Space-like Submanifolds in the de Sitter Spaces*

Liu Ximin
Department of Applied Mathematics,
Dalian University of Technology, Dalian 116024, China
and
Max-Planck-Institut für Mathematik,
Vivatsgasse 7, 53111 Bonn, Germany
E-mail: xmliu@dlut.edu.cn

Abstract

In this paper, by using Cheng-Yau's self-adjoint operator $\Box$, we study the space-like submanifolds in the de Sitter spaces and obtain some general rigidity results.
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1 Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p and simply a space form when $p = 0$. If $c > 0$, we call it as a de Sitter space of index p, denote it by $S_p^{n+p}(c)$. The study of space-like hypersurfaces in de Sitter space has been recently of substantial interest from both physics and mathematical points of view. Akutagawa [2] and Ramanathan [16] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature H satisfies $H^2 \leq c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geq 3$. Later, Cheng [5] generalized this result to general submanifolds in a de Sitter space.

On the other hand, Cheng and Ishikawa [6] have recently shown that the totally umbilical round spheres are the only compact space-like hypersurfaces in $S_1^{n+1}(1)$ with constant scalar curvature $S < n(n-1)$. Some other authors, such as Liu [12] and Zheng [18, 19], have also obtained interesting results related to the characterization of the totally umbilical round spheres as the only compact space-like hypersurfaces in the de Sitter space with constant scalar curvature.

In [11], H. Li first studied the Cheng-Yau's self-adjoint operator $\Box$ for a given codazzi tensor field $\phi = \sum_{i,j} \phi_{ij}\omega_i\omega_j$ on an n-dimensional compact Riemannian manifold and obtain a general rigidity theorem which generalized Cheng-Yau's work [8]. By using this result he studied the hypersurfaces in real space form and obtain general rigidity

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results which unify some existing results. Also he studied the space-like hypersurfaces in Lorentzian space form and obtained some rigidity theorems which naturally generalize the existing results of Akutagawa [2], Ramanathan [16] and Montiel [13] about Goddard's conjecture [9].

In the present paper, we would like to use Cheng-Yau's self-adjoint operator $\Box$ to study the space-like submanifolds in de Sitter space and obtain some general rigidity results, also in the last section of this paper we discuss the space-like hypersurfaces in the de Sitter spaces and get some very interesting rigidity results.

2 Cheng-Yau's self-adjoint operator $\Box$

In this section, we review the fundamental results about the Cheng-Yau's self-adjoint operator $\Box$, for details see [11].

Let M^n be an n-dimensional Riemannian manifold, $e_1, \ldots, e_n$ a local orthonormal frame field on M^n, and let $\omega_1, \ldots, \omega_n$ be its dual coframe field. Then the structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{1}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \tag{2}$$

where

$$\Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{3}$$

and

$$R_{ijkl} + R_{ijlk} = 0,$$

where ω_{ij} is the Levi-Civita connnection form and R_{ijkl} are the components of the curvature tensor of M^n.

For any C^2-function f defined on M^n, we defined its gradient and Hessian by the following formulas

$$df = \sum_i f_i \omega_i, \tag{4}$$

$$\sum_j f_{ij} \omega_j = df_i + \sum_j f_j \omega_{ji}. \tag{5}$$

We know that $f_{ij} = f_{ji}$ by exterior differentiation of (4).

Let $\phi = \sum_{ij} \phi_{ij} \omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n. The covariant derivative of ϕ_{ij} is defined by

$$\sum_k \phi_{ijk} \omega_k = d\phi_{ij} + \sum_k \phi_{kj} \omega_{ki} + \sum_k \phi_{ik} \omega_{kj}. \tag{6}$$

