-operators can be studied by various methods. In particular, they satisfy the Destri-de-Vega non-linear integral equations [18–20]. We solve these

[1]The existence of the two different sets of operators $\mathbf{T}_\mu(t)$ and $\overline{\mathbf{T}}_\mu(t)$ is related to the fact that there are two non-equivalent "evaluation maps" from $U_q(\widehat{sl}(3))$ to $U_q(gl(3))$.

[2]It should noted that some particular cases of (1.12) were previously obtained [13,14] from the properties of the corresponding eigenvalues determined by the so-called "nested Bethe Ansatz" solutions [15].

equations exactly in a particular case when $\Delta_2, \Delta_3 \to \infty$, where Δ_2, Δ_3 are the highest weight in (1.5). On the basis of this result in Section 6 we conjecture exact asymptotic asymptotic expansions of the eigenvalues of the **Q**-operators at large values of t. These expansions exhibit remarkable duality properties of the **Q**-operators with respect to the substitution $g \to g^{-1}$ similar to the case of $c < 1$ CFT [3]. We support our conjectures by consideration of certain exactly solvable cases and numerical calculation of the zeroes of eigenvalues of the **Q**-operators.

As shown in [2] the **T** and **Q**-operators define integrable boundary conditions [21] for CFT in the geometry of the half-infinite cylinder. In particular, the vacuum eigenvalues of these operators can be interpreted [3] as finite-temperature partition functions of certain integrable QFT models with a non-trivial boundary interaction. These models are of interest in many branches of physics. For example, the integrable boundary sine-Gordon model [21] (with zero bulk mass) finds important application in dissipative quantum mechanics [22–25]; as explained in [26, 27] it also describes the universal current through the point contact in the quantum Hall system [3, 28–30]. Here we consider a simplest generalisation of this model, namely, the boundary $A_2^{(1)}$ affine Toda theory with zero bulk mass. Its Hamiltonian has the form

$$\mathbf{H} = \frac{1}{4\pi g} \int_{-\infty}^{0} dx \, (\mathbf{\Pi}^2 + (\partial_x \mathbf{\Phi})^2) - \frac{\kappa}{2g} \sum_{j=1}^{3} y_j \, e^{i(\boldsymbol{e}_j, \mathbf{\Phi}(0))} \,, \tag{1.14}$$

where g, κ are parameters and $\mathbf{\Phi}(x) = (\Phi_1(x), \Phi_2(x))$, $\mathbf{\Pi}(x) = (\Pi_1(x), \Pi_2(x))$ are field operators obeying canonical commutation relations

$$[\Pi_a(x) \, , \Phi_b(x')] = -2\pi i g \; \delta(x - x')\delta_{ab}, \qquad a, b = 1, 2. \tag{1.15}$$

and $\boldsymbol{e}_j$, $j = 1, 2, 3$ are the simple roots of the algebra $A_2^{(1)}$. The "amplitudes" y_1, y_2, y_3 in (1.14) are not c-numbers, but quantum operators describing some additional boundary degrees of freedom. They satisfy the commutation relations

$$y_1 y_2 = q y_2 y_1, \qquad y_2 y_3 = q y_3 y_2, \qquad y_3 y_1 = q y_1 \, y_3, \qquad y_1 y_2 y_3 = 1, \tag{1.16}$$

where $q = e^{i\pi g}$. It should be noted that the integrability of the boundary affine Toda systems has been studied previously both on classical [31] and quantum levels [32]. In [31] it was shown that (with the exception of the boundary sine-Gordon model) the integrability in the boundary affine Toda theory imposes severe restrictions on the amplitudes of the boundary interaction. Namely, the boundary interaction should vanish when the bulk mass is zero. This conclusion was then confirmed [32] with the perturbation theory calculations in the quantum case. We would like to stress here that the model (1.14) does not fall under this "no-go theorem" since (1.14) involves additional boundary interaction, which was not considered in [31, 32]. We claim here that the theory (1.14) is integrable, but postpone detailed considerations of this question to a separate publication. Actually, the sufficient condition for the integrability of (1.14) is that operators y_j obey the Serre relations of quantum affine algebra $U_q(A_2^{(1)})$

$$y_i^2\, y_j - (q+q^{-1})\, y_i\, y_j\, y_i + y_j\, y_i^2 = 0, \qquad i \neq j. \tag{1.17}$$

and not necessarily restricted to the case (1.16). A Similar statement holds for generalisations of (1.14) to other affine algebras. In Section 7.1 we consider a non-equilibrium variant of (1.14) with a particular time-dependent boundary interaction. We show that expectation values of boundary exponential fields over a stationary non-equilibrium state in this theory are simply related to the vacuum eigenvalues of the **Q**-operators of the $\mathcal{W}_3$ CFT. This calculation generalises the corresponding result of [30] for the boundary sine-Gordon model.

2 Quantum Boussinesq theory

2.1 Extended $\mathcal{W}_3$ conformal symmetry and local integrals of motion

In this paper we consider CFT with extended conformal symmetry generated by the simplest W-algebra $\mathcal{W}_3$ (sometimes called WA_2) [8, 33]. Such a CFT involves additional local spin-3 fields $W(u)$ and $\overline{W}(\overline{u})$ besides the usual left and right components $T(u)$ and $\overline{T}(\overline{u})$ of the stress-energy tensor. The left chiral component $\mathcal{W}_3$ of the full symmetry algebra $\mathcal{W}_3 \times \overline{\mathcal{W}}_3$ is generated by the fields $T(u)$ and $W(u)$ along with all composite fields built from products of $T(u)$ and $W(u)$ and their derivatives. We will interpret the variable u as a complex coordinate on a 2D cylinder of circumference 2π and impose the periodic boundary conditions (1.1). The Fourier mode expansions of $T(u)$ and $W(u)$

$$T(u) = -\frac{c}{24} + \sum_{-\infty}^{+\infty} L_{-n}\, e^{inz}, \qquad W(u) = \sum_{-\infty}^{+\infty} W_{-n}\, e^{inz}, \tag{2.1}$$

are expressed in terms of the operators L_n and W_n satisfying the commutation relations [8]

$$\begin{aligned}
[L_n, L_m] &= (n-m)\, L_{n+m} + \frac{c}{12}\,(n^3-n)\, \delta_{n+m,0}, \\
[L_n, W_m] &= (2n-m)\, W_{n+m}, && (2.2) \\
[W_n, W_m] &= \frac{1}{3}(n-m)\Lambda_{n+m} + \frac{(n-m)}{3b^2}\left[\frac{1}{15}(n+m+2)(n+m+3)\right. && (2.3) \\
&\quad \left. -\frac{1}{6}(n+2)(m+2)\right] L_{n+m} + \frac{c}{1080\, b^2}(n^2-4)(n^2-1)n\delta_{n+m,0},
\end{aligned}$$

where c is the central charge,

$$b^2 = 16/(22+5c),$$

and the operators Λ_n are expressed through L_n as

$$\Lambda_n = \sum_{k=-\infty}^{+\infty} : L_k L_{n-k} : + \frac{1}{5} x_n L_n, \tag{2.4}$$

with

$$x_{2l} = (1+l)(1-l), \qquad x_{2l+1} = (2+l)(1-l).$$

The symbol : : in (2.4) denotes the normal ordering such that operators L_n with bigger n are placed to the right.

The chiral space of state $\mathcal{H}_{chiral}$ is built up from a suitable collection of highest weight modules

$$\mathcal{H}_{chiral} = \bigoplus_a \mathcal{V}_{(\Delta_2^{(a)}, \Delta_3^{(a)})}, \tag{2.5}$$

where the parameters Δ_2 and Δ_3 are the highest weights, and the associated highest weight vectors $|\Delta_2, \Delta_3\rangle \in \mathcal{V}_{(\Delta_2,\Delta_3)}$ satisfy the equations

$$L_n|\Delta_2, \Delta_3\rangle = 0, \qquad W_n|\Delta_2, \Delta_3\rangle = 0, \qquad n > 0; \tag{2.6}$$

$$L_0|\Delta_2, \Delta_3\rangle = \Delta_2|\Delta_2, \Delta_3\rangle, \qquad W_0|\Delta_2, \Delta_3\rangle = \Delta_3|\Delta_2, \Delta_3\rangle. \tag{2.7}$$

The algebra $\mathcal{W}_3$ possesses an infinite Abelian subalgebra [9,10] generated by the local integrals of motion (IM), $\mathbf{I}_k \in \mathcal{W}_3$

$$\mathbf{I}_k = \int_0^{2\pi} \frac{du}{2\pi}\, T_{k+1}(u), \qquad k \in Z_+, \qquad k \neq 0 \pmod 3 \tag{2.8}$$

where the densities $T_k(u)$ are appropriately regularized polynomials in $T(u)$ and $W(u)$ and their derivatives. The integer $k+1$ in (2.8) denotes the spin of the local field T_{k+1}

$$\oint_{\mathcal{C}} \frac{dw}{2\pi i}\, (w-u)\mathcal{T}(T(w)T_{k+1}(u)) = (k+1)\; T_{k+1}(u), \tag{2.9}$$

where $\mathcal{T}$ stands for the chronological ordering

$$\mathcal{T}(A(w)B(u)) = \begin{cases} A(w)B(u)\;, & \text{if} \quad \Im m\; u > \Im m\; w\;; \\ B(u)A(w)\;, & \text{if} \quad \Im m\; w > \Im m\; u\;. \end{cases} \tag{2.10}$$

The set of possible values of k in (2.8) is determined by the allowed values of spins of the densities $T_{k+1}(u)$ [9]

$$\text{spin}[T_{k+1}(u)] = k+1, \qquad k \in \mathbb{Z}_+, \qquad k \neq 3 \pmod 3 \tag{2.11}$$

The first few densities $T_k(u)$ can be written as [10][3]

$$T_2(u) = T(u), \qquad T_3(u) = W(u), \qquad T_5(u) =: T(u)W(u):, \tag{2.12}$$

$$T_6(u) =: T(u)^3 : +9 : W(u)^2 : + \frac{c-10}{32} : T'(u)^2 :, \tag{2.13}$$

where the prime denotes the derivative and : : denotes the regularized product of the fields, such that the product $: A(u)B(u) :$ is defined as the regular term (of order $O((u-v)^0)$ in the operator product expansion of $A(u)B(v)$.[4] The densities (2.13) are defined up to total derivatives which, obviously, have no effect in (2.8). A general expression for the densities (2.13) is not known, however, once the first two densities in (2.13) are fixed all other higher densities can be uniquely determined (up to a normalisation) by the requirement of commutativity

$$[\mathbf{I}_k, \mathbf{I}_l] = 0, \tag{2.14}$$

and the spin assignment (2.11). Alternatively one can formulate the last requirement as the "degree assignment", namely, if one assigns degrees 2 and 3 to the fields $T(u)$ and $W(u)$, respectively, and the degree 1 to each derivative then $T_{k+1}(u)$ is a homogeneous polynomial of the degree $k+1$. We normalise the densities $T_{k+1}(u)$ by fixing coefficients to the terms containing highest power of $T(u)$,

$$T_{2n}(u) = (T(u))^n + \ldots, \qquad T_{2n+1}(u) = (T(u))^{n-1} W(u) + \ldots, \qquad n = 1, 2, 3, \ldots \tag{2.15}$$

The integrals (2.8) define operators $\mathbf{I}_k$: $\mathcal{V}_{\Delta_2,\Delta_3} \to \mathcal{V}_{\Delta_2,\Delta_3}$ which can be expressed in terms of the generators L_n and W_n, for example

[3] A misprint in the expression for $T_6(u)$ in [10] is corrected here.

[4] We assume that these regularized product are calculated in the cylinder geometry, as they are not invariant under conformal transformations.

$$
\begin{aligned}
\mathbf{I}_1 &= L_0 - \frac{c}{24}\ , \\
\mathbf{I}_2 &= W_0\ , \\
\mathbf{I}_4 &= \sum_{n=1}^{+\infty}(L_{-n}W_n + W_{-n}L_n) + W_0 L_0 - \frac{c+6}{24}W_0\ , \\
\mathbf{I}_5 &= \sum_{n_1+n_2+n_3=0} :L_{n_1}L_{n_2}L_{n_3}: + \sum_{n=1}^{\infty}\left(\frac{14+c}{16}n^2 - 2 - \frac{c}{4}\right)L_{-n}L_n + \frac{3}{2}\sum_{r=1}^{\infty}L_{1-2r}L_{2r-1} \\
&\quad +18\sum_{n=1}^{\infty}W_{-n}W_n + 9W_0^2 - \frac{c+8}{8}L_0^2 + \frac{(c+15)(c+2)}{192}L_0 - \frac{c(c+23)(7c+30)}{96768}.
\end{aligned}
\tag{2.16}
$$

Here the symbol : : denotes the same normal ordering as in (2.4) (where L_n with bigger n are placed to the right). Note that although these operators are not polynomials in L_n and W_n their action in $\mathcal{V}_{\Delta_2,\Delta_3}$ is well defined.

2.2 Relation to classical Boussinesq theory

In this paper we will study the problem of simultaneous diagonalization of the infinite set of the operators $\mathbf{I}_k$ in $\mathcal{V}_{(\Delta_2,\Delta_3)}$. This problem can be regarded as the quantum version of the related theory of the classical Boussinesq equation [34], which is a well known example of a completely integrable Hamiltonian system possessing an infinite set of local IM [35]. As is known [9] the $\mathcal{W}$-algebras can regarded as the quantization of the Gelfand-Dickey Poisson bracket algebras [36]. The classical limit of (2.2) corresponds to $c \to -\infty$. In this limit the substitution

$$
T(u) \to \frac{c}{24}U(u), \qquad W(u) \to i\Big(-\frac{c}{24}\Big)^{\frac{3}{2}}V(u), \qquad [\ ,\ ] \to \frac{48\pi}{ic}\{\ ,\ \}, \tag{2.17}
$$

where

$$
U(u+2\pi) = U(u), \qquad V(u+2\pi) = V(u), \tag{2.18}
$$

reduces the algebra (2.2) to the following Poisson bracket algebra [37]

$$
\begin{aligned}
\{U(u),U(v)\} &= -(U(u)+U(v))\ \delta'(u-v) + 2\ \delta'''(u-v), \\
\{U(u),V(v)\} &= -(V(u)+2V(v))\ \delta'(u-v), \\
\{V(u),V(v)\} &= -\Big[\frac{1}{4}(U''(u)+U''(v)) + \frac{1}{3}(U^2(u)+U^2(v))\Big]\ \delta'(u-v) \\
&\quad + \frac{5}{12}(U(u)+U(v))\delta'''(u-v) - \frac{1}{6}\delta^{(5)}(u-v),
\end{aligned}
\tag{2.19}
$$

which is a particular example of the Gelfand-Dickey bracket algebra. The substitution (2.17) brings the local IM (2.8) to their classical counterparts

$$
\mathbf{I}_k \to i^{-k-1}\Big(-\frac{c}{24}\Big)^{\frac{k+1}{2}} I_k^{(cl)}, \qquad c \to -\infty \tag{2.20}
$$

which form a commutative Poisson bracket algebra $\{I_k^{(cl)}, I_l^{(cl)}\} = 0$. For example, the first few classical IM $I_k^{(cl)}$ read

$$I_1^{(cl)} = \int_0^{2\pi} \frac{du}{2\pi}\, U(u)\ , \tag{2.21}$$

$$I_2^{(cl)} = \int_0^{2\pi} \frac{du}{2\pi}\, V(u), \tag{2.22}$$

$$I_4^{(cl)} = \int_0^{2\pi} \frac{du}{2\pi}\, U(u)V(u), \tag{2.23}$$

$$I_4^{(cl)} = \int_0^{2\pi} \frac{du}{2\pi}\, \Big(U(u)^3 + 9V(u)^2 + \frac{3}{4}U'(u)^2\Big). \tag{2.24}$$

The Poisson bracket algebra (2.19) is known to describe the *second* Hamiltonian structure of the Boussinesq equation [37,38] provided one of the infinite set of the classical IM $I_k^{(cl)}$ is taken as a Hamiltonian.

The evolution equations[5]

$$\partial_{t_k} U = \{I_k^{(cl)}, U\}, \qquad \partial_{t_k} V = \{I_k^{(cl)}, V\}, \qquad t_1 = u, \qquad k \neq 0 \pmod 3, \tag{2.25}$$

describe isospectral deformations of the third-order differential operator

$$L = \partial_u^3 - U(u)\,\partial_u + V(u) - \frac{1}{2}U'(u) - \lambda^3, \tag{2.26}$$

where λ is the spectral parameter. In particular, if one defines the monodromy matrix $\mathbf{M}(\lambda)$, which is an element of the group $SL(3)$, as

$$(\psi_1(u+2\pi), \psi_2(u+2\pi), \psi_3(u+2\pi)) = (\psi_1(u), \psi_2(u), \psi_3(u))\mathbf{M}(\lambda), \tag{2.27}$$

where $\psi_1(u), \psi_2(u)$ and $\psi_3(u)$ are three linearly independent solutions of the equation $L\psi = 0$, then the eigenvalues of $\mathbf{M}(\lambda)$ are in involution among themselves and with all local IM (2.24) (with respect to the Poisson bracket algebra (2.19)). The trace of the monodromy matrix

$$\mathbf{T}^{(cl)}(\lambda) = \mathrm{Tr}\ \mathbf{M}(\lambda), \tag{2.28}$$

can be regarded as a generating function for the local IM (2.24) as it expands in the asymptotic series [38]

$$\frac{1}{2\pi}\log \mathbf{T}^{(cl)}(\lambda) = \lambda + \sum_{k=1}^{\infty} c_k\ I_k^{(cl)} \lambda^{-k}, \qquad \lambda \to +\infty \tag{2.29}$$

[5]The original Boussinesq equation of [34]

$$\partial_t U = -2\partial_u V, \qquad \partial_t V = \frac{1}{3}\,\partial_u\left[\frac{1}{2}\partial_u^2 U - U^2\right],$$

corresponds to $k = 2$ in (2.25)

where c_k ($c_k = 0$ for $k = 0 \pmod 3$) are some numerical coefficients. The simplest way to calculate these coefficients is to consider the case when both $U(u)$ and $V(u)$ are constant (i.e., independent of u) and take real values. In this case (2.29) reduces to the $\lambda \to +\infty$ expansion of the positive root of the cubic equation $x^3 + 2Ux + V - \lambda^3 = 0$ which can be easily calculated. Similarly to (2.15) we normalise the classical IM $I_k^{(cl)}$ by the term containing the highest power of $U(u)$

$$I_{2n-1}^{(cl)} = \int_0^{2\pi} \frac{du}{2\pi} (U(u))^n + \ldots, \qquad I_{2n}^{(cl)} = \int_0^{2\pi} \frac{du}{2\pi} (U(u))^{n-1} V(u) + \ldots, \qquad n = 1, 2, 3, \ldots \tag{2.30}$$

Note that this normalisation is consistent with (2.15), (2.17) and (2.20). It follows then that

$$c_{2n-1} = \frac{4\pi \cos\left(\frac{(n-2)\pi}{3}\right) \Gamma\left(\frac{2n-1}{3}\right)}{3\, n! \Gamma\left(\frac{2-n}{3}\right)}, \qquad c_{2n+2} = \frac{4\pi\, \cos\left(\frac{(n+1)\pi}{3}\right) \Gamma\left(\frac{2(n+1)}{3}\right)}{n!\, \Gamma\left(-\frac{n+1}{3}\right)}, \tag{2.31}$$

where $n = 0, 1, 2, \ldots$. The required calculations are given in Appendix C; the above expressions are simple corollaries of (C.9).

For our purposes it will be more convenient to work with a first order differential operator instead of (2.26). To define it we need some additional notation. The algebra $gl(3)$ is generated by the elements H_i, $i = 1, 2, 3$ and E_{ij}, $(i, j) = (1, 2), (2, 1), (2, 3), (3, 2)$, for which we use also the notation

$$E_\alpha = E_{12}, \quad E_\beta = E_{23}, \quad F_\alpha = E_{21}, \quad F_\beta = E_{32},$$

and

$$H_\alpha = H_1 - H_2, \qquad H_\beta = H_2 - H_3, \qquad H_{\alpha+\beta} = H_1 - H_3.$$

They satisfy the relations

$$[H_i, H_j] = 0, \qquad [H_i, E_{kl}] = (\delta_{ik} - \delta_{il}) E_{kl}, \qquad [E_{\alpha_i}, F_{\alpha_j}] = \delta_{\alpha_i, \alpha_j} H_{\alpha_i}, \tag{2.32}$$

$$\begin{aligned} E_{\alpha_i}^2 E_{\alpha_j} - 2 E_{\alpha_i} E_{\alpha_j} E_{\alpha_i} + E_{\alpha_j} E_{\alpha_i}^2 &= 0, \\ F_{\alpha_i}^2 F_{\alpha_j} - 2 F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} + F_{\alpha_j} F_{\alpha_i}^2 &= 0, \end{aligned} \qquad \alpha_i \neq \alpha_j. \tag{2.33}$$

where in the last three relations greek indices α_i and α_j take two values α or β.

Introduce two-dimensional vectors

$$\begin{aligned} \mathbf{e}_1 &= \left(\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right), & \mathbf{e}_2 &= (-1, \ 0), & \mathbf{e}_3 &= \left(\ \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \\ \boldsymbol{w}_1 &= \left(1, \ \tfrac{1}{\sqrt{3}}\right), & \boldsymbol{w}_2 &= \left(\ 0, -\tfrac{2}{\sqrt{3}}\right), & \boldsymbol{w}_3 &= \left(-1, \tfrac{1}{\sqrt{3}}\right), \end{aligned} \tag{2.34}$$

such that

$$\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 = 0, \qquad \boldsymbol{w}_1 + \boldsymbol{w}_2 + \boldsymbol{w}_3 = 0, \tag{2.35}$$

and

$$\boldsymbol{e}_i = \frac{1}{2}(\boldsymbol{w}_j - \boldsymbol{w}_k), \qquad \boldsymbol{w}_i = -\frac{2}{3}(\boldsymbol{e}_j - \boldsymbol{e}_k), \tag{2.36}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$. Moreover,

$$\boldsymbol{e}_i\,\boldsymbol{e}_j = \frac{1}{2}a_{ij}, \qquad \boldsymbol{w}_i\,\boldsymbol{w}_j = \frac{2}{3}a_{ij}, \qquad \boldsymbol{e}_i\,\boldsymbol{w}_j = \epsilon_{ij}, \qquad i,j = 1,2,3, \tag{2.37}$$

where $\boldsymbol{ab}$ denotes the (Euclidean) scalar product, a_{ij}, $i,j = 1,2,3$, is the Cartan matrix of the algebra $A_2^{(1)}$

$$a_{ii} = 2, \qquad a_{ij} = -1, \qquad i \neq j, \qquad i,j = 1,2,3, \tag{2.38}$$

and

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = 0, \qquad \epsilon_{12} = \epsilon_{23} = \epsilon_{31} = 1, \qquad \epsilon_{21} = \epsilon_{32} = \epsilon_{13} = -1. \tag{2.39}$$

It is also useful to define

$$E_{31} = [F_\beta, F_\alpha], \qquad \boldsymbol{H} = \boldsymbol{w}_1 H_1 + \boldsymbol{w}_2 H_2 + \boldsymbol{w}_3 H_3. \tag{2.40}$$

For any $\mu = (\mu_1, \mu_2, \mu_3)$, such that $\mu_\alpha = \mu_1 - \mu_2$ and $\mu_\beta = \mu_2 - \mu_3$ are non-negative integers, denote by π_μ the finite dimensional irreducible representation of $gl(3)$ with the highest weight μ and highest weight vector $|\,0\rangle$ defined as

$$E_\alpha|\,0\rangle = E_\beta|\,0\rangle = 0, \qquad H_i|\,0\rangle = \mu_i|\,0\rangle, \qquad i = 1,2,3. \tag{2.41}$$

The dimension of such a representation is $\dim[\pi_\mu] = \frac{1}{2}(\mu_\alpha + 1)(\mu_\beta + 1)(\mu_\alpha + \mu_\beta + 2)$.

Consider the following differential operator

$$\mathcal{L} = \partial_u - \boldsymbol{\phi}'(u)\,\boldsymbol{H} - \lambda(E_{23} + E_{31} + E_{12}), \tag{2.42}$$

where the components of the vector $\boldsymbol{\phi}(u) = (\phi_1(u), \phi_2(u))$ are the canonical variables with Poisson bracket

$$\{\phi_i(u), \phi_j(v)\} = \frac{1}{2}\delta_{ij}\epsilon(u-v), \qquad \epsilon(u) = \frac{u}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nu)}{n}, \qquad 0 \le u, v < 2\pi; \tag{2.43}$$

which are related to $U(u)$ and $V(u)$ by the generalised Miura transform [39]

$$\begin{aligned} U(u) &= \phi_1'(u)^2 + \phi_2'(u)^2 + 2\phi_1''(u), \\ \sqrt{3}V(u) &= \frac{2}{3}\phi_2'(u)^3 - 2\phi_1'(u)^2\phi_2'(u) - 3\phi_1'(u)\phi_2''(u) - \phi_1''(u)\phi_2'(u) - \phi_2'''(u). \end{aligned} \tag{2.44}$$

For the periodic potentials $U(u)$ and $V(u)$, (2.18) the field $\boldsymbol{\phi}(u)$ is, in general, quasiperiodic

$$\boldsymbol{\phi}(u + 2\pi) = \boldsymbol{\phi}(u) + 2\pi i\boldsymbol{p}. \tag{2.45}$$

The monodromy matrix associated with the equation $\pi_\mu[\mathcal{L}]\Psi(u) = 0$ (where π_μ denotes a representation of $gl(3)$) is given by

$$\mathbf{M}_\mu(\lambda) = \pi_\mu \left[e^{2\pi i \boldsymbol{pH}} \mathcal{P} \exp\left(\lambda \int_0^{2\pi} (V_1 E_{23} + V_2(u) E_{31} + V_3(u) E_{12}) du \right) \right], \tag{2.46}$$

where the symbol $\mathcal{P}$ denotes the "path ordered" exponential and

$$V_i(u) = e^{-2\,\boldsymbol{e}_i\,\boldsymbol{\phi}(u)}\,, \qquad i = 1, 2, 3. \tag{2.47}$$

The associated L-operator

$$\mathbf{L}^{(cl)}_\mu(\lambda) = \pi_\mu[e^{-\pi i \boldsymbol{pH}}]\mathbf{M}_\mu(\lambda), \tag{2.48}$$

satisfies the "r-matrix" Poisson bracket algebra [40, 41]

$$\{\mathbf{L}^{(cl)}_\mu(\lambda) \underset{,}{\otimes} \mathbf{L}^{(cl)}_{\mu'}(\lambda')\} = [\mathbf{r}_{\mu\mu'}(\lambda/\lambda'), \mathbf{L}^{(cl)}_\mu(\lambda) \otimes \mathbf{L}^{(cl)}_{\mu'}(\lambda')]\ , \tag{2.49}$$

The quantity $\mathbf{r}_{\mu\mu'}(\lambda) = \pi_\mu \otimes \pi_{\mu'}[\ \mathbf{r}\ ]$ is the "classical r-matrix" given by

$$\mathbf{r}(\lambda) = \frac{\lambda^{\frac{3}{2}} + \lambda^{-\frac{3}{2}}}{\lambda^{\frac{3}{2}} - \lambda^{-\frac{3}{2}}} \sum_{i=1}^{3} H_i \otimes H_i + \frac{2}{\lambda^{\frac{3}{2}} - \lambda^{-\frac{3}{2}}} \sum_{(ij)} \left(\lambda^{-\frac{1}{2}} E_{ij} \otimes E_{ji} + \lambda^{\frac{1}{2}} E_{ji} \otimes E_{ij} \right), \tag{2.50}$$

where the second sum is taken over the values $(ij) = (12), (23), (31)$.

It follows from (2.49) that the quantities

$$\mathbf{T}^{(cl)}_\mu(\lambda) = \mathrm{Tr}_{\pi_\mu} \mathbf{M}_\mu(\lambda), \tag{2.51}$$

are in involution with respect to the Poisson brackets (2.19),(2.43)

$$\{\mathbf{T}^{(cl)}_\mu(\lambda), \mathbf{T}^{(cl)}_{\mu'}(\lambda')\} = 0. \tag{2.52}$$

Note, in particular, that $\mathbf{T}^{(cl)}_{(1,0,0)}(\lambda)$ coincides with (2.28).

3 Quantum Yang-Baxter equation

3.1 Universal R-matrix

The quantum Kac-Moody algebra $\mathcal{A} = U_q(\widehat{sl}(3))$ is generated by the elements x_i, y_i and h_i, $i = 1, 2, 3$, and the grading element d, satisfying the commutation relations

$$[h_i, h_j] = 0\,, \qquad [h_i, x_j] = -a_{ij} x_j\,, \qquad [h_i, y_j] = a_{ij} y_j\,, \tag{3.1}$$

$$[d, y_i] = \delta_{i,3} y_i\,, \qquad [d, x_i] = -\delta_{i,3} x_i\,, \qquad [y_i, x_j] = \delta_{ij}\, \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,, \tag{3.2}$$

and the cubic Serre relations

$$\begin{aligned} x_i^2\, x_j - [2]_q\, x_i\, x_j\, x_i + x_j\, x_i^2 &= 0, \\ y_i^2\, y_j - [2]_q\, y_i\, y_j\, y_i + y_j\, y_i^2 &= 0, \end{aligned} \qquad i \neq j. \tag{3.3}$$

Here the indices i, j take three values $i, j = 1, 2, 3$; the Cartan matrix a_{ij} is the same as (2.38) and the "q-numbers" $[n]_q$ are defined as

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3.4}$$

The element $k = h_1 + h_2 + h_3$ is the central element of the algebra $\mathcal{A}$. In the following we will always work with the representations of the subalgebra of $\mathcal{A}$ with suppressed grading element d on which the central element k acts by zero:

$$h_1 + h_2 + h_3 = 0. \tag{3.5}$$

The algebra $\mathcal{A} = U_q(\widehat{sl}(3))$ is a Hopf algebra with the co-multiplication

$$\delta : \qquad \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A},$$

defined as

$$\begin{aligned} \delta(x_i) &= x_i \otimes 1 + q^{-h_i} \otimes x_i\,, \\ \delta(y_i) &= y_i \otimes q^{h_i} + 1 \otimes y_i\,, \\ \delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i\,, \\ \delta(d) &= d \otimes 1 + 1 \otimes d\,, \end{aligned} \tag{3.6}$$

where $i = 1, 2, 3$. As usual, we introduce the "twisted" co-multiplication

$$\delta' = \sigma \circ \delta\,, \qquad \sigma \circ (a \otimes b) = b \otimes a \qquad (\forall a, b \in \mathcal{A})\,. \tag{3.7}$$

Define also two Borel subalgebras $\mathcal{B}_- \subset \mathcal{A}$ and $\mathcal{B}_+ \subset \mathcal{A}$ generated by $h_i,\ x_i,\ i = 1, 2, 3$, and $h_i,\ y_i,\ i = 1, 2, 3$, respectively. Let $\widehat{\mathcal{B}}_\pm$ be their extensions by d, that is, the subalgebras of $\mathcal{A}$, generated by $\mathcal{B}_\pm$ and d .

