On Burnside's Theorem in Infinite-dimensional Spaces

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Abstract

The classical Burnside's Theorem guarantees in a finite dimensional space the existence of invariant subspaces for a proper subalgebra of the matrix algebra. In this paper we give an extension of Burnside's Theorem for a general Banach space, which also gives new results on invariant subspaces.

Introduction.

This preprint is an extension of the paper (10). New results are theorems 4 and 5.

The fundamental Burnside Theorem for finite-dimensional representations says that an algebra of linear operators on a finite-dimensional space F, without invariant subspace must be the algebra $L(F)$ of all linear operators on the space. In other words, if an algebra R is strictly contained in $L(F)$ then there exists $x \in F$ and $\varphi \in F^*$ s.t. for every $A \in R$, $(Ax, \varphi) = 0$.

In this paper we will extend this result to Banach spaces in the following way. In Theorem 1 below, let B be a Banach space, $L(B)$ the bounded linear operators on B and $K(B)$ the compact operators on B. Let $|||A|||$ denote the essential norm of A, i.e. the distance from A to the space of compact operators. A weakly closed algebra is an algebra closed in weak operator topology.

Theorem 1. *Let R be a weakly closed subalgebra of $L(B)$, $R \neq L(B)$. Then there exists $x \in B^{**}$ and $y \in B^*$, $x \neq 0$ and $y \neq 0$, s.t. for every $A \in R$*

$$|(x,\ A^*y)| \leq |||A|||. \tag{1}$$

If B is finite-dimensional then $|||A||| = 0$, so Theorem 1 gives Burnside's theorem.

For a given subset $S \subset L(B)$ and a vector $x \in B$ put $Sx = \bigcup\limits_{A \in S} Ax$ and $S' = \cup A^*$. We say that S is transitive if it doesn't have a nontrivial invariant subspace and S is essentially transitive if the conclusion of Theorem 1 is false for S. We say that a subalgebra $R \subset L(B)$ has the Pearcy-Salinas (PS) property if there exists net $\{A_\alpha\} \subset R$ and a nonzero operator $A \in L(B)$ such that:

$$\lim_\alpha (x, A_\alpha^* y) = (x, A^* y) \tag{2}$$

for every vector $x \in B^{**}$ and every functional $y \in B^*$ and

$$\lim_\alpha |||A_\alpha||| = 0 \tag{3}$$

Of course, every bounded operator is a weak limit of finite dimensional ones, so it is the assumption $\{A_\alpha\} \subset R$ that makes the condition nontrivial.

Corollary 1. Let R be a weakly closed proper subalgebra of $L(B)$ with PS property. Then the algebra R' is nontransitive.

Proof. Let x and y be as in Theorem 1. Then for every pair of operators T_1, T_2 in R we have

$$\begin{aligned} &|(x,\, T_2^* A^* T_1^* y)| = \\ &= \lim_\alpha |(x,\, T_2^* A_\alpha T_1^* y)| \le \overline{\lim_\alpha} |||T_2 A_\alpha T_1||| \le \\ &\le \overline{\lim_\alpha} ||T_2\|\;\|T_1\|\;|||A_\alpha||| = 0 \end{aligned}$$

It is easy to see that one of the three subspaces $\bigcap\limits_{T\in R} \ker(T^*)$, $\overline{R'y}$, $\overline{\mathrm{span}_{T_1,T_2\in R}(T_2^* A^* T_1^* y)}$ is a nontrivial invariant subspace for the algebra R'.

If the algebra R contains a non-zero compact operator then the PS property is trivially true for R. So we get the results obtained by the author's earlier techniques [8].

Corollary 2. If a weakly closed proper subalgebra R of $L(B)$ contains a nonzero compact operator, then the algebra R' is nontransitive.

Of course the existence of a hyperinvariant subspace in B^* for operators that commute with a compact operator follows from Corollary 2. It should be mentioned that the following theorem by Pearcy and Salinas [6] is not a consequence of the author's earlier techniques [see 7].

Theorem (Pearcy - Salinas): *Let T be a bounded operator on Hilbert space. Assume that there is a sequence of rational functions (s_n) s.t. $s_n(T)$ converges weakly to a non-zero operator A and s.t.*

$$|||s_n(T)||| \to 0 \quad \text{as} \quad n \to \infty.$$

Then T has a non-trivial invariant subspace.