We call the symmetric tensor $\phi = \sum_{i,j} \phi_{ij} \omega_i \otimes \omega_j$ a Codazzi tensor, if

$$\phi_{ijk} = \phi_{ikj}. \tag{7}$$

The second covariant derivative of ϕ_{ij} is defined by

$$\sum_l \phi_{ijkl} \omega_l = d\phi_{ijk} + \sum_m \phi_{mjk} \omega_{mi} + \sum_m \phi_{imk} \omega_{mj} + \sum_m \phi_{ijm} \omega_{mk}. \tag{8}$$

By exterior differentiation of (6), we have

$$\sum_{l,k} \phi_{ijkl}\omega_j \wedge \omega_k = \sum_m \phi_{mj}\Omega_{mi} + \sum_m \phi_{im}\Omega_{mj}. \tag{9}$$

Then we have the following Ricci identities

$$\phi_{ijkl} - \phi_{ijlk} = \sum_m \phi_{mj}R_{mikl} + \sum_m \phi_{im}R_{mjkl}. \tag{10}$$

The definition of the following self-adjoint operator $\square$ was first introduced by Cheng-Yau in [8].

Definition 2.1. Let $\phi = \sum_{i,j} \omega_i \otimes \omega_j$ be a Codazzi tensor field on a Riemanian manifold M^n. We define the opertator $\square$ associated to ϕ by

$$\square f = \sum_{i,j}((\sum_k \phi_{kk})\delta_{ij} - \phi_{ij})f_{ij},$$

for any C^2-function f defined on M^n.

The Laplacian of the tensor ϕ_{ij} is defined to be $\sum_k \phi_{ijkk}$, and therefore

$$\begin{aligned}
\Delta\phi_{ij} &= \sum_k \phi_{ijkk} \\
&= \sum_k(\phi_{ijkk} - \phi_{ikjk}) + \sum_k(\phi_{ikjk} - \phi_{ikkj}) + \sum_k(\phi_{ikkj} - \phi_{kkij}) + \sum_k \phi_{kkij} \\
&= \sum_{m,k} \phi_{mk}R_{mijk} + \sum_{m,k} \phi_{im}R_{mkjk} + \sum_k(\phi_{ijkk} - \phi_{ikjk}) \\
&\quad + \sum_k(\phi_{ikkj} - \phi_{kkij}) + (\sum_k \phi_{kk})_{ij}. \qquad (11)
\end{aligned}$$

By use of (7), we have

$$\Delta\phi_{ij} = (\sum_k \phi_{kk})_{ij} + \sum_{m,k} \phi_{mk}R_{mijk} + \sum_{m,k} \phi_{im}R_{mkjk} \tag{12}$$

Let $|\phi|^2 = \sum_{i,j} \phi_{ij}^2$, $|\nabla\phi|^2 = \sum_{i,j,k} \phi_{ijk}^2$ and $tr\ \phi = \sum_k \phi_{kk}$. Then from (12) we have

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \sum_{i,j} \phi_{ij}(tr\ \phi)_{ij} + \sum_{i,j,m,k} \phi_{ij}\phi_{mk}R_{mijk} + \sum_{i,j,m,k} \phi_{ij}\phi_{im}R_{mkjk}. \tag{13}$$

Near a given point $p \in M$, we choose a local orthonormal frame field $\{e_1, \ldots, e_n\}$ and its dual frame field $\{\omega_1, \ldots, \omega_n\}$ such that $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$, $\phi_{ij} = \lambda_i\delta_{ij}$ at p. Then (13) is simplified to

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \sum_i \lambda_i(tr\ \phi)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{14}$$

Denoting the second symmetric function of ϕ_{ij} by m, we have

$$m = \sum_{i\neq j} \lambda_i\lambda_j = (tr\ \phi)^2 - |\phi|^2. \tag{15}$$

From (14) and (15) we have

$$\frac{1}{2}\Delta(tr\,\phi)^2 = \frac{1}{2}\Delta m + |\nabla\phi|^2 + \sum_i \lambda_i (tr\,\phi)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{16}$$

From Definition 2.1 of $\square$, from (16) we have

$$\square(tr\,\phi) = \frac{1}{2}\Delta m + |\nabla\phi|^2 - |\nabla(tr\,\phi)|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2. \tag{17}$$

Since $\square$ is self-adjoint and M^n is compact, we obtain by integration of (17)

$$\int_{M^n} [|\nabla\phi|^2 - |\nabla(tr\,\phi)|^2] + \int_{M^n} \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 = 0. \tag{18}$$

So we have the following theorem.