There exists a unique element [42, 43]

$$\mathcal{R} \in \widehat{\mathcal{B}}_+ \otimes \widehat{\mathcal{B}}_-\,, \tag{3.8}$$

satisfying the following relations

$$\begin{aligned}
\delta'(a)\,\mathcal{R} &= \mathcal{R}\,\delta(a) \qquad (\forall\ a\in\mathcal{A})\,, &(3.9)\\
(\delta\otimes 1)\,\mathcal{R} &= \mathcal{R}^{13}\,\mathcal{R}^{23}\,, &(3.10)\\
(1\otimes\delta)\,\mathcal{R} &= \mathcal{R}^{13}\,\mathcal{R}^{12}\,, &(3.11)
\end{aligned}$$

where $\mathcal{R}^{12}$, $\mathcal{R}^{13}$, $\mathcal{R}^{23}\in\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$ and $\mathcal{R}^{12}=\mathcal{R}\otimes 1$, $\mathcal{R}^{23}=1\otimes\mathcal{R}$, $\mathcal{R}^{13}=(\sigma\otimes 1)\,\mathcal{R}^{23}$. The element $\mathcal{R}$ is called the universal R-matrix. It satisfies the Yang-Baxter equation

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23}=\mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}\ , \tag{3.12}$$

which is a simple corollary of the definitions (3.9)-(3.11). The universal R-matrix is understood as a formal series in generators in $\widehat{\mathcal{B}}_+\otimes\widehat{\mathcal{B}}_-$. Its dependence on the Cartan elements can be isolated as a simple factor

$$\mathcal{R}=q^{\mathsf{t}}\,\overline{\mathcal{R}}, \tag{3.13}$$

following where t is an invariant tensor in the tensor square of the extended Cartan subalgebra. It can be chosen to be [44,45] $\mathsf{t}=h_1\otimes h^1+h_2\otimes h^2+k\otimes d+d\otimes k$,

$$h^1=\frac{2}{3}h_1+\frac{1}{3}h_2,\quad h^2=\frac{2}{3}h_2+\frac{1}{3}h_1,$$

and q^{t} satisfies the following defining properties:

$$\begin{aligned}
q^{\mathsf{t}}(y_i\otimes 1) &= (y_i\otimes q^{h_i})q^{\mathsf{t}}, & q^{\mathsf{t}}(x_i\otimes 1) &= (x_i\otimes q^{-h_i})q^{\mathsf{t}},\\
q^{\mathsf{t}}(1\otimes y_i) &= (q^{h_i}\otimes y_i)q^{\mathsf{t}}, & q^{\mathsf{t}}(1\otimes x_i) &= (q^{-h_i}\otimes x_i)q^{\mathsf{t}}.
\end{aligned} \tag{3.14}$$

The factor $\overline{\mathcal{R}}$ is the "reduced" universal R-matrix $\overline{\mathcal{R}}\subset\mathcal{B}_+\otimes\mathcal{B}_-$ which is a formal power series in $y_i\otimes 1$ and $1\otimes x_i$ only

(with no dependence on the Cartan elements)

$$\overline{\mathcal{R}}=1+(q-q^{-1})(y_1\otimes x_1+y_2\otimes x_2+y_3\otimes x_3)+\ldots \tag{3.15}$$

Below we will use the following important property of the reduced universal R-matrix.

Proposition 1 *Suppose elements I, A_i and B_i, $i=1,2,3$, satisfy the relations*

$$[I,x_i]=\frac{A_i-B_i}{q-q^{-1}}, \tag{3.16}$$

$$A_i\,x_j=q^{a_{ij}}\,x_j\,A_i,\qquad B_i\,x_j=q^{-a_{ij}}\,x_j\,B_i, \tag{3.17}$$

where $i,j=1,2,3$. Then

$$[1\otimes I,\overline{\mathcal{R}}]=(\sum_{i=1}^{3}y_i\otimes A_i)\ \overline{\mathcal{R}}-\overline{\mathcal{R}}\ (\sum_{i=1}^{3}y_i\otimes B_i). \tag{3.18}$$

The proof of this statement is given in Appendix A.

3.2 Free field representation and the vertex operators

The quantum version of the Miura transformation (2.44) is the free field representation [46] of the algebra $\mathcal{W}_3$,

$$\begin{aligned} -gT(u) &= \,:\phi_1'(u)^2: + :\phi_2'(u)^2: + 2(1-g)\phi_1''(u) + \frac{g}{12}, \\ -\frac{(3g)^{\frac{3}{2}}i}{2}W(u) &= \,:\phi_2'(u)^3: -3:\phi_1'(u)^2\phi_2'(u): + \frac{3}{2}(g-1):\phi_1''(u)\phi_2'(u): \\ &+ \frac{9}{2}(g-1):\phi_1'(u)\phi_2''(u): - \frac{3}{2}(g-1)^2\phi_2'''(u), \end{aligned} \tag{3.19}$$

in terms of a two-component free chiral Bose field $\boldsymbol{\varphi}(u) = (\varphi_1(u), \varphi_2(u))$,

$$\boldsymbol{\varphi}(u) = i\mathbf{X} + i\boldsymbol{P}\,u + \sum_{n\neq 0}\frac{\mathbf{b}_{-n}}{n}e^{inu}. \tag{3.20}$$

The mode operators $\boldsymbol{X} = (X_1, X_2)$, $\boldsymbol{P} = (P_1, P_2)$ and $\mathbf{b}_n = (b_{1,n}, b_{2,n})$ $(n = \pm1, \pm2, \ldots)$ satisfy the commutation relations of the Heisenberg algebra

$$[X_i, P_j] = \frac{ig}{2}\delta_{i,j}, \qquad [b_{i,n}, b_{j,m}] = \frac{ng}{2}\delta_{n+m,0}\,\delta_{i,j}, \qquad i,j = 1,2 \tag{3.21}$$

where the parameter g is related to the central charge c as

$$g = \frac{50 - c - \sqrt{(2-c)(98-c)}}{48}, \qquad c = 50 - 24(g + g^{-1}). \tag{3.22}$$

Let $\mathcal{F}_{\boldsymbol{p}}$ be the highest weight module over the algebra (3.21), with highest weight vector $|\,\boldsymbol{p}\rangle$, $\boldsymbol{p} = (p_1, p_2)$ defined as

$$P_i|\,\boldsymbol{p}\rangle = p_i|\,\boldsymbol{p}\rangle, \qquad a_{i,n}|\,\boldsymbol{p}\rangle = 0, \qquad n > 0, \qquad i = 1,2. \tag{3.23}$$

For generic c and $\boldsymbol{p}$, the Fock space $\mathcal{F}_{\boldsymbol{p}}$ is isomorphic to the space $\mathcal{V}_{\Delta_2,\Delta_3}$ with the highest weights

$$\Delta_2 = \frac{p_1^2 + p_2^2}{g} + \frac{c-2}{24}, \qquad \Delta_3 = \frac{2p_2\,(p_2^2 - 3p_1^2)}{(3g)^{3/2}}. \tag{3.24}$$

The space

$$\hat{\mathcal{F}}_{\boldsymbol{p}} = \bigoplus_{n_1,n_2=-\infty}^{\infty} \mathcal{F}_{\boldsymbol{p}+g(n_1\boldsymbol{e}_1+n_2\boldsymbol{e}_2)}, \tag{3.25}$$

supports the action of the vertex operators

$$V_i(u) = q^{\frac{1}{2}} : e^{-2\,\boldsymbol{e}_i\,\boldsymbol{\varphi}(u)} : , \qquad i = 1,2,3, \tag{3.26}$$

defined as

$$: e^{-2\,\boldsymbol{e}_i\boldsymbol{\varphi}(u)} :\equiv \exp\Big(-2\sum_{n=1}^{\infty}\frac{\boldsymbol{e}_i\,\boldsymbol{b}_{-n}}{n}e^{inu}\Big)\exp\big(-2i\boldsymbol{e}_i(\boldsymbol{X} + \boldsymbol{P}u)\big)\exp\Big(+2\sum_{n=1}^{\infty}\frac{\boldsymbol{e}_i\,\mathbf{b}_n}{n}e^{-inu}\Big), \tag{3.27}$$

where the root vectors $\boldsymbol{e}_i$ are given in (2.34) and

$$q = e^{i\pi g} \; . \tag{3.28}$$

Using (3.21) it is easy to check that

$$\begin{aligned} V_i(u_1)V_j(u_2) &= q^{a_{ij}}V_j(u_2)V_i(u_1), \qquad u_1 > u_2, \\ [\mathbf{P}\,,\,V_j(u)] &= -g\,\mathbf{e}_j\,V_j(u), \end{aligned} \tag{3.29}$$

where the a_{ij} are given by (2.38) and

$$V_j(u+2\pi) = q^{-2}e^{-4i\pi \boldsymbol{P}\mathbf{e}_j}V_j(u). \tag{3.30}$$

It will sometimes be convenient to use the operators

$$z_j = e^{2\pi i \boldsymbol{w}_j \boldsymbol{P}}, \qquad z_1\,z_2\,z_3 = 1, \qquad j = 1,2,3, \tag{3.31}$$

where the vectors $\boldsymbol{w}_j$ are defined in (2.34). For example, with these operators the relations (3.29),(3.30) take the form

$$z_i V_j(u) = q^{2\epsilon_{ij}}V_j(u)z_i, \tag{3.32}$$

where ϵ_{ij} is defined in (2.39) and

$$V_i(u+2\pi) = q^{-2}\,z_j^{-1}\,z_k\,V_i(u), \tag{3.33}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

3.3 The universal L-operator

Consider the following operator [47,48]

$$\mathcal{L} = e^{i\pi \boldsymbol{P}\boldsymbol{h}}\mathcal{P}\exp\left(\int_0^{2\pi} K(u)\,du\right), \tag{3.34}$$

where

$$\boldsymbol{h} = \frac{2}{3}(\boldsymbol{e}_1\,h_1 + \boldsymbol{e}_2\,h_2 + \boldsymbol{e}_3\,h_3), \tag{3.35}$$

with the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ defined in (2.34) and

$$K(u) = V_1(u)y_1 + V_2(u)y_2 + V_3(u)y_3. \tag{3.36}$$

Here h_i, y_i, $i = 1,2,3$, $h_1 + h_2 + h_3 = 0$, are the generators of the Borel subalgebra $\mathcal{B}_+ \subset U_q(\widehat{sl}(3))$ defined above. The ordered exponential in (3.34) (the symbol $\mathcal{P}$ denotes the path ordering) is defined as a series

$$\mathcal{L} = e^{i\pi \boldsymbol{P}\boldsymbol{h}}\sum_{n=0}^{\infty}\int_{2\pi\ge u_1\ge u_2\ge \ldots \ge u_n \ge 0} K(u_1)K(u_2)\cdots K(u_n)du_1du_2\ldots du_n. \tag{3.37}$$

The integrals in (3.37) converge when

$$-\infty < c < -2, \tag{3.38}$$

(which corresponds to the range $0 < g < 2/3$). For the values of c outside this *quasiclassical domain*, (3.38), the integrals in (3.37) can be defined via analytic continuation in c [3]. The operator (3.34) is an element of the algebra $\mathcal{B}_+$ whose coefficients are operators acting in the quantum space (3.25). Following the arguments of [4] it is not difficult to show that this L-operator satisfies the Yang-Baxter equation

$$\mathcal{R}\,(\,\mathcal{L}\otimes 1\,)\,(\,1\otimes\mathcal{L}\,) = (\,1\otimes\mathcal{L}\,)\,(\,\mathcal{L}\otimes 1\,)\,\mathcal{R}\;. \tag{3.39}$$

where $\mathcal{R}$ is the universal R-matrix for the algebra $U_q(\widehat{sl}(3))$ defined above. The only property of the universal R-matrix required for (3.39) to hold is relation (3.9).

An important remark is in order. As the reader might expect, the L-operator (3.34) can be obtained as a specialization of the universal R-matrix and the equation (3.39) is, in fact, a special case of the Yang-Baxter equation (3.12). The exact correspondence (first established in [4] for the case of $U_q(\widehat{sl}(2))$) is formulated as follows. Each side of the equation (3.12) is an element of the direct product $\mathcal{B}_+\otimes\mathcal{A}\otimes\mathcal{B}_-$. Therefore to reduce (3.12) to (3.39) it is enough to find an appropriate realization of the Borel subalgebra $\mathcal{B}_-$ (appearing in the third factor of the product) in the quantum space (3.25). Let us identify the generators $x_i\in\mathcal{B}_-$, $i=1,2,3$, of this Borel subalgebra with the integrals of the vertex operators

$$x_i = \frac{1}{q-q^{-1}}\int_0^{2\pi} V_i(u)du, \qquad i=1,2,3. \tag{3.40}$$

One can check that these generators satisfy [49] the Serre relations (3.3). Substituting (3.40) into the reduced universal R-matrix, (3.15), one gets an element of $\mathcal{B}_+$ whose co-ordinates are operators acting in the quantum space (3.25). As expected, it exactly coincides with the $\mathcal{P}$-ordered exponent from (3.34).

Proposition 2 *The specialization of the "reduced" universal R-matrix $\overline{\mathcal{R}}\in\mathcal{B}_+\otimes\mathcal{B}_-$ for the case when $x_i\in\mathcal{B}_-$, $i=1,2,3$, are realized as in (3.40) coincide with the $\mathcal{P}$-exponent*

$$\overline{\mathcal{R}}\Big|_{x_i=(3.40)} = \mathcal{P}\exp\Big(\int_0^{2\pi}\mathcal{K}(u)\,du\Big)\equiv\overline{\mathcal{L}}, \tag{3.41}$$

where $\mathcal{K}(u)$ is defined in (3.36).

This is precisely the conjecture stated in ref. [4]. It was formulated there for $U_q(\widehat{sl}(2))$, but the statement is rather general and (with obvious modifications) holds for any quantized Kac-Moody algebra. A proof of the conjecture of [4] based on combinatorial properties of the representations of quantum groups has been obtained by E.Frenkel and

F.A.Smirnov [50]. Here we present an alternative (and, in fact, rather elementary) proof of (3.41) which is based on the defining properties of the universal R-matrix.

The reduced universal R-matrix (3.15) is a formal power series in the generators $\overline{y}_i = y_j \otimes 1$ and $\overline{x}_i = 1 \otimes x_i$, $i = 1, 2, 3$, such that $\overline{x}_i \overline{y}_j = \overline{y}_j \overline{x}_i$, therefore we can write it as $\overline{\mathcal{R}} = \overline{\mathcal{R}}(\overline{y}_i, \overline{x}_i)$. Then, substituting (3.13) into (3.11) one obtains

$$\overline{\mathcal{R}}(\overline{y}_j, x_i' + x_i'') = \overline{\mathcal{R}}(\overline{y}_j, x_i')\, \overline{\mathcal{R}}(\overline{y}_j, x_i'') \tag{3.42}$$

where $\overline{y}_i = y_i \otimes 1 \otimes 1$, $x_i' = 1 \otimes q^{-h_i} \otimes x_i$, $x_i'' = 1 \otimes x_i \otimes 1$. Both sets x_i' and x_i'' satisfy the Serre relations (3.3) of $\mathcal{B}_-$, commute with $\overline{y}_j$

$$x_i' \overline{y}_j = \overline{y}_j x_i', \qquad x_i'' \overline{y}_j = \overline{y}_j x_i'' \tag{3.43}$$

and also obey the relations

$$x_i' x_j'' = q^{a_{ij}} x_j'' x_i' \tag{3.44}$$

Conversely, we can regard (3.42) as an identity for $\overline{\mathcal{R}}(\overline{y}_i, \overline{x}_i)$ which should be valid once all the arguments involved there are subjected to the above relations (namely, (3.3), (3.43) and (3.44)).

Consider integrals more general than (3.40) which also satisfy the Serre relations (3.3)

$$\overline{x}_i = \frac{1}{q - q^{-1}} \int_{u_2}^{u_1} V_i(u) du, \qquad i = 1, 2, 3, \tag{3.45}$$

where $2\pi \geq u_1 > u_2 \geq 0$. Denote by $\overline{\mathcal{L}}(u_1, u_2)$ the result of substituting these generators $\overline{x}_i \in \mathcal{B}_-$ into the reduced universal R-matrix

$$\overline{\mathcal{L}}(u_1, u_2) = \overline{\mathcal{R}}(\overline{y_i}, \overline{x}_i)\Big|_{\overline{x}_i = (3.45)} \tag{3.46}$$

Note that from (3.15) it follows that for $(u_1 - u_2) \to 0$

$$\overline{\mathcal{L}}(u_1, u_2) = 1 + \int_{u_2}^{u_1} K(u)\, du + O((u_1 - u_2)^2) \tag{3.47}$$

where $K(u)$ is defined in (3.36).

Now, let $2\pi > u_1 \geq u_2 \geq u_3 \geq 0$, then the identity (3.42) implies that

$$\overline{\mathcal{L}}(u_1, u_3) = \overline{\mathcal{L}}(u_1, u_2) \overline{\mathcal{L}}(u_2, u_3), \tag{3.48}$$

since two sets of quantities

$$x_i' = \frac{1}{q - q^{-1}} \int_{u_2}^{u_1} V_i(u) du,$$

$$x_i'' = \frac{1}{q - q^{-1}} \int_{u_3}^{u_2} V_i(u) du,$$

satisfy all the required relations (3.3), (3.43) and (3.44). In particular, the relation (3.44) readily follows from (3.29). With an account of (3.47) the functional equation (3.48) has a unique solution

$$\overline{\mathcal{L}}(u_1, u_2) = \mathcal{P} \exp\left(\int_{u_2}^{u_1} K(u)\, du\right). \tag{3.49}$$

Finally, setting $u_2 = 0$ and $u_1 = 2\pi$ one arrives at the statement of Proposition 2.

4 Quantum T- and Q-operators

4.1 Symmetry relations

Consider the symmetry properties of the L-operator (3.34). Any linear orthogonal operator $\mathbf{U}$ in two-dimensional Euclidean space induces an automorphism U of the two-component Heisenberg algebra (3.21) by the rule $U[\boldsymbol{\varphi}(u)] = \mathbf{U}\boldsymbol{\varphi}(u)$. In particular, the Weyl reflections associated with the vectors $\boldsymbol{w}_i$, $i = 1, 2, 3$, generate the symmetric group S_3 permuting the vertex operators

$$\sigma_{ij}[V_i(u)] = V_j(u), \qquad \sigma_{ij}[V_j(u)] = V_i(u), \qquad \sigma_{ij}[V_k(u)] = V_k(u), \tag{4.1}$$

$$\sigma_{ij}[z_i] = z_j^{-1}, \qquad \sigma_{ij}[z_j] = z_i^{-1}, \qquad \sigma_{ij}[z_k] = z_k^{-1}. \tag{4.2}$$

where (i, j, k) is any permutation of $(1, 2, 3)$. Define also the cyclic automorphism $\tau = \sigma_{12}\sigma_{23}$ corresponding to the rotation of the Dynkin diagram of $A_2^{(1)}$

$$\tau[V_i(u)] = V_{i+1 \pmod 3}(u). \tag{4.3}$$

The weights (3.24) can be written in a symmetric form

$$\Delta_2 = -\frac{r_1 r_2 + r_1 r_3 + r_2 r_3}{g} + \frac{c-2}{24}, \qquad \Delta_3 = -\frac{r_1 r_2 r_3}{g^{3/2}}\ , \tag{4.4}$$

where

$$r_i = \boldsymbol{w}_i \boldsymbol{p}, \qquad i = 1, 2, 3, \tag{4.5}$$

with $\boldsymbol{w}_i$ given in (2.34). From the symmetry properties

$$\tau[r_i] = r_{i+1 \pmod 3}, \qquad \sigma_{ij}[r_i] = -r_j, \qquad \sigma_{ij}[r_j] = -r_i, \qquad \sigma_{ij}[r_k] = -r_k, \tag{4.6}$$

it is obvious that the cyclic automorphism τ transforms the Fock module $\mathcal{F}_{\boldsymbol{p}}$ to an isomorphic module

$$\tau[\mathcal{F}_{\boldsymbol{p}}] \sim \mathcal{F}_{\boldsymbol{p}}, \tag{4.7}$$

while the Weyl reflections σ_{ij} negate $\boldsymbol{p}$ and the weight Δ_3

$$\sigma_{ij}[\mathcal{F}_{\boldsymbol{p}}] \sim \mathcal{F}_{-\boldsymbol{p}} \sim \mathcal{V}_{\Delta_2, -\Delta_3}. \tag{4.8}$$

Further, the algebra of outer automorphisms of $\mathcal{B}_+ \subset U_q(\widehat{sl}(3))$ also contains the symmetric group S_3: $\sigma[y_i] = y_{\sigma(i)}$, $\sigma[h_i] = h_{\sigma(i)}$. Again by τ we denote the automorphism corresponding to rotation of the Dynkin diagram of $A_2^{(1)}$

$$\tau[y_i] = y_{i+1 \pmod 3}, \qquad \tau[h_i] = h_{i+1 \pmod 3}\ , \tag{4.9}$$

and by σ_{ij} the transpositions

$$\begin{aligned} \sigma_{ij}[y_i] &= y_j, & \sigma_{ij}[y_j] &= y_i, & \sigma_{ij}[y_k] &= y_k, \\ \sigma_{ij}[h_i] &= h_j, & \sigma_{ij}[y_h] &= h_i, & \sigma_{ij}[h_k] &= h_k. \end{aligned} \tag{4.10}$$

The L-operator (3.34) is invariant with respect to the diagonal action of the automorphisms

$$(\sigma \otimes \sigma)[\mathcal{L}] = \mathcal{L}, \qquad \sigma \in S_3. \tag{4.11}$$

4.2 The T-operators

We will need several specializations of the universal L-operator (3.34) corresponding to various matrix representations of the Borel subalgebra $\mathcal{B}_+ \subset U_q(\widehat{sl}(3))$. First consider the so-called *evaluation maps* from $\mathcal{B}_+$ to the finite dimensional algebra $U_q(gl(3))$. This latter algebra is generated by the elements H_i , $i = 1, 2, 3$ and E_{ij}, where $(i, j) = (1, 2)$, $(2, 1)$, $(2, 3)$, $(3, 2)$, for which we use also the notations

$$E_\alpha = E_{12}, \quad E_\beta = E_{23}, \quad F_\alpha = E_{21}, \quad F_\beta = E_{32},$$

and

$$H_\alpha = H_1 - H_2, \qquad H_\beta = H_2 - H_3, \qquad H_{\alpha+\beta} = H_1 - H_3.$$

They satisfy the relations

$$\begin{gathered} [H_i, H_j] = 0, \qquad [H_i, E_{kl}] = (\delta_{ik} - \delta_{il})E_{kl}, \\ [E_{\alpha_i}, F_{\alpha_j}] = \delta_{\alpha_i,\alpha_j} \frac{q^{H_{\alpha_i}} - q^{-H_{\alpha_i}}}{q - q^{-1}}, \\ \left.\begin{aligned} E^2_{\alpha_i} E_{\alpha_j} - [2]_q E_{\alpha_i} E_{\alpha_j} E_{\alpha_i} + E_{\alpha_j} E^2_{\alpha_i} &= 0, \\ F^2_{\alpha_i} F_{\alpha_j} - [2]_q F_{\alpha_i} F_{\alpha_j} F_{\alpha_i} + F_{\alpha_j} F^2_{\alpha_i} &= 0, \end{aligned}\right. \qquad \alpha_i \neq \alpha_j, \end{gathered} \tag{4.12}$$

where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ and the greek indices α_i and α_j in the last three relations take two values α or β. The algebra $U_q(gl(3))$ has an outer automorphism σ_{13}, induced by the automorphism of its Dynkin diagram:

$$\begin{gathered} \sigma_{13}[E_\alpha] = E_\beta, \qquad \sigma_{13}[E_\beta] = E_\alpha, \qquad \sigma_{13}[F_\alpha] = F_\beta, \qquad \sigma_{13}[F_\beta] = F_\alpha, \\ \sigma_{13}[H_1] = -H_3, \qquad \sigma_{13}[H_2] = -H_2, \qquad \sigma_{13}[H_3] = -H_1. \end{gathered} \tag{4.13}$$

Let us introduce the following root vectors:

$$F_{\alpha+\beta} = q^{\frac{1}{2}} F_\beta F_\alpha - q^{-\frac{1}{2}} F_\alpha F_\beta, \qquad \overline{F}_{\alpha+\beta} = q^{-\frac{1}{2}} F_\beta F_\alpha - q^{\frac{1}{2}} F_\alpha F_\beta,$$

and

$$E_{31} = F_{\alpha+\beta} q^{H_1+H_3}, \qquad \overline{E}_{31} = \overline{F}_{\alpha+\beta} q^{-H_1-H_3}.$$

There are two non-equivalent evaluation maps $\mathcal{B}_+ \to U_q(gl(3))$. The first one is

$$\mathcal{E}v_t(y_1) = E_{23}, \qquad \mathcal{E}v_t(y_2) = tE_{31}, \qquad \mathcal{E}v_t(y_3) = E_{12}, \tag{4.14}$$

$$\mathcal{E}v_t(h_1) = H_2 - H_3, \qquad \mathcal{E}v_t(h_2) = H_3 - H_1, \qquad \mathcal{E}v_t(h_3) = H_1 - H_2. \tag{4.15}$$

where t is a spectral parameter. The second evaluation map $\overline{\mathcal{E}v}_t : \mathcal{B}_+ \to U_q(gl(3))[t]$ has the form

$$\overline{\mathcal{E}v}_t = \sigma_{13} \cdot \mathcal{E}v_{-t} \cdot \sigma_{13}, \tag{4.16}$$

where the automorphisms σ_{13} entering this formula are defined in (4.10), (4.13). Explicitly one has

$$\overline{\mathcal{E}v}_t(y_1) = E_{23}, \qquad \overline{\mathcal{E}v}_t(y_2) = t\overline{E}_{31}, \qquad \overline{\mathcal{E}v}_t(y_3) = E_{12}, \tag{4.17}$$

$$\overline{\mathcal{E}v}_t(h_1) = H_2 - H_3, \qquad \overline{\mathcal{E}v}_t(h_2) = H_3 - H_1, \qquad \overline{\mathcal{E}v}_t(h_3) = H_1 - H_2. \tag{4.18}$$

One can easily check that both these maps are algebra homomorphisms as all the defining relations (3.3), (3.1) of the algebra $\mathcal{B}_+$ become simple corollaries of (4.12).

For any $\mu = (\mu_1, \mu_2, \mu_3)$, such that $\mu_\alpha = \mu_1 - \mu_2$ and $\mu_\beta = \mu_2 - \mu_3$ are nonnegative integers denote by π_μ the finite dimensional representation of $U_q(gl(3))$ with the highest weight μ and highest weight vector $|\,0\rangle$ defined as

$$E_{12}|\,0\rangle = E_{23}|\,0\rangle = 0, \qquad H_i|\,0\rangle = \mu_i|\,0\rangle, \qquad i = 1,2,3. \tag{4.19}$$

Further let $\pi_\mu(t)$ be the representation of $\mathcal{B}_+$ given by the composition of π_μ with the evaluation homomorphism $\mathcal{E}v_t$. Evidently, the operator-valued matrices

$$\mathbf{L}_\mu(t) = \pi_\mu(t)[\mathcal{L}], \tag{4.20}$$

(whose matrix elements are operators acting in (3.25)) satisfy suitable specializations of the quantum Yang-Baxter equation (3.39). These operators (4.20) are quantum analogues of the classical L-operators (2.48) whereas the operators

$$\mathbf{T}_\mu(t) = \mathrm{Tr}_{\pi_\mu(t)}\,[e^{i\pi \boldsymbol{P}\,h}\mathcal{L}], \tag{4.21}$$

are the quantum analogues of $\mathbf{T}^{(cl)}(\lambda)$ in (2.51) (the spectral parameter t is related to λ in (2.48) and (2.51) as $t = \lambda^3$). ¿From the standard arguments based on the Yang-Baxter equation it follows that

$$[\mathbf{T}_\mu(t), \mathbf{T}_{\mu'}(t')] = 0. \tag{4.22}$$

Let us show that they also commute with all local IM (2.8),

$$[\mathbf{T}_\mu(t), \mathbf{I}_k] = 0. \tag{4.23}$$

The characteristic property of the local IM is that their commutators with vertex operators (3.26) reduce to total derivatives [FF]

$$[\,\mathbf{I}_k\,, V_i(u)\,] = \partial_u \Big\{ : O_k^{(i)}(u) V_i(u) : \Big\} \equiv \partial_u \Theta_k^{(i)}(u), \tag{4.24}$$

where $O_k^{(i)}(u)$ are some polynomials with respect to the field $\partial_u \boldsymbol{\phi}(u)$ and its derivatives. Obviously, the operators $\Theta_k^{(i)}(u)$ in (4.24) obey the same periodicity properties as the vertex operators $V_i(u)$, (3.30),

$$\Theta_k^{(j)}(u+2\pi) = q^{-2} e^{-4i\pi \boldsymbol{P}\mathbf{e}_j} \Theta_k^{(j)}(u), \tag{4.25}$$

and satisfy the commutation relations

$$\begin{aligned} \Theta_k^{(i)}(u_1) V_j(u_2) &= q^{a_{ij}} V_j(u_2) \Theta_k^{(i)}(u_1), \qquad u_1 > u_2, \\ [\boldsymbol{P}\,,\, \Theta_k^{(j)}(u)] &= -g\, \mathbf{e}_j\, \Theta_k^{(j)}(u)\ , \end{aligned} \tag{4.26}$$

where a_{ij} is defined in (2.38). It is easy to check that the triple

$$I = \mathbf{I}_k, \qquad A_i = \Theta_k^{(i)}(2\pi), \qquad B_i = \Theta_k^{(i)}(0), \tag{4.27}$$

satisfies the conditions (3.16),(3.17) of Proposition 1 with the generators $x_i \in \mathcal{B}_+$ realized as in (3.40) Therefore taking into account (3.41) and applying (3.18) one obtains

$$[\mathbf{I}_k, \overline{\mathcal{L}}] = \Theta_k(2\pi)\, \overline{\mathcal{L}} - \overline{\mathcal{L}}\, \Theta_k(0), \tag{4.28}$$

where

$$\Theta_k(u) = \sum_{i=1}^{3} y_i \Theta_k^{(i)}(u). \tag{4.29}$$

Now substitute (4.21) into the RHS of (4.23). Using the relation (4.28), the commutation relations (3.1), (4.26) and the cyclic property of the trace it is not difficult to show that the commutator (4.23) actually vanishes.

