ON INNER PRODUCTS OF EIGENFUNCTIONS FOR THE HYPERBOLIC PLANE

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ABSTRACT. We present an improved bound on the rate of decay of the Fourier coefficients of the square of an eigenfunction ϕ of the Laplace operator for a compact hyperbolic surface. If the surface has finite area, we get a new bound for the Rankin-Selberg convolution $L(\phi\otimes\phi, s)$ on its critical line for arbitrary cofinite subgroups of $SL(2,\mathbb{R})$ and a new bound for the Fourier coefficients of ϕ: $a_m = O(|m|^{2/5+\epsilon})$. The method follows the same steps as [11].

1. INTRODUCTION

Various applications of L-series require knowledge of their behavior on their critical line. One usually needs to know the location of the poles and the various gamma factors that appear in the functional equation of the L-series. In [11] the special case of the Rankin-Selberg convolution $L(\phi\otimes\phi, s)$ is treated without such knowledge, where ϕ is an L^2-eigenfunction of the Laplace operator on the surface $\Gamma\setminus\mathbb{H}$, Γ a cofinite subgroup of $SL(2,\mathbb{R})$. The following bound is proved in [11]

$$(1) \qquad \int_T^{T+1} |(\phi^2, E(z, 1/2+it))|^2\, dt \ll (T\log T)^2 e^{-\pi T}$$

as $T\to\infty$. The notation $\ll$ means that the left-hand side is (for sufficiently large T) less than a constant multiple of the right-hand side. Here $E(z,s)$ is the Eisenstein series corresponding to a cusp of Γ. Eq.(1) implies that the Fourier coefficients a_n of an arbitrary Maaß cusp form ϕ satisfy the bound

$$(2) \qquad |a_n| \ll_{\epsilon,\phi} |n|^{5/12+\epsilon}$$

for all $\epsilon>0$. Here $\Delta\phi + (\frac14+\lambda^2)\phi = 0$, $\lambda\in\mathbb{R}$ or $\lambda\in i[-1/2,1/2]$ and ϕ has the following Fourier expansion at the cusp $S^1\times[a,\infty)$ with coordinates x, y:

$$(3) \qquad \phi(x+iy) = \sum_n a_n y^{1/2} K_{i\lambda}(2\pi|n|y) e^{2\pi i n x}$$

We can naturally assume that ϕ is real-valued and $\lambda\neq\pm i/2$, since we are not interested in the constant eigenfunction.

In [11] the case of compact surfaces is discussed as well. If ϕ_j are an orthonormal basis for $L^2(\Gamma\setminus\mathbb{H})$, Γ a cocompact subgroup of $SL(2,\mathbb{R})$ and $\Delta\phi_j + (1/4+r_j^2)\phi_j = 0$, then

$$(4) \qquad (\phi^2, \phi_j) \ll (r_j \log r_j) e^{-\pi r_j/2}$$

as $j \to \infty$.

In the case that ϕ is a holomorphic cusp form of even integral weight $k > 2$ Good [3] proved that $a_n \ll n^{k/2-1/6}$ for arbitrary cofinite subgroup of $SL(2,\mathbb{R})$. We see that (2) falls short of the corresponding bound for the holomorphic cusp forms. This raises the issue of improving (1), (2) and (4). In this work we prove:

Theorem 1. *If Γ is a cocompact subgroup of $SL(2,\mathbb{R})$ and $\lambda \neq 0$, i.e. the eigenvalue corresponding to ϕ is not $1/4$, then*

$$(\phi^2, \phi_k) \ll r_k^{1/2} e^{-\pi r_k/2} \tag{5}$$

as $k \to \infty$.

Theorem 2. *If Γ is cofinite subgroup of $SL(2,\mathbb{R})$ and the eigenvalue corresponding to ϕ is not $1/4$, then there exists an $\epsilon > 0$ such that*

$$\int_t^{t+\epsilon} |(\phi^2, E(z, 1/2+is))|^2 \, ds \ll te^{-\pi t}. \tag{6}$$

as $t \to \infty$.

Corollary 1. *The Fourier coefficients a_n of Maaß cusp form ϕ satisfy*

$$|a_n| \ll_{\epsilon,\phi} |n|^{2/5+\epsilon} \tag{7}$$

for all $\epsilon > 0$.

In the case Γ is an arithmetic group of a special kind, like $SL(2,\mathbf{Z})$, much better bounds than (7) are known, see [1]. Even in these cases the bounds do not prove the Ramanujan conjecture $|a_n| \ll |n|^\epsilon$. In [11] it is suggested that the Ramanujan conjecture may hold for arbitrary cofinite subgroups and is not a special feature of arithmetic. This makes improvements on (2) and (7) interesting to pursue.

The restriction $\lambda \neq 0$, which follows from $i\lambda \notin \frac{1}{2}\mathbf{Z}$, is purely technical. Lemma 1 is not necessarily true for the eigenvalue 1/4. Eq.(14) that gives the analytic continuation of the hypergeometric function fails for $i\lambda \in \frac{1}{2}\mathbf{Z}$. Similarly Eq.(2.17) in [11] fails for $i\lambda \in \frac{1}{2}\mathbf{Z}$. This problem first showed up in [5] and [7]. However, even in the case $\lambda = 0$, the sequence a_n in (12) increases at most polynomially in n, see [8, Th. 7.3]. The author has not investigated the order of growth of the a_n that follows from [8, Th. 7.3] for $\lambda = 0$. In any case, a generic cofinite or cocompact subgroup of $SL(2,\mathbb{R})$ does not have 1/4 in its L^2 spectrum, see [10].

2. Some remarks on point-pair invariants on $\mathbb{H}$

If $t(z,z') = \frac{|z-z'|^2}{yy'}$, z, $z' \in \mathbb{H}$, then the hyperbolic distance $r = r(z,z')$, between z and z' satisfies: $t = 4\sinh^2 \frac{r}{2} = 2\cosh r - 2$. A point-pair invariant is a $K = SO(2)$ bi-invariant function $\tilde{k}(t) = k(r)$ and its Selberg-Harish-Chandra transform is given by

$$\begin{aligned} h(s) &= \int_{\mathbf{H}} \tilde{k}(t(i,z)) y^{\frac{1}{2}+is} \frac{dx\,dy}{y^2} \\ h(s) &= \int_{-\infty}^{\infty} e^{isu} g(u)\, du \\ g(u) &= Q(e^u + e^{-u} - 2) \\ Q(w) &= \int_w^\infty \frac{\tilde{k}(t)}{\sqrt{t-w}} dt \end{aligned} \tag{8}$$

and $h(s)$ and $g(u)$ are even functions (see [12, pp. 72]). It is important to understand that the transform $h(s)$ is the integral of $k(r)$ with a spherical function. This is seen as follows:

$$\begin{aligned} h(s) &= 2\int_0^\infty \cos(su)g(u)du \\ &= 2\int_0^\infty \cos(su)\int_{e^u+e^{-u}-2}^\infty \frac{\tilde{k}(t)}{\sqrt{t-(e^u+e^{-u}-2)}}dt\,du \\ &= 2\int_0^\infty \cos(su)\int_u^\infty \frac{2}{\sqrt{2}}\frac{k(r)\sinh r}{\sqrt{\cosh r-\cosh u}}dr\,du \\ &= \frac{4}{\sqrt{2}}\int_0^\infty\int_0^r \frac{\cos(su)}{\sqrt{\cosh r-\cosh u}}du\ k(r)\sinh r\,dr. \end{aligned}$$

Using the integral representation for the Legendre function [2, 3.7 (8), pp. 156], we see that

$$P_{-\frac{1}{2}+is}(\cosh r) = \frac{\sqrt{2}}{\pi}\int_0^r \frac{\cos(su)}{\sqrt{\cosh r-\cosh u}}du$$

so

$$h(s) = 2\pi\int_0^\infty P_{-\frac{1}{2}+is}(\cosh r)k(r)\sinh r\,dr \tag{9}$$

exactly as in [13, 3.27, pp. 149].