Theorem 2.1 [11]. Let $\phi = \sum_{i,j} \omega_i \otimes \omega_j$ be a Codazzi tensor field on a Riemanian manifold M^n. We assume the following condition

$$|\nabla\phi|^2 \geq |\nabla(tr\,\phi)|^2. \tag{19}$$

(1) If M^n has positive sectional curvature, then all the eigenvalues of ϕ_{ij} are the same on M^n.

(2) If M^n has nonnegative sectional curvature, then we have $|\nabla\phi|^2 = |\nabla(tr\,\phi)|^2$ and $R_{ijij} = 0$, when $\lambda_i \neq \lambda_j$ on M^n.

Proposition 2.1 [11]. If

$$tr\,\phi = constant, \tag{20}$$

then (19) holds.

Proposition 2.2 [11]. If the second symmetric function of ϕ_{ij} is a nonnegative constant, i.e.

$$m = \sum_{i\neq j} \lambda_i\lambda_j = (tr\,\phi)^2 - |\phi|^2 = constant \geq 0,$$

then (19) holds.

M. Okumura [15] established the following lemma (see also [3]).

Lemma 2.1. Let μ_i, $i = 1, \ldots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = constant \geq 0$. then

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{21}$$

and the equality holds in (21) if and only if at least $(n-1)$ of the μ_i are equal.

3 Space-like submanifolds in de Sitter space

Let $S_p^{n+p}(c)$ be an $(n+p)$-dimensional de Sitter of constant curvature c whose index is p. Let M^n be an n-dimensional Riemannian manifold immersed in $S_p^{n+p}(c)$. As the semi-Riemannian metric of $S_p^{n+p}(c)$ induces the Riemannian metric of M^n, M^n is called a space-like submanifold. We choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in $S_p^{n+p}(c)$ such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of

M^n and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \le A, B, C, \ldots \le n+p; \quad 1 \le i, j, k, \ldots \le n; \quad n+1 \le \alpha, \beta, \gamma \le n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $S_p^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$. Then the structure equations of $S_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{22}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{23}$$

$$K_{ABCD} = c\, \epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{24}$$

Restrict these form to M^n, we have

$$\omega_\alpha = 0, \qquad n+1 \le \alpha \le n+p, \tag{25}$$

the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. From Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{26}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{27}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{28}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{29}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_\alpha h_\alpha e_\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha \tag{30}$$

is the second fundamental form of M^n.

For indefinite Riemannian manifolds in detail, refer to O'Neill [14].

Denote $L_\alpha = (h_{ij}^\alpha)_{n\times n}$ and $H_\alpha = (1/n)\sum_i h_{ii}^\alpha$ for $\alpha = n+1, \cdots, n+p$. Then the mean curvature vector field ξ, the mean curvature H and the square of the length of the second fundamental form S are expressed as

$$\xi = \sum_\alpha H_\alpha e_\alpha, \quad H = |\xi|, \quad S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2, \tag{31}$$

respectively. Moreover, the normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the normalized scalar curvature R are expressed as

$$\begin{aligned} R_{\alpha\beta kl} &= \sum_m (h^\alpha_{km}h^\beta_{ml} - h^\alpha_{lm}h^\beta_{mk}), \\ R_{ik} &= (n-1)\,c\,\delta_{ik} + n\sum_\alpha (H_\alpha)h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk}, \\ R &= c + \frac{1}{n(n-1)}(S - n^2H^2). \end{aligned} \tag{32}$$

If there exist p functions ρ_α such that $h^\alpha_{ij} = \rho_\alpha\delta_{ij}$ at each point of M^n, we call M^n a totally umbilical submanifold. For a totally umbilical submanifold, we have

$$\rho_\alpha = \frac{1}{n} tr\ h_\alpha = \frac{1}{n}\sum_i h^\alpha_{ii}. \tag{33}$$

If $R_{\alpha\beta kl} = 0$ at point x of M^n we say that the normal connection of M^n is flat at x and it is well known [4] that $R_{\alpha\beta kl} = 0$ at x if and only if h_α are simultaneously diagonalizable at x.