However, from Corollary 1 we get the following strengthening of the Pearcy - Salinas Theorem as

Corollary 3. Let T be a bounded operator on Banach space. Assume that there is a nonzero operator A and net $\{A_\alpha\}$ of operators that commute with T s.t. (2) and (3) holds. Then

T^* has a nontrivial hyperinvariant subspace. The commutant of T is a proper subalgebra of $L(B)$, so Corollary 3 is an immediate consequence of Corollary 1.

Corollary 4. Let $S_\phi \subset L(B)$ be defined by $S_\phi = \{A \in R,\ |||A||| \leq 1\}$. Then there exists a nonzero functional $y \in B^*$ s.t. the set $S'_\phi y$ is not dense in B^*.

Proof. Let x and y be as in Theorem 1. Then $\sup\limits_{|||A^*|||<1} (x, A^*y)| \leq 1$, so $S'_\phi y$ is not dense in B.

This corollary was pointed out to us by S.W. Brown, who has obtained it and similar results in the case of a commutative algebra.

In Hilbert space Theorem 1 has the following equivalent formulation.

Theorem 2. *Let R be as in Theorem 1. Let B be Hilbert space, then there exist two bounded nets (x_α) in B and (y_α) in B s.t. $x_\alpha \overset{w}{\to} x \neq 0,\ y_\alpha \overset{w}{\to} y \neq 0$ and for every $A \in R$*

$$< Ax_\alpha,\ y_\alpha > \to 0.$$

Proof of theorems 1 and 2. Let B be a Banach space, Q a compact Hausdorff space, R a subalgebra of $L(B)$, WQ the Banach space of weak*-continuous functions $Q \to B^*$ with sup-norm, SQ the subspace of WQ consisting of the strongly continuous functions, NR the algebra of norm-continuous functions $Q \to R'$. Let $C = C(Q)$ be the algebra of all complex-valued continuous functions on Q. Every function $f \in C(Q)$ is a linear operator on the space WQ. We let f^* denote its adjoint.

Definition. Let M be a subspace of WQ, invariant under the algebra C. We say that $\theta \in M^*$ is a point functional if there exists a point $q \in Q$ s.t.

$$f^*\theta = \overline{f(q)}\theta \tag{4}$$

for every function $f \in C$.

If E is a subspace in a Banach space let U_E denote the unit ball of E and $E^\perp$ denote the annihilator of E in the dual space. We say that a functional $\theta \in M^*$ is a $\delta(q, x)$ functional or simply δ-functional if there exists a point $q \in Q$ and an element $x \in B^{**}$ s.t. if $h \in M$ then

$$\theta(h) = (x, h(q)) \tag{5}$$

Lemma 1. Let $\theta \in M^*$ be a point functional and assume $M \supset SQ$. Then the restriction, $\chi = \theta|_{SQ}$ is a δ-functional.

Proof. It is known [3] that the functional χ has the form

$$\chi(h) = \int_Q (d\chi(q), h(q)) \tag{6}$$

where $d\chi$ is a regular Borel measure on Q with values in B^{**}. Let $q \in Q$ be as in (4). For an open set $V = Q/\{q\}$ and for a given $\epsilon > 0$ there exists a closed set $F \subset V$ s.t. $\|d\chi|_V - d\chi|_F\| < \epsilon$. Then there exist functions $f \in C(Q)$ s.t. $f(q) = 0$, $f|_F \equiv 1$. If G is given Borel set and $G \subset F$, then $d\chi(G) = \int_G d\chi = \int_G \overline{f} d\chi = f^*\chi = \overline{f(q)}\chi = 0$, so $d\chi|_F = 0$. This give for every $\epsilon > 0$ $\|d\chi|_V\| < \epsilon$, and so $d\chi|_V = 0$. Q.E.D.

Let H be a Hilbert space with orthogonal basis $\{e_n\}$, $LIM(\alpha_n)$ a Banach limit in the space l^∞, Q compactification of integer numbers by $\{\infty\}$ .Then $\theta(h) = LIM(h(n), e_n)$ is an example of a point functional that is not δ-functional.

Let $Q \times U_B$ be the topological product of the compact Q and the ball U_B with norm topology. The functions $s \in WQ$ and $f \in C(Q)$ define complex valued functions $L(s)$ and $L_1(f)$ on $Q \times U_B$ for the formulas $L(s)(q,x) = (x, s(q)), L_1(f)(q,x) = f(q)$.