The most important **T**-operators correspond to the three-dimensional (fundamental) representations of π_{ω_α}, and π_{ω_β} of $U_q(gl(3))$ with the highest weights $\omega_\alpha = (1,0,0)$ and $\omega_\beta = (1,1,0)$ (The corresponding representation matrices are given explicitly in Appendix D). We will introduce special notation for these **T**-operators (adjusting the normalisation of the spectral parameter for later convenience)

$$\mathbb{T}(t) \equiv \mathbf{T}_{(1,0,0)}(tq^{-\frac{3}{2}}), \qquad \overline{\mathbb{T}}(t) \equiv \mathbf{T}_{(1,1,0)}(tq^{-\frac{1}{2}}). \tag{4.30}$$

Computing the trace in (4.21) in this case one obtains

$$\mathbb{T}(t)=\sum_{n=0}^{\infty} t^n\,\mathbf{G}_n\ ,\qquad \overline{\mathbb{T}}(t)=\sum_{n=0}^{\infty} t^n\,\overline{\mathbf{G}}_n. \tag{4.31}$$

Here

$$\mathbf{G}_0=z_1+z_2+z_3,\qquad \overline{\mathbf{G}}_0=z_1^{-1}+z_2^{-1}+z_3^{-1}\ , \tag{4.32}$$

with z_1, z_2 and z_3 defined in (3.31) and for $n\geq 1$

$$\begin{aligned} \mathbf{G}_n &= z_1\,J^{(n)}_{312}+z_2\,J^{(n)}_{123}+z_3\,J^{(n)}_{231},\\ (-1)^n\overline{\mathbf{G}}_n &= z_1^{-1}J^{(n)}_{213}+z_2^{-1}J^{(n)}_{321}+z_3^{-1}J^{(n)}_{132}, \end{aligned} \tag{4.33}$$

$$\begin{aligned} J^{(n)}_{ijk} &= \int_{2\pi\geq u_1\geq u_2\ldots\geq u_{3n}} \Big(V_i(u_1)V_j(u_2)V_k(u_3)\ V_i(u_4)V_j(u_5)V_k(u_6)\cdots \\ &\quad \cdots V_i(u_{3n-2})V_j(u_{3n-1})V_k(u_{3n})\Big)\ du_1\,du_2\ldots du_{3n}, \end{aligned} \tag{4.34}$$

where (i,j,k) is a permutation of $(1,2,3)$.

It follows from (4.22) and (4.23) that the operators $\mathbf{G}_n$ and $\overline{\mathbf{G}}_n$ can be regarded as the "nonlocal" IM which commute among themselves and with all the local IM (2.8)

$$[\mathbf{G}_k,\mathbf{G}_l]=[\mathbf{G}_k,\overline{\mathbf{G}}_l]=[\overline{\mathbf{G}}_k,\overline{\mathbf{G}}_l]=[\mathbf{G}_k,\mathbf{I}_l]=[\overline{\mathbf{G}}_k,\mathbf{I}_l]=0. \tag{4.35}$$

Definitions similar to those in (4.21) but with the map (4.18) instead of (4.15) lead to a new set of $\mathbf{T}$-operators

$$\overline{\mathbf{T}}_\mu(t)=\mathrm{Tr}_{\overline{\pi}_\mu(t)}\,[e^{i\pi \boldsymbol{P}\,h}\mathcal{L}], \tag{4.36}$$

where $\overline{\pi}_\mu(t)$ denotes the representation of $\mathcal{B}_+$ obtained by the composition of the representation π_μ of $U_q(gl(3))$ with the evaluation map (4.18). The operators $\overline{\mathbf{T}}_\mu(t)$ with different values of t commute among themselves and with all $\mathbf{T}_\mu(t)$,

$$[\overline{\mathbf{T}}_\mu(t),\overline{\mathbf{T}}_{\mu'}(t')]=[\overline{\mathbf{T}}_\mu(t),\mathbf{T}_{\mu'}(t')]=0. \tag{4.37}$$

In general for the same π_μ the formulae (4.21) and (4.36) define different operators. However, when π_μ is restricted to symmetric powers of a fundamental representation π_{ω_α} and π_{ω_β} then the two sets of $\mathbf{T}$-operators coincide (up to a rescaling of spectral parameter). For future references it will be convenient to define

$$\mathbb{T}_m(t)\equiv \mathbf{T}_{(m,0,0)}(tq^{-m-\frac{1}{2}})=\overline{\mathbf{T}}_{(m,0,0)}(tq^{m+\frac{1}{2}}), \tag{4.38}$$

and

$$\overline{\mathbb{T}}_m(t)\equiv \mathbf{T}_{(m,m,0)}(tq^{-m+\frac{1}{2}})=\overline{\mathbf{T}}_{(m,m,0)}(tq^{m-\frac{1}{2}}), \tag{4.39}$$

where $m=0,1,2,\ldots,\infty$. In particular, $\mathbb{T}_0(t)=\overline{\mathbb{T}}_0(t)\equiv 1$ while $\mathbb{T}_1(t)\equiv\mathbb{T}(t)$, $\overline{\mathbb{T}}_1(t)\equiv\overline{\mathbb{T}}(t)$ are exactly the same as in (4.31).

4.3 The Q-operators

The operators (4.21) and (4.36) are obviously the CFT versions of Baxter's transfer-matrices of the lattice models related to the $U_q(\widehat{sl}(3))$ algebra. It is well known that the theory of solvable lattice models involves other important objects, namely Baxter's Q-matrix. Similarly to the standard CFT case [3] appropriate $\mathcal{W}_3$ CFT versions of $\mathbf{Q}$-operators can be obtained as certain specializations of the universal L-operator (3.34).

The generators y_i, h_i, $i = 1,2,3$, $h_1 + h_2 + h_3 = 0$, of the Borel subalgebra $\mathcal{B}_+ \subset U_q(\widehat{sl}(3))$ satisfy the commutation relations (3.1)

$$[h_i, h_j] = 0, \qquad [h_i, y_i] = a_{ij}, \tag{4.40}$$

and the Serre relations (3.3). The later can be written as

$$y_i y_{ij} = q^{-1} y_{i,j} y_i, \qquad i,j = 1,2,3, \qquad i \neq j, \tag{4.41}$$

where y_{ij} stands for the q-commutator of y_i and y_j:

$$y_{ij} = y_i y_j - q y_j y_i. \tag{4.42}$$

Let $\mathbf{a}_i^+, \mathbf{a}_i^-, \mathcal{H}_i$, $i = 1,2$ be the generators of two independent q-oscillator algebras H_q,

$$[\mathcal{H}_i, \mathbf{a}_i^\pm] = \pm \mathbf{a}_i^\pm, \qquad q\mathbf{a}_i^+\mathbf{a}_i^- - q^{-1}\mathbf{a}_i^-\mathbf{a}_i^+ = \frac{1}{q-q^{-1}}. \tag{4.43}$$

Define the following homomorphism $\rho_t : \mathcal{B}_+ \to \mathsf{H}_q \otimes \mathsf{H}_q$

$$\begin{aligned} \rho_t(y_1) &= q^{\frac{1}{2}}\, t(q-q^{-1})\, q^{-\mathcal{H}_2}\mathbf{a}_1^-\mathbf{a}_2^+, \qquad \rho_t(y_2) = \mathbf{a}_2^-, \qquad \rho_t(y_3) = \mathbf{a}_1^+ q^{\mathcal{H}_2}, \\ \rho_t(h_1) &= -\mathcal{H}_1 + \mathcal{H}_2, \qquad \rho_t(h_2) = -\mathcal{H}_1 - 2\mathcal{H}_2, \qquad \rho_t(h_3) = 2\mathcal{H}_1 + \mathcal{H}_2, \end{aligned} \tag{4.44}$$

where t is a spectral parameter. Note that this map subjects the generators y_i, $i = 1,2,3$ to some additional relations which are not implied by (4.40) and the Serre relations (4.41).

$$\rho_t\left(q y_3 y_{21} - q^{-1} y_{21} y_3\right) = \rho_t\left(q y_{13} y_2 - q^{-1} y_2 y_{13}\right) = -\frac{q^{\frac{1}{2}}\, t}{q-q^{-1}}, \qquad \rho_t\left(y_{23}\right) = 0, \tag{4.45}$$

where y_{ij} is defined in (4.42). Define also the maps $\rho_i(t)$ and $\overline{\rho}_i(t)$: $\mathcal{B}_+ \to \mathsf{H}_q \otimes \mathsf{H}_q$, $i = 1,2,3$ obtained as compositions of (4.44) with the automorphisms (4.9),(4.10) of the algebra $\mathcal{B}_+$

$$\rho_i(t) = \rho_t \cdot \tau^{-i+1}, \qquad \overline{\rho}_i(t) = \rho_{-t} \cdot \tau^{-i+1} \cdot \sigma_{jk} \qquad i = 1,2,3, \tag{4.46}$$

where ρ_t is given by (4.44) and ρ_{-t} is obtained from (4.44) by the replacement $t \to -t$. Let π_i and $\overline{\pi}_i$ be any representations of $\mathsf{H}_q \otimes \mathsf{H}_q$ such that the traces

$$Z_i = \mathrm{Tr}_{\pi_i \rho_i(t)}[e^{2i\pi \boldsymbol{p}\, h}] \qquad \text{and} \qquad \overline{Z}_i = \mathrm{Tr}_{\overline{\pi}_i \overline{\rho}_i(t)}[e^{2i\pi \boldsymbol{p}\, h}], \tag{4.47}$$

converge and do not vanish for

$$\Im m\, \boldsymbol{e}_1 \boldsymbol{p} < 0, \qquad \Im m\, \boldsymbol{e}_3 \boldsymbol{p} < 0, \tag{4.48}$$

where $\boldsymbol{p} = (p_1, p_2) \in \mathbb{C}^2$.

Define the operators

$$\mathbf{A}_i(t) = Z_i^{-1}\ \mathrm{Tr}_{\pi_i \rho_i^+(t)}[e^{i\pi \boldsymbol{P}\, h}\mathcal{L}] \qquad \text{and} \qquad \overline{\mathbf{A}}_i(t) = \overline{Z}_i^{-1}\ \mathrm{Tr}_{\overline{\pi}_i \overline{\rho}_i^+(t)}[e^{i\pi \boldsymbol{P}\, h}\mathcal{L}], \tag{4.49}$$

where $\mathcal{L}$ is given in (3.34). Since we are interested in the action of these operators in $\mathcal{F}_{\boldsymbol{p}}$ the operator $\boldsymbol{P}$ in (4.49) can be substituted by its eigenvalue $\boldsymbol{p} = (p_1, p_2)$. The definitions (4.49) obviously apply to the domain (4.48). However they can be extended for all complex $\boldsymbol{p} \in \mathbb{C}^2$ (except for some set of singular points for real values p_1 and p_2) by an analytic continuation in $\boldsymbol{p}$.

Obviously, the operators (4.49) can be written as a power series. For example,

$$\mathbf{A}_i(t) = 1 + \sum_{n=1}^{\infty} \sum_{\{\sigma_i = 1,2,3\}} a_{3n}^{(i)}(\sigma_1, \sigma_2, \ldots, \sigma_{3n}) \mathbb{J}_{3n}(\sigma_1, \sigma_2, \ldots, \sigma_{3n}), \tag{4.50}$$

where

$$\mathbb{J}_{3n}(\sigma_1, \sigma_2, \ldots, \sigma_{3n}) = \int_{2\pi \geq u_1 \geq \cdots \geq u_{3n}} V_{\sigma_1}(u_1) V_{\sigma_2}(u_2) \cdots V_{\sigma_{3n}}(u_{3n}), \tag{4.51}$$

and

$$a_{3n}^{(i)}(\sigma_1, \sigma_2, \ldots, \sigma_{3n}) = Z_i^{-1} \mathrm{Tr}_{\pi_i \rho_i(t)} \Big(e^{2\pi i \boldsymbol{P} h} y_{\sigma_1} y_{\sigma_2} \cdots y_{\sigma_{3n}} \Big). \tag{4.52}$$

Note that these coefficients vanish unless the "total charge" of the product of vertex operators in (4.51)

$$\boldsymbol{e}_{\sigma_1} + \boldsymbol{e}_{\sigma_2} + \cdots + \boldsymbol{e}_{\sigma_{3n}} = 0, \tag{4.53}$$

is zero. The spectral parameter t is absorbed in the coefficients $a_{3n}^{(i)} \sim O(t^n)$ so that (4.50) is a power series in t.

A remarkable feature of the definitions (4.49) is that the coefficients (4.52) are completely determined by the commutation relations (4.40), (4.41), (4.45) and the cyclic property of the trace, so the specific choice of the representations π_i and $\overline{\pi}_i$ is not significant as long as the above convergence property is maintained. The operator coefficients in the power series (4.50) can be expressed in terms of basic nonlocal IM (4.33). For example, calculating the first nontrivial coefficient in (4.50) one gets

$$\begin{aligned}
\mathbf{A}_i(t) &= 1 + \frac{q^{\frac{5}{2}}(q z_i \mathbf{G}_1 - \overline{\mathbf{G}}_1)}{(q^2-1)(z_j - q^2 z_i)(z_k - q^2 z_i)}\, t + O(t^2), \\
\overline{\mathbf{A}}_i(t) &= 1 - \frac{q^{\frac{5}{2}}(z_i \mathbf{G}_1 - q\overline{\mathbf{G}}_1)}{(q^2-1)(q^2 z_j - z_i)(q^2 z_k - z_i)}\, t + O(t^2),
\end{aligned} \tag{4.54}$$

where z_i, $i = 1,2,3$, defined in (3.31). For further references introduce an alternative set of nonlocal IM, $\mathbf{H}_i^{(n)}$, $\overline{\mathbf{H}}_i^{(n)}$, $i = 1,2,3$, $n = 1,2\ldots\infty$ defined as coefficients in the expansions

$$\log \mathbf{A}_j(t) = -\sum_{n=1}^{\infty} s^n\, \mathbf{H}_n^{(j)}, \qquad \log \overline{\mathbf{A}}_j(t) = -\sum_{n=1}^{\infty} s^n\, \overline{\mathbf{H}}_n^{(j)}, \tag{4.55}$$

where

$$s = i\,(g^{-1}\Gamma(1-g))^3\, t. \tag{4.56}$$

The operators $\mathbf{A}_i(t)$ and $\overline{\mathbf{A}}_i(t)$ simplify considerably when acting in the space $\mathcal{F}_{\boldsymbol{p}}$ with values of $2\boldsymbol{p}$ belonging to the weight lattice of the algebra sl_3

$$2\boldsymbol{p} = n_1\boldsymbol{w}_1 + n_2\boldsymbol{w}_2, \qquad n_1, n_2 \in \mathbb{Z}. \tag{4.57}$$

In this case the coefficients (4.52) can be written as

$$a_{3n}(\sigma_1, \sigma_2, \ldots, \sigma_{3n}) = \mathrm{Tr}_u\Big(u_{\sigma_1}u_{\sigma_2}\cdots u_{\sigma_{3n}}\Big), \tag{4.58}$$

where u_1, u_2 are generators of the Weyl algebra

$$u_1u_2 = q u_2 u_1, \tag{4.59}$$

with the trace defined as [51]

$$\mathrm{Tr}_u(u_1^n u_2^m) = \delta_{n,0}\delta_{m,0}, \qquad n, m \in \mathbb{Z} \tag{4.60}$$

and

$$u_3 = \frac{q^{\frac{1}{2}}\, t}{(q-q^{-1})^3}\, u_2^{-1}u_1^{-1}\ . \tag{4.61}$$

Obviously, the expression (4.58) is invariant with respect to the cyclic automorphism τ in (4.9) therefore all three operators $\mathbf{A}_i(t)$, $i = 1,2,3$ coincide. With the help of (4.50), (4.58) they can be written as

$$\mathbf{A}_i(t)\Big|_{2\boldsymbol{p}=(4.57)} = \mathrm{Tr}_u\Big[\mathcal{P}\exp\int_0^{2\pi}\Big(u_1V_1(v)+u_2V_2(v)+u_3V_3(v)\Big)dv\Big], \qquad i = 1,2,3. \tag{4.62}$$

Exactly the same expression but with u_1 and u_2 interchanged (which is equivalent to the replacement $q \to q^{-1}$ in (4.59)) holds for the operators $\overline{\mathbf{A}}_i(t)$.

The $\mathbf{Q}$-operators (the CFT analogues of Baxter's Q-matrix for $U_q(\widehat{sl}(3)$ related lattice models) are defined as

$$\mathbf{Q}_i(t) = t^{\frac{\boldsymbol{w}_i \boldsymbol{P}}{g}}\ \mathbf{A}_i(t), \qquad \overline{\mathbf{Q}}_i(t) = t^{-\frac{\boldsymbol{w}_i \boldsymbol{P}}{g}}\ \overline{\mathbf{A}}_i(t), \tag{4.63}$$

where $\boldsymbol{w}_i$, $i = 1,2,3$, are defined in (2.34).

Evidently, the operators $\mathbf{Q}_i(t)$ and $\overline{\mathbf{Q}}_i(t)$ with different spectral parameters commute among themselves and with all the operators $\mathbf{T}_\mu(t)$ and $\overline{\mathbf{T}}_\mu(t)$ constructed above. They also commute with all the local IM defined in (2.8)

$$[\mathbf{Q}_i(t), \mathbf{I}_k] = 0, \qquad [\overline{\mathbf{Q}}_i(t), \mathbf{I}_k] = 0. \tag{4.64}$$

Indeed, the proof of the similar statement (4.23) for the $\mathbf{T}$-operators given above does not depend on a particular choice of the representation of $\mathcal{B}_+$ and relies only on the commutation relations (4.40), (4.41) and the cyclic property of the trace. Therefore this proof applies to (4.64) as well.

As is known [52–54] the local IM $\mathbf{I}_k$ do not change under the transformation $g \to g^{-1}$, provided one makes simultaneous substitution $\boldsymbol{\phi}(u) \to g^{-1}\boldsymbol{\phi}(u)$

$$\mathbf{I}_k\{g, \boldsymbol{\phi}(u)\} = \mathbf{I}_k\{g^{-1}, g^{-1}\boldsymbol{\phi}(u)\}. \tag{4.65}$$

Obviously, the nonlocal IM $\mathbf{G}_n$ and $\overline{\mathbf{G}}_n$ (as well as $\mathbf{H}_i^{(n)}$ and $\overline{\mathbf{H}}_i^{(n)}$) do change and there exists an infinite set of *dual nonlocal IM* [3] obtained from (2.8) by the above substitution. For instance,

$$\widetilde{\mathbf{G}}_n = z_1^{\frac{1}{g}}\ \widetilde{J}^{(n)}_{312} + z_2^{\frac{1}{g}}\ \widetilde{J}^{(n)}_{123} + z_3^{\frac{1}{g}}\ \widetilde{J}^{(n)}_{231}, \tag{4.66}$$

$$\begin{aligned} \widetilde{J}^{(n)}_{ijk} \quad = \quad & \int_{2\pi \geq u_1 \geq u_2 \ldots \geq u_{3n}} \Big(\widetilde{V}_i(u_1)\widetilde{V}_j(u_2)\widetilde{V}_k(u_3)\ \widetilde{V}_i(u_4)\widetilde{V}_j(u_5)\widetilde{V}_k(u_6)\cdots \\ & \cdots \widetilde{V}_i(u_{3n-2})\widetilde{V}_j(u_{3n-1})\widetilde{V}_k(u_{3n})\Big)\ du_1\, du_2 \ldots du_{3n}, \end{aligned} \tag{4.67}$$

where (i, j, k) is a permutation of $(1, 2, 3)$ and

$$\widetilde{V}_i(u) = \widetilde{q}^{\frac{1}{2}} e^{-2\boldsymbol{e}_i\boldsymbol{\phi}(u)/g}, \qquad \widetilde{q} = e^{i\pi/g}. \tag{4.68}$$

Similarly, define

$$\widetilde{\overline{\mathbf{G}}}_n\{g, \boldsymbol{\phi}(u)\} = \overline{\mathbf{G}}_n\{g^{-1}, g^{-1}\boldsymbol{\phi}(u)\}, \tag{4.69}$$

and

$$\widetilde{\mathbf{H}}_i^{(n)}\{g, \boldsymbol{\phi}(u)\} = \mathbf{H}_i^{(i)}\{g^{-1}, g^{-1}\boldsymbol{\phi}(u)\}, \qquad \widetilde{\overline{\mathbf{H}}}_i^{(n)}\{g, \boldsymbol{\phi}(u)\} = \overline{\mathbf{H}}_i^{(n)}\{g^{-1}, g^{-1}\boldsymbol{\phi}(u)\}. \tag{4.70}$$

For $g < 3/2$ the above definitions require analytic continuation in g as described in [3].

4.4 Vacuum eigenvalues

The Fock space (3.25) naturally splits into level subspaces $\mathcal{F}_{\boldsymbol{p}}^{(\ell)}$, $\ell = 0, 1, 2, \ldots$, defined as

$$\mathcal{F}_{\boldsymbol{p}} = \sum_{\ell=0}^{\infty} \mathcal{F}_{\boldsymbol{p}}^{(\ell)}, \qquad L_0\mathcal{F}_{\boldsymbol{p}}^{(\ell)} = (\Delta_2 + \ell)\mathcal{F}_{\boldsymbol{p}}^{(\ell)}. \tag{4.71}$$

All the IM (both local and nonlocal) act invariantly in each level subspace $\mathcal{F}_{\boldsymbol{p}}^{(\ell)}$. In particular the vacuum state is an eigenstate of all these operators,

$$\mathbf{I}_k\,|\boldsymbol{p}\rangle = I_k^{(vac)}\,|\boldsymbol{p}\rangle, \qquad \mathbf{G}_n\,|\boldsymbol{p}\rangle = G_n^{(vac)}\,|\boldsymbol{p}\rangle, \qquad \overline{\mathbf{G}}_n\,|\boldsymbol{p}\rangle = \overline{G}_n^{(vac)}\,|\boldsymbol{p}\rangle. \tag{4.72}$$

Below we list a few vacuum eigenvalues $I_k^{(vac)}$ which readily follow from (2.16)

$$\begin{aligned} I_1^{(vac)} &= \Delta_2 - \frac{c}{24}\,, \\ I_2^{(vac)} &= \Delta_3\,, \\ I_4^{(vac)} &= (\Delta_2 - \frac{c+6}{24})\,\Delta_3\,, \\ I_5^{(vac)} &= \Delta_2^3 + \frac{4}{3}\Delta_3^2 - \frac{c+8}{8}\Delta_2^2 + \frac{(c+15)(c+2)}{192}\Delta_2 - \frac{c(c+23)(7c+30)}{96768}. \end{aligned} \tag{4.73}$$

Using (4.33) one can express the vacuum eigenvalues of the nonlocal IM as Coulomb-type integrals in a circle. In general they cannot be evaluated in a closed form. Even the simplest eigenvalues $G_1^{(vac)}$ and $\overline{G}_1^{(vac)}$ reduce to the generalised hypergeometric integral

$$\begin{aligned} J(x,y) &= \int_0^1 \frac{d\omega_1}{\omega_1}\int_0^1 \frac{d\omega_2}{\omega_2}\frac{\omega_1^{x+g}\omega_2^{y+g}}{(1-\omega_1)^g(1-\omega_2)^g(1-\omega_1\omega_2)^g} \\ &= \frac{\Gamma(g+x)\Gamma(g+y)\Gamma^2(1-g)}{\Gamma(1+x)\Gamma(1+y)}\;{}_3F_2\left(\begin{matrix} g,\ g+x,\ g+y \\ 1+x,\ 1+y\end{matrix}\,;1\right). \end{aligned} \tag{4.74}$$

For example,

$$G_1^{(vac)} = z_2\int_0^{2\pi} du_3 \int_{u_3}^{u_3+2\pi} du_1 \int_{u_3}^{u_1} du_2 \frac{e^{-2i\boldsymbol{p}(\boldsymbol{e}_1u_1+\boldsymbol{e}_2u_2+\boldsymbol{e}_3u_3)}}{\Big[(2\sin\frac{u_1-u_2}{2})(2\sin\frac{u_1-u_3}{2})(2\sin\frac{u_2-u_3}{2})\Big]^g}, \tag{4.75}$$

can be expressed through (4.74) in the following alternative forms

$$\begin{aligned} G_1^{(vac)} &= -2\pi q^{-\frac{3}{2}}\Big[q(q^2z_1-z_2)J_{32} - (q^2z_2-z_3)J_{12}\Big], \\ &= -2\pi q^{-\frac{3}{2}}\Big[q(q^2z_2-z_3)J_{13} - (q^2z_3-z_1)J_{23}\Big], \\ &= -2\pi q^{-\frac{3}{2}}\Big[q(q^2z_3-z_1)J_{21} - (q^2z_1-z_2)J_{31}\Big], \end{aligned} \tag{4.76}$$

where

$$J_{ij} = J(2\boldsymbol{e}_i\boldsymbol{p}, -2\boldsymbol{e}_j\boldsymbol{p}), \tag{4.77}$$

with $J(x,y)$ defined in (4.74) and z_1, z_2 and z_3 denote the vacuum eigenvalues of the corresponding operators in (3.31). The vacuum eigenvalue of $\overline{G}_1^{(vac)}$ is obtained from $G_1^{(vac)}$ by the negation of the overall sign in (4.75), (4.76) and the replacement

$$z_1 \to z_3^{-1}, \qquad z_2 \to z_2^{-1}, \qquad z_3 \to z_1^{-1}, \qquad \boldsymbol{e}_1 \leftrightarrow \boldsymbol{e}_3, \qquad \boldsymbol{e}_2 \to \boldsymbol{e}_2, \tag{4.78}$$

induced by the automorphism σ_{13},

$$\overline{G}_1^{(vac)} = 2\pi q^{-\frac{3}{2}}\Big[q(q^2 z_3^{-1} - z_2^{-1})J_{12} - (q^2 z_2^{-1} - z_1^{-1})J_{32}\Big],$$

$$= 2\pi q^{-\frac{3}{2}}\Big[q(q^2 z_2^{-1} - z_1^{-1})J_{31} - (q^2 z_1^{-1} - z_3^{-1})J_{21}\Big], \tag{4.79}$$

$$= 2\pi q^{-\frac{3}{2}}\Big[q(q^2 z_1^{-1} - z_3^{-1})J_{23} - (q^2 z_3^{-1} - z_2^{-1})J_{13}\Big]. \tag{4.80}$$

Using (4.54), (4.76) and (4.79) one can obtain the vacuum eigenvalues of the nonlocal IM $\mathbf{H}_i^{(1)}$ and $\overline{\mathbf{H}}_i^{(1)}$ defined in (4.55),

$$H_1^{(i)(vac)} = -\frac{\pi\, g^3\, J_{kj}}{\Gamma^3(1-g)\,\sin(\pi g)}, \qquad \overline{H}_1^{(i)(vac)} = \frac{\pi\, g^3\, J_{jk}}{\Gamma^3(1-g)\,\sin(\pi g)}, \tag{4.81}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

It will be useful to consider the special choice of vacuum parameter $\boldsymbol{p} = (p,0)$. In this case, we have

$$(\boldsymbol{e}_1\boldsymbol{p}) = \frac{p}{2}, \qquad (\boldsymbol{e}_2\boldsymbol{p}) = -p, \qquad (\boldsymbol{e}_3\boldsymbol{p}) = \frac{p}{2}, \tag{4.82}$$

and

$$z_1 = z_3^{-1} = e^{2\pi i p}, \qquad z_2 = 1. \tag{4.83}$$

Both $G_1^{(vac)}$ and $H_1^{(1)(vac)}$ simplify considerably in this case,

$$G_1^{(vac)}(p,0) = \frac{4\pi^3\Gamma^2(1-g)\Gamma(1-\frac{3g}{2})}{\Gamma(1-\frac{g}{2})\Gamma(1-g+p)\Gamma(1-g-p)\Gamma(1-g/2+p)\Gamma(1-g/2-p)}, \tag{4.84}$$

$$H_1^{(1)(vac)}(p,0) = -\frac{\pi}{2\sin(\pi g)}\frac{g^3\Gamma\left(1-\frac{3g}{2}\right)}{\Gamma\left(1-\frac{g}{2}\right)\Gamma\left(1-g\right)}\frac{\Gamma(g+p)\Gamma(g/2+p)}{\Gamma(1-g+p)\Gamma(1-g/2+p)}, \tag{4.85}$$

where we have used eq. (4.4(5)) of [55] to simplify the hypergeometric functions J_{ij}.

5 Functional Relations

The **T**- and **Q**-operators introduced in the previous Section enjoy a variety of important functional relations (FR). In fact, many such functional relations have been previously obtained in the context of the solvable lattice models associated with $U_q(\widehat{sl}(3))$ algebra (see, e.g. [12–14, 16, 17]). It turns out that the explicit construction of the **Q**-operators given in the previous Section, (which defines them as special quantum transfer matrices associated with the q-oscillator algebra), can be used to considerably simplify the derivation of these FR. In fact, we found it rather remarkable that all known FR relevant here (as well as some new ones) can be obtained as elementary corollaries of just two basic relations which express the operators $\mathbf{T}_\mu(t)$, (4.21), and $\overline{\mathbf{T}}_\mu(t)$, (4.36), corresponding to finite dimensional representations π_μ of $U_q(gl(3))$ (i.e. those with integral non-negative weights $\mu_\alpha = \mu_1 - \mu_2 \geq 0$ and $\mu_\beta = \mu_2 - \mu_3 \geq 0$) in terms of the operators $\mathbf{Q}_i(t)$ and $\overline{\mathbf{Q}}_i(t)$

$$\mathbf{T}_{(\mu_1,\mu_2,\mu_3)}(t) \;=\; z_0^{-1} \det \left\| \, \mathbf{Q}_i(t\, q^{+2\mu'_j}) \right\|_{i,j=1,2,3} \tag{5.1}$$

$$-\overline{\mathbf{T}}_{(\mu_1,\mu_2,\mu_3)}(t) \;=\; z_0^{-1} \det \left\| \, \overline{\mathbf{Q}}_i(t\, q^{-2\mu'_j}) \right\|_{i,j=1,2,3} \tag{5.2}$$

where $\mu'_1 = \mu_1 + 1$, $\mu'_2 = \mu_2$, $\mu'_3 = \mu_3 - 1$, and

$$z_0 = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3), \tag{5.3}$$

with z_1, z_2 and z_3 defined in (3.31). The operators $\mathbf{Q}_i(t)$, $\overline{\mathbf{Q}}_i(t)$ are not functionally independent. They satisfy the relations

$$\begin{aligned} c_i\, \overline{\mathbf{Q}}_i(t) &= \mathbf{Q}_j(tq)\mathbf{Q}_k(tq^{-1}) - \mathbf{Q}_k(tq)\mathbf{Q}_j(tq^{-1}), && (5.4)\\ -c_i\, \mathbf{Q}_i(t) &= \overline{\mathbf{Q}}_j(tq)\overline{\mathbf{Q}}_k(tq^{-1}) - \overline{\mathbf{Q}}_k(tq)\overline{\mathbf{Q}}_j(tq^{-1}), && (5.5) \end{aligned}$$

where $(i,j,k) = \mathrm{cycle}(1,2,3)$, and

$$c_i = z_i^{\frac{1}{2}}(z_j - z_k). \tag{5.6}$$

The proof of these relations is given in Appendix B (see also the discussion at the end of this Section). Below we present some simple corollaries of (5.1), (5.2), (5.4), and (5.5).