Define the first and the second covariant derivatives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$ by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_k h^\alpha_{ik}\omega_{kj} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}, \tag{34}$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_m h^\alpha_{mjk}\omega_{mi} + \sum_m h^\alpha_{imk}\omega_{mj} + \sum_m h^\alpha_{ijm}\omega_{mk} + \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{35}$$

Then, by exterior differentiation of (26), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj}. \tag{36}$$

It follows from Ricci's identity that

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{37}$$

Suppose $H > 0$ on M^n and choose $e_{n+1} = \xi/H$. Then it follows that

$$H_{n+1} = H; \quad H_\alpha = 0, \qquad \alpha > n+1. \tag{38}$$

We set

$$S_\alpha = \sum_{i,j}(h^\alpha_{ij})^2, \quad S_I = \sum_{\beta > n+1} S_\beta. \tag{39}$$

Suppose that the normal bundle of M^n is flat, then we can choose $e_1, \ldots, e_n$ such that

$$h^\alpha_{ij} = \lambda^\alpha_i\delta_{ij}, \quad \alpha = n+1, \cdots, n+p. \tag{40}$$

Let $\phi_{ij} = h^\alpha_{ij}$. From (18) we have

$$\int_{M^n}[|\nabla h|^2 - n^2|\nabla H|^2] + \int_{M^n}\frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda^\alpha_i - \lambda^\alpha_j)^2 = 0. \tag{41}$$

By use of Gauss equation, we have

$$\frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^{n+1}-\lambda_j^{n+1})^2 = ncS_{n+1} - n^2H^2c + S_{n+1}^2 - nH\sum_i(\lambda_i^{n+1})^3. \tag{42}$$

Let $\mu_i^{n+1} = \lambda_i^{n+1} - H$ and $|Z|^2 = \sum_i(\mu_i^{n+1})^2$. We have

$$\sum_i \mu_i^{n+1} = 0, \quad |Z|^2 = S_{n+1} - nH^2, \tag{43}$$

$$\sum_i(\lambda_i^{n+1})^3 = \sum_i(\mu_i^{n+1})^3 + 3H|Z|^2 - nH^3, \tag{44}$$

Putting (43) and (44) into (42), we get

$$\frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^{n+1}-\lambda_j^{n+1})^2 = |Z|^2(nc - nH^2 + |Z|^2) - nH\sum_i(\mu_i^{n+1})^3. \tag{45}$$

By Using of Lemma 2.1, we have

$$\begin{aligned} &\frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^{\alpha}-\lambda_j^{\alpha})^2 \\ \geq\ & (S_{n+1}-nH^2)(nc - 2nH^2 + S_{n+1} - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_{n+1}-nH^2}) \end{aligned} \tag{46}$$

Putting (46) into (41), we hve

$$\begin{aligned} &\int_{M^n}[(|\nabla h|^2 - n^2|\nabla H|^2) \\ &+(S_{n+1}-nH^2)(nc - 2nH^2 + S_{n+1} - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S-nH^2})] \leq 0. \end{aligned} \tag{47}$$

Note that

$$\begin{aligned} &nc - 2nH^2 + S_{n+1} - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{S_{n+1}-nH^2} \\ &= (\sqrt{S_{n+1}-nH^2} + \frac{1}{2}(n-2)H\sqrt{\frac{n}{n-1}})^2 + n(c - \frac{n^2}{4(n-1)}H^2). \end{aligned} \tag{48}$$

From (47) and (48), we have the following result.