Lemma 2 $L(s), L_1(f)$ are continuous functions on

$$Q \times U_B$$

Proof. Let $q_0 \in Q$ and $x_0 \in U_B$. Let V_{g_0}, V_{x_0} be open subsets defined by

$$V_{q_0} = \{q \in Q : |(x_o, s(q) - s(q_o))| < \tfrac{\epsilon}{2}\}$$
$$V_{x_0} = \{x \in U_B : \|x - x_o\| < \tfrac{\epsilon}{2|s|}\}$$

then for all $(q,x) \in V_{x_0} \times V_{q_0}$ we have

$$|(x, s(q)) - (x_o, s(q_o))| \leq |((x - x_o, s(q))| +$$
$$+|(x_o, s(q) - s(q_o))| \leq \tfrac{\epsilon}{2|s|}|s| + \tfrac{\epsilon}{2} = \epsilon,$$

so $L(s)$ is continuous. The continuity of $L_1(f)$ is obvious. U_B is a completely regular space, so there exists the Cheh compactification F of the space $Q \times U_B$. The functions $L_1(f)$ and $L(s)$ in lemma 2 have continuous extensions to all of F. Thus $L(M), L(N), L_1(C(Q))$ are subspaces of $C(F)$, $L(M) \supset L(N)$, and $L(M)$ and $L(N)$ are invariant for $L_1(C(Q))$. Moreover the operators $L : M \to C(F)$ and $L_1 : C(Q) \to C(F)$ are isometries.

Lemma 3. Let $\varphi \in L(M)^*$, $f \in C(Q)$ and

$$(L_1 f)^* \varphi = \gamma \varphi$$

then $f^*(L^*\varphi) = \gamma(L^*\varphi)$

Proof. For a given function $h \in WQ$ we have $(h, f^*(L^*\varphi)) = (fh, L^*\varphi) = (L(fh), \varphi) = (L_1(f)L(h), \varphi) = (L(h), L_1(f)^*\varphi) = (L(h), \gamma\varphi) = (h, \gamma L^*\varphi)$. Q.E.D.

Lemma 4. Let M, N be subspaces of WQ, invariant for $C(Q)$. Let $M \supset N$ and let $\theta \in M^*$ be an extreme point of $U_{N^\perp}$. Then θ is a point-functional.

The proof of this lemma is similar to De Branges famous proof of Stone-Weierstrass Theorem [2]

Proof. Let $\hat{\theta}$ be a functional in $L(M)^*$ s.t. $L^*(\hat{\theta}) = \theta$. Then, obviously $1 = |\hat{\theta}| = |\theta|$. By Hahn-Banach and Riesz Theorems there exists a measure $d\mu \in C(F)^*$ which is an extension of $\hat{\theta}$ to space $C(F)$. Let f be a function in $C(Q)$, s.t.

$$0 \leq f \leq 1 \tag{7}$$

and $\hat{f} = L_1(f), d\mu_1 = \hat{f} d\mu, d\mu_2 = (1-\hat{f}) d\mu, m_1 = \|d\mu_1\|, m_2 = \|d\mu_2\|$. Then $\{d\mu_1, d\mu_2, d\mu\} \subset L(N)^\perp$ and $m_1 + m_2 = \int_F \hat{f} |d\mu| + \int_F (1 - \hat{f}) |d\mu| = \int_F |d\mu| = 1$. If $\theta_1 = d\mu_1|_{L(M)}, \theta_2 = d\mu_2|_{L(M)}$, then $|\theta_1| + |\theta_2| \geq |\theta_1 + \theta_2| = 1$ and $|\theta_1| + |\theta_2| \leq m_1 + m_2 = 1$, so we have $|\theta_2| = m_2, |\theta_1| = m_1$. If $m_1 = 0$, then $(L_1 f)^* \hat{\theta} = 0$, so by lemma 3 $f^*\theta = 0$. If $m_2 = 0$ then $(1 - f)^*\theta = 0$. If $m_1 \neq 0$ and $m_2 \neq 0$ then we have

$$\hat{\theta} = m_1 \frac{\theta_1}{m_1} + m_2 \frac{\theta_2}{m_2}, \|\frac{\theta_1}{m_1}\| = 1, \|\frac{\theta_2}{m_2}\| = 1$$

The functional $\hat{\theta}$ is an extreme point in the ball $U_{L(N)^\perp}$, so we get $\frac{\theta_1}{m_1} = \hat{\theta}$ or by lemma 3 $\frac{f^*\theta}{m_1} = \theta$. So for any function f with property (7) the functional θ is an eigenvector and the functions of this type obviously generate $C(Q)$. The corresponding eigenvalue is a multiplicative functional on $C(Q)$, so Gelfand's theorem now gives the lemma. Q.E.D.