The operators $\mathbb{T}(t)$ and $\overline{\mathbb{T}}(t)$ defined in (4.31) corresponding to the three-dimensional fundamental representations of $U_q(gl(3))$ satisfy the following relations with the $\mathbf{Q}$-operators

$$\begin{aligned} \mathbf{Q}_i(tq^3) - \mathbb{T}(tq^{\frac{1}{2}})\, \mathbf{Q}_i(tq) + \overline{\mathbb{T}}(tq^{-\frac{1}{2}})\, \mathbf{Q}_i(tq^{-1}) - \mathbf{Q}_i(tq^{-3}) &= 0, && (5.7)\\ \overline{\mathbf{Q}}_i(tq^3) - \overline{\mathbb{T}}(tq^{\frac{1}{2}})\, \overline{\mathbf{Q}}_i(tq) + \mathbb{T}(tq^{-\frac{1}{2}})\, \overline{\mathbf{Q}}_i(tq^{-1}) - \overline{\mathbf{Q}}_i(tq^{-3}) &= 0, && (5.8) \end{aligned}$$

where $i = 1,2,3$, These relations are obviously the $U_q(\widehat{sl}(3))$-analogues [12] of Baxter's famous T-Q relation [11]. Given $\mathbb{T}(t)$ and $\overline{\mathbb{T}}(t)$ each of the equations (5.7) is a third order difference equation for $\mathbf{Q}$'s which should have three linearly independent solutions. Since $\mathbb{T}(t)$ and $\overline{\mathbb{T}}(t)$ are single-valued functions of t, (i.e. periodic functions of $\log t$) the operators $\mathbf{Q}_i(t)$ and $\overline{\mathbf{Q}}_i(t)$ can be interpreted as the "Bloch wave" solutions of (5.7) which satisfy the "quantum Wronskian" condition

$$z_0 = \det \left\| \mathbf{Q}_i(tq^{-2j}) \right\|_{i,j=1,2,3} = -\det \left\| \overline{\mathbf{Q}}_i(tq^{2j}) \right\|_{i,j=1,2,3} \tag{5.9}$$

which is just a particular case of (5.1), (5.2) for $\mu_i = 0$.

The equations (5.7) and (5.8) imply more familiar expressions

$$\mathbb{T}(t)=\frac{\mathbf{Q}(t\,q^{\frac{5}{2}})}{\mathbf{Q}(t\,q^{\frac{1}{2}})}+\frac{\mathbf{Q}(t\,q^{-\frac{3}{2}})\,\overline{\mathbf{Q}}\,(t\,q^{\frac{3}{2}})}{\mathbf{Q}(t\,q^{\frac{1}{2}})\,\overline{\mathbf{Q}}\,(t\,q^{-\frac{1}{2}})}+\frac{\overline{\mathbf{Q}}\,(t\,q^{-\frac{5}{2}})}{\overline{\mathbf{Q}}\,(t\,q^{-\frac{1}{2}})}, \tag{5.10}$$

$$\overline{\mathbb{T}}(t)=\frac{\overline{\mathbf{Q}}(t\,q^{\frac{5}{2}})}{\overline{\mathbf{Q}}(t\,q^{\frac{1}{2}})}+\frac{\overline{\mathbf{Q}}(t\,q^{-\frac{3}{2}})\,\mathbf{Q}\,(t\,q^{\frac{3}{2}})}{\overline{\mathbf{Q}}(t\,q^{\frac{1}{2}})\,\mathbf{Q}\,(t\,q^{-\frac{1}{2}})}+\frac{\mathbf{Q}\,(t\,q^{-\frac{5}{2}})}{\mathbf{Q}\,(t\,q^{-\frac{1}{2}})}, \tag{5.11}$$

where $\{\mathbf{Q}(t),\overline{\mathbf{Q}}(t)\}$ is any of the six pairs $\{\mathbf{Q}_i(t),\overline{\mathbf{Q}}_j(t)\}$ with $i\neq j$. More general expressions [16] of this type for the $\mathbf{T}$-operators with arbitrary weight μ can be obtained, for example, from (5.10), (5.11) and the relation (5.16) given below.

Further, using (5.4) and (5.5) in (5.1) and (5.2) one can easily deduce that

$$\mathbb{T}_m(t) \;=\; z_0^{-1}\sum_{i=1}^{3} c_i\,\mathbf{Q}_i(tq^{m+\frac{3}{2}})\overline{\mathbf{Q}}_i(tq^{-m-\frac{3}{2}}), \tag{5.12}$$

$$\overline{\mathbb{T}}_m(t) \;=\; z_0^{-1}\sum_{i=1}^{3} c_i\,\mathbf{Q}_i(tq^{-m-\frac{3}{2}})\overline{\mathbf{Q}}_i(tq^{m+\frac{3}{2}}), \tag{5.13}$$

Using these two results, the well known *fusion relations*, [17], can be deduced,

$$\mathbb{T}_m(tq)\mathbb{T}_m(tq^{-1})=\overline{\mathbb{T}}_m(t)+\mathbb{T}_{m-1}(t)\mathbb{T}_{m+1}(t), \tag{5.14}$$

$$\overline{\mathbb{T}}_m(tq)\overline{\mathbb{T}}_m(tq^{-1})=\mathbb{T}_m(t)+\overline{\mathbb{T}}_{m-1}(t)\overline{\mathbb{T}}_{m+1}(t), \tag{5.15}$$

Note that the formulae (5.1) and (5.2) strongly resemble the first Weyl formula for characters of $gl(3)$. Note also, that the corresponding analogue of the second Weyl formula for the transfer matrices (sometimes called the quantum Jacobi-Trudi identity [17]) has been known for some time [16]

$$\mathbf{T}_{(\mu_1,\mu_2,0)}(t)=\det\left\|\,\boldsymbol{\tau}^{(\mu_i^T-i+j)}(t\,q^{2(j-1)})\,\right\|_{1\le i,j\le\mu_1} \tag{5.16}$$

where

$$\boldsymbol{\tau}^{(0)}(t)=\boldsymbol{\tau}^{(3)}(t)\equiv 1,\qquad \boldsymbol{\tau}^{(1)}(t)=\mathbf{T}_{(1,0,0)}(t),\qquad \boldsymbol{\tau}^{(2)}(t)=\mathbf{T}_{(1,1,0)}(t), \tag{5.17}$$

and $\boldsymbol{\tau}^{(k)}(t)\equiv 0$, for $k<0$ and $k>3$. The integers μ_i^T, $i=1,\ldots,\mu_1$, read

$$\mu_i^T=2,\quad 1\le i\le\mu_2;\qquad \mu_i^T=1,\quad \mu_2<i\le\mu_1.$$

Again, the formula (5.16) easily follows from (5.1). Also, a similar expression for $\overline{\mathbf{T}}_\mu(t)$ is obtained from (5.16) by the replacement $q\to q^{-1}$.

The proof of the basic relations (5.1) and (5.2) given in Appendix B is a straightforward (but rather more technically complicated) generalisation of the corresponding

$U_q(\widehat{sl}(2))$ results of [4]. We consider more general **T**-operators which correspond to the infinite dimensional highest weight representations of $U_q(gl(3))$. These new **T**-operators are defined by the same formulae as (4.21)

$$\mathbf{T}^+_\mu(t) = \mathrm{Tr}_{\pi^+_\mu}\,[e^{i\pi\mathbf{P}\,\mathbf{h}}\mathbf{L}^+_\mu(t)], \qquad \mathbf{L}^+_\mu(t) = \pi^+_\mu(t)[\mathcal{L}], \tag{5.18}$$

except the trace is now taken over the infinite dimensional representations π^+_μ of $U_q(gl(3))$ with arbitrary highest weights $\mu = (\mu_1, \mu_2, \mu_3)$ the highest weight vector $|\,0\rangle$ defined by (2.41). Similar modifications to (4.36) allow us to define the operators $\overline{\mathbf{T}}^+_\mu(t)$ related to the second evaluation map (4.18). Then, by considering the tensor product of three q-oscillator algebras corresponding to the product of three $\mathbf{Q}_i(t)$ operators, we show that

$$\mathbf{T}^+_{\mu_1,\mu_2,\mu_3}(t) = z_0^{-1}\mathbf{Q}_1(tq^{2\mu_1+2})\mathbf{Q}_2(tq^{2\mu_2})\mathbf{Q}_3(tq^{2\mu_3-2}), \tag{5.19}$$

and similarly

$$\overline{\mathbf{T}}^+_{\mu_1,\mu_2,\mu_3}(t) = -z_0^{-1}\overline{\mathbf{Q}}_1(tq^{-2\mu_1-2})\overline{\mathbf{Q}}_2(tq^{-2\mu_2})\overline{\mathbf{Q}}_3(q^{-2\mu_3+2}). \tag{5.20}$$

Then, we make use of the Bernstein-Gel'fand-Gel'fand (BGG) resolution of the finite-dimensional modules. The BGG result allows one to express the finite dimensional highest weight modules in terms of a direct sum of infinite dimensional highest weight modules. This implies that the **T**-operators for a finite dimensional module can be written in terms of (5.19), (5.20) as

$$\mathbf{T}_\mu(t) = \sum_{\sigma\in S_3}(-1)^{l(\sigma)}\mathbf{T}^+_{\sigma(\mu+\rho)-\rho}(t), \tag{5.21}$$

where $\rho = (1, 0, -1)$, the summation is over all permutations $\sigma \in S_3$ of three elements, and $l(\sigma)$ is the parity of the permutation. There an analogous result for the operators $\overline{\mathbf{T}}_{\mu_1,\mu_2,\mu_3}(t)$. The determinant formulas (5.1) and (5.2) follow immediately.

6 Conjectures: Exact Asymptotic Expansions

6.1 Assumptions

We will make the following assumptions

i) Analyticity. The operators $\mathbf{A}_i(t)$ and $\overline{\mathbf{A}}_i(t)$ (and hence all the **T**-operators introduced above) are entire functions of the complex variable t in the sense that all their matrix elements and eigenvalues are entire functions of this variable. Their leading asymptotics at large t are given by

$$\log \mathbf{A}_i(t) \sim M\,(+it)^{\frac{1}{3(1-g)}}, \qquad \log \overline{\mathbf{A}}_i(t) \sim M\,(-it)^{\frac{1}{3(1-g)}}, \qquad |t| \to \infty \tag{6.1}$$

where M is some c-number constant depending on the central charge c only.

ii) Location of zeroes. For real $\boldsymbol{p}$ and large t the zeros of the eigenvalues of $A_i(t)$ and $\overline{A}_i(t)$ accumulate along the positive imaginary and negative imaginary axes of t respectively. The asymptotics (6.1) are valid everywhere except in the vicinity of the lines of zeroes (i.e. when $|\arg(it)| < \pi$ for $\mathbf{A}_i(t)$ and when $|\arg(-it)| < \pi$ for $\overline{\mathbf{A}}_i(t)$).

As is known the zeroes of the eigenvalues of $\mathbf{Q}$-operators satisfy the Bethe Ansatz equations. In our case there are six sets of these equations involving different pairs of eigenvalues $\{A_i(t), \overline{A}_j(t)\}$ with $i \neq j$. To simplify the notation let us choose a particular pair of eigenvalues, say $\{A_1(t), \overline{A}_3(t)\}$, and denote by $\{t_k\}|_{k=1}^{\infty}$ and $\{\overline{t}_k\}|_{k=1}^{\infty}$ their sets of zeroes

$$A_1(t_k) = 0, \qquad \overline{A}_3(\overline{t}_k) = 0, \tag{6.2}$$

ordered as $|t_1| \leq |t_2| \leq \ldots$ and similarly $\overline{t}_k$. The equations (5.10), (5.11) (together with the standard arguments based on the analyticity of $\mathbf{T}$-operators) imply the following Bethe-Ansatz equations

$$\begin{aligned} \frac{1}{2\pi i} \log \left[\frac{A_1(t_k q^2)}{A_1(t_k q^{-2})} \frac{\overline{A}_3(t_k q^{-1})}{\overline{A}_3(t_k q)} \right] &= -2\boldsymbol{e}_3 \boldsymbol{p} - n_k^{(1)} + \tfrac{1}{2}, \\ \frac{1}{2\pi i} \log \left[\frac{\overline{A}_3(\overline{t}_k q^2)}{\overline{A}_3(\overline{t}_k q^{-2})} \frac{A_1(\overline{t}_k q^{-1})}{A_1(\overline{t}_k q)} \right] &= -2\boldsymbol{e}_1 \boldsymbol{p} - n_k^{(3)} + \tfrac{1}{2}, \end{aligned} \tag{6.3}$$

where $n_k^{(1)}, n_k^{(3)} \in \mathbb{Z}$. Different eigenvalues of the operators $\mathbf{A}_1(t)$ and $\overline{\mathbf{A}}_3(t)$ correspond to different sets of integer phases $\{n_k^{(1)}, n_k^{(3)}\}_{k=1}^{\infty}$.

The following assumption concerns the structure of the vacuum eigenvalues $A_1^{(vac)}(t)$ and $\overline{A}_3^{(vac)}(t)$ of operators $\mathbf{A}_1(t)$ and $\overline{\mathbf{A}}_3(t)$ acting in $\mathcal{F}_{\mathbf{p}}$.

iii). For real values of $2(\boldsymbol{w}_1\boldsymbol{p}) < g$ and $2(\boldsymbol{w}_3\boldsymbol{p}) > -g$ all zeroes of $A_1(t)$ ($\overline{A}_3(t)$) located on the positive (negative) imaginary axis of t and the integer phases in (6.3) run over consecutive positive integers

$$n_k^{(1)} = n_k^{(3)} = k, \qquad k = 1, 2, \ldots, \infty. \tag{6.4}$$

6.2 Results

Following [3] consider the following operator valued function

$$\boldsymbol{\Psi}_i(\nu) = \frac{\Gamma(1-g)^{-i\nu(1+\xi)}}{\Gamma(i\nu\xi/3)\Gamma(-i\nu(\xi+1)/3)\Gamma((-1+i\nu)/3)\Gamma((1+i\nu)/3)} \int_{-i\infty}^{0} \frac{dt}{t} (it)^{-i\nu(1+\xi)/3}\, \mathbf{A}_i(t)\,, \tag{6.5}$$

where the integration contour goes along negative imaginary axis and

$$\xi = \frac{g}{1-g}. \tag{6.6}$$

This integral converges for $3(g-1) < \Im m\,\nu < -1$, however the definition of $\mathbf{\Psi}_j(\nu)$ can be extended to the whole complex plane of ν by analytic continuation. For example, using the product representation which holds for any eigenvalue $A_j(t)$ of $\mathbf{A}_j(t)$

$$A_j(t) = \prod_{n=1}^{\infty}\Big(1 - \frac{t}{t_n^{(j)}}\Big), \tag{6.7}$$

one can write the corresponding eigenvalue of $\mathbf{\Psi}_j(\nu)$ as a Dirichlet series

$$\Psi_j(\nu) = \frac{\Big(\Gamma(1-g)\Big)^{-i\nu(1+\xi)}\Gamma(i\nu(\xi+1)/3)}{\Gamma(i\nu\xi/3)\Gamma((-1+i\nu)/3)\Gamma((1+i\nu)/3)}\sum_{n=1}^{\infty}(-it_n^{(j)})^{-i\nu(1+\xi)/3}. \tag{6.8}$$

For $0 < g < 2/3$ this series converges absolutely for $\Im m\,\nu < -1$ and can be analytically continued to the whole complex ν-plane by means of the standard technique [56]. We expect that the function $\mathbf{\Psi}(\nu)$ thus defined enjoys the following analytic properties

Conjecture 1 *The function $\mathbf{\Psi}(\nu)$ is an entire function of the variable ν.*

Converting the integral transform in (6.5) one obtains

$$\begin{aligned}\log \mathbf{A}_i(t) \quad &= \quad \frac{1}{2\pi i}\int_{C_\nu}\frac{d\nu}{\nu}\,\Gamma(i\nu\xi/3)\Gamma(1-i\nu(\xi+1)/3)\Gamma((-1+i\nu)/3)\,\times \\ &\qquad \Gamma((1+i\nu)/3)\Big(\Gamma(1-g)\Big)^{i\nu(1+\xi)}(it)^{i\nu(1+\xi)/3}\ \mathbf{\Psi}(\nu),\end{aligned} \tag{6.9}$$

where the integration contour C_ν goes along the line $\Im m\,\nu = -1-\varepsilon$ with arbitrary small positive ε.

The values of $\mathbf{\Psi}_j(\nu)$ at special points on the imaginary axis of ν (where the Gamma-functions display poles) are of particular interest. For example, using (6.8), (6.7) and the definition of the nonlocal IM $\mathbf{H}_n^{(j)}$ in (4.55) it is easy to see that

$$\mathbf{\Psi}_j(-3in(1-g)) = \frac{(-1)^n\, n!\, g^{-3n}}{\Gamma(ng)\Gamma(-1/3+n(1-g))\Gamma(1/3+n(1-g))}\ \mathbf{H}_n^{(j)}. \tag{6.10}$$

The other special values of $\mathbf{\Psi}_j(\nu)$ of the imaginary axis are determined by the following

Conjecture 2 *The function $\mathbf{\Psi}(\nu)$ has the following special values on the positive imaginary axis*

$$\mathbf{\Psi}_j((2n-1)i) \quad = \quad \frac{1}{6\sqrt{3}\pi}\Big(\frac{2^{\frac{2}{3}}}{3}\Big)^{2n-1}\frac{\Gamma(\frac{1}{6}-\frac{n}{3})}{\Gamma(\frac{1}{2}-n)}\ g^n\ \mathbf{I}_{2n-1}, \qquad n = 0,1,2,\ldots \tag{6.11}$$

$$\mathbf{\Psi}_j(2ni) \quad = \quad \frac{1}{2\sqrt{3}\pi}\Big(\frac{2^{\frac{2}{3}}}{3}\Big)^{2n}\frac{\Gamma(\frac{1}{2}-\frac{n}{3})}{\Gamma(-\frac{1}{2}-n)}\ g^{n+\frac{1}{2}}\ \mathbf{I}_{2n}, \qquad n = 1,2,\ldots \tag{6.12}$$

$$\boldsymbol{\Psi}_j(3in(g^{-1}-1)) = \frac{(-1)^n\, n!\, g^{\frac{3n}{g}+1}\, \widetilde{\mathbf{H}}_n^{(j)}}{\Gamma(ng^{-1})\Gamma(-\frac{1}{3}+n(1-g^{-1}))\Gamma(\frac{1}{3}+n(1-g^{-1}))}\,, \qquad n=1,2,\dots \tag{6.13}$$

$$\boldsymbol{\Psi}_j(0) = \frac{(\boldsymbol{w}_j\boldsymbol{P})}{2\pi\sqrt{3}}, \tag{6.14}$$

where $j=1,2,3$ and $\mathbf{I}_{-1}\equiv\mathbf{I}$ is the identity operator.

Below it will be more convenient to work with the variable s introduced in (4.56) instead of the spectral parameter t and exhibit all arguments of $\mathbf{A}(s)$

$$\mathbf{A}_j\{s,g,\boldsymbol{\phi}(u)\}\equiv\exp\Big(-\sum_{n=1}^{\infty}\mathbf{H}_n^{(j)}s^n\Big). \tag{6.15}$$

The above conjectures allow us to derive the asymptotic expansion of the operators $\mathbf{A}_j(t)$ for large t.

Calculating the integral (6.9) as a formal sum over residues in the upper half-plane $\Im m\,\nu\geq-1$ one has

$$\mathbf{A}_j\{s,g,g\boldsymbol{\phi}(u)\}\simeq\mathbf{C}_j\{g,g\boldsymbol{\phi}(u)\}\,s^{-\frac{\boldsymbol{w}_j\boldsymbol{P}}{g}}\exp\Big\{\sum_{k=-1}^{\infty}B_k\,s^{-k/3(1-g)}\,\mathbf{I}_k\Big\}\mathbf{A}_j\{s^{-\frac{1}{g}},g^{-1},g^{-1}\boldsymbol{\phi}(u)\}, \tag{6.16}$$

where

$$\mathbf{A}_j\{s^{-\frac{1}{g}},g^{-1},g^{-1}\boldsymbol{\phi}(u)\}\equiv\exp\Big(-\sum_{n=1}^{\infty}\widetilde{\mathbf{H}}_n^{(j)}s^{-\frac{n}{g}}\Big), \tag{6.17}$$

and the coefficients B_n read

$$B_{2n-1} = \frac{(-1)^{n+1}\Gamma\Big(\frac{2n-1}{3(1-g)}\Big)\Gamma\Big(\frac{2n-1}{3(1-g^{-1})}\Big)}{3\,(1-g)\,n!\,\Gamma\Big(\frac{2-n}{3}\Big)}\,g^{\frac{n(1+g)-1}{g-1}}, \qquad n=0,1,2,\dots \tag{6.18}$$

$$B_{2n} = \frac{(-1)^{n+1}\Gamma\Big(\frac{2n}{3(1-g)}\Big)\Gamma\Big(\frac{2n}{3(1-g^{-1})}\Big)}{(1-g)\,n!\,\Gamma\Big(-\frac{n}{3}\Big)}\,g^{\frac{n(1+g)}{g-1}+\frac{1}{2}}, \qquad n=1,2,\dots \tag{6.19}$$

An exact form for the operator $\mathbf{C}_j\{g,g\boldsymbol{\phi}(u)\}$ is unknown. However, an asymptotic vacuum eigenvalue in a special case can be calculated (see (C.56)). Note, in particular, that the coefficient M in the leading asymptotics (6.16) is given by

$$M=\frac{\Gamma\left(\frac{g}{3(1-g)}\right)\Gamma\left(\frac{2-3g}{3(1-g)}\right)}{\Gamma\left(\frac{2}{3}\right)}\left(\Gamma(1-g)\right)^{\frac{1}{1-g}}. \tag{6.20}$$

Naturally, considerations similar to those above are valid for the operators $\overline{\mathbf{A}}_j(t)$ as well. The corresponding formulae are obtained from (6.9) simply by a replacement of $\mathbf{A}_j(t)$, $\mathbf{\Psi}_j(\nu)$, $\mathbf{H}_n^{(j)}$ by their "barred" counterparts

$$\mathbf{A}_j(t) \to \overline{\mathbf{A}}_j(t), \qquad \mathbf{\Psi}_j(\nu) \to \overline{\mathbf{\Psi}}_j(\nu), \qquad \mathbf{H}_n^{(j)} \to \overline{\mathbf{H}}_n^{(j)}, \tag{6.21}$$

accompanied by sign changes

$$\boldsymbol{w}_j \to -\boldsymbol{w}_j, \qquad \mathbf{I}_{2n} \to -\mathbf{I}_{2n}, \tag{6.22}$$

where $j = 1, 2, 3$ and $n = 1, 2, \ldots, \infty$ (the sign of the "odd" local IM $\mathbf{I}_{2n-1}$ remains the same). Also, in the integrand of (6.9), the change $t \to -t$ needs to be made.

We can use these asymptotic results for the operators $\mathbf{A}_i(t)$, $\overline{\mathbf{A}}_i(t)$ to calculate an asymptotic expansion for $\mathbb{T}(t)$. Using (5.10), we can write $\mathbb{T}(t)$ in the following form

$$\mathbb{T}(t) = \mathbf{\Lambda}_i(q^{\frac{3}{2}}t) + \mathbf{\Lambda}_i^{-1}(q^{-\frac{1}{2}}t)\overline{\mathbf{\Lambda}}_j(q^{\frac{1}{2}}t) + \overline{\mathbf{\Lambda}}_j^{-1}(q^{-\frac{3}{2}}t), \qquad i \neq j, \tag{6.23}$$

where

$$\mathbf{\Lambda}_j(t) = \frac{\mathbf{Q}_j(qt)}{\mathbf{Q}_j(q^{-1}t)}, \qquad \overline{\mathbf{\Lambda}}_j = \frac{\overline{\mathbf{Q}}_j(qt)}{\overline{\mathbf{Q}}_j(q^{-1}t)}. \tag{6.24}$$

Then, using (6.9) we have

$$\begin{aligned} \log \mathbf{\Lambda}_j &= 2\pi i(\boldsymbol{w}_j \boldsymbol{P}) + \int_{C_\nu} \frac{d\nu}{\nu} \frac{\Gamma(1 - i\nu(\xi+1)/3)\Gamma((-1+i\nu)/3)\Gamma((1+i\nu)/3)}{\Gamma(1 - i\nu\xi/3)} \times \\ & \Big(\Gamma(1-g)\Big)^{i\nu(1+\xi)} (it)^{i\nu(1+\xi)/3}\ \mathbf{\Psi}_j(\nu), \end{aligned} \tag{6.25}$$

which gives via the sums of residues in the upper half-plane $\Im m\, \nu \geq -1$

$$\log \mathbf{\Lambda}_j(t) \simeq i\, m\, \mathbf{I}\, (it)^{\frac{1}{3(1-g)}} + \sum_{k=1}^{\infty}{}' C_k\, (it)^{\frac{-k}{3(1-g)}}\ \mathbf{I}_k, \qquad t \to +\infty \tag{6.26}$$

where

$$m = \frac{2\pi\Gamma(\frac{2}{3} - \frac{\xi}{3})}{\Gamma(1 - \frac{\xi}{3})\Gamma(\frac{2}{3})} \Big(\Gamma(1-g)\Big)^{\frac{1}{1-g}}, \tag{6.27}$$

and

$$C_k = -2i\, g^{k(1+\xi)} \sin\Big(\frac{\pi\xi k}{3}\Big) \Big(\Gamma(1-g)\Big)^{-\frac{k}{1-g}} B_k. \tag{6.28}$$

A similar expression is obtained for $\log \overline{\mathbf{\Lambda}}_i(t)$, using (6.21), (6.22). It is easy then to see that only the second term in (6.24) will contribute in the asymptotic limit, as the other two will be exponentially small. Thus we have

$$\begin{aligned} \log \mathbb{T}(t) &\simeq m t^{\frac{1}{3(1-g)}} \mathbf{I} - 2 \sum_{n=1}^{\infty} C_{2n} \cos\Big(\frac{\pi n}{3}\Big) (t)^{-\frac{2n}{3(1-g)}} \mathbf{I}_{2n} \\ &+ 2i \sum_{n=1}^{\infty} C_{2n-1} \sin\Big(\frac{\pi(2n-1)}{6}\Big) (t)^{-\frac{2n-1}{3(1-g)}} \mathbf{I}_{2n-1}. \end{aligned} \tag{6.29}$$

The corresponding formula for $\overline{\mathbb{T}}(t)$ is obtained from the above one by the replacement (6.22). This result can be regarded as the the quantum version of (2.29). In fact, if we take the limit $\xi \to 0$ (equivalently $q \to 1$), then, using the correspondence (2.20), we recover exactly the classical expression (2.29).[6]

As in [3] the main motivation to the conjectures given above came from the study of the large p asymptotics of the vacuum eigenvalues of the operators $\mathbf{A}_j(t)$ (see Appendix C) as well as consideration of the classical limit $g \to 0$ in the $\mathbf{T}$-operator (6.23). In this way we obtain

$$\Psi_j(\nu)|_{\boldsymbol{p}\to\infty} = \frac{2^{-\frac{2i\nu}{3}}\Big(-\frac{g_2}{3}\Big)^{(1-i\nu)/2}}{\sqrt{3\pi}\Gamma(1/2+i\nu/3)}\,{}_2F_1\Big(-\frac{1}{6}+\frac{i\nu}{6},\frac{1}{6}+\frac{i\nu}{6},\frac{1}{2}+\frac{i\nu}{3},1+\frac{27g_3^2}{4g_2^3}\Big)|_{j-\text{th}}, \quad (6.30)$$

where the subscript $j-$th means that an appropriate branch of the hypergeometric function is chosen such that (6.14) holds, and g_2, g_3 are defined by

$$g_2 = x_1x_2 + x_1x_3 + x_2x_3, \qquad g_3 = -x_1x_2x_3, \quad (6.31)$$

where

$$x_j = (\boldsymbol{w}_j \boldsymbol{P}), \qquad j = 1, 2, 3. \quad (6.32)$$

The expression (6.30) together with (6.9) allows one to derive the asymptotic expansion (6.16) in the limit $\boldsymbol{p} \to \infty$ which provided a basis for the conjectures (6.11), (6.12), (6.13), (6.14).