Theorem 3.1. Let M^n be an n-dimentional compact space-like submanifold in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that M^n has flat normal bundle, if

$$|\nabla h|^2 \geq n^2|\nabla H|^2 \tag{49}$$

and

$$H^2 < 4\frac{(n-1)c}{n^2}, \tag{50}$$

then $S_{n+1} \equiv nH^2$ and M^n is a pseudo-umbilical submanifold.

Corollary 3.1 Let M^n be an n-dimentional compact space-like submanifold with constant scalar curvature R in an $(n+p)$-dimensional de Sitter space $S_p^{n+p}(c)$. Suppose that M^n has flat normal bundle, if $R - c \leq 0$ and

$$H^2 < 4\frac{(n-1)c}{n^2}, \tag{51}$$

then $S_{n+1} \equiv nH^2$ and M^n is a pseudo-umbilical submanifold.

Proof. From (32), we have $n^2H^2 - S = n(n-1)(c-R) \geq 0$. Taking the covariant derivative on both sides of this equality, we get

$$n^2 H\, H_k = \sum_{i,j,\alpha} h_{ij}^{\alpha}\, h_{ijk}^{\alpha}, \quad k = 1, \cdots, n.$$

For every k, it follows from Cauchy-Schwarz's inequality that

$$n^4 H^2 H_k^2 = (\sum_{i,j,\alpha} h_{ij}^{\alpha}\, h_{ijk}^{\alpha})^2 \leq S \sum_{i,j,\alpha} (h_{ijk}^{\alpha})^2, \tag{52}$$

Taking sum on both sides of (52) with respect to k, we have

$$n^4 H^2 |\nabla H|^2 = n^4 H^2 \sum_k H_k^2 \leq S \sum_{(i,j,k,\alpha)} (h_{ijk}^{\alpha})^2 \leq n^2 H^2 \sum_{(i,j,k,\alpha)} (h_{ijk}^{\alpha})^2, \tag{53}$$

i.e. $|\nabla h|^2 \geq n^2 |\nabla H|^2$, then Corollary 3.1 follows from Theorem 3.1.

Now consider the quadratic form $Q(u,t) = u^2 - \frac{n-2}{\sqrt{n-1}} ut - t^2$. By the orthogonal transformation

$$\begin{cases} \bar{u} = \frac{1}{\sqrt{2n}}\{(1+\sqrt{n-1})u + (1-\sqrt{n-1})t\} \\ \bar{t} = \frac{1}{\sqrt{2n}}\{(\sqrt{n-1}-1)u + (\sqrt{n-1}+1)t\} \end{cases}$$

$Q(u,t)$ turns into $Q(u,t) = \frac{n}{2\sqrt{n-1}}(\bar{u}^2 - \bar{t}^2)$, where $\bar{u}^2 + \bar{t}^2 = u^2 + t^2$.

Take $u = \sqrt{\bar{S}_{n+1}}$, $t = \sqrt{n}H$, then

$$\begin{aligned} & nc - nH^2 - n(n-2)H\sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} + \bar{S}_{n+1} \\ & = nc + Q(u,t) = nc + \frac{n(\bar{u}^2 - \bar{t}^2)}{2\sqrt{n-1}} \\ & = nc + \frac{n(-\bar{u}^2 - \bar{t}^2)}{2\sqrt{n-1}} + \frac{n\bar{u}^2}{\sqrt{n-1}} \\ & \geq nc - \frac{n\bar{S}_{n+1}}{2\sqrt{n-1}} \end{aligned} \tag{54}$$

From (54) and (47) we have

$$\begin{aligned} 0 \geq & \int_{M^n} (|\nabla h|^2 - n^2|\nabla H|^2) \\ & + (S_{n+1} - nH^2)\{nc - \frac{n(S_{n+1} - nH^2}{2\sqrt{n-1}}\} \end{aligned} \tag{55}$$

Therefore we have the following result

Theorem 3.2. Let M^n $(n \geq 3)$ be a compact space-like submanifold in the de Sitter space $S_p^{n+p}(c)$. Suppose that $|\nabla h|^2 \geq n^2|\nabla H|^2$. If M^n has flat normal bundle and

$$S_{n+1} \, < \, nH^2 + 2\sqrt{n-1}c, \tag{56}$$

then $S_{n+1} = nH^2$ and M^n is a pseudo-umbilical submanifold.