Lemma 5. Let $T \in K(B)$ and $h \in WQ$. Then $T^*h \in SQ$.

We omit the simple proof.

Let R be an algebra in $L(B)$ and let $h \in WQ$, $\|h\| = 1$. By applying the algebra NR to h we get subspaces

$$N = NR(h), \ M = \text{ span } (N, SQ).$$

then $M \supset N$ and M and N are invariant for the algebra $C(Q)$. Let φ be a point functional, $\varphi \in M^*$. Then, by Lemma 1 , $\varphi|_{SQ}$ is a $\delta(q, x)$ functional

Lemma 6. If functional $\varphi \in N^\perp$ then for every operator $A \in R$

$$|(x, A^*h(q))| \leq 2|\varphi| \ |||A||| \ |h| \tag{8}$$

Proof. Let T be a compact operator s.t. $\|A - T\| \leq |||A||| + \epsilon$. Then we have

$$\begin{aligned} &|(x, A^*h(q))| = |\delta A^*h| = \\ &= |\delta(A^*h) - \varphi(A^*h)| = |(\delta - \varphi)A^*h| = \\ &\leq |(\delta - \varphi)(A^* - T^*)h| + |(\delta - \varphi)T^*h| \end{aligned}$$

By Lemma 5 $T^*h \in SQ$, so by Lemma 1 $\delta(T^*h) - \varphi(T^*h) = 0$. It is clear that $|\delta| \leq |\varphi|$, so we get

$$|(x, A^*h(q))| \leq 2|\varphi| \ \|A - T\| \ |h| < 2|\varphi|(|||A||| + \epsilon)|h|.$$

Since ϵ is arbitrary,

$$|(x, A^*h(q))| \leq 2|\varphi| \ |||A||| \ |h| \qquad Q.E.D.$$

Let $\sigma_F(T)$ be the subset of the spectrum of an operator $T \in L(B)$ consisting of the isolated points of finite multiplicity.

Lemma 7. Let the subspace $L \subset B$ have finite codimension. Assume $\alpha > 0$ and

$$\sup_{x \in L} \frac{\|Tx\|}{\|x\|} < \alpha \tag{8}$$

Assume $\gamma \in \sigma(T)$ and $|\gamma| > \alpha$. Then $\gamma \in \sigma_F(T)$

Proof. It is sufficient to prove this lemma in the case that γ is a boundary point of $\sigma(T)$. Let P be a bounded projection on L. Then the operator $T_1 = (T - \gamma I)P$ is semi-Fredholm.

It is clear that $kerT_1 = kerP$ and if $x \in L$, then $|T_1 x| = |(T - \gamma I)Px| = |(T - \gamma I)x| \geq (|\gamma| - \alpha)|x|$. So T_1 has finite dimensional kernel and closed range. $T - \gamma I$ is a finite dimensional perturbation of T_1, so by [1, Corollary 1.3.7] $T - \gamma I$ is also semi-Fredholm. Moreover, $T - \gamma I$ is a limit of invertible operators so it is a Fredholm operator with index zero by [1, Theorem 4.2.1., Corollary 3.2.10]. Now by [9, Corollary 3.11] $T - \gamma I$ is a compact perturbation of an invertible operator, so by Weil's theorem γ is isolated and has finite multiplicity Q.E.D.

Lemma 8. If R is an essentially transitive subalgebra, then R contains an operator T with the following properties: $1 \in \sigma_F(T)$ and if $\gamma \in \sigma(T) \backslash \sigma_F(T)$ then $|\gamma| < \frac{1}{2}$.