The formulae (6.16) allow us to obtain asymptotic expansions for the zeroes of the eigenvalues of the operators $\mathbf{A}_j(t)$ and $\overline{\mathbf{A}}_j(t)$. Let us introduce new (rescaled) variables $E_k^{(j)}$, $\overline{E}_k^{(j)}$, $k = 1, 2, \ldots$, (whose meaning will become clear in Section 7.2) to denote the positions of these zeroes

$$E_k^{(j)} = -i\rho\, t_k^{(j)}, \qquad \overline{E}_k^{(j)} = i\rho\, \bar{t}_k^{(j)}, \qquad A_j(t_k^{(j)}) = \overline{A}_j(\bar{t}_k^{(j)}) = 0, \qquad k = 1, 2, \ldots, \infty \quad (6.33)$$

where

$$\rho = \Big[(3/g)^{1-g}\Gamma(1-g)\Big]^3. \quad (6.34)$$

Substituting (6.16) into the Bethe Ansatz equations one obtains

$$N_k^{(j)} \equiv n_k^{(j)} - \frac{1}{2} + \frac{\boldsymbol{w}_j \boldsymbol{p}}{g} \simeq \sum_{n=-1}^{\infty} \beta_n\, I_n\, \Big(E_k^{(j)}\Big)^{-\frac{n}{3(1-g)}} + \sum_{n=1}^{\infty} \gamma_n\, \widetilde{H}_n^{(j)}\, \Big(E_k^{(j)}\Big)^{-\frac{n}{g}}, \quad (6.35)$$

$$\overline{N}_k^{(j)} \equiv \overline{n}_k^{(j)} - \frac{1}{2} - \frac{\boldsymbol{w}_j \boldsymbol{p}}{g} \simeq \sum_{n=-1}^{\infty} (-1)^{n+1}\beta_n\, I_n\, \Big(\overline{E}_k^{(j)}\Big)^{-\frac{n}{3(1-g)}} + \sum_{n=1}^{\infty} \gamma_n\, \widetilde{\overline{H}}_n^{(j)}\, \Big(\overline{E}_k^{(j)}\Big)^{-\frac{n}{g}}, \quad (6.36)$$

[6] Recall the correspondence of variables $t = \lambda^3$.

where

$$\beta_n = -\pi^{-1}\ 3^n\ g^{gn/(1-g)}\ \sin(\pi n/3(1-g))\ B_n, \qquad \gamma_k = \pi^{-1}\ 3^{3(1-g)n/g}\ g^{3n}\ \sin(\pi n/g), \tag{6.37}$$

the coefficients B_n are given in (6.18), (6.19) and I_k, $\widetilde{H}_n^{(j)}$ and $\overline{\widetilde{H}}_n^{(j)}$ are the corresponding eigenvalues of the local IM (2.8) and the dual nonlocal IM, defined in (4.70). Inverting the series (6.35), (6.36) one can express $E_k^{(j)}$ and $E_k^{(j)}$ as asymptotic series

$$E_k^{(j)} \simeq \Upsilon(N_k^{(j)}), \qquad \overline{E}_k^{(j)} \simeq \overline{\Upsilon}(\overline{N}_k^{(j)}), \tag{6.38}$$

where, for example, for $g < 3/5$ one has

$$\Upsilon(x) \simeq \alpha_0\ x^{3-3g}\ \Big(1 + \alpha_2 x^{-2} + \alpha_3 x^{-3} + O(x^{-4}) + O(x^{2-3/g})\Big), \qquad x \to \infty \tag{6.39}$$

$$\overline{\Upsilon}(x) \simeq \alpha_0\ x^{3-3g}\ \Big(1 + \alpha_2 x^{-2} - \alpha_3 x^{-3} + O(x^{-4}) + O(x^{2-3/g})\Big), \qquad x \to \infty \tag{6.40}$$

$$\alpha_0 = (\beta_{-1})^{3(g-1)} = \left[\frac{3\Gamma(\frac{2}{3})\Gamma((\xi+1)/3)}{g\Gamma(\frac{\xi}{3})}\right]^{3(1-g)}, \tag{6.41}$$

$$\alpha_2 = \beta_1\beta_{-1}\ I_1 = \frac{3(1-g)I_1}{\pi(\cot(\pi(\xi+1)/3) - \cot\pi/3)}, \tag{6.42}$$

$$\alpha_3 = \beta_2\beta_{-1}^2\ I_2 = \frac{3\, g^{\frac{3}{2}}\,(1-g)\,\Gamma\Big(-\frac{2}{3}\xi\Big)\Gamma\Big(\xi/3\Big)^2}{2\Gamma\Big(\frac{2}{3}\Big)^3\,\Gamma\Big(-\frac{2}{3}(\xi+1)\Big)\,\Gamma\Big((\xi+1)/3\Big)}\ I_2. \tag{6.43}$$

The Bethe Ansatz equations (6.3) can be solved numerically. We have performed such calculations for vacuum eigenvalues of $\mathbf{A}_1(t)$ and $\overline{\mathbf{A}}_3(t)$ for a range of different values of g and $\boldsymbol{p}$ and compared the results with the asymptotic expansions (6.38). A typical example of these calculations with

$$g = \frac{1}{5}, \qquad \boldsymbol{p} = (\ \tfrac{1}{2},\ \ \tfrac{3}{10}\ ), \tag{6.44}$$

is presented in Table 1 where the numerical values for E's are compared with the corresponding values given by six terms of the asymptotic series (6.38) (the value of the leading term in (6.38) is also shown).

6.3 Special cases

The following special cases are related to a particular choice of the vacuum parameter $\boldsymbol{p} = (p, 0)$ such that

$$(\boldsymbol{w}_1\boldsymbol{p}) = p, \qquad (\boldsymbol{w}_2\boldsymbol{p}) = 0, \qquad (\boldsymbol{w}_3\boldsymbol{p}) = -p, \tag{6.45}$$

n	$E_n^{(1)}$(BA)	$E_n^{(1)}$(AS)	$E_n^{(1)}$(LT)	$\overline{E}_n^{(3)}$(BA)	$\overline{E}_n^{(3)}$(AS)	$\overline{E}_n^{(3)}$(LT)
1	426.44998	429.22624	615.63	42.72214	25.28503	147.89
2	871.85882	872.59835	1069.31	232.30688	229.38814	371.98
3	1466.71208	1466.98579	1674.60	563.02921	562.24812	723.05
4	2224.92354	2225.04588	2443.32	1039.72260	1039.43412	1216.12
5	3157.57647	3157.63845	3386.10	1673.43618	1673.30798	1864.09
6	4274.50955	4274.54393	4512.70	2474.97610	2474.91149	2678.44
7	5584.74586	5584.76628	5832.14	3454.37424	3454.33857	3669.55
8	7096.68686	7096.69967	7352.86	4620.94608	4620.92497	4846.97
9	8818.22784	8818.23622	9082.79	5983.39245	5983.37925	6219.54
10	10756.83708	10756.84278	11029.43	7549.88941	7549.88079	7795.56
11	12919.61477	12919.61876	13199.92	9328.16145	9328.15561	9582.83
12	15313.33879	15313.34165	15601.06	11325.54050	11325.53642	11588.77
13	17944.50152	17944.50363	18239.38	13549.01388	13549.01096	13820.41
14	20819.34016	20819.34174	21121.13	16005.26365	16005.26150	16284.47
15	23943.86187	23943.86307	24252.34	18700.69926	18700.69765	18987.41
16	27323.86522	27323.86615	27638.82	21641.48502	21641.48379	21935.42
17	30964.95845	30964.95918	31286.21	24833.56334	24833.56239	25134.47
18	34872.57528	34872.57586	35199.94	28282.67468	28282.67393	28590.32
19	39051.98867	39051.98914	39385.30	31994.37469	31994.37410	32308.54
20	43508.32291	43508.32328	43847.43	35974.04923	35974.04875	36294.55
21	48246.56426	48246.56456	48591.33	40226.92739	40226.92700	40553.57
22	53271.57046	53271.57072	53621.85	44758.09307	44758.09276	45090.72
23	58588.07919	58588.07940	58943.76	49572.49523	49572.49497	49910.94
24	64200.71562	64200.71580	64561.67	54674.95691	54674.95669	55019.08
25	70113.99930	70113.99945	70480.11	60070.18348	60070.18330	60419.85

Table 1: Comparison of the numerical solution of the Bethe Ansatz Equations (6.3), (6.4) for $E_n^{(1)}$, $E_n^{(3)}$ with g, $\boldsymbol{p}$ given by (6.44), to the value from asymptotic series (6.38), $E_n^{(i)}$(AS), and the leading term of the series $E_n^{(i)}$(LT).

and the weights (3.24) read

$$\Delta_2 = \frac{p^2}{g} + \frac{c-2}{24}, \qquad \Delta_3 = 0. \tag{6.46}$$

The vacuum eigenvalues in this case obey the symmetry

$$A_1^{(vac)}(t) = \overline{A}_3^{(vac)}(-t), \qquad A_2^{(vac)}(t) = \overline{A}_2^{(vac)}(-t), \qquad A_3^{(vac)}(t) = \overline{A}_1^{(vac)}(-t), \tag{6.47}$$

$$\mathbb{T}^{(vac)}(t) = \overline{\mathbb{T}}^{(vac)}(-t), \tag{6.48}$$

while the expression (4.81) reduces to

$$H_1^{(1)(vac)} = -\frac{\pi}{2\sin(\pi g)} \frac{g^3\Gamma\left(1-\frac{3g}{2}\right)}{\Gamma\left(1-\frac{g}{2}\right)\Gamma\left(1-g\right)} \frac{\Gamma(g+p)\Gamma(g/2+p)}{\Gamma(1-g+p)\Gamma(1-g/2+p)}. \tag{6.49}$$

Further the asymptotic expansion of the vacuum eigenvalues $\mathbb{T}^{(vac)}(t)$ given by (6.29) now reads

$$\mathbb{T}^{(vac)}(t) = m\, t^{\frac{1}{3(1-g)}} + 2i \sum_{n=1}^{\infty} \cos((n-2)\pi/3)\, C_{2n-1}\, I_{2n-1}^{(vac)} t^{\frac{1-2n}{3(1-g)}}, \qquad |\arg(t)| < \frac{\pi}{2}(1-g). \tag{6.50}$$

6.3.1 The case $g = 1/2$, $c = -10$

Consider the case when

$$c = -10, \qquad g = 1/2, \qquad q = e^{i\pi g} = i. \tag{6.51}$$

Using (4.57) and (6.47) it easy to see that the Bethe Ansatz equations (6.3) in this case reduce to those for the $U_q(\widehat{sl}_2)$ related **Q**-operators studied in [3]. This fact has been noticed in [20] and recently discussed in [57]. Therefore the functions $A_1^{(vac)}(t)$ and $A_3^{(vac)}(t)$ coincide (up to a normalisation of the spectral parameter t) with $A_+^{(vac)}(t)$ and $A_-^{(vac)}(t)$ from ref. [3] [7]. The relevant parameters used therein are related to our notation as follows

$$c^{sl_2} = -\frac{25}{2}, \qquad \beta^2 = \frac{1}{4}, \qquad p^{sl_2} = \frac{p}{2}, \qquad \Delta = \frac{\Delta_2}{2} = p^2 - 9/16. \tag{6.52}$$

Comparing now the asymptotic expansion (6.16) with the corresponding result of ref [3] (the eq.(4.19) therein) and using the expressions (4.73) and (6.49) one can reproduce the values of first three local IM $I_k^{(sl_2)(vac)}(\Delta, c^{(sl_2)} = -25/2)$ given in eq.(40) of ref. [2]

$$I_1^{(sl_2)(vac)}(\Delta, c^{sl_2}) = \frac{1}{2} I_1^{(vac)}(\Delta_2, 0, c) = \Delta + \frac{25}{48}, \tag{6.53}$$

$$I_3^{(sl_2)(vac)}(\Delta, c^{sl_2}) = -\frac{3}{32} H_1^{(1)(vac)}(1/g, p/g) = \Delta^2 + \frac{7}{8}\Delta + \frac{45}{256}, \tag{6.54}$$

$$I_5^{sl(2)(vac)}(\Delta, c^{sl_2}) = \frac{1}{8} I_5^{(vac)}(\Delta_2, 0, c) = \Delta^3 + \frac{17}{16}\Delta^2 - \frac{175}{98304}\Delta + \frac{15275}{774144}. \tag{6.55}$$

[7] This correspondence extends to certain excited states as well (see also Section 8.2), but we will not elaborate this point here

6.3.2 The case $g = 2/3$, $p = 1/6$

Consider now the vacuum eigenvalue (6.50) of the operator $\mathbb{T}(t)$ when

$$c = -2, \qquad g = 2/3, \qquad p = 1/6. \tag{6.56}$$

The value $g = 2/3$ does lie in the domain (3.38) therefore the results of the previous Sections do not directly apply. In particular the definitions (4.31)-(4.34) require a renormalization since the nonlocal IM (4.33) diverge logarithmically at $g = 2/3$. To obtain the asymptotic of $\mathbb{T}(t)$ at large t in this case we set $g = 2/3 - \epsilon$ in (6.50) and let $\epsilon \to 0$ (this corresponds to an analytic regularization of divergent integrals). The first few terms of the formula (6.50) then read

$$\log \mathbb{T}^{(vac)}(t)|_{g=2/3,p=1/6} = -\sqrt{3}\,at\,\log t + \mathbb{C}t + \frac{\sqrt{3}}{4!}(at)^{-1} - \frac{31\sqrt{3}}{8!}(at)^{-5} + \ldots \tag{6.57}$$

where $\mathbb{C}$ is a renormalization constant and

$$a = \Gamma(1-g)^3 = \Gamma(1/3)^3. \tag{6.58}$$

On the other hand the eigenvalues of $\mathbb{T}(t)$ can be computed independently. Using the results of [58] one can show that in the case (6.56) the fusion equations (5.14) for the eigenvalues of $\mathbb{T}(t)$ simplify to a single equation

$$\mathbb{T}(tq^{\frac{1}{2}})\mathbb{T}(tq^{-\frac{1}{2}}) = 2\cosh(a\pi t)\ \mathbb{T}(t), \qquad q = e^{2\pi i/3} \tag{6.59}$$

which with an assumption that $\mathbb{T}(t)$ is an entire function of t and has the leading asymptotics at large t as in (6.57) can be completely solved

$$\mathbf{T}(t) = \mathbb{T}^{(vac)}(t)\prod_{k=1}^{L}\frac{(1+2n_k-2ate^{i\pi/6})(1+2n_k-2ate^{-i\pi/6})}{(1+2n_k+2ate^{i\pi/6})(1+2n_k+2ate^{-i\pi/6})}, \tag{6.60}$$

$$T^{(vac)}(t) = \frac{2\pi e^{\mathbb{C}'\,t}}{\Gamma(\frac{1}{2}+ate^{i\pi/6})\Gamma(\frac{1}{2}+ate^{-i\pi/6})}. \tag{6.61}$$

where the integer $L \geq 0$; $\{n_k\}$ is an arbitrary increasing sequence of integers $0 \leq n_1 < n_2 < \ldots < n_L$ and $\mathbb{C}'$ is an arbitrary constant. The vacuum eigenvalue $\mathbb{T}^{(vac)}(t)$ is a unique solution of (6.59) with no zeroes in the strip $|\arg(t)| < \pi/3$. One can easily check that its asymptotic expansion at large t perfectly agrees with (6.57).

6.3.3 The case $g = 5/6$

Below we will compare the asymptotic expansion (6.50) for $g = 5/6$ against the numerical results of [58] for several values of p

$$p = 1/6, \qquad p = 1/3, \qquad p = 2/3, \tag{6.62}$$

which corresponds to

$$\Delta_2 = 0, \qquad \Delta_2 = 1/10, \qquad \Delta_2 = 1/2, \tag{6.63}$$

respectively. Again, for $g = 5/6$ the formula (6.50) requires a renormalization. Using the analytic regularization as in the previous subsection one obtains

$$T^{(vac)}(t)|_{g=5/6} = -\frac{2}{\sqrt{3}}(at)^2 \log t + \mathbb{C}\, t - \frac{3\sqrt{3}}{2}\,(at)^{-2}\, I_1^{(vac)} + \frac{32805\sqrt{3}}{43472}(at)^{-10}\, I_5^{(vac)} + \dots \tag{6.64}$$

where

$$a = \Gamma(1-g)^3 = \Gamma(1/6)^3, \tag{6.65}$$

and $I_1^{(vac)}$ and $I_5^{(vac)}$ are given in (4.73).

$I_k^{(vac)}(c,\Delta_2,\Delta_3)$			Numerical values [58]
$I_1^{(vac)}(6/5,0,0)$	$= -\frac{1}{20}$	$= -0.05$	$-0.499999999599\ 10^{-1}$
$I_5^{(vac)}(6/5,0,0)$	$= -\frac{121}{10500}$	$= -0.115238095238\dots\ 10^{-1}$	$-0.115238095240\ 10^{-1}$
$I_1^{(vac)}(6/5,1/10,0)$	$= \frac{1}{20}$	$= 0.05$	$0.500000000002\ 10^{-1}$
$I_5^{(vac)}(6/5,1/10,0)$	$= \frac{209}{42000}$	$= 0.497619047619\dots\ 10^{-2}$	$0.497619047634\ 10^{-2}$

Table 2: Exact values of $I^{(vac)}(c,\Delta_2,\Delta_3)$ for $g = 5/6$, and p given by (6.62), compared with the corresponding numerical results from [58].

As shown in [58] the eigenvalues of $\mathbb{T}(t)$ in this case satisfy the equation

$$\mathbb{T}(tq)\mathbb{T}(tq^{-1}) = e^{\frac{a^2\pi}{3}t^2}\mathbb{T}(-t) + e^{-\frac{a^2\pi}{3}t^2}\mathbb{T}(t). \tag{6.66}$$

In [58] this equation was transformed to TBA-like integral equations, due to Klümper and Pearce [59], and studied numerically for a few vacuum eigenvalues of $\mathbb{T}(t)$. In particular, the numerical values of the coefficients in (6.64) were calculated. We have verified that the values of the local IM extracted from this numerical data are in a good agreement with the corresponding exact values determined by (4.73) (see Table 2).

7 Applications

7.1 Boundary Affine Toda Theory

The Hamiltonian corresponding to the (non-equilibrium) boundary affine Toda theory (BAT) with zero bulk mass reads

$$\mathbf{H}_{\rm BAT} = \mathbf{H}_0 + \mathbf{H}_1, \tag{7.1}$$

$$\mathbf{H}_0 = \frac{1}{4\pi g}\int_{-\infty}^{0} dx\,(\mathbf{\Pi}^2 + \mathbf{\Phi}_x^2\,), \qquad \mathbf{H}_1 = -\frac{\kappa}{2g}\sum_{i=1}^{3}\, u_i\; e^{i(\boldsymbol{e}_i, \mathbf{\Phi}_B + \mathbf{V}t)}\;, \tag{7.2}$$

where g, κ and $\boldsymbol{V} = (V_1, V_2)$ are parameters; $\mathbf{\Phi}(x) = (\Phi_1(x), \Phi_2(x))$, $\mathbf{\Pi}(x) = (\Pi_1(x), \Pi_2(x))$ are field operators obeying canonical commutation relations

$$[\Pi_a(x)\,,\Phi_b(x')] = -2\pi i g\;\delta(x-x')\delta_{ab}, \qquad a, b = 1, 2 \tag{7.3}$$

the variables u_1, u_2, u_3

$$u_1u_2 = qu_2u_1, \qquad u_2u_3 = qu_3u_2, \qquad u_3u_1 = qu_1u_3, \qquad u_1u_2u_3 = q^{\frac{1}{2}}, \tag{7.4}$$

describe boundary degree of freedom and $\Phi_B \equiv \Phi(0)$. The way g enters the Hamiltonian (7.1) allows one to interpret it as a quantum parameter, because it always appears in the combination $g\hbar$; in what follows we will set $\hbar = 1$. We will also assume that the parameter $\boldsymbol{V}$ satisfies the requirement

$$(\mathbf{e}_1\,\boldsymbol{V}\,) > 0, \qquad (\mathbf{e}_2\,\boldsymbol{V}\,) > 0. \tag{7.5}$$

At a nonzero $\boldsymbol{V}$ and a temperature T the system (7.1) develops a stationary non-equilibrium state which can be thought of as the result of an infinite time evolution of the equilibrium state of the corresponding "free" system, with the interaction term (the last term in (7.1)) adiabatically switched on. We will denote by $\langle\,\mathbf{A}\,\rangle_{\mathrm{N}}$ the expectation value of an observable $\mathbf{A}$ over this non-equilibrium stationary state.

The density matrix $\mathbf{P}(t)$ of the system is determined by the Schrödinger equation

$$\partial_t\,\mathbf{P}(t) = -i[\mathbf{H}(t),\,\mathbf{P}], \tag{7.6}$$

with the initial condition

$$\mathbf{P}(t)|_{t=-\infty} = \mathbf{P}_0 = Z_0^{-1}\,e^{-R\mathbf{H}_0}, \qquad R = g/T, \tag{7.7}$$

where $\mathbf{P}_0$ is the equilibrium density matrix of the free system at the temperature T. Using the interaction representation corresponding to the Hamiltonian $\mathbf{H}_0$ in (7.1) one can write the density matrix $\mathbf{P}(t)$ as

$$\mathbf{P}(t) = e^{-i\mathbf{H}_0 t}\;\mathbf{S}(t,-\infty)\;\mathbf{P}_0\;\mathbf{S}(-\infty,t)\;e^{i\mathbf{H}_0 t}\;, \tag{7.8}$$

where

$$\begin{aligned} \mathbf{S}(t,t_0) \;&=\; \mathcal{T}\exp\Big\{-i\int_{t_0}^{t} d\tau\;\mathbf{H}_1^{(int)}(\tau)\Big\} = \\ &=\; 1 + \sum_{k=1}^{\infty}(-i)^k\int_{t_0}^{t}\mathcal{D}_k(\{\tau\})\;\mathbf{H}_1^{(int)}(\tau_1)\,\mathbf{H}_1^{(int)}(\tau_2)\cdots\mathbf{H}_1^{(int)}(\tau_n), \end{aligned} \tag{7.9}$$

with

$$\mathbf{H}_1^{(int)}(t) = e^{i\mathbf{H}_0 t}\ \mathbf{H}_1\ e^{-i\mathbf{H}_0 t}\ . \tag{7.10}$$

In (7.9) and below the shorthand notation for the multiple ordered integrals

$$\int_{t_0}^{t} \mathcal{D}_k(\{\tau\}) = \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k. \tag{7.11}$$

is used. The expectation value of an arbitrary operator $\mathbf{A}$ over the stationary non-equilibrium state (7.8) (which will be marked with the subscript N) reads

$$\langle \mathbf{A} \rangle_{\mathrm{N}} = \mathrm{Tr}_{\mathcal{H}}[\,\mathbf{P}(t)\,\mathbf{A}]. \tag{7.12}$$

The trace is taken over the space $\mathcal{H} = \mathcal{F} \otimes \mathcal{U}$ where $\mathcal{F}$ is the Fock space representing the Bose commutation relations (7.3) and $\mathcal{U}$ is the space of states of the boundary variables u_i, $i = 1, 2, 3$ (with the trace over these states defined exactly as in (4.60)). Substituting (7.8) into (7.12) one gets

$$\langle \mathbf{A} \rangle_{\mathrm{BAT}} = \mathrm{Tr}_{\mathcal{F}}[\,\mathbf{P}_0\,\mathbf{S}(-\infty, t)\,\mathbf{A}^{(int)}(t)\,\mathbf{S}(t, -\infty)\,] = \langle\, \mathbf{S}(-\infty, t)\,\mathbf{A}^{(int)}(t)\,\mathbf{S}(t, -\infty)\,\rangle_0\,, \tag{7.13}$$

where $\langle \ldots \rangle_0$ denotes the expectation value over the equilibrium state (7.7) of the free system, and the superscript "(int)" means that this operator is taken in the interaction representation, i.e. $\mathbf{A}^{(int)}(t) = e^{i\mathbf{H}_0 t}\,\mathbf{A}\,e^{-i\mathbf{H}_0 t}$. Equivalently, one may write the above expectation value as

$$\langle\, \mathbf{A}\,\rangle_{\mathrm{N}} = \mathrm{Tr}_{\mathcal{F}}[\,\mathbf{P}\,\mathbf{A}(t)\,]\ , \tag{7.14}$$

where $\mathbf{P}$ stands for density matrix of the system at $t = 0$, i.e. $\mathbf{P} = \mathbf{P}(0)$, and $\mathbf{A}(t)$ is the full Heisenberg operator

$$\mathbf{A}(t) = \mathbf{S}(0, t)\,\mathbf{A}^{(int)}(t)\,\mathbf{S}(t, 0)\ . \tag{7.15}$$

All above formulae are very well known (see e.g. [60]); the notation used here is exactly the same as in [30]. We are interested in the expectation values $\langle\, \mathbb{V}_j\,\rangle_{\mathrm{N}}$ of the Heisenberg operators

$$\mathbb{V}_j\,(t) = \mathbf{S}(0, t)\,u_j\,\mathbf{V}_j^{(int)}(t)\,\mathbf{S}(t, 0)\ . \tag{7.16}$$

where

$$\mathbf{V}_j^{(int)}(t) = e^{i(\mathbf{e}_j, \boldsymbol{\Phi}_B^{(int)}(t) + \mathbf{V}t)}, \qquad j = 1, 2, 3 \tag{7.17}$$

and $\boldsymbol{\Phi}_B^{(int)}(t)$ is the boundary field $\boldsymbol{\Phi}_B$ in the interaction representation. It is convenient to introduce also auxiliary operators

$$\mathbb{V}_j\,(t, t_0) = \mathbf{S}(t_0, t)\,u_j\,\mathbf{V}_j^{(int)}(t)\,\mathbf{S}(t, t_0)\ , \tag{7.18}$$

where t_0 is a parameter. For $t_0 = 0$ (7.18) coincides with the Heisenberg operators (7.16), and according to (7.13) the expectation values of (7.16) can be expressed through (7.18) as follows

$$\langle\, \mathbb{V}_\sigma(t)\,\rangle_{\rm N} = \lim_{t_0\to-\infty} \langle\, \mathbb{V}_\sigma(t,t_0)\,\rangle_0\ , \tag{7.19}$$

where $\sigma = 1,2,3$.

Using the series expansion for the evolution operators (7.9) and the commutation relations

$$\mathbf{V}^{(int)}_i(t_1)\mathbf{V}^{(int)}_j(t_2) = q^{2(\mathbf{e}_i\mathbf{e}_j)}\mathbf{V}^{(int)}_j(t_2)\mathbf{V}^{(int)}_i(t_1), \qquad t_1 > t_2 \tag{7.20}$$

one can represent (7.18) as a series of time-ordered integrals of products of $\mathbf{V}^{(int)}_j$

$$\begin{aligned}&\mathbb{V}_\sigma(t,t_0) = u_\sigma\, \mathbf{V}^{(int)}_\sigma(t)\\ &+\mathbf{V}^{(int)}_\sigma(t)\sum_{k=1}^{\infty}\ \sum_{\sigma_1,\ldots,\sigma_k=1,2,3} \mathbb{C}_k(\sigma,\,\sigma_1,\ldots,\sigma_k)\int_{t_0}^{t}\mathcal{D}_k(\{t\})\ \mathbf{V}^{(int)}_{\sigma_1}(t_1)\cdots\mathbf{V}^{(int)}_{\sigma_k}(t_k)\ ,\end{aligned} \tag{7.21}$$

where the sum is taken over all arrangements of the "charges" $\sigma_1,\ldots,\sigma_k = 1,2,3$. The coefficients $C_k(\sigma|\,\sigma_1,\ldots,\sigma_k)$ can be easily calculated following the arguments of ref. [30]. From the definition (7.18) it follows that

$$i\frac{\partial}{\partial t_0}\mathbb{V}_\sigma(t,t_0) = [\mathbf{H}^{(int)}_1(t_0),\mathbb{V}_\sigma(t,t_0)]. \tag{7.22}$$

Substituting the the series (7.21) into this differential equation one gets a recurrence relation for the coefficients $\mathbb{C}_k(\sigma|\,\sigma_1,\ldots,\sigma_k)$, which (with the initial condition $\mathbb{C}_0(\sigma) = u_\sigma$) has a unique solution

$$\mathbb{C}_k(\sigma,\,\sigma_1,\ldots,\sigma_k) = u_1^{(\boldsymbol{w}_2\boldsymbol{X}_k)}u_2^{-(\boldsymbol{w}_1\boldsymbol{X}_k)}\ \Big(\frac{q^{\frac{1}{2}}\kappa}{g}\Big)^k\ q^{F(\boldsymbol{X}_k)}\prod_{j=1}^{k}\sin(\pi g\,\boldsymbol{w}_{\sigma'_j}\boldsymbol{X}_{j-1})\ , \tag{7.23}$$

where

$$\sigma'_j = \sigma_j - 1 \pmod 3, \qquad \boldsymbol{X}_j = \sum_{s=0}^{j}\boldsymbol{e}_{\sigma_s}, \qquad \sigma_0\equiv\sigma \tag{7.24}$$

and

$$F(\boldsymbol{X}) = -\frac{1}{2}\Big(\boldsymbol{w}_1\boldsymbol{X}\Big)^2 - \frac{1}{2}\Big(\boldsymbol{w}_2\boldsymbol{X}\Big)^2 + \frac{1}{2}\ . \tag{7.25}$$

Using (7.21), (7.23) one obtains an infinite series representation for the expectation values (7.19)[8]

$$\langle\mathbb{V}_\sigma\rangle_{\rm N} = \frac{2\pi T}{\kappa\,\sin\pi g}\sum_{n=1}^{\infty}(-it)^n\sum_{\sigma_1,\ldots,\sigma_{3n-1}}\Big(\prod_{j=1}^{3n-1}\frac{\sin(\pi g\,\boldsymbol{w}_{\sigma'_j}\boldsymbol{X}_{j-1})}{\sin(\pi g)}\Big)\ J(\sigma,\sigma_1,\ldots,\sigma_{3n-1}|\boldsymbol{p}), \tag{7.26}$$

[8]Representations of this type for a similar problem related to the boundary sine-Gordon model were first derived in [61] by combinatorial methods.

$$J(\sigma,\sigma_1,\ldots,\sigma_{3n-1}|\boldsymbol{p})=\int_{-\infty}^{0}\mathcal{D}_{3n-1}(\{\tau\})e^{-2\boldsymbol{p}\sum_{j=1}^{3n-1}\boldsymbol{e}_{\sigma_j}\tau_j}\prod_{0\le j\le k\le 3n-1}\Big(2\sin\big(\frac{\tau_j-\tau_k}{2}\big)\Big)^{g\,a_{jk}}. \tag{7.27}$$

where $\sigma_i = 1,2,3$ and the internal sum taken over all possible of values of $\{\sigma_i\}$, $i = 1,\ldots,3n-1$ such that

$$\boldsymbol{e}_{\sigma_0}+\boldsymbol{e}_{\sigma_1}+\ldots+\boldsymbol{e}_{\sigma_{3n-1}}=0\ . \tag{7.28}$$