Theorem 3.3. Let M^n be an n-dimensional compact orientable space-like submanifold with constant scalar curvature and with nonnegative sectional curvature immersed in a de Sitter space $S_n^{n+p}(c)$. Suppose that M^n has flat normal bundle, if the normalized scalar curvature R of M^n satisfies $R < c$, then M^n is totally umbilical.

Proof. We have from (32)

$$n^2H^2 - \|h\|^2 = n(n-1)(c-R), \tag{57}$$

where R is the normalized scalar curvature of M^n. Taking the covariant derivative of (57) and using the fact that R =const, we obtain

$$n^2HH_k = \sum_{i,j,\alpha} h_{ij}^\alpha \cdot h_{ijk}^\alpha$$

and hence, by Cauchy-Schwarz inequality, we have

$$\sum_k n^4H^2(H_k)^2 = \sum_k(\sum_{i,j,\alpha} h_{ij}^\alpha \cdot h_{ijk}^\alpha)^2 \leq \sum_{i,j,\alpha}(h_{ij}^\alpha)^2 \cdot \sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2,$$

that is

$$n^4H^2\|\nabla H\|^2 \leq \|h\|^2 \cdot \|\nabla h\|^2. \tag{58}$$

From (41) and (58), we have

$$0 \geq \int_{M^n}\{\|h\|^{-2}n^3(n-1)(c-R)\|\nabla H\|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2\}. \tag{59}$$

Thus, by hypothesis, $\|\nabla H\|^2 = 0$, so H is constant on M^n. Therefore Theorem 3.3 follows immediately from a result of Aiyama [1] and this completes the proof of the Theorem 3.3.

4 Space-like hypersurfaces in de Sitter space

In this section, we will study the Space-like hypersurfaces in de Sitter space and prove some interesting rigidity results.

Let S_1^{n+1} be an $(n+1)$-dimensional de Sitter of constant curvature 1 whose index is 1. Let M be an n-dimensional compact space-like hypersurface in S_1^{n+1}. We choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+1}$ in S_1^{n+1} such that at each point of M, $e_1, \dots, e_n$ span the tangent space of M and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+1; \quad 1 \leq i, j, k, \dots \leq n.$$

Let $\omega_1, \dots, \omega_{n+1}$ be its dual frame field so that the semi-Riemannian metric of S_1^{n+1} is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2 = \sum_A \epsilon_A\omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$. Then the structure equations of S_1^{n+1} are given by

$$d\omega_A = \sum_B \epsilon_B\omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{60}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \tag{61}$$

$$K_{ABCD} = \epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{62}$$

Restrict these form to M^n, we have

$$\omega_{n+1} = 0, \tag{63}$$

the Riemannian metric of M^n is written as $ds^2 = \sum_i \omega_i^2$. From Cartan's lemma we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{64}$$

From these formulas, we obtain the structure equations of M:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{65}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{66}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{67}$$

where R_{ijkl} are the components of the curvature tensor of M and

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j \tag{68}$$

is the second fundamental form of M. We also have

$$R_{ij} = (n-1)\delta_{ij} - nHh_{ij} + \sum_k h_{ik}h_{kj}, \tag{69}$$

$$n(n-1)(R-1) = -n^2H^2 + |h|^2, \tag{70}$$

where R is the normalized scalar curvature, and H the mean curvature.