Let Γ be a cocompact or cofinite subgroup of $PSL(2,\mathbb{R})$. We set

$$K(w,w') = \sum_{\gamma\in\Gamma}\tilde{k}(t(\gamma w,w')) = \sum_{\gamma\in\Gamma}k(r(\gamma w,w')).$$

Then the conditions on k to assure absolute convergence of the series above and, therefore, be able to define an operator K by

$$\begin{aligned} (Kf)(w) &= \int_{\Gamma\backslash\mathbf{H}} K(w,w')f(w')\,dw' \\ &= \int_{\mathbf{H}} \tilde{k}(t(w,w'))f(w')\,dw' \end{aligned}$$

are explained in [12, pp. 60]. They are: $k(r(x,y))$ should have a majorant, $k_1(x,y)$ such that (a) $\int_{\mathbf{H}} k_1(x,y)dy < \infty$ and (b) there are constants $\delta > 0$, $A > 0$ such that for all x and y

$$k_1(x,y) \le A\int_{r(y,y')<\delta} k_1(x,y')dy'.$$

The Fourier expansion of $K(w,w')$ is

(10)

$$K(w,w') = \sum_{j=0}^\infty h(r_j)\phi_j(w)\overline{\phi_j(w')} + \int_0^\infty h(s)E(w,1/2+is)E(w',1/2-is)\,ds,$$

where the ϕ_j's are an orthonormal basis of eigenfunctions for the discrete spectrum with corresponding eigenvalues $\frac{1}{4}+r_j^2$.

3. Some remarks about Fourier expansions of eigenfunctions on $\mathbb{H}$

The metric on the disc model B^2 of hyperbolic space is $ds^2 = 4\frac{dx^2+dy^2}{(1-(x^2+y^2))^2}$, $z = x + iy$. We use polar coordinates on B^2: $r = r(0, z) = \log\left(\frac{1+|z|}{1-|z|}\right)$, $0 \le r < \infty$, $\theta \in [0, 2\pi]$. Denote an eigenfunction on B^2 again by ϕ, which satisfies

$$\Delta\phi + \left(\frac{1}{4} + \lambda^2\right)\phi = 0.$$

We separate variables in polar coordinates. Since the metric in polar coordinates is given by: $ds^2 = dr^2 + (\sinh r)^2 d\theta^2$, the Laplace operator is

$$\Delta = \frac{\partial^2}{\partial r^2} + \coth r\frac{\partial}{\partial r} + \frac{1}{(\sinh r)^2}\frac{\partial^2}{\partial \theta^2}.$$

We write: $\phi(r, \theta) = \sum_{-\infty}^{\infty} f_j(r)e^{ij\theta}$. Then for all $j \in \mathbf{Z}$

$$\frac{\partial^2 f_j(r)}{\partial r^2} + \coth r\frac{\partial f_j(r)}{\partial r} - \frac{j^2 f_j(r)}{(\sinh r)^2} + \left(\frac{1}{4} + \lambda^2\right) f_j(r) = 0.$$

We put $q = -\sinh^2 r$ and we get, since $\frac{dq}{dr} = -\sinh(2r)$, $\frac{\partial}{\partial r} = (-\sinh(2r))\frac{\partial}{\partial q}$ and $\frac{\partial^2}{\partial r^2} = 4q(q-1)\frac{\partial^2}{\partial q^2} - 2(1-2q)\frac{\partial}{\partial q}$,

$$q^2(q-1)\frac{\partial^2 f_j}{\partial q^2} + q\left(\frac{3}{2}q - 1\right)\frac{\partial f_j}{\partial q} + \left(\frac{j^2}{4} + \frac{1}{16} + \frac{\lambda^2}{4}q\right) f_j = 0.$$

Since the equation

$$x^2(x-1)y'' + [(a+b+1)x + (\alpha+\beta-1)]xy' + (abx - \alpha\beta)y = 0$$

has the solution $y = x^\alpha F(a+\alpha, b+\alpha, \alpha-\beta+1, x)$ (see [6, (19), pp. 470]), where F is the Gauss hypergeometric function, we get

$$f_j(q) = c_j q^{\frac{|j|}{2}} F(1/4 + i\lambda/2 + |j|/2, 1/4 - i\lambda/2 + |j|/2, |j| + 1, q)$$

and, since $q = -4\frac{\tanh^2(\frac{r}{2})}{(1-\tanh^2(\frac{r}{2}))^2}$, we use the quadratic transformation formula [2, 2.11 (1), pp. 110] to get

$$f_j(r) = a'_j \tanh^{|j|}(r/2)\,(\cosh(r/2))^{-1-2i\lambda}\, F(1/2+i\lambda+|j|, 1/2+i\lambda, 1+|j|, \tanh^2(r/2))$$

for some constants a'_j.

4. Outline of the proof

Assume Γ is cocompact. If K is an integral operator as in Sect. 2, then

$$\begin{aligned}(K\phi^2)(w) &= \int_{\Gamma\backslash\mathbf{H}} K(w, w')\phi^2(w')\, dw' \\ &= \sum_{j=0}^{\infty} h(r_j)(\phi^2, \phi_j)\phi_j(w),\end{aligned}$$

which gives (using Parseval's equality)

$$||K\phi^2||_2^2 = \sum_{j=0}^{\infty} |h(r_j)(\phi^2, \phi_j)|^2.$$

We will choose a family of operators K_t given by point pair invariants $k_t(r)$ such that the corresponding transforms localize at r_j, i.e. $|h_{r_j}(r_j)| \geq c$ for all sufficiently large r_j, where c is a constant independent of j. Then

$$(11) \qquad ||K_t\phi^2||_\infty^2 \cdot \mathrm{vol}\,(\Gamma \setminus \mathbb{H}) \geq ||K_t\phi^2||_2^2 = \sum_{j=0}^{\infty} |h_t(r_j)(\phi^2, \phi_j)|^2.$$

If we prove that $||K_t(\phi^2)||_\infty \ll t^{1/2}\exp(-\pi t/2)$, then

$$||K_{r_k}(\phi^2)||_\infty \ll (r_k^{1/2})\exp(-\pi r_k/2)$$

and, by looking at one summand in (11),

$$r_k \exp(-\pi r_k) \gg \sum_{j=0}^{\infty} \left|h_{r_k}(r_j)(\phi^2, \phi_j)\right|^2 \geq |h_{r_k}(r_k)|^2 \left|(\phi^2, \phi_k)\right|^2 \geq c^2 \left|(\phi^2, \phi_k)\right|^2,$$

which implies that $(\phi^2, \phi_k) \ll r_k^{1/2}\exp(-\pi r_k/2)$ and proves theorem 1. The choice of k_t will be explained later. The issue is to estimate the L^∞ norm of $K_t(\phi^2)$. Fix $w \in \mathbb{H}$. We switch to the disc model of hyperbolic space by a transformation that maps w to 0. All bounds will be uniform in w.

For $0 \leq r < \infty$ we define

$$B(r) = \int_{S^1} |\phi(r, \theta)|^2 \, d\theta$$

We set

$$C_j(r) = (\cosh(r/2))^{-1-2i\lambda} F(1/2 + i\lambda + |j|, 1/2 + i\lambda, 1 + |j|, \tanh^2(r/2))$$

so that

$$\phi(r, \theta) = \sum_{j=-\infty}^{\infty} a_j' \tanh^{|j|}(r/2) C_j(r) e^{ij\theta}.$$

The functions $\tanh^{|j|}(r/2)C_j(r)$ are the associated spherical functions. Parseval's equality then gives

$$(12) \qquad B(r) = \sum_{j=0}^{\infty} |a_j|^2 \tanh^{2j}(r/2) |C_j(r)|^2,$$

where $|a_j|^2 = |a_j'|^2 + |a_{-j}'|^2$, $j \geq 0$. The function $B(r)$ extends on the real line as an even function and, if $|\phi(r, \theta)| \leq M$, then $B(r) \leq 2\pi M$. It extends to an analytic function for $|\Im r| < \pi/2$, see Lemma 2. The crucial point is a lower bound of $C_j(r_j)$ as $j \to \infty$ for a certain sequence r_j that gives a rather sharp bound on a_j, see Lemma 1 and an upper bound on $C_j(r)$ as $j \to \infty$ for all r with $|\Im r| < \pi/2$, see Lemma 3. This lemma is a sharper version of [11, Eq. 2.21] and is the crucial new ingredient. Its proof is included in Appendix A.