It is convenient to represent the sequence of the "charges" $\sigma_0,\sigma_1,\ldots,\sigma_k$ for each term in (7.21) by a path staring from the origin $\mathbf{X}=0$ of the triangular lattice

$$\mathbf{X}=n_1\mathbf{e}_1+n_2\mathbf{e}_2,\qquad n_1,n_2\in\mathcal{Z}, \tag{7.29}$$

and formed by the sequence of the "step" vector $\mathbf{e}_{\sigma_0},\mathbf{e}_{\sigma_2},\ldots,\mathbf{e}_{\sigma_k}$. The vectors $\mathbf{X}_j$ defined in (7.24) then represent the end point of the path after $j+1$ steps. It is important to note, that not every such path contributes to (7.21). For example, for $\sigma_0=1$ only paths which are

i) always contained in the sector (see Fig 1)

$$\boldsymbol{w}_1\boldsymbol{X}_m\ge 0,\qquad \boldsymbol{w}_2\boldsymbol{X}_m\ge 0,\qquad m=0,1,\ldots k \tag{7.30}$$

ii) do not return to the origin (except for the final step), i.e.,

$$\boldsymbol{X}_m\neq 0,\qquad m=0,1,\ldots k-1 \tag{7.31}$$

correspond to nonzero coefficients $C_k(\sigma_0,\sigma_1,\ldots,\sigma_k)$, since otherwise one of the factors $\sin(\pi g\,\boldsymbol{w}_{\sigma'_j}\boldsymbol{X}_{j-1})$ in (7.23) vanishes. Obviously, the paths (7.28) contributing to (7.26) return to the origin at the last step. As a result all integrals appearing in (7.26) converge at large τ's for

$$2(\mathbf{e}_1\boldsymbol{p})>-g,\qquad 2(\mathbf{e}_2\boldsymbol{p})<g. \tag{7.32}$$

The expectation values (7.26) coincide with *equilibrium* expectation values of certain operators in the system which is similar to (7.1) but contains different boundary degrees of freedom obeying the q-oscillator algebra (4.43). Consider the Hamiltonian

$$\overline{\mathbf{H}}=\frac{1}{4\pi g}\int_{-\infty}^{0}dx\,(\,\boldsymbol{\Pi}^2+\boldsymbol{\Phi}_x^2\,)+\frac{1}{2}\boldsymbol{h}\boldsymbol{V}-\frac{\kappa}{2g}\sum_{i=1}^{3}\,y_i\;e^{i\boldsymbol{e}_i\boldsymbol{\Phi}_B}\ , \tag{7.33}$$

where $\boldsymbol{\Phi}(x)$, $\boldsymbol{\Pi}(x)$ are again the Bose field operators obeying the same commutation relations (7.3)

$$\boldsymbol{h}=\frac{2}{3}(h_1\boldsymbol{e}_1+h_2\boldsymbol{e}_2+h_3\boldsymbol{e}_3), \tag{7.34}$$

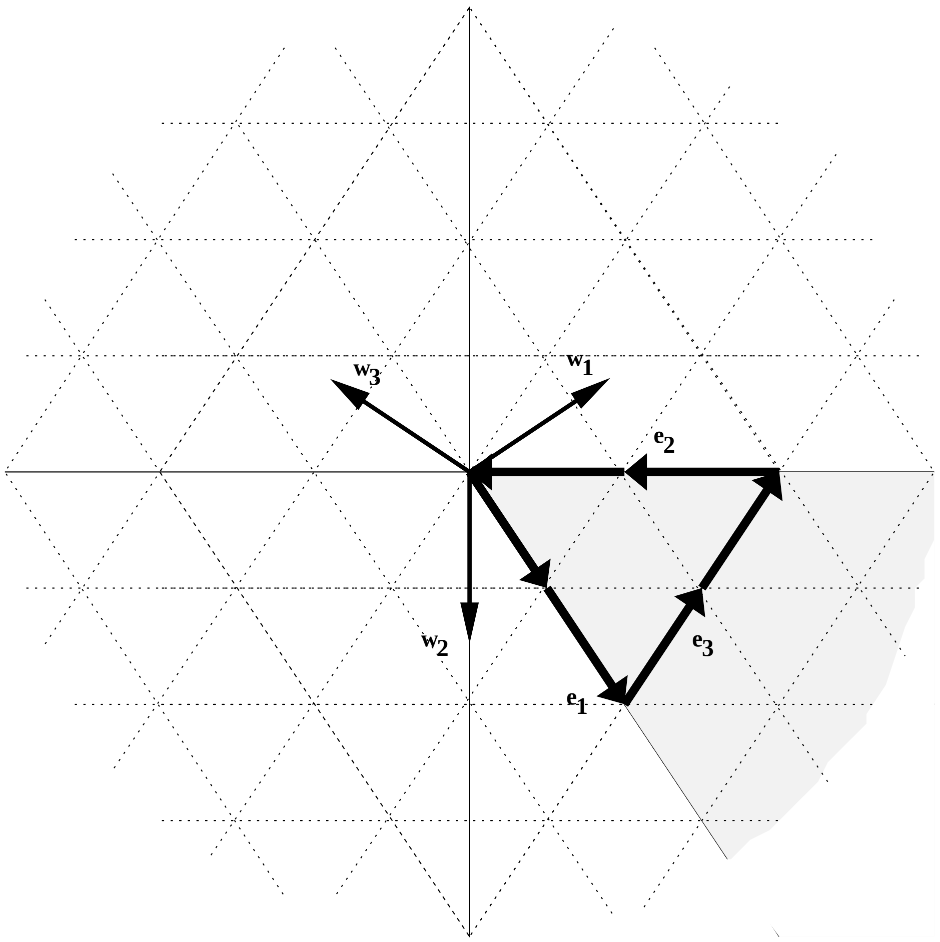

Figure 1: The shaded area is the region defined by (7.30). We also show an admissible path $\boldsymbol{X}_5 = \boldsymbol{e}_1 + \boldsymbol{e}_1 + \boldsymbol{e}_3 + \boldsymbol{e}_3 + \boldsymbol{e}_2 + \boldsymbol{e}_2$.

and

$$\begin{aligned} y_1 &= (q-q^{-1})\mathbf{a}_2^-, \qquad y_2 = (q-q^{-1})\mathbf{a}_1^+ q^{\mathcal{H}_2}, \qquad y_3 = q^{\frac{1}{2}}(q-q^{-1})^2 q^{-\mathcal{H}_2}\mathbf{a}_1^-\mathbf{a}_2^+, \\ h_1 &= -\mathcal{H}_1 - 2\mathcal{H}_2, \qquad h_2 = 2\mathcal{H}_1 + \mathcal{H}_2, \qquad h_1 = -\mathcal{H}_1 + \mathcal{H}_2, \end{aligned} \tag{7.35}$$

where the operators $\mathbf{a}_i^\pm$ and $\mathcal{H}_i$, $i = 1, 2$, commute with $\boldsymbol{\Phi}(x)$, $\boldsymbol{\Pi}(x)$ and satisfy the commutation relations (4.43). Let ρ be any representation of the algebra (4.43) such that the spectra of $\mathcal{H}_1$ and $\mathcal{H}_2$ are both real and bounded from above. The Hamiltonian (7.1) acts in the space

$$\overline{\mathcal{H}} = \mathcal{F} \otimes \rho, \tag{7.36}$$

where $\mathcal{F}$ is the Fock space representing the Bose commutation relations (7.3) and for the values of $\boldsymbol{V}$ satisfying (7.5) this Hamiltonian is bounded from below. Then the system

(7.1) has a thermal equilibrium state described by the standard density matrix

$$\overline{\mathbf{P}} = \overline{Z}(\kappa,\, \boldsymbol{V})^{-1} e^{-R\overline{\mathbf{H}}}, \qquad R = g/T \tag{7.37}$$

where

$$\overline{Z}(\kappa,\, \boldsymbol{V}) = \mathrm{Tr}_{\overline{\mathcal{H}}} e^{-R\overline{\mathbf{H}}}, \tag{7.38}$$

is the partition function. Denote by $\langle \mathbf{A} \rangle_{\mathrm{E}}$ the expectation value of an observable $\mathbf{A}$ over this thermal equilibrium state

$$\langle \mathbf{A} \rangle_{\mathrm{E}} = \mathrm{Tr}_{\overline{\mathcal{H}}}\,(\mathbf{A}\overline{\mathbf{P}}). \tag{7.39}$$

We will be particularly interested in the expectation value $\langle \overline{\mathbb{V}}_1 \rangle_E$ of the operator

$$\overline{\mathbb{V}}_1 = y_1\, e^{i e_1 \boldsymbol{\Phi}_B}, \tag{7.40}$$

and will show below that

$$\langle \overline{\mathbb{V}}_1 \rangle_E = \langle \mathbb{V}_1 \rangle_N, \tag{7.41}$$

where $\langle \mathbb{V}_1 \rangle_N$ is given by (7.26) above.

The proof is closely parallel to that used in the proof of the analogous result for the Boundary Sine-Gordon model, found in [30]. We will use the interaction representation with the Hamiltonian

$$\overline{\mathbf{H}}_0 = \frac{1}{4\pi g} \int_{-\infty}^{0} dx\, (\,\boldsymbol{\Pi}^2 + \boldsymbol{\Phi}_x^2\,) + \frac{1}{2} \boldsymbol{h}\, \boldsymbol{V}, \tag{7.42}$$

corresponding to $\kappa = 0$ in (7.33). Consider an auxiliary operator

$$\overline{\mathbb{V}}_1(t, t_0) = \overline{\mathbf{S}}(t_0, t)\, e^{i\overline{\mathbf{H}}_0 t}\, \overline{\mathbb{V}}_1\, e^{-i\overline{\mathbf{H}}_0 t} \overline{\mathbf{S}}(t, t_0), \tag{7.43}$$

where $\overline{\mathbf{S}}(t, t_0)$ is defined as in (7.9) but with the interaction Hamiltonian replaced by

$$\overline{\mathbf{H}}_1^{(int)}(t) = e^{i\overline{\mathbf{H}}_0 t}\, \overline{\mathbf{H}}_1\, e^{-i\overline{\mathbf{H}}_0 t}, \qquad \overline{\mathbf{H}}_1 = -\frac{\kappa}{2g} \sum_{i=1}^{3}\; y_i\; e^{i e_i \boldsymbol{\Phi}_B}. \tag{7.44}$$

The operator (7.43) can be represented in a form similar to (7.21)

$$\begin{aligned} &\overline{\mathbb{V}}_1(t, t_0) = \mathbf{a}_2^-\, \mathbf{V}_1^{(int)}(t) \\ &+ \mathbf{V}_\sigma^{(int)}(t) \sum_{k=1}^{\infty} \sum_{\sigma_1, \dots, \sigma_k = 1,2,3} \overline{\mathbb{C}}_k(\sigma,\, \sigma_1, \dots, \sigma_k) \int_{t_0}^{t} \mathcal{D}_k(\{t\})\; \mathbf{V}_{\sigma_1}^{(int)}(t_1) \cdots \mathbf{V}_{\sigma_k}^{(int)}(t_k)\;\;, \end{aligned} \tag{7.45}$$

where

$$\overline{\mathbb{C}}_k(\sigma,\, \sigma_1, \dots, \sigma_k) = (\mathbf{a}_2^-)^{(\boldsymbol{w}_2 \boldsymbol{X}_k)} (\mathbf{a}_1^- q^{-\mathcal{H}_2})^{-(\boldsymbol{w}_1 \boldsymbol{X}_k)}\, \Big(\frac{q^{\frac{1}{2}}\kappa}{g}\Big)^k\, q^{F(\boldsymbol{X}_k)} \prod_{j=1}^{k} \sin(\pi g\, \boldsymbol{w}_{\sigma_j'} \boldsymbol{X}_{j-1})\;, \tag{7.46}$$

and the rest of notation is the same as in (7.21). In particular, the same remarks about the paths contributing to (7.21) apply to (7.45). We can now use the representation (7.45) in (7.39). Repeating the arguments which used in derivation of eq.(3.15) in ref. [30] one obtains

$$\langle \overline{\mathbb{V}}_1 \rangle_{\mathrm{E}} = \lim_{t_0 \to -\infty} \frac{\mathrm{Tr}_{\overline{\mathcal{H}}}[e^{-R\overline{H}_0} \overline{\mathbb{V}}_1(0, t_0)]}{\mathrm{Tr}_{\overline{\mathcal{H}}}[e^{-R\overline{H}_0}]} \tag{7.47}$$

Using now (7.45) and the simple property of the trace

$$\mathrm{Tr}_{\overline{\mathcal{H}}}[e^{-R\overline{H}_0} (\mathbf{a}_1^-)^n (\mathbf{a}_2^-)^m] = \delta_{n,0}\delta_{m,0} \mathrm{Tr}_{\overline{\mathcal{H}}}[e^{-R\overline{H}_0}] \tag{7.48}$$

one arrives exactly to the series (7.26) for the expectation value (7.47). This proves the relation (7.41).

Further, comparing the Matsubara representation for the partition function with the vacuum eigenvalues of the operators $\mathbf{A}_i(t)$ defined by (4.49) one can conclude that

$$\frac{\overline{Z}(\kappa,\, \boldsymbol{V})}{\overline{Z}(0,\, \boldsymbol{V})} = A_3^{(vac)}(t, \boldsymbol{p}), \tag{7.49}$$

where the parameters t and $\boldsymbol{p}$ related to κ, $\boldsymbol{V}$ and the temperature T as

$$t = i \Big[\frac{\kappa \sin \pi g}{2\pi T} \Big(\frac{g}{2\pi T} \Big)^{-g} \Big]^3, \qquad \boldsymbol{p} = -i \frac{g}{4\pi T} \boldsymbol{V}. \tag{7.50}$$

Expressing now the equilibrium expectation value $\langle \overline{\mathbb{V}}_1 \rangle_E$ through the partition function (7.38)

$$< \overline{\mathbb{V}_1} >_E = \frac{2}{3} T \partial_\kappa \log \overline{Z}(\kappa,\, \boldsymbol{V}), \tag{7.51}$$

and using (7.41) we obtain

$$\langle \mathbb{V}_1 \rangle_N = \frac{2}{3} T \partial_\kappa \log A_3^{(vac)}(t, \boldsymbol{p}). \tag{7.52}$$

In a similar way one can show that

$$\langle \mathbb{V}_2 \rangle_N = \frac{2}{3} T \partial_\kappa \log A_1^{(vac)}(t, \boldsymbol{p}), \qquad \langle \mathbb{V}_3 \rangle_N = \frac{2}{3} T \partial_\kappa \log A_2^{(vac)}(t, \boldsymbol{p}). \tag{7.53}$$

Finally note the $g \to 1/g$ duality properties of the expectation values (7.52), (7.53). Introduce the following combinations expectation values

$$J_\alpha = i\pi\kappa \, \langle \, \mathbb{V}_{\sigma-1} \rangle_N - i\pi\kappa \, \langle \mathbb{V}_\sigma \rangle_N, \qquad \alpha = 1, 2, 3 \tag{7.54}$$

then

$$J_\alpha = 2\pi i T \, t \partial_t \log \frac{A_\beta^{(vac)}(t, \boldsymbol{p})}{A_\gamma^{(vac)}(t, \boldsymbol{p})}, \tag{7.55}$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

According to (6.16) the quantities (7.54) satisfy the following "strong weak" duality relations

$$J_\alpha(g, \boldsymbol{V}, -\kappa) = -\mathbf{e}_\alpha \boldsymbol{V} - g^{-1} J_\alpha(g^{-1}, g\boldsymbol{V}, -C(g)(-\kappa)^{-\frac{1}{g}}), \tag{7.56}$$

$$C(g) = \frac{\Gamma(g^{-1})}{\pi}\Big(\frac{\Gamma(g)}{\pi}\Big)^{\frac{1}{g}}. \tag{7.57}$$

7.2 Relation to third order differential equations

Consider the differential equation

$$\Big\{\partial_x^3 + \frac{\ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3 - 2}{x^2}\partial_x - \frac{\ell_1\ell_2\ell_3}{x^3} + x^{3\alpha}\Big\}\Psi(x) = E\Psi(x), \tag{7.58}$$

where

$$\ell_1 + \ell_2 + \ell_3 = 3. \tag{7.59}$$

For

$$\Re e(\ell_1 + 3) > \Re e(\ell_2), \qquad \Re e(\ell_1 + 3) > \Re e(\ell_3), \tag{7.60}$$

the equation (7.58) has a unique solution satisfying the condition

$$\psi(x, E, \ell_1, \ell_2, \ell_3) = x^{\ell_1} + O(x^{\ell_1+3}), \tag{7.61}$$

This solution can be analytically continued outside the domain (7.60). Obviously the functions $\psi(x, E, \ell_2, \ell_1, \ell_3)$ and $\psi(x, E, \ell_3, \ell_1, \ell_2)$ defined in this way satisfy the same equation (7.58) and for generic ℓ_1, ℓ_2, ℓ_3 the solutions

$$\psi_1(x) = \psi(x, E, \ell_1, \ell_2, \ell_3), \qquad \psi_2(x) = \psi(x, E, \ell_2, \ell_1, \ell_3), \qquad \psi_3(x) = \psi(x, E, \ell_3, \ell_1, \ell_2), \tag{7.62}$$

are linearly independent since

$$\mathrm{Wr}\,[\psi_1(x), \psi_2(x), \psi_3(x)] = (\ell_1 - \ell_2)(\ell_2 - \ell_3)(\ell_3 - \ell_1), \tag{7.63}$$

where Wr denotes the usual Wronskian.

Further, for all E the equation (7.58) has a unique solution $\chi(x, E)$ which decays at $x \to +\infty$. We normalise this solution as

$$\chi(x) = x^{-\alpha} \exp\Big\{ - \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha-2})\Big\}. \tag{7.64}$$

It can be expanded in the basis (7.62)

$$\chi(x, E) = R_1(E)\psi_1 + R_2(E)\psi_2 + R_3(E)\psi_3. \tag{7.65}$$

It was conjectured in [20] that the connection coefficients $R_i(E)$ are simply related to the vacuum eigenvalues of the $\mathbf{Q}$-operators (4.49). The exact correspondence is as follows:

$$\frac{R_i(E)}{R_i(0)} = A_i^{(vac)}(t, \boldsymbol{p}), \qquad i = 1, 2, 3 \tag{7.66}$$

where

$$\alpha = \frac{1}{g} - 1, \qquad E = i\,\rho\,t, \qquad \ell_i = -\frac{3(\boldsymbol{p}\boldsymbol{w}_i)}{g} + 1, \tag{7.67}$$

with the parameter ρ and the vectors $\boldsymbol{w}_i$ defined in (2.34) and (6.34) respectively. The conjecture (7.66) was supported in [20] by extensive numerical work, but no analytical proof is currently known (see Section 8.3 below for a discussion).

n	$E_n^{(1)}$ [20]	$E_n^{(1)}$ (AS)	$E_n^{(3)}$ [20]	$E_n^{(3)}$ (AS)
1	2.88682816	2.8787210	0.890944821	0.66892875
2	5.15225193	5.1513633	3.643901890	3.64527385
3	7.13037330	7.1300266	5.817136367	5.81718647
4	8.94036211	8.9401849	7.737709470	7.73772076
5	10.6356087	10.6355029	9.508485338	9.50848916
6	12.2451641	12.2450946	11.17453999	11.17454166
7	13.7870634	13.7870146	12.76111150	12.76111235
8	15.2734900	15.2734540	14.28420813	14.28420862
9	16.7131734	16.7131460	15.75481968	15.75481998

Table 3: Numerical values for $E_n^{(1)}$ and $E_n^{(3)}$ from Table 2 in [20], truncated to 8 decimal places, compared with $E_n^{(1)}$(AS) and $E_n^{(3)}$(AS), obtained from our asymptotic formula, (7.71).

Note that, due to the correspondence (7.66), the formulae (6.16) provide asymptotic expansions for the zeroes of the coefficients $R_i(E)$. As an example consider the following particular case of (7.58),

$$\partial_x^3 \Psi(x) + x\;\Psi(x) = E\Psi(x), \tag{7.68}$$

which corresponds to

$$\alpha = 1/3, \qquad \ell_1 = 0, \qquad \ell_2 = 1, \qquad \ell_3 = 2, \tag{7.69}$$

and

$$g = 3/4, \qquad (\boldsymbol{p}\boldsymbol{w}_1) = 1/4, \qquad (\boldsymbol{p}\boldsymbol{w}_2) = 0, \qquad (\boldsymbol{p}\boldsymbol{w}_3) = -1/4. \tag{7.70}$$

The zeroes $R_1(E)$ and $R_3(E)$ in this case have been extensively studied numerically both from the solutions of the differential equation as well as from the non-linear integral equation [18–20] which is equivalent to the Bethe Ansatz equations (6.3), (6.4) for the vacuum eigenvalues $A_i^{(vac)}(t)$. We have compared this numerical data against the asymptotic expansion (6.38). For $g=3/4$ the expressions (6.35), (6.36) allow us to derive only one non-trivial term in (6.39), (6.40) (note that the first dual nonlocal IM also contribute). For the values of $\boldsymbol{p}$ given in (7.70) the corresponding formulae read

$$\begin{aligned} E_n^{(1)} &= \left(\frac{8\pi}{3\sqrt{3}}\right)^{\frac{3}{4}} (n-\tfrac{1}{6})^{3/4}\left[1+\tfrac{21}{256\pi^2}(n-\tfrac{1}{6})^{-2}+O(n^{-4})\right], \\ E_n^{(3)} &= \left(\frac{8\pi}{3\sqrt{3}}\right)^{\frac{3}{4}} (n-\tfrac{5}{6})^{3/4}\left[1-\tfrac{15}{256\pi^2}(n-\tfrac{5}{6})^{-2}+O(n^{-4})\right]. \end{aligned} \tag{7.71}$$

The numerical values are presented in Table 3.

As shown in [62] the correspondence (7.66) allows us to derive a certain duality relation for the vacuum eigenvalues $A_i^{(vac)}(t,\boldsymbol{p})$. It is convenient to make a change of variables $\Psi(x)=e^y\tilde{\Psi}(y)$, $x=e^y$ which brings the equation (7.58) to the form

$$\left\{\partial_y^3-(\nu_1\nu_2+\nu_1\nu_3+\nu_2\nu_3)\partial_y+i\nu_1\nu_2\nu_3+e^{3(\alpha+1)y}-Ee^{3y}\right\}\tilde{\Psi}(y)=0, \tag{7.72}$$

where

$$\nu_j=-i(\ell_j-1)=\frac{3i(\boldsymbol{p}\boldsymbol{w}_j)}{g}. \tag{7.73}$$

It is easy to see that the change of variables

$$y=gy'+y_0, \qquad e^{-3y_0}=-g^3E, \tag{7.74}$$

transform (7.72) to an equation of the same form but with the parameters g, $\{\nu_i\}$ and replaced by

$$g\to g^{-1}, \qquad \nu_i=g\nu_i, \qquad E\to -g^3g^{-\frac{3}{g}}(-E)^{-\frac{1}{g}}. \tag{7.75}$$

Using this fact in (7.65) one obtains the following duality relation for the connection coefficients R_i

$$\frac{e^{iy_0\nu_\alpha}\,R_\alpha(g,\nu_1,\nu_2,\nu_3,E)}{e^{iy_0\nu_\beta}\,R_\beta(g,\nu_1,\nu_2,\nu_3,E)}=\frac{R_\alpha(g^{-1},g\nu_1,g\nu_2,g\nu_3,-c(g)\,(-E)^{-\frac{1}{g}})}{R_\beta(g^{-1},g\nu_1,g\nu_2,g\nu_3,-c(g)\,(-E)^{-\frac{1}{g}})}, \qquad c(g)=g^{3(1-g^{-1})}, \tag{7.76}$$

where $\alpha,\beta=1,2,3$. Note that due to (7.66), this relation is in agreement with (6.16) and (7.56).

8 Discussion

In conclusion we make a few remarks on some open problems relevant to the topics of this paper.

8.1 Truncation of the functional relations

It is well known that the fusion equation for the transfer matrices in solvable lattice models truncate when q^2 is a root unity. A similar truncation, of course takes place for the CFT $\mathbf{T}$-operators. For the theories related to the quantum affine $U_q(\widehat{sl}(2))$ this truncation is well understood. For example, consider the CFT $\mathbf{T}$-operators $\mathbf{T}_j(\lambda)$, $j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$, of ref. [2], associated with $2j+1$-dimensional representations of $U_q(\widehat{sl}(2))$. If $q = e^{i\pi\beta^2}$ is a primitive root of 1 or -1, such that $q^N = \pm 1$ with some integer $N \geq 2$ then one can easily show (see, e.g., [4]) that

$$\mathbf{T}_{\frac{N}{2}}(\lambda) = 2\cos(2\pi N p) + \mathbf{T}_{\frac{N}{2}-1}(\lambda) \ , \tag{8.77}$$

where p and β are related to the Virasoro highest weight Δ and the central charge $\hat{c}$ as

$$\Delta = \frac{p^2}{\beta^2} + \frac{\hat{c}-1}{24}, \qquad \hat{c} = 1 - 6(\beta - \beta^{-1})^2, \qquad q = e^{i\pi\beta^2} \tag{8.78}$$

As is shown in [3] this allows one to bring the fusion relations for $\mathbf{T}_j(\lambda)$ to the form identical to the functional TBA equations (the Y-system) of D_N type [29, 63].

Further reduction occurs at particular values of the vacuum parameter p

$$\mathbf{T}_{\frac{N}{2}-\frac{1}{2}}(\lambda) = 0, \qquad \text{when} \qquad z^N = \pm 1, \qquad z = e^{2\pi i p} \tag{8.79}$$

which corresponds to the RSOS reduction in the six-vertex lattice model [64]. This latter RSOS-type reduction is also well known for the case of $U_q(\widehat{sl}(3))$ [16]. For example, it readily follows from the determinant formulae (5.1), (5.2): when the vacuum parameters z_i defined in (3.31) become roots of unity $z_i^N = 1$, $i = 1, 2, 3$ along with $q^N = 1$ and either $\mu_1 + 2 = N$ or $\mu_2 + 1 = N$ the determinants (5.1), (5.2) vanish. However, no simple analogues of the reduction (8.77) valid for arbitrary values of z_i is known (except for the particular case of Sect 6.3 which accidentally reduces to $U_q(\widehat{sl}(2))$). Similar remarks apply to related problems such as the string structure of the Bethe Ansatz equations and the TBA equations. In particular, no $U_q(\widehat{sl}(3))$ analogues of the D_N type TBA equations are known.

8.2 Connection with $U_q(A_2^{(2)})$ related IQFT

The set of classical IM (1.2) admits consistent reductions $V(u) = 0$ or $V(u) = U'(u)/2$ causing all even IM $I_k^{(cl)}$, $k = 2, 4 \pmod 6$ to vanish. Remarkably, the non-vanishing IM $I_k^{(cl)}$, $k = 1, 5 \pmod 6$ remain in involution [10] with respect to Virasoro Poisson bracket of the field $U(u)$. The quantization of these conserved quantities leads to new sets (with respect to the quantum KdV case [2, 52, 53, 65]) of local IM in $c < 1$ CFT.

Actually, there are two new non-equivalent sets of IM related to the different reductions mentioned above. This structure of the local IM naturally arises in the conformal limit of the massive IQFT associated with the twisted quantum affine $U_q(A_2^{(2)})$ [65, 66]. The universal CFT L-operator for this case reads

$$\mathcal{L} = e^{i\pi Ph}\mathcal{P}\exp\Big\{\int_0^{2\pi}\Big(:e^{-2\phi(u)}:y_0 + :e^{\phi(u)}:y_1\Big)du\Big\}, \tag{8.80}$$

where $\phi(u)$ is the chiral Bose field normalised exactly as in [2] and h, y_0 and y_1 are the generators of the Borel subalgebra of $U_q(A_2^{(2)})$ where q is related to the Virasoro central charge $\hat{c}$ as in (8.78). In ref. [67] Fioravanti, Ravanini and Stanishkov constructed an infinite set of finite-dimensional representations $\pi_s(t)$, $s = 0, 1, 2, \ldots$, of this subalgebra (with the dimensions $\dim[\pi_s(t)] = (s+1)(s+2)/2$) and defined a set of $\mathbf{T}$-operators

$$\mathbf{T}_s(t) = \mathrm{Tr}_{\pi_s(t)}[e^{i\pi Ph}\mathcal{L}], \qquad s = 0, 1, 2, \ldots \tag{8.81}$$

Here we use the spectral parameter t which is related to λ of [67] as $t = \lambda^3$. These operators act in the Virasoro highest weight module $\mathcal{V}_\Delta$ with Δ defined in (8.78); they are entire functions of t and their leading asymptotics at large t can be easily estimated

$$\log \mathbf{T}_s(t) \sim m_s\, t^{2/(6-3\beta^2)}, \qquad |t| \to \infty \tag{8.82}$$

where m_s are some numerical constants. Further, at small t one has

$$\mathbf{T}_1(t) = 1 + 2\cos(2\pi p) + \mathbf{G}_1\, t + O(t^2), \tag{8.83}$$

where the vacuum eigenvalue $\mathbf{G}_1$ in $\mathcal{V}_\Delta$ can be explicitly calculated

$$G_1^{(vac)}|_{A_2^{(2)}} = 2\pi\, e^{-2\pi i p}\int_0^{2\pi} du_1 \int_0^{u_1} du_2 \frac{e^{ip(u_1+u_2)}\Big(2\sin\big(\frac{u_1-u_2}{2}\big)\Big)^{\frac{\beta^2}{2}}}{\Big(2\sin\frac{u_1}{2}\Big)^{\beta^2}\Big(2\sin\frac{u_2}{2}\Big)^{\beta^2}}$$

$$= \frac{4\pi^3\Gamma(1+\beta^2/2)\Gamma(1-3\beta^2/4)\Gamma(1-\beta^2)}{\Gamma(1+\beta^2/4)\Gamma(1+p-\beta^2/2)\Gamma(1-p-\beta^2/2)\Gamma(1+p-\beta^2/4)\Gamma(1-p-\beta^2/4)}. \tag{8.84}$$

In [67] it was shown[9] that these operators satisfy the functional relation [68]

$$\mathbf{T}_s(tq^{1/2})\mathbf{T}_s(tq^{-1/2}) = \mathbf{T}_s(-t) + \mathbf{T}_{s-1}(t)\mathbf{T}_{s+1}(t), \qquad q = e^{i\pi\beta^2} \tag{8.85}$$

Now turn back to the $\mathcal{W}_3$ theory. Consider the $\mathcal{W}_3$ module $\mathcal{V}_{\Delta_2,0}$, with the weight $\Delta_3 = 0$. Let $\mathcal{V}_{sym}$ be a subspace $\mathcal{V}_{sym} \in \mathcal{V}_{\Delta_2,0}$ of this module invariant with respect to

[9] Although the calculations in [67] have been performed in the lowest non-trivial order in t the relation (8.85) is expected to hold in all orders in t.

the action of the automorphism σ_{13} defined in Sect. 4.1 Then from the definitions (4.21), (4.36) and the relations (4.18), (4.11) one concludes that the operators $\mathbb{T}_m(t)$ and $\overline{\mathbb{T}}_m(t)$ from (4.38), (4.39) obey the relation

$$\mathbb{T}_m(t) = \overline{\mathbb{T}}_m(-t) \tag{8.86}$$

when acting in the subspace $\mathcal{V}_{sym}$. It follows then from (6.29) and the discussion there that all even local IM $\mathbf{I}_k$, $k = 2, 4 \pmod 6$, vanish in this subspace. Denote by $\mathbb{T}_m(t|g,p))$ the eigenvalues $\mathbb{T}_m(t)$ in $\mathcal{V}_{sym} \in \mathcal{V}_{\Delta_2,0}$ where g is related to the central charge c of the $\mathcal{W}_3$ algebra (3.22) and p parameterises the weight $\Delta_2 = p^2/g + (c-2)/24$. Taking into account (8.86) the functional equations (5.14) become equivalent to (8.85) provided one makes a replacement $q \to q^{\frac{1}{2}}$. Thus we have good reasons to expect that with a suitable redefinition of parameters the above eigenvalues $\mathbb{T}_m(t|g,p))$ of $\mathbb{T}_m(t)$ are identical to the eigenvalues $T_s(t|\beta^2,p)$ of the operators (8.81) acting in the Virasoro module $\mathcal{V}_\Delta(\hat{c})$ with p and $\hat{c}$ already defined in (8.78). The exact correspondence is

$$T_s(t|\beta^2,p) = \mathbb{T}_s(\rho_1 t|\beta^2/2,p), \qquad s = 0,1,2,\ldots \tag{8.87}$$

where the factor

$$\rho_1 = \frac{\Gamma(1-\frac{\beta^2}{4})\,\Gamma(1+\frac{\beta^2}{2})\,\Gamma(1-\beta^2)}{\Gamma(1+\frac{\beta^2}{4})\,\Gamma^2(1-\frac{\beta^2}{2})}, \tag{8.88}$$

is calculated from the comparison of (8.84) with (4.84). Further, the subspace $\mathcal{V}_{sym}$ should support the action of the Virasoro algebra with the central charge $\hat{c}$. It would be interesting to obtain an explicit realization of this Virasoro algebra in terms of the generators of the $\mathcal{W}_3$ algebra. It would also be interesting to obtain an explicit construction of the $\mathbf{Q}$ operators for the $U_q(A_2^{(2)})$-related $\mathbf{T}$ operators (8.81).