Define the first and the second covariant derivatives of h_{ij}, say h_{ijk} and h_{ijkl} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{71}$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}. \tag{72}$$

Then we have the Codazzi equation

$$h_{ijk} = h_{ikj}, \tag{73}$$

and the Ricci's identity

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{74}$$

We choose a local frame field $e_1, \dots . e_n$ at any point $p \in M$, suth that $h_{ij} = \lambda_i \delta_{ij}$. Now let $\phi_{ij} = h_{ij} = \lambda_i \delta_{ij}$, from (18) we have

$$\int_{M^n} [|\nabla h|^2 - n^2 |\nabla H|^2 + \frac{1}{2} \sum_{i,j} R_{ijij} (\lambda_i - \lambda_j)^2] = 0. \tag{75}$$

From (67), we have $R_{ijij} = 1 - \lambda_i \lambda_j, \quad i \neq j$, and by putting this into (75), we obtain

$$\int_{M^n} [|\nabla h|^2 - n^2 |\nabla H|^2 + n|h|^2 - n^2 H^2 + |h|^4 - nH \sum_i \lambda_i^3] = 0. \tag{76}$$

Let $\mu_i = \lambda_i - H$ and $|Z|^2 = \sum_i \mu_i^2$, we have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2, \tag{77}$$

$$\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 - nH^3, \tag{78}$$

From (76) -(78), we get

$$\int_{M^n} [|\nabla h|^2 - n^2 |\nabla H|^2 + |Z|^2 (n - nH^2 + |Z|^2) - nH \sum_i \mu_i^3] = 0. \tag{79}$$

By Using of Lemma 2.1, we have

$$\begin{aligned} &\int_{M^n} [|\nabla h|^2 - n^2 |\nabla H|^2 + (|h|^2 - nH^2) \\ &\times (n - 2nH^2 + |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H \sqrt{|h|^2 - nH^2})] = 0. \end{aligned} \tag{80}$$

Now consider the quadratic form $Q(u,t) = u^2 - \frac{n-2}{\sqrt{n-1}} ut - t^2$. By the orthogonal transformation

$$\begin{cases} \bar{u} = \frac{1}{\sqrt{2n}} \{(1 + \sqrt{n-1})u + (1 - \sqrt{n-1})t\} \\ \bar{t} = \frac{1}{\sqrt{2n}} \{(\sqrt{n-1} - 1)u + (\sqrt{n-1} + 1)t\} \end{cases}$$

$Q(u,t)$ turns into $Q(u,t) = \frac{n}{2\sqrt{n-1}} (\bar{u}^2 - \bar{t}^2)$, where $\bar{u}^2 + \bar{t}^2 = u^2 + t^2$.
Take $u = |Z|$, $t = \sqrt{n} H$, then

$$\begin{aligned} & n - nH^2 - \frac{n(n-2)}{\sqrt{n(n-1)}} H|Z| + |Z|^2 \\ &= n + Q(u,t) = nc + \frac{n(\bar{u}^2 - \bar{t}^2)}{2\sqrt{n-1}} \\ &= n + \frac{n(-\bar{u}^2 - \bar{t}^2)}{2\sqrt{n-1}} + \frac{n\bar{u}^2}{\sqrt{n-1}} \\ &\geq n - \frac{n}{2\sqrt{n-1}} |h|^2 \end{aligned} \tag{81}$$

From (81) and (80) we have

$$0 \geq \int_{M^n} \{(|\nabla h|^2 - n^2 |\nabla H|^2) + |Z|^2 [n - \frac{n}{2\sqrt{n-1}} |h|^2]\}. \tag{82}$$

Therefore we have the following result.

Theorem 4.1. Let M be an n-dimensional ($n \geq 3$) compact space-like hypersurface in an $(n+1)$-dimensional de Sitter space S_1^{n+1}. If

$$|\nabla h|^2 \geq n^2 |\nabla H|^2, \tag{83}$$

and

$$0 \leq |h|^2 \leq 2\sqrt{n-1}, \tag{84}$$

then M is a totally umbilical hypersurface.