The proof of Lemma 1 is exactly the same as in [5, Th. 4.24, pp. 66] and [7] and is included only as a convenience to the reader. The main point is that if an eigenfunction does not increase more than exponentially in the distance from the origin of the hyperbolic disc, then it corresponds to a distribution on the boundary of the disc and Lemma 1 gives a bound on the order, as long as $i\lambda \notin \frac{1}{2}\mathbb{Z}$. This is issue is related to the question whether such an eigenfunction can be represented as a Poisson transform of its boundary values, see [5] and [7]. For general affine symmetric spaces (no restriction to the rank) this is true for λ outside certain hyperplanes, see [9].

5. Proof of Theorem 1

5.1. We state the lemmas mentioned in Sect. 4.

Lemma 1. (see also [5, Th. 4.24, pp. 66] and [7]) *Assume that $i\lambda \notin \frac{1}{2}\mathbf{Z}$. Then the sequence a_j is square integrable.*

Proof. Using [2, 2.10(1), pp. 108] we get

$$F(1/2+i\lambda+j, 1/2+i\lambda, 1+j, \tanh^2(r/2)) =$$
$$(13) = \tfrac{\Gamma(1+j)\Gamma(-2i\lambda)}{\Gamma(1/2-i\lambda)\Gamma(1/2+j-i\lambda)} F(1/2+i\lambda+j, 1/2+i\lambda, 1+2i\lambda, \cosh^{-2}(r/2))$$
$$+(\cosh(r/2))^{4i\lambda} \tfrac{\Gamma(2i\lambda)\Gamma(1+j)}{\Gamma(1/2+i\lambda+j)\Gamma(1/2+i\lambda)} F(1/2-i\lambda, 1/2+j-i\lambda, 1-2i\lambda, \cosh^{-2}(r/2))$$

We have: $\lim_{m\to\infty} F(a+m, b, c, \frac{x}{m}) = {}_1F_1(b,c,x)$, where ${}_1F_1$ is the confluent hypergeometric function. We construct a sequence $r_j \to \infty$ by setting $\cosh^2(r_j/2) = j/a$ where $a > 0$ is to be determined later. We also have

$$\lim_{j\to\infty} \frac{\Gamma(x+j)}{\Gamma(j)} j^{-x} = 1 \tag{14}$$

and

$$\lim_{j\to\infty} (\tanh(r_j/2))^j = e^{-a/2}.$$

Therefore

$$\lim_{j\to\infty} \tanh^j(r_j/2) C_j(r_j) = e^{-a/2} \tfrac{\Gamma(-2i\lambda)}{\Gamma(1/2-i\lambda)}\, {}_1F_1(1/2+i\lambda, 1+2i\lambda, a) a^{1/2+i\lambda}$$
$$+ e^{-a/2} \tfrac{\Gamma(2i\lambda)}{\Gamma(1/2+i\lambda)}\, {}_1F_1(1/2-i\lambda, 1-2i\lambda, a) a^{1/2-i\lambda}.$$

Since

$$\lim_{x\to\infty} {}_1F_1(b,c,x) e^{-x} x^{b-c} = \frac{\Gamma(c)}{\Gamma(b)}$$

[2, 6.13.1 (3), pp. 278], the terms on the right hand side of the previous equation behave differently for a large a. In particular we can select a such that the last limit is different from 0. Then for a suitable constant $c_0 \neq 0$ and j sufficiently large, say $j \geq j_0$, we have

$$\tanh^{2j}(r_j/2)|C_j(r_j)|^2 \geq c_0$$

so

$$2\pi M \geq B(r_j) = \sum_{j=0}^{j_0} + \sum_{j=j_0+1}^{\infty} |a_j|^2 \tanh^{2j}(r_j/2)|C_j(r_j)|^2 \geq \sum_{j=j_0+1}^{\infty} |a_j|^2 c_0$$

and the sequence a_j is in l^2. □

Lemma 2. *The function $B(r)$ extends to an even analytic function of r in the strip $|\Im r| < \frac{\pi}{2}$ and satisfies the bound*

$$B(r) \ll |\cosh(r/2)|^4.$$

Proof. We have

$$B(r) = \sum_{j=0}^{\infty} |a_j|^2 \tanh^j(r/2)\overline{\tanh^j(\overline{r}/2)} C_j(r)\overline{C_j(\overline{r})}$$

and we note that the hypergeometric function $F(a,b,c,z)$ is holomorphic in the region $|z| < 1$ and that the map $z = \tanh\left(\frac{r}{2}\right)$ is a conformal map from $|\Im r| < \frac{\pi}{2}$ to $|z| < 1$. So we have a series of holomorphic functions and we will prove that

(15)
$$\left|(\cosh(r/2))^{-3-2i\lambda} F(1/2 + i\lambda + j, 1/2 + i\lambda, 1 + j, \tanh^2(r/2))\right| \ll 1$$

i.e. $|C(r)| \ll |\cosh(r/2)|^2$. Eq.(15) is equivalent to Eq.(16), which captures the behavior of the hypergeometric function in (15) for large j:

Lemma 3. *The following bound holds for* $|z| < 1$

(16)
$$\left|(1-z)^{3/2+i\lambda} F\left(1/2 + i\lambda + j, 1/2 + i\lambda, 1 + j, z\right)\right| \ll 1.$$

We also have for $t > 0$

(17)
$$\left|(1-z)^{3/2} F\left(1/2, 1/2, 1 - it, z\right)\right| \ll 1.$$

The proof of lemma 3 is included in Appendix A.

We now have

$$B(r) \ll \sum_{j=0}^{\infty} |a_j|^2 \, |\tanh(r/2)|^{2j} \, |\cosh(r/2)|^4.$$

Since $|\tanh r/2| < 1$ and the sequence a_j is in l^2 by lemma 1, the proof of lemma 2 is complete. □

Remark 1. We note that on the horizontal lines $r = x + i\left(\frac{\pi}{2} - \frac{1}{t}\right)$ and $r = x - i\left(\frac{\pi}{2} - \frac{1}{t}\right)$, $x \in \mathbb{R}$, we have the bound

(18)
$$B(r) \ll e^{2|x|}.$$

This is so because a trivial calculation gives

$$|\cosh(r/2)|^2 = \cos^2(\pi/4 - 1/2t) + \sinh^2(x/2).$$

5.2. We now come to the choice of the point-pair invariants $k_t(r)$:

(19)
$$k_t(r) = \frac{tP_{-\frac{1}{2}+it}(\cosh r)\sinh^2 r}{\cosh^8(r)}.$$

Remark 2. The intuition behind the choice of the point-pair invariants is as follows: The inversion formula for the Harish-Chandra transform

$$k(r) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} h(\lambda) P_{-\frac{1}{2}+i\lambda}(\cosh r)|c(\lambda)|^{-2} \, d\lambda,$$

where $c(\lambda)$ is the Harish-Chandra c function, suggests that in order to localize $h_t(\lambda)$ at t, say $h_t(\lambda) = \delta_t(\lambda)$, $k_t(r)$ has to be essentially $P_{-1/2+it}(\cosh r)|c(t)|^{-2}$. One sees that $|c(t)|$ is asymptotic to $\pi^{-1/2}t^{-1/2}$, see Eq.(52). However, since we do not want to work with distributions and need to define integral operators as in Sect. 2, we use a factor to make $k_t(r)$ rapidly decreasing. We choose it to be $\sinh^m r/\cosh^n r$, for m and n natural numbers. These point-pairs are smooth at $r = 0$ and can be odd or even functions of r and have order of vanishing at 0 as high as needed by adjusting m and n. These options are important in order to generalize to the other rank one symmetric spaces. Moreover the fact that we incorporate in the point-pair invariants the spherical function, which is a hypergeometric function, allows to avoid the fractional integral in (8). With the exception of the odd dimensional real hyperbolic spaces all other rank one spaces have Harish-Chandra transform that involves fractional integration and multiple integrals, see [14, pp. 31]. The advantage of using (19) is now obvious.