Proof. Let y be a functional in B^* $|y| = 3$, and let Q be a ball of a radius 2 with center in y. Then Q is weak* compact and $\{0\} \notin Q$. The function $h(q) \equiv q$ if obviously weak* continuous, so $h(q) \in WQ$ and $|h(q)| = 5$. Let M and N are subspaces of WQ as in Lemma 6. If N is not dense in M, then there exists a functional $\theta \in M^*$ which is an extreme point of $U_{N^\perp}$. Then by Lemma 6 there exists a point $q_0 \in Q$ and a nonzero element $x \in B^{**}$ s.t. the inequality

$$|(x, A^* h(q_0))| \leq 2|||A||| \ |h|$$

holds for every operator $A \in R$. For $y = \frac{h(q_0)}{2|h|}$ we get inequality (1). This contradiction gives that N is dense in M. Let g be a function in SQ defined by $g(q) \equiv y$. Then there exists an operator-valued function $A(q) \in NR$ s.t. $|A(q)h(q) - g(q)| < \frac{1}{8}$. Thus the function $A(q)$ generates a continuous map $\Psi : Q \to Q$ by the formula $\Psi(q) = A(q)q$. By Tychonoff's fixed point theorem there exists a point $q_0 \in Q$ s.t. $A(q_0)q_0 = q_0$. We let T denote $A(q_0)$. Let V

be open set in Q defined by

$$V = \{q \in Q : \|A(q) - T\| < \frac{1}{48}\}.$$

Then

$$\sup_{q\in V}|Tq - y| \leq \sup_{q\in V}|(T - A(q))q| +$$
$$+\sup_{q\in V}|(A(q)q) - y| \leq \frac{5}{48} + \frac{1}{8} < \frac{1}{4}$$

If $V_1 = \{z : |z| \leq 1,\ z + q_0 \in V\}$ then V_1 is a weak* open subset of the unit ball U_{B^*} that contains $\{0\}$ and

$$\sup_{z\in V_1}\|Tz\| = \sup_{z\in V_1}\|T(z+q_0) - Tq_0\| \leq$$
$$\leq \sup_{q\in V}|Tq - Tq_0| \leq \sup_{q\in V}|Tq - y| + |Tq_0 - y|$$
$$< \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

By definition of weak* topology there exists a subspace $L \subset B^*$ of finite codimension s.t. $L \cap U_{B^*} \subset V_1$. We have $\sup_{z\in U_L}|Tz| \leq \sup_{z\in V_1}|Tz| < \frac{1}{2}$, so by Lemma 7 every point $\gamma \in \sigma(T)$, $|\gamma| \geq \frac{1}{2}$ is an isolated point in $\sigma(T)$ of finite multiplicity. Q.E.D.

Lemma 9. Let R be a uniformly closed essentially transitive subalgebra $L(B)$, then R contains a nonzero finite dimensional projection.

Proof. Let T be an operator as in Lemma 8. Since $\{\gamma :\ (T-\gamma I)^{-1} \in R\}$ is a component of the resolvent set of T it follows that if $(T-\gamma I)^{-1}$ exists and $|\gamma| \geq \frac{1}{2}$ then $(T-\gamma I)^{-1} \in R$.

By Lemma 8 there exists circle $\alpha \subset C$ s.t. 1 is the only point of $\sigma(T)$ inside α and for every point $\gamma \in \alpha$ $(T-\gamma I)^{-1} \in R$. By Riesz theorem $P = \frac{1}{-2\pi i}\int_\alpha (T-\gamma I)^{-1} d\gamma$ is a non-zero finite dimensional projection and $P \in R'$. If $P = P_1^*$ then P_1 is finite dimensional projection in R. Q.E.D.

Now to finish the proof of Theorem 1 we need the folloving well-known fact [8].

Lemma 10. If a transitive algebra R contains a non-zero finite dimensional operator, then R is weakly dense in $L(B)$.

Essential transitivity implies transitivity, so by combining Lemma 9 and Lemma 10 we finish the proof of Theorem 1.

Of course Lemma 9 gives the possibility to obtain different results on density. For example one of them is

Theorem 3. *If H is a Hilbert space and R is uniformly closed essentially transitive subalgebra of $L(H)$ then R contains all compact operators in $L(H)$.*

Now we'll prove the equivalence of Theorem 1 and Theorem 2. in Hilbert space. We first prove that Theorem 2 implies Theorem 1 if B is Hilbert space.