Proof. By the assumption of theorem, the right hand side is non-negative. Thus, either $|Z|^2 \equiv 0$, that is , M is totally umbilical; or

$$|h|^2 = 2\sqrt{n-1}. \tag{85}$$

In the latter case, equality holds in Lemma 2.1, and it follows that M has two distinct constant principle curvatures. We conclude that M is totally umbilical from the compactness of M. This completes the proof of Theorem 4.1.

Corollary 4.1. Let M be an n-dimensional ($n \geq 2$) compact space-like hypersurface with constant normalized scalar curcature R in an $(n+1)$-dimensional de Sitter space S_1^{n+1}. If $R \leq 1$ and

$$0 \leq |h|^2 \leq 2\sqrt{n-1}, \tag{86}$$

then M is a totally umbilical hypersurface.

Proof. From (70),

$$-n^2H^2 + \sum_{i,j} h_{ij}^2 = n(n-1)(R-1).$$

Taking the covariant derivative of the above expression, and using the fact $R = constant$, we get

$$n^2 H H_k = \sum_{i,j} h_{ij} h_{ijk}.$$

By Cauchy-Schwarz inequality, we have

$$\sum_k n^4 H^2 (H_k)^2 = \sum_k (\sum_{i,j} h_{ij} h_{ijk})^2 \leq (\sum_{i,j} h_{ij}^2) \sum_{i,j,k} h_{ijk}^2, \tag{87}$$

that is

$$n^4 H^2 \|\nabla H\|^2 \leq |h|^2 |\nabla h|^2. \tag{88}$$

On the other hand, from $R \leq 1$, we have $n^2H^2 - |h|^2 \geq 0$. Thus

$$|\nabla h|^2 \geq n^2 |\nabla H|^2,$$

so from Theorem 4.1 the Corollary 4.1 follows.

Lemma 4.1. Let M be an n-dimensional compact space-like hypersurface in an $(n+1)$-dimensional de Sitter space S_1^{n+1}. Suppose that the normalized scalar curvature R is proportional to the mean curvature H of M, that is

$$R = aH, \tag{89}$$

where a is any constant. Then $|\nabla h|^2 \geq n^2 |\nabla H|^2$.

Proof. By use of Gauss equation $-n^2H^2+|h|^2 = n(n-1)(R-1)$ and the assumption (89), we have

$$|h|^2 = n^2H^2 - n(n-1)(1-aH). \tag{90}$$

Taking the covariant derivative of (90), we have for every k

$$2\sum_{i,j} h_{ij}h_{ijk} = (2n^2H + n(n-1)a)H_k.$$

It follows that

$$4|h|^2|\nabla h|^2 \geq 4\sum_k(\sum_{i,j} h_{ij}h_{ijk})^2 = (2n^2H+n(n-1)a)^2|\nabla H|^2. \tag{91}$$

By (89) and (90), we have

$$\begin{aligned}&(2n^2H+n(n-1)a)^2 - 4n^2|h|^2\\ &= (4n^4H^2 + n^2(n-1)^2a^2 + 4n^3(n-1)Ha) - n^2(4n^2H^2 - 4n(n-1)(1-aH))\\ &= n^2(n-1)((n-1)a^2+4n) > 0.\end{aligned} \tag{92}$$

Combining (91) with (92), we have $|\nabla h|^2 \geq n^2|\nabla H|^2$. This completes the proof of Lemma 4.1.

From Theorem 4.1 and Lemma 4.1, we have

Corollary 4.1. Let M be an n-dimensional ($n \geq 2$) compact space-like hypersurface in an $(n+1)$-dimensional de Sitter space S_1^{n+1}. Suppose that the normalized scalar curvature R is proportional to the mean curvature H of M, that is, there exists a constant a such that

$$R = aH. \tag{93}$$

If

$$0 \leq |h|^2 \leq 2\sqrt{n-1}, \tag{94}$$

then M is a totally umbilical hypersurface.

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