We need to know that the point-pair invariants (19) satisfy the conditions explained in Sect. 2 in order that the series

$$K_t(w,w') = \sum_{\gamma\in\Gamma} k_t(r(w,\gamma w'))$$

converges absolutely. We will show that k_t has majorant $k_1(x,y) = te^{-\frac{13}{2}r}$, $r = r(x,y)$. We study the behavior of $P_{-\frac{1}{2}+it}(\cosh r)$ for $r \in \mathbb{R}$, which we need later too. The Legendre function of the first kind $P_{-\frac{1}{2}+it}(z) = F(1/2+it, 1/2-it, 1, (1-z)/2)$, $|1-z| < 2$ is real for z real, since the hypergeometric function is symmetric in its first two arguments and for $z \in \mathbf{R}$ we have

$$\overline{F(1/2+it, 1/2-it, 1, (1-z)/2)} = F(1/2-it, 1/2+it, 1, (1-z)/2).$$

Consequently $P_{-\frac{1}{2}+it}(z)$ is real for $z \geq -1$. Formula (26), [2, pp. 128] gives

(20)
$$\begin{aligned} P_{-\frac{1}{2}+it}(\cosh r) = & \tfrac{1}{\sqrt{\pi}} \tfrac{\Gamma(-it)}{\Gamma(1/2-it)} \tfrac{e^{(1/2-it)r}}{(e^{2r}-1)^{1/2}} F(1/2, 1/2, 1+it, \tfrac{1}{1-e^{2r}}) \\ & + \tfrac{1}{\sqrt{\pi}} \tfrac{\Gamma(it)}{\Gamma(1/2+it)} \tfrac{e^{(1/2+it)r}}{(e^{2r}-1)^{1/2}} F(1/2, 1/2, 1-it, \tfrac{1}{1-e^{2r}}) \end{aligned}$$

for $r > \frac{1}{2}\ln 2$. We note that the two hypergeometric functions in (20) are conjugate numbers for $r \in \mathbb{R}$. We also have for $r > \ln 2$ and $t > 1$

$$F(1/2, 1/2, 1+it, \frac{1}{1-e^{2r}}) = 1 + \xi(t,r),$$

where

(21)
$$|\xi(t,r)| \leq 2e^{-2r}.$$

This follows from the series expansion of the hypergeometric function as follows:

$$F(1/2, 1/2, 1+it, z) = 1 + \sum_{k=1}^{\infty} \frac{(1/2)_k^2}{(1+it)_k k!} z^k,$$

where we use Pochhammer's notation: $(a)_k = a(a+1)\ldots(a+k-1)$. We have $\left(\frac{1}{2}\right)_k < k!$, $\left(\frac{1}{2}\right)_k < |(1+it)_k|$ and

$$\sum_{k=1}^{\infty} |z|^k = \frac{|z|}{1-|z|} = \frac{1}{e^{2r}-2} < 2e^{-2r}.$$

We note that the bound on $\xi(t,r)$ is independent of t. From (20) we deduce that, for fixed t,

$$P_{-\frac{1}{2}+it}(\cosh r) \ll e^{-\frac{1}{2}r},$$

as $r \to \mathbb{R}$ and the integral

$$\int_0^\infty \frac{e^{-\frac{1}{2}r}}{e^{6r}} \sinh r \, dr < \infty$$

converges. This proves condition (a) in [12]. For fixed δ small and $r = r(x,y)$

$$\int_{\{y' | r(y,y') < \delta\}} k_1(r(x,y'))\, dy' > e^{-\frac{13}{2}(r+\delta)} \text{vol } B(\delta)$$

since $k_1(r)$ is decreasing. On the other hand, $k_1(r(x,y)) = e^{-\frac{13}{2}r} \le e^{\frac{13}{2}\delta} e^{-\frac{13}{2}(r+\delta)}$, so condition (b) is [12] is satisfied too.

5.3. Now we prove that $h_t(t) \ge c_0$ for all t sufficiently large, i.e. the Selberg-Harish-Chandra transform of k_t localizes at t. Using (9) and the fact that the spherical function $P_{-\frac{1}{2}+it}(\cosh r)$ is real, we get for $m > m_0$

$$\frac{1}{\pi} h_t(t) \ge t \int_m^\infty \left[P_{-\frac{1}{2}+it}(\cosh r) \right]^2 (\cosh r)^{-5}\, dr,$$

where m is to be determined later independently of t. The issue is to show that the integral giving $h_t(t)$, which is positive and decreases as $t \to \infty$, decreases at most like $1/t$ and not more quickly. Using (20) and (21) we get

$$\begin{aligned} P_{-\frac{1}{2}+it}(\cosh r) = \quad & \tfrac{1}{\sqrt{\pi}} \tfrac{\Gamma(-it)}{\Gamma(1/2-it)} \tfrac{e^{(1/2-it)r}}{(e^{2r}-1)^{1/2}} (1 + \xi(t,r)) \\ & + \tfrac{1}{\sqrt{\pi}} \tfrac{\Gamma(it)}{\Gamma(1/2+it)} \tfrac{e^{(1/2+it)r}}{(e^{2r}-1)^{1/2}} \left(1 + \overline{\xi}(t,r)\right) \end{aligned} \tag{22}$$

and

(23)

$$\begin{aligned} h_t(t) \ge \quad & t \textstyle\int_m^\infty \tfrac{\Gamma^2(-it)}{\Gamma^2(1/2-it)} \tfrac{1}{\cosh^5 r} \tfrac{e^{(1-2it)r}}{e^{2r}-1} (1 + \xi(t,r))^2 \, dr \\ & + t \textstyle\int_m^\infty \tfrac{\Gamma^2(it)}{\Gamma^2(1/2+it)} \tfrac{1}{\cosh^5 r} \tfrac{e^{(1+2it)r}}{e^{2r}-1} \left(1 + \overline{\xi}(t,r)\right)^2 dr \\ & + 2t \textstyle\int_m^\infty \tfrac{\Gamma(-it)\Gamma(it)}{\Gamma(1/2-it)\Gamma(1/2+it)} \tfrac{1}{\cosh^5 r} \tfrac{e^{r}}{e^{2r}-1} (1 + \xi(t,r)) \left(1 + \overline{\xi}(t,r)\right) dr. \end{aligned}$$

The idea suggested by the asymptotics of $P_{-\frac{1}{2}+it}(\cosh r)$, as given by Eq.(22), is that the main contribution comes from the integral

$$2t \int_m^\infty \frac{\Gamma(-it)\Gamma(it)}{\Gamma(1/2-it)\Gamma(1/2+it)} \frac{1}{\cosh^5 r} \frac{e^r}{e^{2r}-1}\, dr.$$