8.3 Concluding remarks

The study of the integrable structure of CFT in [2–4] and in this paper once again demonstrates the fundamental role of Baxter's $\mathbf{Q}$-operators in the theory of integrable quantum systems. Some further applications of $\mathbf{Q}$-operators in CFT can be found in [69].

Here we adopted the approach of [3,4] to construct the $\mathbf{Q}$-operators as traces of certain quantum monodromy matrices associated with infinite-dimensional representations of the q-oscillator algebra. As remarked in [4] this construction is rather general and can be applied to the lattice models as well. In particular, as the structure of the functional relations considered in Section 5 is completely determined by the decomposition properties of the products of representation of the quantum affine algebra $U_q(\widehat{sl}(3))$, all these functional relations remain valid for the associated lattice models, with minor modifications related to the normalisation conventions of the lattice transfer matrices. Note that for certain lattice models there exists an alternative approach to **Q**-operators [70] (based on Baxter's original work [71]) which allows the explicit calculation of their matrix elements in a simple product form (see also [72–78] for further examples). It would be interesting to understand a relationship between the above two approaches to the **Q**-operators.

The conjecture (7.66) of [20] is a generalisation of an earlier conjecture of the same authors [79] which established an exact relation between the spectral determinants of the (second order) Schrödinger operator with homogeneous potentials [80,81] and the vacuum eigenvalues of the **Q**-operators in the $c < 1$ CFT [3]. This earlier conjecture of [79] has been extended for the more general Schrödinger operator (which includes the "angular momentum" term)

$$\partial_x^2 \Psi(x) + \Big\{ E - x^{2\alpha} - \frac{l(l+1)}{x^2} \Big\} \Psi(x) = 0 \tag{8.89}$$

and proved in [62]. Here the parameters l and α are related to the Virasoro vacuum parameter p and central charge $\hat{c}$ defined in (8.78) as $l = 2p/\beta^2 - 1/2$ and $\alpha = 1/\beta^2 - 1$. The most essential ingredients of this proof can be summarized as follows:

(i) *The analytic properties (both in the spectral parameter E and the angular momentum l).* For the spectral determinants of (8.89) these properties are deduced from the standard theory of the Schrödinger equations (see, e.g., [82]). On the other hand, in [4] and [30] the same analytic properties for the vacuum eigenvalues of the **Q**-operators were establish in $c < 1$ CFT. Note that the analysis of these analytic properties in the variable p in [30] (corresponding to l in (8.89)) was based on the "real time perturbation theory" results (analogous to the representations (7.26) of this paper).

(ii) *The quasiclassical approximation.* The same real time perturbation theory results were used in [30] to obtain the leading large p asymptotics of the coefficient of the expansion of the eigenvalues of the **Q**-operators in the spectral parameter. These asymptotics exactly matched [62] the corresponding asymptotics for the spectral determinants of (8.89) calculated from the standard quasiclassical approximation to (8.89) for large values of l.

(iii) *Quantum Wronskian relation and the Riemann-Hilbert problem.* The final step of the proof of [62] was based on the key observation of [79] that the functional relation for spectral determinants of the Schrödinger equation obtained in [80,81] (see also [62] for the case of general l) exactly coincide with the quantum Wronskian relation for the eigenvalues of the **Q**-operators in $c < 1$ CFT obtained in [3]. In fact, it was shown [30] that the above properties (i) and (ii) transform this functional relation to a simple Riemann-Hilbert problem which has a unique solution.

It should be noted that no additional assumptions (like the phase assignment in the Bethe Ansatz equations for vacuum eigenvalues, similar to (6.4)) were used in [62].

It would be interesting to see whether is it possible to modify the above arguments to prove the conjecture (7.66) related to the third order differential equation (7.58). It seems that there no difficulties in generalising the above properties (i) and (ii). In particular, we have verified that the asymptotics of the connection coefficients $R_i(E)$ in (7.65) obtained from the quasiclassical approximation to the differential equation (7.58) for large values of ℓ_i exactly match the $\boldsymbol{p} \to \infty$ asymptotics of the eigenvalues of the $\mathbf{Q}$-operators obtained from (6.9) and (6.30). It is also easy to prove that the connection coefficients $R_i(E)/R_i(0)$ satisfy the same quantum Wronskian relation (1.13). However, it is currently not clear how transform this relation to a tractable Riemann-Hilbert problem which now should involve two complex "angular momentum" variables.

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Appendix A. Universal R-matrix

Here we give a short proof of Proposition 1, which allows us to derive the basic properties of the universal L-operator (3.34).

Due to (3.6) and (3.14) the basic property (3.9) of the universal R-matrix implies the following relations:

$$(1 \otimes y_i + y_i \otimes q^{-h_i})\overline{\mathcal{R}} = \overline{\mathcal{R}}(1 \otimes y_i + y_i \otimes q^{h_i}), \tag{A.1}$$

that is

$$[1 \otimes y_i, \overline{\mathcal{R}}] = \overline{\mathcal{R}}(y_i \otimes q^{h_i}) - (y_i \otimes q^{-h_i})\overline{\mathcal{R}}. \tag{A.2}$$

Sum up (A.2) over i and put $\overline{I} = \sum_i y_i$. We have

$$[\overline{I}, \overline{\mathcal{R}}] = \overline{\mathcal{R}}(\sum_{i=1}^{3} y_i \otimes q^{h_i}) - (\sum_{i=1}^{3} y_i \otimes q^{-h_i})\overline{\mathcal{R}}. \tag{A.3}$$

Let us imagine now that we check equality (A.3) directly. For this we compute the commutator on the left hand side using the relations

$$[y_i, x_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}. \tag{A.4}$$

After application of the Leibnitz rule we have certain series over $y_i \otimes 1$ and $1 \otimes x_i$ with q^{h_i} and q^{-h_i} inserted somewhere in the right tensor space.

We move all q^{-h_i} to the left and all q^{h_i} to the right using the relations

$$q^{h_i} x_j = q^{-a_{ij}} x_j q^{h_i}, \qquad q^{-h_i} x_j = q^{a_{ij}} x_j q^{-h_i}. \tag{A.5}$$

The result should coincide with right hand side of (A.3).

Suppose now that we have some I, A_i and B_i, satisfying the conditions (3.16), (3.17) of Proposition 1, and we try to compute $[I \otimes 1, \overline{\mathcal{R}}]$. Again, we first use (3.16), to get monomials over $y_i \otimes 1$ and $1 \otimes x_i$ with A_i and B_i inserted in the second tensor component, then move A_i to the left and B_i to the right by means of (3.17). Clearly, we should get the same result as in the first calculation with $q^{h_i} \otimes 1$ replaced by $-B_i \otimes 1$ and $q^{-h_i} \otimes 1$ replaced by $-A_i \otimes 1$, since we used identical commutation relations. This proves (3.18).

Appendix B. Functional relations

In this appendix we derive the determinant formulas (5.1), (5.2) for the operators $\mathbf{T}_\mu(t)$, and $\overline{\mathbf{T}}_\mu(t)$ and the functional relations (5.4), (5.5). The main ingredient of the proof is similar to that of [4] and consists of decomposition of the tensor product of certain degenerated Verma modules into a sum of shifted evaluation Verma modules.

Let $\pi^+_\mu(t)$ be the evaluation Verma module of the algebra $\mathcal{B}_+$ with highest weight $\mu = (\mu_1, \mu_2, \mu_3)$ with respect to evaluation map (4.15). We fix the basis $v_{\overline{n}}$, where $\overline{n} \in \mathbb{Z}^3_{\geq 0}$, that is $\overline{n} = (n_1, n_2, n_3)$ and $n_i \geq 0$, $n_i \in \mathbb{Z}$, which is given by the action of the generators F_α,F_β and $F_{\alpha+\beta}$ of $U_q(gl(3))$:

$$v_{\overline{n}} = t^{n_2 + \frac{n_1+n_3}{2}} q^{-\frac{n_2}{2}} F_\alpha^{n_1} F_{\alpha+\beta}^{n_2} F_\beta^{n_3} v_0, \tag{B.1}$$

where v_0 is the vacuum vector. Then the action of the generators of the algebra $\mathcal{B}_+$ is given by the relations:

$$\begin{aligned}
y_1 v_{\overline{n}} &= t^{\frac{1}{2}} [n_3]_q [\mu_\beta - n_3 + 1]_q v_{\overline{n} - \overline{\varepsilon}_3} + t^{\frac{1}{2}} q^{\mu_\beta - 2n_3} [n_2]_q v_{\overline{n} + \overline{\varepsilon}_1 - \overline{\varepsilon}_2}, \\
y_2 v_{\overline{n}} &= q^{n_3 + \mu_1 + \mu_3 + \frac{1}{2}} v_{\overline{n} + \overline{\varepsilon}_2}, \\
y_3 v_{\overline{n}} &= t^{\frac{1}{2}} [n_1]_q [\mu_\alpha - n_1 - n_2 + n_3 + 1]_q v_{\overline{n} - \overline{\varepsilon}_1} - t^{\frac{1}{2}} q^{n_2 - n_3 - 2 - \mu_\alpha} [n_2]_q v_{\overline{n} - \overline{\varepsilon}_2 + \overline{\varepsilon}_3}, \\
h_1 v_{\overline{n}} &= (\mu_\beta + n_1 - n_2 - 2n_3) v_{\overline{n}}, \\
h_2 v_{\overline{n}} &= (-\mu_{\alpha+\beta} + n_1 + 2n_2 + n_3) v_{\overline{n}}, \\
h_3 v_{\overline{n}} &= (\mu_\alpha - 2n_1 - n_2 + n_3) v_{\overline{n}},
\end{aligned} \tag{B.2}$$

where

$$
\begin{aligned}
\mu_\alpha &= \mu_1-\mu_2, & \mu_\beta &= \mu_2-\mu_3, & \mu_{\alpha+\beta} &= \mu_1-\mu_3,\\
\overline{\varepsilon}_1 &= (1,0,0), & \overline{\varepsilon}_2 &= (0,1,0), & \overline{\varepsilon}_3 &= (0,0,1).
\end{aligned}
\tag{B.3}
$$

Any weight μ defines a shift automorphism p_μ of the algebra $\mathcal{B}_+$:

$$
\begin{aligned}
p_\mu(y_i) &= y_i, & p_\mu(h_1) &= h_1-\mu_2+\mu_3,\\
p_\mu(h_2) &= h_2+\mu_1-\mu_3, & p_\mu(h_3) &= h_2-\mu_1+\mu_2.
\end{aligned}
\tag{B.4}
$$

For a representation π of $\mathcal{B}_+$ denote by $\pi[\mu]$ the shifted representation $\pi\cdot p_\mu$. In particular, denote by $\tilde{\pi}^+_\mu(t)$ the shifted evaluation Verma module $\pi^+_\mu(t)[\mu]$. In this module the action of generators y_i is given by (B.2) while the vacuum vector is of zero weight and

$$
\begin{gathered}
h_1 v_{\overline{n}} = (n_1-n_2-2n_3)v_{\overline{n}}, \qquad h_2 v_{\overline{n}} = (n_1+2n_2+n_3)v_{\overline{n}},\\
h_3 v_{\overline{n}} = (n_3-2n_1-n_2)v_{\overline{n}},
\end{gathered}
\tag{B.5}
$$

so the traces are related by

$$
\mathrm{Tr}_{\pi^+_\mu(t)}\, e^{i\pi\mathbf{P}\,\mathbf{h}}\mathcal{L} = z_1^{\mu_1} z_2^{\mu_2} z_3^{\mu_3}\mathrm{Tr}_{\tilde{\pi}^+_\mu(t)}\, e^{i\pi\mathbf{P}\,\mathbf{h}}\mathcal{L}. \tag{B.6}
$$

Let us define three degenerations $\tilde{\pi}^i(t)$, $i=1,2,3$ of shifted evaluation Verma module $\tilde{\pi}^+_\mu(t)$. They correspond to the limit $q\to+\infty$ in three different asymptotic zones

$$
\mu_\alpha \gg 0,\ \mu_\beta \gg 0; \qquad \mu_\alpha \ll 0,\ \mu_\beta \gg 0 \qquad \text{and} \qquad \mu_\alpha \ll 0,\ \mu_\beta \ll 0 \tag{B.7}
$$

under assumptions that all the numbers are real.

All the degenerated Verma modules $\tilde{\pi}^i(t)$ are equipped with bases $v_{\overline{n}}$, $\overline{n}_i\in\mathbb{Z}^3_{\geq 0}$, on which the Cartan generators act as in (B.5) and

$$
\begin{aligned}
y_1 v_{\overline{n}} &= (q-q^{-1})^{-1} q^{x-n_3+1}[n_3]_q v_{\overline{n}-\overline{\varepsilon}_3} + q^{x-2n_3}[n_2]_q v_{\overline{n}+\overline{\varepsilon}_1-\overline{\varepsilon}_2},\\
y_2 v_{\overline{n}} &= q^{n_3+\frac{1}{2}} v_{\overline{n}+\overline{\varepsilon}_2},\\
y_3 v_{\overline{n}} &= (q-q^{-1})^{-1} q^{x-n_1-n_2+n_3+1}[n_1]_q v_{\overline{n}-\overline{\varepsilon}_1},
\end{aligned}
\tag{B.8}
$$

for the module $\tilde{\pi}^1(t)$, where we use the notation $t=q^{2x}$;

$$
\begin{aligned}
y_1 v_{\overline{n}} &= (q-q^{-1})^{-1} q^{x-n_3+1}[n_3]_q v_{\overline{n}-\overline{\varepsilon}_3} + q^{x-2n_3}[n_2]_q v_{\overline{n}+\overline{\varepsilon}_1-\overline{\varepsilon}_2},\\
y_2 v_{\overline{n}} &= q^{n_3+\frac{1}{2}} v_{\overline{n}+\overline{\varepsilon}_2},\\
y_3 v_{\overline{n}} &= -(q-q^{-1})^{-1} q^{x+n_1+n_2-n_3-1}[n_1]_q v_{\overline{n}-\overline{\varepsilon}_1} - q^{x+n_2-n_3-2}[n_2]_q v_{\overline{n}-\overline{\varepsilon}_2+\overline{\varepsilon}_3},
\end{aligned}
\tag{B.9}
$$

for the module $\tilde{\pi}^2(t)$ and

$$\begin{aligned} y_1 v_{\overline{n}} &= -(q-q^{-1})^{-1} q^{x+n_3-1} [n_3]_q v_{\overline{n}-\overline{\varepsilon}_3}, \\ y_2 v_{\overline{n}} &= q^{n_3+\frac{1}{2}} v_{\overline{n}+\overline{\varepsilon}_2}, \\ y_3 v_{\overline{n}} &= -(q-q^{-1})^{-1} q^{x+n_1+n_2-n_3-1} [n_1]_q v_{\overline{n}-\overline{\varepsilon}_1} - q^{x+n_2-n_3-2} [n_2]_q v_{\overline{n}-\overline{\varepsilon}_2+\overline{\varepsilon}_3}, \end{aligned} \tag{B.10}$$

for the module $\tilde{\pi}^3(t)$.

The determinant formula (5.1) is based on the decomposition of the tensor product

$$\tilde{\pi}^1(tq^{2\mu_1}) \otimes \tilde{\pi}^2(tq^{2\mu_2}) \otimes \tilde{\pi}^2(tq^{2\mu_3}). \tag{B.11}$$

The claim is that this tensor product admits a filtration, labelled by six arbitrary non-negative integers k_1, k_2, l_1, l_2, m_1, m_2 with factormodules, isomorphic to

$$\tilde{\pi}^+_\mu(t)[(k_1+k_2)\alpha + (l_1+l_2)\beta + (m_1+m_2)(\alpha+\beta)],$$

where $\alpha = (1,-1,0)$, $\beta = (0,1,-1)$ are simple roots for the Lie algebra $sl(3)$.

On the level of characters this means an equality of the form

$$\begin{gathered} \tilde{\pi}^1(tq^{2\mu_1}) \otimes \tilde{\pi}^2(tq^{2\mu_2}) \otimes \tilde{\pi}^2(tq^{2\mu_3}) = \\ \sum_{k_i,l_i,m_i\geq 0} \tilde{\pi}^+_\mu(t)[(k_1+k_2+m_1+m_2)\alpha + (l_1+l_2+m_1+m_2)\beta], \end{gathered} \tag{B.12}$$

or, if we put $\widetilde{\mathbf{A}}_i(t) = \mathrm{Tr}_{\tilde{\pi}^i(t)}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L}$, which converges in a region (4.48),

$$\widetilde{\mathbf{A}}_1(tq^{2\mu_1}) \widetilde{\mathbf{A}}_2(tq^{2\mu_2}) \widetilde{\mathbf{A}}_3(tq^{2\mu_3}) = \overline{\Delta}^{-2} \mathrm{Tr}_{\tilde{\pi}^+_\mu(t)}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L}, \tag{B.13}$$

where

$$\overline{\Delta} = (1 - z_2/z_1)(1 - z_3/z_1)(1 - z_3/z_2). \tag{B.14}$$

We construct the decomposition of tensor product (B.11) in two steps. First, consider the second order tensor product $\tilde{\pi}^1(t_1) \otimes \tilde{\pi}^2(t_2)$, where we choose the basis $v^{\overline{m}}_{\overline{n}}$, $\overline{n}, \overline{m} \in \mathbb{Z}^3_{\geq 0}$

$$v^{\overline{m}}_{\overline{n}} = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \binom{n_1}{k_1}_q \binom{n_2}{k_2}_q q^{\Omega(\overline{n},\overline{m},\overline{k})} (t_1/t_2)^{k_1/2}\, v_{\overline{m}+\overline{k}} \otimes v_{\overline{n}-\overline{k}}, \tag{B.15}$$

where $\overline{k} = (k_1, k_2, 0)$,

$$(n)_q = \frac{q^{2n}-1}{q^2-1}, \qquad \binom{n}{k}_q = \frac{(n)_q!}{(k)_q!(n-q)_q!}, \tag{B.16}$$

and the phase $\Omega(\overline{n},\overline{m},\overline{k})$ is

$$\Omega(\overline{n},\overline{m},\overline{k})=(n_1-k_1)k_2+(k_2-k_1)m_3+k_2m_2. \tag{B.17}$$

Define the following partial order in $\mathbb{Z}^3_{\geq 0}$: $\overline{m}'\leq\overline{m}$ if

$$m'_3\leq m_3,\ m'_2+m'_3\leq m_2+m_3 \quad \text{and} \quad m'_1+m'_2+m'_3\leq m_1+m_2+m_3. \tag{B.18}$$

Let $V^{\overline{m}}$ be $\mathcal{B}_+$ submodules of $\tilde{\pi}^1(t_1)\otimes\tilde{\pi}^2(t_2)$, generated by the free action of the y_i, $i=1,2,3$,

$$V^{\overline{m}}=\sum_{\overline{m}'\leq\overline{m}}\mathcal{B}_+(v_{\overline{m}'}\otimes v_0). \tag{B.19}$$

Then the vectors $v^{\overline{m}}_{\overline{n}}$, $\overline{n}\in\mathbb{Z}^3_{\geq 0}$ form a basis of the factormodule

$$\widetilde{V}^{\overline{m}}=V^{\overline{m}}/\sum_{\overline{m}'\leq\overline{m}}^{\overline{m}'\neq\overline{m}}V^{\overline{m}'} \tag{B.20}$$

and the action of the generators of $\mathcal{B}_+$ in $\widetilde{V}^{\overline{m}}$ is given by

$$\begin{aligned}
y_1v^{\overline{m}}_{\overline{n}} &= (q-q^{-1})^{-1}q^{x_2-n_3+1}[n_3]_q v^{\overline{m}}_{\overline{n}-\overline{\varepsilon}_3}+q^{x_2-2n_3}[n_2]_q v^{\overline{m}}_{\overline{n}+\overline{\varepsilon}_1-\overline{\varepsilon}_2},\\
y_2v^{\overline{m}}_{\overline{n}} &= q^{n_3+\frac{1}{2}}v^{\overline{m}}_{\overline{n}+\overline{\varepsilon}_2},\\
y_3v^{\overline{m}}_{\overline{n}} &= q^{x_1}[x_1-x_2-n_1-n_2+n_3+1]_q[n_1]_q v^{\overline{m}}_{\overline{n}-\overline{\varepsilon}_1}-\\
&\quad -q^{x_2+n_2-n_3-2}[n_2]_q v^{\overline{m}}_{\overline{n}-\overline{\varepsilon}_2+\overline{\varepsilon}_3},\\
h_1v^{\overline{m}}_{\overline{n}} &= (n_1-n_2-2n_3+m_1-m_2-2m_3)v^{\overline{m}}_{\overline{n}},\\
h_2v^{\overline{m}}_{\overline{n}} &= (n_1+2n_2+n_3+m_1+2m_2+m_3)v^{\overline{m}}_{\overline{n}},\\
h_3v^{\overline{m}}_{\overline{n}} &= (-2n_1-n_2+n_3-2m_1-m_2+m_3)v^{\overline{m}}_{\overline{n}}.
\end{aligned} \tag{B.21}$$

where $t_1=q^{2x_1},\ t_2=q^{2x_2}$.

Let $\tilde{\pi}^{12}(t_1,t_2)$ be the following representation of $\mathcal{B}_+$ with a basis $v_{\overline{n}}$, $\overline{n}\in\mathbb{Z}^3_{\geq 0}$:

$$\begin{aligned}
y_1v_{\overline{n}} &= (q-q^{-1})^{-1}q^{x_2-n_3+1}[n_3]_q v_{\overline{n}-\overline{\varepsilon}_3}+q^{x_2-2n_3}[n_2]_q v_{\overline{n}+\overline{\varepsilon}_1-\overline{\varepsilon}_2},\\
y_2v_{\overline{n}} &= q^{n_3+\frac{1}{2}}v_{\overline{n}+\overline{\varepsilon}_2},\\
y_3v_{\overline{n}} &= q^{x_1}[x_1-x_2+1-n_1-n_2+n_3]_q[n_1]_q v_{\overline{n}-\overline{\varepsilon}_1}-q^{x_2+n_2-n_3-2}[n_2]_q v_{\overline{n}-\overline{\varepsilon}_2+\overline{\varepsilon}_3},
\end{aligned} \tag{B.22}$$

(the Cartan generators act as in (B.5)). Then, the relations (B.21) mean that the tensor product $\tilde{\pi}^1(t_1)\otimes\tilde{\pi}^2(t_2)$ admits a filtration with factormodules isomorphic to

$$\tilde{\pi}^{12}(t_1,t_2)[(m_1+m_2)\alpha+(m_2+m_3)\beta], \qquad m_i\geq 0$$

and

$$\widetilde{\mathbf{A}}_1(t_1)\widetilde{\mathbf{A}}_2(t_2)=\overline{\Delta}^{-1}\mathrm{Tr}_{\tilde{\pi}^{12}(t_1,t_2)}\, e^{i\pi\mathbf{P}\,\mathbf{h}}\mathcal{L}. \tag{B.23}$$

Next, we decompose the tensor product $\tilde{\pi}^{1,2}(t_1,t_2)\otimes\tilde{\pi}^3(t_3)$. Here we use the basis $v_{\overline{n}}^{\overline{m}}$, $\overline{n},\overline{m}\in\mathbb{Z}^3_{\geq 0}$,

$$\begin{aligned} v_{\overline{n}}^{\overline{m}} &= \sum_{j=0}^{\min(n_1,n_3)}\sum_{k_2=0}^{n_2}\sum_{k_3=0}^{n_3-j}(-1)^j(j)_q!\binom{n_2}{k_2}_q\binom{n_3-j}{k_3}_q\binom{n_3}{j}_q\binom{n_1}{j}_q\times \\ &\qquad \times q^{\Omega_j(\overline{n},\overline{m},\overline{k})}\,(t_3/t_2)^{k_3/2}\,v_{\overline{n}-\overline{k}-j(\overline{\varepsilon}_1+\overline{\varepsilon}_3)}\otimes v_{\overline{m}+\overline{k}+j\overline{\varepsilon}_2}, \end{aligned} \tag{B.24}$$

where $\overline{k}=(0,k_2,k_3)$ and the phase $\Omega_j(\overline{n},\overline{m},\overline{k})$ is

$$\begin{aligned}\Omega_j(\overline{n},\overline{m},\overline{k}) &= j(n_2-k_2-n_3+k_3-m_1-2m_2-m_3)+k_2(k_3-n_3-m_1-2m_2+m_3)\\ &\quad +n_1(2m_1+m_2-m_3)+n_2(m_1+2m_2)+n_3(m_2-m_1+2m_3)+\tfrac{3j(j+1)}{2}.\end{aligned}$$

In this case we define the another partial order in $\mathbb{Z}^3_{\geq 0}$: $\overline{m}'\leq\overline{m}$ if

$$m'_1\leq m_1,\quad m'_1+m'_2\leq m_1+m_2\quad\text{and}\quad m'_1+m'_2+m'_3\leq m_1+m_2+m_3.$$

Let $V^{\overline{m}}$ be the following $\mathcal{B}_+$ submodules of $\tilde{\pi}^{12}(t_1,t_2)\otimes\tilde{\pi}^3(t_3)$:

$$V^{\overline{m}}=\sum_{\overline{m}'\leq\overline{m}}\mathcal{B}_+(v_0\otimes v_{\overline{m}'}). \tag{B.25}$$

The vectors $v_{\overline{n}}^{\overline{m}}$, $\overline{n}\in\mathbb{Z}^3_{\geq 0}$ form a basis of the factormodule

$$\widetilde{V}^{\overline{m}}=V^{\overline{m}}/\sum_{\overline{m}'\leq\overline{m}}^{\overline{m}'\neq\overline{m}}V^{\overline{m}'} \tag{B.26}$$

in which

$$\begin{aligned} y_1v_{\overline{n}}^{\overline{m}} &= q^{x_3}[x_2-x_3-n_3+1]_q[n_3]_q v_{\overline{n}-\overline{\varepsilon}_3}^{\overline{m}}+q^{x_2-2n_3}[n_2]_q v_{\overline{n}+\overline{\varepsilon}_1-\overline{\varepsilon}_2}^{\overline{m}},\\ y_2v_{\overline{n}}^{\overline{m}} &= q^{n_3+\frac{1}{2}}v_{\overline{n}+\overline{\varepsilon}_2}^{\overline{m}},\\ y_3v_{\overline{n}}^{\overline{m}} &= q^{x_1}[x_1-x_2-n_1-n_2+n_3+1]_q[n_1]_q v_{\overline{n}-\overline{\varepsilon}_1}^{\overline{m}}-\\ &\quad -q^{x_2+n_2-n_3-2}[n_2]_q v_{\overline{n}-\overline{\varepsilon}_2+\overline{\varepsilon}_3}^{\overline{m}}, \end{aligned} \tag{B.27}$$

with $t_i = q^{2x_i}$ and the Cartan generators acting as in (B.21). We see that the factormodules are shifts of the same $\mathcal{B}_+$-module which is isomorphic to $\tilde{\pi}^+_\mu(t)$, if we put $t_i = tq^{2\mu_i}$. This in combination with (B.23), proves the main statements (B.12) and (B.13).

In order to deduce (5.1) from (B.13) we notice that the degenerated evaluation Verma modules $\tilde{\pi}^i(t_i)$ are not irreducible; each of them admits a filtration indexed by a single nonnegative integer such that all the factors are the shifts of oscillator representations: the representation $\tilde{\pi}^1(t)$ admits an increasing filtration by submodules V^n, where V^n is generated over $\mathcal{B}_+$ by the vectors $v_{(0,0,k)}$, $k \le n$ with factormodules isomorphic to representation $\rho_1(q^2t)[n\beta]$,$n \ge 0$; the representation $\tilde{\pi}^3(t)$ admits an increasing filtration by submodules V^n, which are generated over $\mathcal{B}_+$ by the vectors $v_{(k,0,0)}$, $k \le n$ with factormodules isomorphic to representation $\rho_3(q^{-2}t)[n\alpha]$,$n \ge 0$ and the representation $\tilde{\pi}^2(t)$ admits a decreasing filtration by submodules V^n, which are generated over $\mathcal{B}_+$ by the vector

$$\sum_{k=0}^{n} \frac{[n]_q!}{[k]_q![n-k]_q!} q^{\frac{-k(k-1)}{2}} (q^{-1} - q)^k v_{(k,n-k,k)}. \tag{B.28}$$

The factormodules are isomorphic to $\rho_3(t)[n(\alpha+\beta)]$, $n \ge 0$.