Assume that Theorem 2 holds. Let $C = \overline{\lim_\alpha} |x_\alpha| \cdot |y_\alpha|$. If $A \in R$ then

$$\begin{aligned}
&0 = \lim_\alpha (Ax_\alpha,\, y_\alpha) = \\
&\lim_\alpha (A(x_\alpha - x),\, y_\alpha - y)) - (Ax,\ y) + \\
&+ \lim_\alpha (Ax,\ y_\alpha) + \lim_\alpha (Ax_\alpha,\ y)
\end{aligned}$$

where

$$\lim_\alpha (Ax,\ y_\alpha) = \lim_\alpha (Ax_\alpha,\ y) = (Ax, y).$$

Thus

$$(Ax,\ y) = \lim_\alpha (A(x - x_\alpha),\ y_\alpha - y)).$$

If K is a compact operator, then

$$\begin{aligned}
|(Ax,\ y)| &\leq \overline{\lim_\alpha} |((A - K)(x - x_\alpha),\ (y_\alpha - y))| \\
&+ \overline{\lim_\alpha} |(K(x - x_\alpha),\ (y_\alpha - y))| \leq \\
&\leq \|A - K\| 4C.
\end{aligned}$$

Now let $z = \frac{y}{4C}$. Then

$$|(Ax,\ z)| \leq \inf_K \|A - K\| = |||A|||$$

and so Theorem 1 follows from Theorem 2.

To prove that Theorem 1 implies Theorem 2 we use the following result (by Glimm). [5] B is assumed to be Hilbert space

Lemma 11 . Let $\Psi \in L(B)^*$ and $\Psi \in K(B)^\perp$. Then there is a pair of bounded nets (x_α) and (y_α) s.t.

1. $w - \lim_\alpha x_\alpha = 0,\ w - \lim_\alpha y_\alpha = 0$

2. For every $A \in L(B), \Psi(A) = \lim_{\alpha}(Ax_\alpha,\ y_\alpha)$.

Now let $\Psi_1(A) = (Ax, y)$ where x, y are as in Theorem 1, $A \in R$. Theorem 1 gives that Ψ_1 can be extended to span $(R, K(B))$ by putting $\Psi_1(T) = 0$ for $T \in K(B)$. Now let Ψ be the Hahn-Banach extension of the Ψ_1 to all of $L(B)$. We now use the Lemma 1 and we get Theorem 2 with $x_\alpha = x + x_\alpha$ and $y_\alpha = y - y_\alpha$.

Theorem 4. *Let H be a Hilbert space and let R be a norm closed proper subalgebra of $L(H)$. Then there exists a nonzero functional φ on $L(H)$ which annihilates R and there exist two bounded nets (x_α) in H and (y_α) in H s.t. for every operator $A \in L(H)$*

$$\varphi(A) = \lim_{\alpha} < Ax_\alpha,\ y_\alpha > . \tag{9}$$

Proof. If R is essentially transitive then by Theorem 3 it contains all compact operators. Let φ be any nonzero functional on $L(H)$ which annihilates R. By Glimm Theorem it has the representation (9). If R is not essentially transitive then the conclusion of Theorem 1 is true for R and the representation (9) follows from the Theorem 2.

It is natural to ask if the previous theorem is true for an arbitrary proper subalgebra of $L(H)$. The answer is negative, because of the following statement.

Theorem 5. *There exists a norm dense proper subalgebra of $L(H)$.*

Proof. Let $\{e_i\}$ be an orthonormal basis in H and let M be the set of finite linear combinations of vectors $\{e_i\}$. Consider the set R of all bounded linear operators on H which leave the set M invariant. R is obviously a subalgebra of $L(H)$. We'll show that R is norm dense in $L(H)$. Let T be a bounded linear operator on H and let ϵ be an arbitrary positive number. For each integer i there exists a vector $x_i \in M$ such that $||x_i - Te_i|| \leq \epsilon\ 2^{-i}$. Let Q be a linear operator on M defined by $Qe_i = Te_i - x_i$. Then $||Qe_i|| \leq \epsilon\ 2^{-i}$, so $\sum ||Qe_i|| < \infty$. For this reason the operator Q has an extension $\overline{Q}$ which is a bounded linear operator on H and $||\overline{Q}|| \leq \epsilon \sum 2^{-i} = \epsilon$. Put $P = T - \overline{Q}$. Then $||T - P|| \leq \epsilon$ and obviously M is invariant under T.

Finally we will mention that in the case of a nonreflexive Banach space Theorem 1 gives invariant subspace corollaries only in the dual space B^*.

Since Enflo in 1976 [4] showed that there are counter examples to the invariant subspace problem in general Banach spaces, this may be a sign that the following result can be true: If A is a bounded linear operator in a Banach space then A^* has a non-trivial invariant subspace.

The known counterexamples do not contradict this conjecture. By corollary 1, this would be true if the following is true: If A is a bounded linear operator in a Banach space then there exists an algebra with PS property which contains the operator A.

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