By expanding the products and the squares in Eq.(23) we get nine integrals, which we estimate using (21):

$$A_1 = \left|\int_m^\infty \frac{e^r\left(\xi(t,r)+\bar{\xi}(t,r)\right)}{(e^{2r}-1)\cosh^5 r}dr\right| < c_1 e^{-8m}$$

$$A_2 = \left|\int_m^\infty \frac{e^r|\xi(t,r)|^2}{(e^{2r}-1)\cosh^5 r}dr\right| < c_2 e^{-10m}$$

$$A_3 = \left|\int_m^\infty \frac{e^{(1+2it)r}2\bar{\xi}(t,r)}{(e^{2r}-1)\cosh^5 r}dr\right| < c_3 e^{-8m}$$

$$A_4 = \left|\int_m^\infty \frac{e^{(1+2it)r}(\bar{\xi}(t,r))^2}{(e^{2r}-1)\cosh^5 r}dr\right| < c_4 e^{-10m}$$

$$A_5 = \left|\int_m^\infty \frac{e^{(1-2it)r}2\xi(t,r)}{(e^{2r}-1)\cosh^5 r}dr\right| < c_3 e^{-8m}$$

$$A_6 = \left|\int_m^\infty \frac{e^{(1-2it)r}(\xi(t,r))^2}{(e^{2r}-1)\cosh^5 r}dr\right| < c_4 e^{-10m}$$

$$A_7 = \int_m^\infty \frac{e^r}{(e^{2r}-1)\cosh^5 r}dr \geq \int_m^\infty e^{-6r}dr = \tfrac{1}{6}e^{-6m}$$

$$A_8 = \int_m^\infty \frac{e^{(1+2it)r}}{(e^{2r}-1)\cosh^5 r}dr = \frac{1}{2it+1}\frac{e^{(2it+1)m}}{(1-e^{2m})\cosh^5 m}$$
$$+\frac{1}{2it+1}\int_m^\infty \frac{e^{(2it+1)r}[5\sinh r(e^{2r}-1)+2e^{2r}\cosh r]}{(e^{2r}-1)^2\cosh^6 r}dr$$
$$\ll c_8 t^{-1}e^{-6m}$$

$$A_9 = \int_m^\infty \frac{e^{(1-2it)r}}{(e^{2r}-1)\cosh^5 r}dr = \overline{A}_8 \ll c_8 t^{-1}e^{-6m}.$$

The asymptotic behavior of the Gamma function $\Gamma(x+iy)$ for large $|y|$ is described by the formula

$$\lim_{|y|\to\infty} |\Gamma(x+iy)|e^{\pi|y|/2}|y|^{\frac{1}{2}-x} = (2\pi)^{1/2}, \tag{24}$$

see [2, (6), pp. 47]. Using (24) we see that

$$\lim_{t\to\infty} \frac{t|\Gamma^2(it)|}{|\Gamma^2(1/2+it)|} = 1,$$
$$\lim_{t\to\infty} \frac{t|\Gamma^2(-it)|}{|\Gamma^2(1/2-it)|} = 1,$$
$$\lim_{t\to\infty} \frac{t|\Gamma(-it)\Gamma(it)|}{|\Gamma(1/2-it)\Gamma(1/2+it)|} = 1.$$

We now choose m such that

$$\frac{1}{6}e^{-6m} > 2\left((c_1+2c_3)e^{-8m} + (c_2+2c_4)e^{-10m}\right)$$

and we use the last fourteen equations together with the triangle inequality and (23) to deduce that

$$\liminf_{t\to\infty} h_t(t) > 0, \tag{25}$$

which concludes the claims about the choice of the point-pair invariants.

5.4. We come back to estimate the L^∞ norm of $K_t(\phi^2)$. We have by using polar coordinates

$$K(\phi^2)(w) = I = \int_0^\infty k_t(r)B(r)\sinh r\,dr = \int_0^\infty \frac{tP_{-\frac{1}{2}+it}(\cosh r)}{\cosh^8(r)}B(r)\sinh^3 r\,dr.$$

Formula 3.3.1 (3) in [2, pp. 140] gives

$$\frac{\tan(-1/2+it)\pi}{\pi}\left(Q_{-\frac{1}{2}+it}(z) - Q_{-\frac{1}{2}-it}(z)\right) = P_{-\frac{1}{2}+it}(z), \tag{26}$$

where $Q_\nu(z)$ is the Legendre function of the second kind. Eq.(26) gives, since $\tan(-\frac{1}{2}+it)\pi = i\coth(t\pi)$,

$$(27)\qquad \begin{aligned} I = & \int_0^\infty \frac{it}{\pi}\coth(t\pi)\frac{Q_{-1/2+it}(\cosh r)}{\cosh^8(r)}B(r)\sinh^3 r\,dr \\ & -\int_0^\infty \frac{it}{\pi}\coth(t\pi)\frac{Q_{-1/2-it}(\cosh r)}{\cosh^8(r)}B(r)\sinh^3 r\,dr \\ & = I_1 - I_2. \end{aligned}$$

The Legendre functions of the second kind $Q_\nu^\mu(z)$ and the Legendre functions of the first kind $P_\nu^\mu(z)$ are not single-valued in the plane. One must introduce a cut from $-\infty$ to 1. However, when μ is an even integer, we can reduce the cut for $P_\nu^\mu(z)$ to $(-\infty,-1]$. This is explained in [2, pp. 143]. We see that in the strip $|\Im r| < \frac{\pi}{2}$ the cut $[0,1]$ corresponds to $i[-\frac{\pi}{2},\frac{\pi}{2}]$ and that the conformal map $z=\cosh r$ opens the cut $[0,1]$ so that approaching $[0,1]$ from above (below) corresponds to approaching $i[0,\frac{\pi}{2}]$ $(i[-\frac{\pi}{2},0])$. We denote the new branches of $Q_\nu^\mu(z)$ when we go around the branch point 1 clockwise (counterclockwise) by $Q_\nu^\mu(z,1-)$ $(Q_\nu^\mu(z,1+))$. The relation between $Q_\nu^\mu(z)$, $Q_\nu^\mu(z,1\pm)$ and $P_\nu^\mu(z)$ is described by the equations

$$(28)\qquad \begin{aligned} Q_\nu^\mu(z,1-) - e^{-i\mu\pi}Q_\nu^\mu(z) &= \pi i e^{i\mu\pi}P_\nu^\mu(z) \\ Q_\nu^\mu(z,1+) - e^{i\mu\pi}Q_\nu^\mu(z) &= -\pi i e^{i\mu\pi}P_\nu^\mu(z), \end{aligned}$$

see [2, 3.3.2 (19), pp. 142]. For completeness we include the proof of the second equation. We have

$$(29)\qquad \begin{aligned} e^{-i\pi\mu}Q_\nu^\mu(z) &= \frac{\Gamma(1+\nu+\mu)\Gamma(-\mu)}{2\Gamma(1-\mu+\nu)}\frac{(z-1)^{\mu/2}}{(z+1)^{\mu/2}}F(-\nu,1+\nu,1+\mu,(1-z)/2) \\ & +\frac{\Gamma(\mu)}{2}\frac{(z+1)^{\mu/2}}{(z-1)^{\mu/2}}F(-\nu,1+\nu,1-\mu,(1-z)/2), \end{aligned}$$

see [2, 3.2 (32), pp.130]. We continue analytically Eq.(29) to get

$$\begin{aligned} e^{-i\pi\mu}Q_\nu^\mu(z,1+) &= \frac{\Gamma(1+\nu+\mu)\Gamma(-\mu)}{2\Gamma(1-\mu+\nu)}e^{i\mu\pi}\frac{(z-1)^{\mu/2}}{(z+1)^{\mu/2}}F(-\nu,1+\nu,1+\mu,(1-z)/2) \\ & +\frac{\Gamma(\mu)}{2}e^{-i\mu\pi}\frac{(z+1)^{\mu/2}}{(z-1)^{\mu/2}}F(-\nu,1+\nu,1-\mu,(1-z)/2). \end{aligned}$$

As a consequence of the last two equations it follows that

$$e^{-i\pi\mu}Q_\nu^\mu(z,1+) - Q_\nu^\mu(z) = (e^{-i\mu\pi}-e^{i\mu\pi})\frac{\Gamma(\mu)}{2}\frac{(z+1)^{\mu/2}}{(z-1)^{\mu/2}}F(-\nu,1+\nu,1-\mu,(1-z)/2)$$

and now the equation: $\Gamma(\mu)\Gamma(1-\mu) = \frac{\pi}{\sin\pi\mu}$ together with

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\frac{(z+1)^{\mu/2}}{(z-1)^{\mu/2}}F(-\nu,1+\nu,1-\mu,(1-z)/2)$$

[2, 3.2 (3), pp. 122] gives the result.