This means that

$$\begin{aligned}
\widetilde{\mathbf{A}}_1(t) &= (1 - z_3/z_2)^{-1} \mathrm{Tr}_{\rho_1(q^2t)}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L} &= \overline{\Delta}^{-1} \mathbf{A}_1(q^2t), \\
\widetilde{\mathbf{A}}_2(t) &= (1 - z_3/z_1)^{-1} \mathrm{Tr}_{\rho_2(t)}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L} &= \overline{\Delta}^{-1} \mathbf{A}_2(t), \\
\widetilde{\mathbf{A}}_3(t) &= (1 - z_2/z_1)^{-1} \mathrm{Tr}_{\rho_3(q^{-2}t)}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L} &= \overline{\Delta}^{-1} \mathbf{A}_3(q^{-2}t).
\end{aligned} \tag{B.29}$$

The relation (B.13) combined with (B.29) gives

$$\mathrm{Tr}_{\tilde{\pi}^+_\mu(t)}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L} = \overline{\Delta}^{-1} \mathbf{A}_1(tq^{2\mu_1+2}) \mathbf{A}_2(tq^{2\mu_2}) \mathbf{A}_3(tq^{2\mu_3-2}). \tag{B.30}$$

The determinant formula (5.1) is a direct corollary of (B.30) and of the Bernstein-Gel'fand-Gel'fand resolution of finite-dimensional module π_μ, which implies the relation

$$\mathbf{T}_\mu(t) = \sum_{\sigma \in S_3} (-1)^{l(\sigma)} \mathbf{T}^+_{\sigma(\mu+\rho)-\rho}(t), \tag{B.31}$$

where $\rho = (1, 0, -1)$.

The determinant formula (5.2) can be deduced in an analogous manner with the use of the second evaluation map or can be deduced from (5.1) in the following way. Due to the definition (4.18), we conjugate the action of the generators of $\mathcal{B}_+$ in the shifted Verma module $\tilde{\pi}^+_\mu(t)$ by σ_{13}. Taking into account the fact that the $U_q(gl(3))$-Verma module $\pi_{\mu_1,\mu_2,\mu_3} \cdot \sigma_{13}$ is isomorphic to the Verma module $\pi_{-\mu_3,-\mu_2,-\mu_1}$, we see that

$$\overline{\mathbf{T}}_\mu(t) = \mathrm{Tr}_{\pi_{-\sigma_{13}(\mu)}(-t)\cdot(1\otimes\sigma_{13})}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L} \qquad \text{and} \qquad \mathbf{T}^+_\mu(t) = \mathrm{Tr}_{\pi^+_{-\sigma_{13}(\mu)}(-t)\cdot(1\otimes\sigma_{13})}\, e^{i\pi \mathbf{P}\,\mathbf{h}} \mathcal{L},$$

which means that we can use the statement of (5.1) for the highest weight $-\sigma_{13}(\mu)$ combined with automorphism σ_{13} of the algebra $\mathcal{B}_+$. Due to the definition (4.46) we get the statement of (5.2).

Let us turn to the relations (5.4), (5.5). Their proof consists of an interpretation of both sides of the equalities as of $\mathcal{B}_+$-modules. Consider, for instance, the tensor product

$$\tilde{\pi}^2(t_2)\otimes\tilde{\pi}^3(t_3),$$

of degenerated evaluation Verma modules. Analogously to (B.22), (B.23) it is isomorphic, on the level of characters, to a sum

$$\sum_{m_i\geq 0}\tilde{\pi}^{23}(t_2,t_3)[(m_1+m_2)\alpha+(m_2+m_3)\beta)],$$

where the module $\tilde{\pi}^{23}(t_2,t_3)$ is the following representation of $\mathcal{B}_+$ with a basis $v_{\overline{n}}$, $\overline{n}\in\mathbb{Z}^3_{\geq 0}$:

$$\begin{aligned} y_1 v_{\overline{n}} &= q^{x_3}[x_2-x_3-n_3+1]_q[n_3]_q v_{\overline{n}-\overline{\varepsilon}_3}+q^{x_2-2n_3}[n_2]_q v_{\overline{n}+\overline{\varepsilon}_1-\overline{\varepsilon}_2},\\ y_2 v_{\overline{n}} &= q^{n_3+\frac{1}{2}}v_{\overline{n}+\overline{\varepsilon}_2},\\ y_3 v_{\overline{n}} &= -(q-q^{-1})^{-1}q^{x_2+n_1+n_2-n_3-1}[n_1]_q v_{\overline{n}-\overline{\varepsilon}_1}-q^{x_2+n_2-n_3-2}[n_2]_q v_{\overline{n}-\overline{\varepsilon}_2+\overline{\varepsilon}_3}, \end{aligned}\tag{B.32}$$

where h_i act as in (B.5), $t_i=q^{2x_i}$. When $t_2=t_3=t$, $\tilde{\pi}^{23}(t_2,t_3)$ has a submodule generated by the vector $v_{\overline{\varepsilon}_3}$ with a basis $\widetilde{v_{\overline{n}}}=v_{\overline{n}+\overline{\varepsilon}_3}$, $\overline{n}\in\mathbb{Z}^3_{\geq 0}$. One can see from (B.33), (B.5) that this submodule is isomorphic to $\tilde{\pi}^{23}(q^{-2}t,q^2t)[\beta]$. The factormodule

$$\tilde{\pi}^{23}(t,t)/\tilde{\pi}^{23}(q^{-2}t,q^2t)[\beta]$$

has a basis $v_{n,m}$, $n,m\geq 0$ with the action of $\mathcal{B}_+$ given by

$$\begin{aligned} y_1 v_{n,m} &= q^{x-2p}[m]_q v_{n+1,m-1},\\ y_2 v_{n,m} &= q^{\frac{1}{2}}v_{n,m+1},\\ y_3 v_{n,m} &= -(q-q^{-1})^{-1}q^{x+n+m-1}[n]_q v_{n-1,m}, \end{aligned}\tag{B.33}$$

$$h_1 v_{n,m}=(n-m)v_{n,m},\qquad h_2 v_{n,m}=(n+2m)v_{n,m},\qquad h_3 v_{n,m}=(-2n-m)v_{n,m}.$$

From (B.33) we can check directly that this factormodule is isomorphic to the representation $\overline{\rho}_1^+(q^{-1}t)$, so

$$\overline{\Delta}\left(\widetilde{\mathbf{A}}_2(t)\widetilde{\mathbf{A}}_3(t)-z_3/z_2\widetilde{\mathbf{A}}_2(q^{-2}t)\widetilde{\mathbf{A}}_3(q^2t)\right)=(1-z_3/z_1)^{-1}(1-z_2/z_1)^{-1}\overline{\mathbf{A}}_1(q^{-1}t),$$

or

$$\mathbf{A}_2(qt)\mathbf{A}_3(q^{-1}t)-z_3/z_2\mathbf{A}_2(q^{-1}t)\mathbf{A}_3(qt)=(1-z_3/z_2)\overline{\mathbf{A}}_1(t),\tag{B.34}$$

which is equivalent to one of the relations (5.4). The others follow from (B.34) after application of the symmetries (4.9) and (4.10).

Appendix C. The $\boldsymbol{p} \to \infty$ asymptotics of $\boldsymbol{\Psi}(\nu)$.

In this Appendix we derive the leading asymptotics of the function $\boldsymbol{\Psi}(\nu)$ at large values of $\boldsymbol{p} = (p_1, p_2)$. We assume p_1 and p_2 tend to infinity while their ratio p_2/p_1 is kept fixed

$$p_1, p_2 \to \infty, \qquad p_2/p_1 = \text{finite}. \tag{C.1}$$

First consider the classical case. Writing the formula (2.28) in full one gets

$$\mathbf{T}^{(cl)}(\lambda) = \boldsymbol{\Lambda}_1^{(cl)}(\lambda) + \boldsymbol{\Lambda}_2^{(cl)}(\lambda) + \boldsymbol{\Lambda}_3^{(cl)}(\lambda), \qquad \boldsymbol{\Lambda}_1^{(cl)}(\lambda)\boldsymbol{\Lambda}_2^{(cl)}(\lambda)\boldsymbol{\Lambda}_3^{(cl)}(\lambda) = 1, \tag{C.2}$$

where $\boldsymbol{\Lambda}_j^{(cl)}(\lambda)$, $j = 1, 2, 3$ are the eigenvalues of the classical monodromy matrix $\mathbf{M}(\lambda)$. The leading $\boldsymbol{p} \to \infty$ contribution to these eigenvalues can be easily calculated directly from the definition (2.27). According to (2.44) the potentials $U(u)$ and $V(u)$ in (2.26) in this limit become independent of u

$$U(u)|_{\boldsymbol{p}\to\infty} \sim x_1x_2 + x_1x_3 + x_2x_3, \qquad V(u)|_{\boldsymbol{p}\to\infty} \sim i x_1x_2x_3, \tag{C.3}$$

where

$$x_j = (\boldsymbol{w}_j \boldsymbol{p}), \qquad j = 1, 2, 3 \tag{C.4}$$

with $\boldsymbol{w}_j$ defined in (2.34). Therefore the eigenvalues of $\mathbf{M}(\lambda)$

$$\boldsymbol{\Lambda}_j^{(cl)}(\lambda) = e^{2\pi i x_j(\lambda)}, \tag{C.5}$$

are expressed though the roots $x_j(\lambda)$, $j = 1, 2, 3$, of the cubic equation

$$x(\lambda)^3 + g_2\, x(\lambda) + g_3 - i\lambda^3 = 0, \tag{C.6}$$

where we have introduced the notation

$$g_2 = x_1x_2 + x_1x_3 + x_2x_3, \qquad g_3 = -x_1x_2x_3. \tag{C.7}$$

The roots of (C.6) can be written as

$$x_j(\lambda) = 2\Big[\Big(-\frac{g_2}{3}\Big)^{\frac{1}{2}}{}_2F_1\big(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; 1 + \frac{27\,(g_3 - i\lambda^3)^2}{4\,g_2^3}\big)\Big]_{j-th}, \tag{C.8}$$

where ${}_2F_1$ stands for the Gauss hypergeometric function. The expression in the square brackets above is a multivalued function and the symbol $[\ldots]_{j-th}$ denotes an appropriate branch of this function fixed by the requirement $x_j(0) = x_j$ where x_j, $j = 1,2,3$ defined in (C.4). Expanding this expression in a series in λ and using the Sommerfeld-Watson transformation one can rewrite (C.8) as

$$\log \boldsymbol{\Lambda}_j^{(cl)}(\lambda) = 2\pi i r_j + \int_{C_\nu} \frac{d\nu}{\nu}\,\Gamma(1-\frac{i\nu}{3})\Gamma(-\frac{1}{3}+\frac{i\nu}{3})\Gamma(\frac{1}{3}+\frac{i\nu}{3})(-i\lambda)^{\frac{i\nu}{3}}\ \boldsymbol{\Psi}_j^{(cl)}(\nu), \quad \text{(C.9)}$$

where

$$\begin{aligned}\boldsymbol{\Psi}_j^{(cl)}(\nu)\Big|_{\boldsymbol{p}\to\infty} &= \frac{2^{-\frac{2i\nu}{3}}}{\sqrt{3\pi}\,\Gamma(1/2+i\nu/3)} \times \\ &\Big[\Big(-\frac{g_2}{3}\Big)^{(1-i\nu)/2}\ {}_2F_1(-\frac{1}{6}+\frac{i\nu}{6},\frac{1}{6}+\frac{i\nu}{6},\frac{1}{2}+\frac{i\nu}{3},1+\frac{27g_3^2}{4g_2^3})\Big]_{j-\text{th}} \quad \text{(C.10)}\end{aligned}$$

where the integration contour goes along the line $\Im m\,\nu = -1-\epsilon$ and the branches of the expression in the square brackets are chosen in accordance with (C.8)

$$\boldsymbol{\Psi}_j^{(cl)}(0) = \frac{(\boldsymbol{w}_j\boldsymbol{p})}{2\pi\sqrt{3}}. \quad \text{(C.11)}$$

Now turn to the quantum case. As before we restrict our considerations to the case $0 < g < \frac{2}{3}$. Examining expressions (6.9) it is easy to conclude that the leading $\boldsymbol{p} \to \infty$ asymptotics of $\boldsymbol{\Psi}(\nu)$ is the same for all its eigenvalues, i.e. $\boldsymbol{\Psi}(\nu)|_{\boldsymbol{p}\to\infty}$ is proportional to the unit operator $\mathbf{I}$. Further we make an assumption (which again can be justified by studying the $\boldsymbol{p} \to \infty$ of the Coulomb integrals (4.75)) that the $\boldsymbol{p}$ and g dependence of $\boldsymbol{\Psi}(\nu)|_{\boldsymbol{p}\to\infty}$ factorizes in the form

$$\boldsymbol{\Psi}(\lambda)|_{\boldsymbol{p}\to\infty} \sim \psi_1(\nu,g)\psi_2(\nu,\boldsymbol{p})\ \mathbf{I}. \quad \text{(C.12)}$$

Comparing (C.9) with the $g \to 0$ limit of (6.9) one gets

$$\boldsymbol{\Psi}_j(\nu)|_{\boldsymbol{p}\to\infty} = \frac{\psi_1(\nu,g)}{\psi_1(\nu,0)}\boldsymbol{\Psi}_j^{(cl)}(\nu)|_{\boldsymbol{p}\to\infty}. \quad \text{(C.13)}$$

To calculate the g-dependent multiplier in the last formula we will use the DDV equation. Since the asymptotics (6.16) are the same for all eigenvalues of $\boldsymbol{\Psi}(\nu)$ it is enough to consider the vacuum eigenvalue. Moreover, since the multiplier we want to calculate does not depend on $\boldsymbol{p}$ we will restrict ourselves to the case $p_2 = 0$ and denote

$$p_1 = p, \qquad x_1 = p, \qquad x_2 = 0, \qquad x_3 = -p. \quad \text{(C.14)}$$

In this case the vacuum eigenvalues $A_1^{(vac)}(t)$ and $\overline{A}_3^{(vac)}(t)$ of the operator $\mathbf{A}_1(t)$ and $\overline{\mathbf{A}}_3(t)$ satisfy the relation

$$\overline{A}_3^{(vac)}(t) = A_1^{(vac)}(-t). \quad \text{(C.15)}$$

Define the function

$$a(t)=e^{2\pi ip}\frac{A_1^{(vac)}(itq^2)}{A_1^{(vac)}(itq^{-2})}\frac{A_1^{(vac)}(-itq^{-1})}{A_1^{(vac)}(-itq)}. \tag{C.16}$$

Recalling the assumption about the structure of the zeros of $A_1(t)$ in the beginning of Section 6 (the zeroes of $A_1^{(vac)}(t)$ lie on the positive imaginary axis of t) and using (6.7) we can show for real p

$$a(t)^* = a(t^*)^{-1}. \tag{C.17}$$

Due to (6.1) $a(t)$ has the asymptotics

$$\log a(t) \sim -2iM\sin\left(\frac{\pi(1+\xi)}{3}\right)\, t^{\frac{1+\xi}{3}}, \qquad t\to\infty, \qquad |\arg(t)|<\min[2\pi g,\pi(1-g)] \tag{C.18}$$

while for small t, we obviously have

$$a(t)=2\pi ip+O(t), \qquad t\to 0. \tag{C.19}$$

Again using (6.7) we have

$$\log a(t)=2\pi ip+\sum_k F(itt_k^{-1}), \tag{C.20}$$

where

$$F(t)=\log\left(\frac{1-tq^2}{1-tq^{-2}}\frac{1+tq^{-1}}{1+tq}\right). \tag{C.21}$$

Using the Bethe Ansatz equations (6.3), (6.4) we can write the sum in (C.20) as a contour integral

$$\log a(t)=2\pi ip+\int_C\frac{dv}{2\pi i}F(tv^{-1})\partial_v\log(1+a(v)), \tag{C.22}$$

where the contour C goes from $+\infty$ to zero above the real axis, winds about zero, and returns to $+\infty$ below the real axis. Integrating by parts (boundary terms vanish) one obtains

$$\log a(t)-2\pi ip=\frac{1}{2\pi i}\int_0^\infty\frac{dv}{v}\; t\partial_t F(tv^{-1})\left\{\log(1+a(v-i0))-\log(1+a(v+i0))\right\},$$

$$=\int_0^\infty\frac{dv}{\pi v}t\partial_t F(tv^{-1})\Im m\log(1+a(v-i0))-\int_0^\infty\frac{dv}{2\pi iv}t\partial_t F(tv^{-1})\log a(v),$$

where we have used (C.17) to obtain the second line.

We now introduce new variables

$$t=e^{\frac{3\theta}{1+\xi}}, \quad v=e^{\frac{3\theta'}{1+\xi}}, \tag{C.23}$$

and recalling that $q = \exp(i\pi g)$, we obtain

$$\log a(\theta) = 2\pi i p \quad - \quad 2i \int_{-\infty}^{\infty} R(\theta - \theta') \Im m \log(1 + a(\theta' - i0))\, d\theta' \\ + \int_{-\infty}^{\infty} R(\theta - \theta') \log a(\theta')\, d\theta', \tag{C.24}$$

where

$$R(\theta) = \frac{3i}{2\pi(1+\xi)} t\partial_t F(t). \tag{C.25}$$

With the standard notation for the convolution integral the Eq. (C.24) becomes

$$K * \log a(\theta) = 2\pi i p - 2iR * \Im m \log(1 + a(\theta - i0)), \tag{C.26}$$

with $K(\theta) = \delta(\theta) - R(\theta)$. We now apply the inverse of the integral operator $K(\theta)$ to both sides of (C.26). Since $K(\theta)$ is a singular operator one has to add an appropriate zero mode of K to the RHS of the resulting equation. The required zero mode is completely fixed by the asymptotic conditions (C.18) and (C.19). In this way one obtains

$$i \log a(\theta) = -\frac{2\pi p}{g} - 2G * \Im m \log(1 + a(\theta - i0)) + 2M \sin\left(\frac{\pi(1+\xi)}{3}\right) e^{\theta}, \tag{C.27}$$

where $G(\theta) = -(K^{-1} * R)(\theta)$.

Now define

$$B(p) = \frac{(1+\xi)}{3} \log(-it_1), \tag{C.28}$$

where t_1 is the smallest (in absolute value) zero of $A_1^{vac}(t)$. The convolution term in (C.27) can be written as

$$G * \Im m \log(1 + a^{vac}(\theta - i0)) \quad = \quad \int_{-\infty}^{B(p)} G(\theta - \theta') \Im m \log(1 + a^{vac}(\theta' - i0))\, d\theta' \\ + \int_{B(p)}^{\infty} G(\theta - \theta') \Im m \log(1 + a^{vac}(\theta' - i0))\, d\theta'. \tag{C.29}$$

Let us make the technical assumption that

$$B(p) \sim \text{const} \log p, \quad p \to \infty. \tag{C.30}$$

Then one can show the second term in (C.29) vanishes when $p \to +\infty$ and therefore can be dropped in the leading large p approximation. The Eq. (C.27) then becomes

$$-\frac{\pi p}{g} + M \sin\left(\frac{\pi(1+\xi)}{3}\right) e^{\theta} = \int_{-\infty}^{B(p)} \frac{d\theta'}{2\pi i} \partial_\theta \log S(\theta - \theta') \Im m \log\left(1 + a(\theta')\right), \tag{C.31}$$

where we have used the notation

$$G(\theta) = \delta(\theta) + \frac{1}{2\pi i} \partial_\theta \log S(\theta), \tag{C.32}$$

$$\log S(\theta) = -i \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \sin(\nu\theta) \frac{\sinh\left(\frac{\pi\nu(1+\xi)}{3}\right)\cosh\left(\frac{\pi\nu}{6}\right)}{\sinh\left(\frac{\pi\nu\xi}{3}\right)\cosh\left(\frac{\pi\nu}{2}\right)}, \tag{C.33}$$

and the relation

$$\log a^{vac}(\theta) = 2i \log \Im m \log(1 + a^{vac}(\theta)), \quad \theta < B(p) \tag{C.34}$$

which holds for $\theta < B(p)$. The function $S(\theta)$ is Smirnov's kink-kink S-matrix for the algebra $A_2^{(2)}$, [66].

The Fourier transform of the integral kernel in (C.31) factorizes as

$$-\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-i\nu\theta} \partial_\theta \log S(\theta) d\theta = \frac{1}{K_+(\nu) K_-(\nu)}, \tag{C.35}$$

where

$$K_+(\nu) = \frac{1}{\sqrt{g}} \frac{\Gamma\left(-\frac{i\nu(1+\xi)}{3}\right)\Gamma\left(\frac{1}{2} - \frac{i\nu}{6}\right) e^{i\nu\delta}}{\Gamma\left(-\frac{i\nu\xi}{3}\right)\Gamma\left(\frac{1}{2} - \frac{i\nu}{2}\right)}, \qquad K_-(\nu) = K_+(-\nu), \tag{C.36}$$

$$3\delta = (1+\xi)\log(1+\xi) - \xi \log \xi + \log \frac{2}{3\sqrt{3}}. \tag{C.37}$$

The function $K_+(\nu)$ is analytic and nonzero in the upper half-plane $\Im m\ \nu > -1$ where it has the asymptotics

$$K_+(\nu) \sim 1 + O\left(\frac{1}{\nu}\right), \qquad \nu \to \infty. \tag{C.38}$$

The Eq.(C.31) is an equation of Wiener-Hopf type, which can be solved by the standard technique. Defining the Fourier transform

$$f_+(\nu) = \int_{-\infty}^{B(p)} \Im m \log(1 + a^{vac}(\theta)) e^{-i\theta\nu}\, d\theta, \tag{C.39}$$

one obtains

$$\frac{f_+(\nu)}{K_+(\nu)} e^{i\nu B(p)} = \frac{1}{2\pi i} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{1}{\nu' - \nu} K_-(\nu') G_+(\nu')\, d\nu', \tag{C.40}$$

where

$$\begin{aligned} G_+(\nu) &= \int_{-\infty}^{0} \left(\frac{\pi p}{g} - M \sin\left(\frac{\pi(1+\xi)}{3}\right) e^{\theta + B(p)}\right) e^{-i\theta\nu}\, d\nu, \\ &= \frac{i\pi p}{g\nu} - \frac{iM}{\nu + i} \sin\left(\frac{\pi(1+\xi)}{3}\right) e^{B(p)}, \end{aligned} \tag{C.41}$$

and ϵ is a small positive constant. Completing the contour in the lower half-plane, we have

$$\frac{f_+(\nu)}{K_+(\nu)} e^{i\nu B(p)} = \frac{i\pi p}{\sqrt{g}\nu} - \frac{iM\pi e^{B(p)-\delta}}{\sqrt{g}(\nu + i)} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{\xi}{3}\right)\Gamma\left(\frac{2-\xi}{3}\right)}. \tag{C.42}$$

As follows from the definition (C.39) the leading $1/\nu$ term in the $\nu \to \infty$ asymptotic expansion of the RHS of (C.42) is independent of p. This gives

$$(-it_1)^{\frac{1+\xi}{3}} = p\frac{\Gamma\left(\frac{\xi}{3}\right)\Gamma\left(\frac{2-\xi}{3}\right)}{M\Gamma\left(\frac{2}{3}\right)}e^{\delta}, \tag{C.43}$$

and thus we have

$$f_+(\nu) = \frac{i\pi p}{2\nu}\frac{\Gamma\left(1-\frac{i\nu(1+\xi)}{3}\right)\Gamma\left(\frac{1}{2}-\frac{i\nu}{6}\right)}{\Gamma\left(1-\frac{i\xi\nu}{3}\right)\Gamma\left(\frac{3}{2}-\frac{i\nu}{2}\right)}e^{i\nu\delta}(-it_1)^{\frac{-i\nu(1+\xi)}{3}}. \tag{C.44}$$

Given the function $f_+(\nu)$, (C.44), we can recover $\log A_1(t)$ from

$$\log A_1(t) = -\frac{i}{2\pi}\int_{C_\nu}\frac{\cosh\left(\frac{\pi\nu}{6}\right)}{\cosh\left(\frac{\pi\nu}{2}\right)\sinh\left(\frac{\pi\xi\nu}{3}\right)}f_+(\nu)(it)^{\frac{i\nu(1+\xi)}{3}}\,d\nu, \tag{C.45}$$

where the contour C_ν is along the line $\Im m\ \nu = -1-\epsilon$, with ϵ is a small positive constant. Thus we have

$$\log A_1^{vac}(t) = -\frac{3p}{4\pi\xi}\int_{C_\nu}\frac{e^{i\nu\delta}}{\nu^2}\frac{\Gamma\left(1-\frac{i\nu(1+\xi)}{3}\right)\Gamma\left(-\frac{1}{2}+\frac{i\nu}{2}\right)\Gamma\left(1+\frac{i\xi\nu}{3}\right)}{\Gamma\left(\frac{1}{2}+\frac{i\nu}{6}\right)}\left(-\frac{t}{t_1}\right)^{\frac{i\nu(1+\xi)}{3}}d\nu. \tag{C.46}$$

This expression can be written in the form of (6.9)

$$\begin{aligned}\log \mathbf{A}_i(t) \quad &= \quad \frac{1}{2\pi i}\int_{C_\nu}\frac{d\nu}{\nu}\,\Gamma(i\nu\xi/3)\Gamma(1-i\nu(\xi+1)/3)\Gamma((-1+i\nu)/3)\times \\ &\qquad \Gamma((1+i\nu)/3)\Big(\Gamma(1-g)\Big)^{i\nu(1+\xi)}(it)^{i\nu(1+\xi)/3}\ \mathbf{\Psi}(\nu)|_{p\to\infty},\end{aligned} \tag{C.47}$$

where

$$\mathbf{\Psi}(\nu)|_{p\to\infty} = \frac{2^{-\frac{2i\nu}{3}}}{\sqrt{3}}\left(\frac{p^2}{3}\right)^{(1-i\nu)/2}\frac{1}{\Gamma(1/3+i\nu/6)\Gamma(2/3+i\nu/6)}. \tag{C.48}$$

Comparing this with the specialization of (C.10) for the case (C.14), one concludes that the factor in (C.13)

$$\frac{\psi_1(\nu,g)}{\psi_1(\nu,0)} \equiv 1. \tag{C.49}$$

Taking into account (C.13), we obtain the expression (6.30) given in the main text.

To get a small t expansion of $A_1^{vac}(t)$, we complete the contour in the lower-half plane, and evaluate the residues at the points $\nu = 3i(g-1)n,\quad n = 1,2,3\ldots$, and we obtain the large p asymptotic equivalent of (4.55). Thus, we have from (C.46)

$$H_1^{(1)(vac)}|_{p\to\infty} = -\frac{p}{2}\frac{e^{3(1-g)\delta}}{\Gamma^3(1-g)}\frac{\Gamma\left(1-\frac{3}{2}g\right)\Gamma(g)}{\Gamma\left(1-\frac{1}{2}g\right)}g^3(-it_1)^{-1}. \tag{C.50}$$

However, we know $H_1^{(1)(vac)}$ exactly (4.85). Taking the large p asymptotic of (4.85), comparing with (C.50), and using (C.43) we see that we must take

$$M = \frac{\Gamma\left(\frac{\xi}{3}\right)\Gamma\left(\frac{2-\xi}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\left(\Gamma(1-g)\right)^{\frac{1}{1-g}}, \tag{C.51}$$

and then

$$H_n^{(1)(vac)}|_{p\to\infty} = \frac{(-1)^n}{2n!}\frac{\Gamma\left(\frac{3(1-g)n-1}{2}\right)\Gamma(gn)}{\Gamma\left(\frac{(1-g)n+1}{2}\right)}g^{3n}p^{1-3(1-g)n}. \tag{C.52}$$

We also generate an asymptotic expansion for large t by completing the contour in the upper half-plane. Here we have poles at the following points:

$$\nu = i(2n-1), \quad n = 0,1,2,\ldots, \quad \nu = 3i(g^{-1}-1)n, \quad n = 0,1,2,3\ldots, \tag{C.53}$$

and we obtain

$$\begin{aligned} A_1^{vac}(t) &= C^{vac}(g,p)(s)^{-\frac{p}{g}}\exp\left\{\sum_{n=0}^{\infty} B_{2n-1}I_{2n-1}^{vac}|_{p\to\infty}(s)^{-\frac{2n-1}{3(1-g)}}\right\} \\ &\times \exp\left\{-\sum_{n=1}^{\infty}\tilde{H}_n^{(1)(vac)}|_{p\to\infty}(s)^{-\frac{n}{g}}\right\}, \end{aligned} \tag{C.54}$$

where

$$B_{2n-1} = \frac{(-1)^{n+1}}{3(1-g)(n)!}\frac{\Gamma\left(\frac{2n-1}{3(1-g)}\right)\Gamma\left(\frac{2n-1}{3(1-g^{-1})}\right)}{\Gamma\left(\frac{2-n}{3}\right)}g^{\frac{n(1+g)-1}{g-1}}, \tag{C.55}$$

$$C^{(vac)}(g,p) = \left(\frac{2^{\frac{2}{3}}p}{e}\right)^{\frac{3p}{g}-3p}(g)^{-\frac{3p}{g}}, \qquad \tilde{H}_n^{(1)(vac)}(g,p) = H_n^{(1)(vac)}(g^{-1},g^{-1}p). \tag{C.56}$$

and $I_{2n-1}^{vac}|_{p\to\infty}$, $n = 0,1,2,\ldots$ are the remaining Integrals of Motion after the $A_2^{(2)}$ specialization has been taken into account ($W(u)\equiv 0$),

$$I_{-1}^{vac} \equiv 1, \qquad I_{2n-1}^{vac}|_{p\to\infty} = \frac{p^{2n}}{g^n}, \quad n = 1,2,3,\ldots \tag{C.57}$$

Note also that $C^{(vac)}(g,p)$ is the vacuum asymptotic value of the operator $\mathbf{C}_j\{g, g\boldsymbol{\phi}(u)\}$ defined in (6.16).

Appendix D. Representation Matrices for the Fundamental Representations

For the first fundamental representation π_{ω_1} we have

$$\pi_{\omega_1}(H_\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \pi_{\omega_1}(E_\alpha) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \pi_{\omega_1}(F_\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{D.1}$$

$$\pi_{\omega_1}(H_\beta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \pi_{\omega_1}(E_\beta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \pi_{\omega_1}(F_\beta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{D.2}$$

while for π_{ω_2} we have

$$\pi_{\omega_2}(H_\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \pi_{\omega_2}(E_\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \pi_{\omega_2}(F_\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{D.3}$$

$$\pi_{\omega_2}(H_\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \pi_{\omega_2}(E_\beta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \pi_{\omega_2}(F_\beta) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{D.4}$$

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