In (28) we pass to the limit $\mu\to 0$ to get

$$\begin{aligned} Q_\nu(z,1-) - Q_\nu(z) &= \pi i P_\nu(z) \\ Q_\nu(z,1+) - Q_\nu(z) &= -\pi i P_\nu(z) \end{aligned}$$

which imply

$$(30)\qquad Q_\nu(z,1+) - Q_\nu(z,1-) = -2\pi i P_\nu(z).$$

Now we shift the contour of integration for I_1, I_2 as follows: For I_1 we first go along the negative real axis from 0 to $-\infty$ and on the lower cut of the plane (called path γ_1) and then along the line γ_2 given by $r = x - i\left(\frac{\pi}{2}-\frac{1}{t}\right)$. For I_2 we first go along the negative real axis from 0 to $-\infty$ and on the upper cut of the plane (called path γ_3) and then along the line γ_4 given by $r = x + i\left(\frac{\pi}{2} - \frac{1}{t}\right)$. We set

$$\tilde{Q}_{-\frac{1}{2}\pm it}(\cosh r) = \begin{cases} Q_{-\frac{1}{2}\pm it}(\cosh r) & \Re r \geq 0 \\ Q_{-\frac{1}{2}\pm it}(\cosh r, 1\mp) & \Re r < 0. \end{cases}$$

Then

$$\begin{aligned} I = \quad & \int_{\gamma_1+\gamma_2} \tfrac{it}{\pi} \coth(t\pi) \tilde{Q}_{-\frac{1}{2}+it}(\cosh r) \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ & - \int_{\gamma_3+\gamma_4} \tfrac{it}{\pi} \coth(t\pi) \tilde{Q}_{-\frac{1}{2}-it}(\cosh r) \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr \end{aligned}$$

so that

$$\begin{aligned} I = \quad & - \int_{-\infty}^{0} \tfrac{it}{\pi} \coth(t\pi) Q_{-\frac{1}{2}+it}(\cosh r, 1-) \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ & + \int_{-\infty}^{0} \tfrac{it}{\pi} \coth(t\pi) Q_{-\frac{1}{2}-it}(\cosh r, 1+) \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ & + \int_{\gamma_2} - \int_{\gamma_4}. \end{aligned}$$

Moreover,

(31)
$$\begin{aligned} I = \quad & \int_0^\infty \tfrac{it}{\pi} \coth(t\pi) \left[Q_{-1/2-it}(\cosh r, 1+) - Q_{-1/2+it}(\cosh r, 1-)\right] \tfrac{-B(r)\sinh^3 r}{\cosh^8 r} dr \\ & + \int_{\gamma_2} - \int_{\gamma_4}, \end{aligned}$$

because $B(r)$ is even. Since

$$P_{-\frac{1}{2}-it}(\cosh r) = \frac{\tan(-1/2 - it)\pi}{\pi} \left[Q_{-\frac{1}{2}-it}(\cosh r) - Q_{-\frac{1}{2}+it}(\cosh r)\right]$$

for $r > 0$, we get by analytic continuation when we cross the cut $i[0, \frac{\pi}{2}]$

$$P_{-\frac{1}{2}-it}(\cosh r) = \frac{\tan(-1/2 - it)\pi}{\pi} \left[Q_{-\frac{1}{2}-it}(\cosh r, 1+) - Q_{-\frac{1}{2}+it}(\cosh r, 1+)\right],$$

which gives together with (30), (31)

$$\begin{aligned} I = \quad & \int_{\gamma_2} - \int_{\gamma_4} \\ & - \int_0^\infty \tfrac{it}{\pi} \coth(t\pi) \tfrac{i\pi}{\coth(\pi t)} P_{-\frac{1}{2}-it}(\cosh r) \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ & - \int_0^\infty \tfrac{it}{\pi} \coth(t\pi) \left[Q_{-\frac{1}{2}+it}(\cosh r, 1+) - Q_{-\frac{1}{2}+it}(\cosh r, 1-)\right] \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr \end{aligned}$$

and, therefore,

$$\begin{aligned} I = \quad & \int_{\gamma_2} - \int_{\gamma_4} \\ & + \int_0^\infty t P_{-\frac{1}{2}-it}(\cosh r) \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr \\ & - \int_0^\infty \tfrac{it}{\pi} \coth(t\pi)(-2\pi i) P_{-\frac{1}{2}+it}(\cosh r) \tfrac{B(r)\sinh^3 r}{\cosh^8 r} dr. \end{aligned} \tag{32}$$

Since $P_{-\frac{1}{2}-it}(z) = P_{-\frac{1}{2}+it}(z)$ (see [2, 3.3.1 (1), pp. 140]) we get from (32)

$$I = \int_{\gamma_2} - \int_{\gamma_4} + I - 2\coth(t\pi) I$$

or, equivalently,

$$I = \frac{e^{t\pi} - e^{-t\pi}}{2(e^{t\pi} + e^{-t\pi})} \left(\int_{\gamma_2} - \int_{\gamma_4}\right).$$

Therefore, it is enough to prove that

$$\int_{\gamma_2} \frac{it}{\pi}\coth(t\pi)\tilde{Q}_{-\frac{1}{2}+it}(\cosh r)\frac{B(r)\sinh^3 r}{\cosh^8 r}dr = O(t^{1/2}e^{-\frac{\pi}{2}t})$$

$$\int_{\gamma_4} \frac{it}{\pi}\coth(t\pi)\tilde{Q}_{-\frac{1}{2}-it}(\cosh r)\frac{B(r)\sinh^3 r}{\cosh^8 r}dr = O(t^{1/2}e^{-\frac{\pi}{2}t}).$$

Since $B(r)$ is real for real $r > 0$, we have $B(r) = \overline{B(\bar{r})}$ on the strip $|\Im r| < \frac{\pi}{2}$ and we see that the integrand $\mathcal{C}(r)$ in $\int_{\gamma_2}$ is $\mathcal{C}(r) = \overline{\mathcal{D}(\bar{r})}$, where $\mathcal{D}(r)$ is the integrand for $\int_{\gamma_4}$. So it is enough to look at $\int_{\gamma_4}\mathcal{D}(r)dr$. We have

$$Q_{-\frac{1}{2}-it}(\cosh r) = \sqrt{\pi/2}\frac{\Gamma(1/2-it)}{\Gamma(1-it)}\frac{e^{itr}}{\sqrt{\sinh r}}F(1/2,1/2,1-it,\frac{1}{1-e^{2r}})$$

for $r > \frac{1}{2}\ln 2$ [2, 3.2 (44), pp. 136]. This formula holds by analytic continuation in the domain: $\{r|\frac{\pi}{4} < \Im r < \frac{\pi}{2}, -\infty < \Re r < \ln 2\} \cup \{r|-\frac{\pi}{4} < \Im r < \frac{\pi}{2}, \Re r \geq \ln 2\}$. On this domain we have: $\left|\frac{1}{1-e^{2r}}\right| < 1$, so we can apply (17). On the line γ_4 we have: $|\sinh r| \ll e^{|x|}$, $|F(1/2,1/2,1-it,\frac{1}{1-e^{2r}})| \ll |1-e^{-2r}|^{3/2} \ll e^{3|x|}$, $|\cosh r| \gg e^{|x|}$ for $t > \frac{3}{\pi}$. Using (24), (18) we finally get

$$\int_{\gamma_4} \frac{it}{\pi}\coth(t\pi)\tilde{Q}_{-\frac{1}{2}-it}(\cosh r)\frac{B(r)\sinh^3 r}{\cosh^8 r}dr \ll t^{1/2}e^{-\frac{\pi}{2}t}\int_{-\infty}^{\infty} e^{-\frac{|x|}{2}}dx$$

which gives the result. This completes the proof of theorem 1.

6. Non compact surfaces

We need the following property of the point-pair invariants $k_t(r)$ defined in (19):

Claim: There exist $\epsilon > 0$, $\epsilon_0 > 0$ *and* $t_0 > 0$ *such that* $|h_t(s)| \geq \epsilon_0$ *for all* $t \geq t_0$ *and* $|s-t| < \epsilon$.

This property will be proved in Appendix B.

The spectral decomposition of the integral kernel is in this case given by (10), where we have assumed that $\Gamma\backslash\mathbb{H}$ has only one cusp and $E(z,s)$ is the corresponding Eisenstein series. The sum in this equation may actually be only a finite sum. Parseval's identity now gives

(33)

$$\||K\varphi^2\||_2^2 = \sum_{j=0}^{\infty}|h(r_j)|^2|(\varphi^2,\varphi_j)|^2 + \frac{1}{4\pi}\int_0^{\infty}|h(s)|^2|(\varphi^2,E(z,1/2+is))|^2\,ds.$$

The rest of the proof remains unchanged and we look now at the integral on the right-hand side of (33) over the short interval $[t, t+\epsilon]$ to deduce (6) and complete the proof of theorem 2. In order to study the Fourier coefficients of Maaß cusp forms for Γ, we follow the method used in [3, pp. 546] to study the Fourier coefficients of holomorphic cusp forms. We define ψ_U, U sufficiently large, to be a C^∞ function on $\mathbb{R}$ with

$$\psi_U(\tau) = \begin{cases} 1 & \text{if } \tau \leq 1-1/U \\ 0 & \text{if } \tau \geq 1+1/U \end{cases}$$

and $\psi_U^{(j)}(\tau) \ll U^j$ for $j = 0, 1, \ldots$. We will work with the Mellin transform of ψ_U given by

$$R_U(s) = \int_0^\infty \psi_U(\tau)\tau^{s-1}\,d\tau$$

for $\sigma = \Re s > 0$. We have

$$\begin{aligned} R_U(s) &= \int_0^{1+1/U} \tau^{s-1}d\tau + \int_0^{1+1/U} (\psi_U(\tau) - 1)\tau^{s-1}d\tau \\ &= \frac{1}{s} + \int_1^{1+1/U} \tau^{s-1}dt + \int_{1-1/U}^{1+1/U} (\psi_U(\tau) - 1)\tau^{s-1}d\tau. \end{aligned}$$

The two integrals in the right-hand side of the last equation are $O(1/U)$ uniformly on vertical strips. Since $\psi_U(\tau)$ is bounded, this follows by applying the mean-value theorem:

$$\int_{1-1/U}^{1+1/U} \tau^{\sigma-1}d\tau = \frac{2}{U}(1+\xi)^{\sigma-1}$$

for some ξ between $-1/U$ and $1/U$ and $(1+\xi)^{\sigma-1}$ is bounded for σ bounded. As a result

$$R_U(s) = \frac{1}{s} + O\left(\frac{1}{U}\right). \tag{34}$$

Integration by parts gives

(35)
$$R_U(s) = \frac{(-1)^j}{s(s+1)\cdots(s+j-1)} \int_0^\infty \tau^{s+j-1}\psi_U^{(j)}(\tau)d\tau \ll \frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^{j-1}$$

for $j = 1, 2, \ldots$. This follows from the estimates

$$\frac{1}{|s+k|} \ll \frac{1}{|s|+1}$$

for $k = 1, \ldots, j-1$ and

$$\int_0^\infty \tau^{s+j-1}\psi_U^{(j)}(\tau)d\tau \ll U^j \int_{1-1/U}^{1+1/U} \tau^{\sigma+j-1}d\tau = U^j\frac{2}{U}(1+\xi')^{\sigma+j-1} = O\left(U^{j-1}\right)$$

for some ξ' between $-1/U$ and $1/U$. The estimates (34) and (35) are uniform for σ bounded. Now by interpolation it is easy to see that for all $c \geq 0$ we have

$$R_U(s) \ll \frac{1}{|s|}\left(\frac{U}{1+|s|}\right)^c \tag{36}$$

again uniformly for σ bounded. We assume the Maaß cusp form $\phi(z)$ has the Fourier expansion (3) at the cusp and its eigenvalue is $1/4 + \lambda^2$. The L-series $D(s) = \sum |a_n|^2|n|^{-s}$ converges absolutely for $\Re s > 2$ by the Hecke bound $a_n = O(|n|^{1/2})$. The Rankin-Selberg method provides the analytic continuation of $D(s)$ to the whole plane. A standard argument gives

$$D(s) = \frac{2\pi^s\Gamma(s)}{\Gamma(s/2)^2\Gamma(s/2+i\lambda)\Gamma(s/2-i\lambda)} \int_{\Gamma\backslash\mathbf{H}} \phi^2 E(z,s)\,dz. \tag{37}$$

On the critical line line $\Re s = 1/2$ the factor $f(s) = 2\pi^s\Gamma(s)\Gamma(s/2)^{-2}\Gamma(s/2+i\lambda)^{-1}\Gamma(s/2-i\lambda)^{-1}$ is asymptotic to $e^{\pi t/2}t$, as $t \to \infty$, as follows from Eq.(24). The inversion formula for the Mellin transform gives

$$\psi_U(|n|/X) = \frac{1}{2\pi i}\int_{\Re s=2+\epsilon} |n|^{-s}X^s R_U(s)\,ds$$

and, therefore,

(38)

$$\sum_{|n|\le X(1-1/U)} |a_n|^2 \le \sum_{|n|} |a_n|^2\psi_U(|n|/X) = \frac{1}{2\pi i}\int_{\Re s=2+\epsilon} D(s)X^s R_U(s)\,ds.$$

We shift the countour of integration in the integral in (38) to the line $\Re s = 1/2$. The function $D(s)$ has poles coming from the residues of the Eisenstein series on the interval $(1/2, 1]$. Let us assume these are at the points s_j with residues the non cuspidal eigenfunctions $r_j(z)$. We estimate the integral along the line $\Re s = 1/2$ as follows: we choose m an integer with $1/m < \epsilon$. Then, using (6) and (24)

$$\begin{aligned}
&\int_{-\infty}^{\infty} D(1/2+it)X^{1/2+it}R_U(1/2+it)dt = \\
= \;& \sum_{n=-\infty}^{\infty}\sum_{k=1}^{m}\int_{n+\frac{k-1}{m}}^{n+\frac{k}{m}} f(1/2+it)(\varphi^2, E(z,1/2+it))X^{1/2+it}R_U(1/2+it)dt \\
\ll \;& \sum_{n,k}\left(\int_{n+\frac{k-1}{m}}^{n+\frac{k}{m}} |(\varphi^2, E(z,1/2+it))|^2 dt \int_{n+\frac{k-1}{m}}^{n+\frac{k}{m}} |f(1/2+it)|^2 X|R_U(1/2+it)|^2 dt\right)^{1/2} \\
\ll \;& \sum_{n=-\infty}^{\infty}\sum_{k=1}^{m} e^{-\pi|n|}|n|^{1/2}\left(\int_{n+\frac{k-1}{m}}^{n+\frac{k}{m}} e^{\pi|t|}t^2Xt^{-2}\left(\frac{U}{1+t}\right)^{2c}dt\right)^{1/2} \\
\ll \;& \sum_{n=-\infty}^{\infty}\sum_{k=1}^{m} e^{-\pi|n|/2}|n|^{1/2}\left(e^{\pi|n|}X\left(\frac{U}{1+n}\right)^{2c}\right)^{1/2} \\
= \;& X^{1/2}\sum_{n=-\infty}^{\infty} n^{1/2}U^c(1+n)^{-c}
\end{aligned}$$

To make the last series converge we choose $c > 3/2$, say, $c = 3/2 + \epsilon'$ with $\epsilon' > 0$. Then the integral is estimated by $X^{1/2}U^{3/2+\epsilon'}$. Therefore,

$$\sum_{|n|\le X(1-1/U)} |a_n|^2 = \sum_{1/2\le s_j\le 1} (r_j, \phi^2)f(s_j)X^{s_j}\frac{1}{s_j} + O(X/U) + O(X^{3/2}U^{1/2+\epsilon'}),$$

since $X^{s_j} \ll X$ and (34) holds. Coming from the pole of Eisenstein series at $s = 1$ we get the constant eigenfunction and we conclude

$$\sum_{|n|\le X(1-1/U)} |a_n|^2 = cX + O(X/U + X^{1/2}U^{3/2+\epsilon'}) \tag{39}$$

We choose U so that the two error terms are equal, i.e. $U = X^{1/(5+2\epsilon')} = X^{1/5-\epsilon}$ and then the error term becomes $O(X^{4/5+\epsilon})$. Then $a_m = O(|m|^{2/5+\epsilon})$. This proves Corollary 1.

7. Appendix A

Proof of Lemma 3. Using the fundamental integral representation for the hypergeometric function [2, 2.1.3(10), p. 59] we get

$$F(1/2+i\lambda+j,1/2+i\lambda,1+j,z) = \Gamma(1+j)\Gamma(1/2+i\lambda+j)^{-1}\Gamma(1/2-i\lambda)^{-1}\times \times\int_0^1 s^{-1/2+i\lambda+j}[(1-s)(1-zs)]^{-1/2-i\lambda}ds \tag{40}$$

We can assume that $0\geq\Im\lambda>-\frac{1}{2}$, which is necessary for the integral representation to be valid. We can also assume that $|\arg(1-z)|<\pi$ and $|\arg(1-sz)|<\pi$. We study the hypergeometric integral in Eq.(40) using Laplace's method. For a similar approach to get uniform asymptotics of hypergeometric integrals see [15]. We fix $\delta>0$ small and set $u(s)=s^{-1/2+i\lambda+j}$, $v(s)=(1-s)^{-1/2-i\lambda}$, $P(s)=(1-zs)^{-1/2-i\lambda}$, $U(s)=s^{1/2+i\lambda+j}/(1/2+i\lambda+j)$. Then $U(0)=0$, $v'(s)=(1/2+i\lambda)(1-s)^{-3/2-i\lambda}$ and $P'(s)=z(1/2+i\lambda)(1-zs)^{-3/2-i\lambda}$. We have

$$\begin{aligned}\int_0^1 uvP &= \int_0^{1-\delta}uvP+\int_{1-\delta}^1 uvP\\ &= U(1-\delta)v(1-\delta)P(1-\delta)-\int_0^{1-\delta}U(v'P+vP')+\int_{1-\delta}^1 uvP.\end{aligned} \tag{41}$$

The first term in (41) is $O(1/j)$, since $|P(1-\delta)|$ is bounded, as $|z|<1$. Since for $|z|<1$ and $0\leq s\leq 1$, $|1-z|\leq 2|1-zs|$, we also have

$$\left|(1-z)^{3/2+i\lambda}P(s)\right|\leq|1-z|c_1$$

$$\left|(1-z)^{3/2+i\lambda}P'(s)\right|\leq c_2$$

and, therefore,

$$\left|(1-z)^{3/2+i\lambda}\int_0^{1-\delta}U(v'P+P'v)\right|\leq C\int_0^{1-\delta}|U||v'|+|U||v|=O(1/j). \tag{42}$$

We now look at the third term in (41).

$$\begin{aligned}\int_{1-\delta}^1 uvP &= \int_{1-\delta}^1 uvP(1)+\int_{1-\delta}^1 uv[P-P(1)]\\ &= P(1)\int_0^1 uv-P(1)\int_0^{1-\delta}uv+\int_{1-\delta}^1 uv[P-P(1)]\\ &= (1-z)^{-1/2-i\lambda}\frac{\Gamma(1/2+i\lambda+j)\Gamma(1/2-i\lambda)}{\Gamma(1+j)}-P(1)U(1-\delta)v(1-\delta)\\ &\quad +P(1)\int_0^{1-\delta}Uv'+\int_{1-\delta}^1 uv[P-P(1)],\end{aligned} \tag{43}$$

where we used the beta integral to evaluate $\int_0^1 uv$. The second and third terms in (43) multiplied by $(1-z)^{3/2+i\lambda}$ are clearly $O(1/j)$. Since

$$\lim_{s\to 1}(P(s)-P(1))/(s-1)=z(1/2+i\lambda)(1-z)^{-3/2-i\lambda},$$

we have

(44)

$$\int_{1-\delta}^{1} uv[P-P(1)] = -(Uv)(1-\delta)[P(1-\delta)-P(1)] - \int_{1-\delta}^{1} U\{v'[P-P(1)]+vP'\}$$

The first term in (44) is $O(1/j)$, when multiplied by $(1-z)^{3/2+i\lambda}$. Moreover,

$$\left|(1-z)^{3/2+i\lambda}\int_{1-\delta}^{1} UvP'\right| \le \int_{1-\delta}^{1} |U||v|, \tag{45}$$

which is $O(1/j)$, since v is integrable on $[1-\delta, 1]$, as $\Re(-1/2-i\lambda) > -1$. The last term to consider in (44) is

$$\int_{1-\delta}^{1} Uv'[P-P(1)] = -(1/2+i\lambda)\int_{1-\delta}^{1} Uv[P-P(1)]/(s-1) \tag{46}$$

and the function $(1-z)^{3/2+i\lambda}[P-P(1)]/(s-1)$ is bounded for s close to 1. This completes the study of the various terms. We now take into account the asymptotics of the Gamma function (14) to see that

$$\frac{\Gamma(1+j)}{\Gamma(1/2+i\lambda+j)} \sim j^{1/2-i\lambda}.$$

as $j \to \infty$. Since $\Re(1/2-i\lambda) \le 1/2$, all the terms in the expansion of the integral representation of the hypergeometric function tend to 0 as $j \to \infty$, when we multiply by $(1-z)^{3/2+i\lambda}$, except

$$(1-z)^{-1/2-i\lambda}\frac{\Gamma(1/2+i\lambda+j)\Gamma(1/2-i\lambda)}{\Gamma(1+j)}$$

which, when multiplied by

$$(1-z)^{3/2+i\lambda}\Gamma(1+j)\Gamma(1/2+i\lambda+j)^{-1}\Gamma(1/2-i\lambda)^{-1},$$

remains bounded. This proves the estimate in (16). The second estimate in lemma 3 is proved similarly. More precisely:

Using the fundamental integral representation for the hypergeometric function [2, 2.1.3(10), p. 59] we get

$$\begin{aligned} F(1/2,1/2,1-it,z) = \;& \Gamma(1-it)\Gamma(1/2)^{-1}\Gamma(1/2-it)^{-1}\times \\ & \times \textstyle\int_0^1 s^{-1/2}(1-s)^{-1/2-it}(1-sz)^{-1/2}ds \end{aligned} \tag{47}$$

We fix δ small, set $u(s) = s^{-1/2}$, $v(s) = (1-s)^{-1/2-it}$, $P(s) = (1-zs)^{-1/2}$, $V(s) = -\frac{(1-s)^{1/2-it}}{1/2-it}$. We have $P'(s) = \frac{z}{2}(1-zs)^{-3/2}$ and $V(1) = 0$. We have

$$\begin{aligned} \textstyle\int_0^1 uvP &= \textstyle\int_0^\delta uvP + \int_\delta^1 uvP \\ &= \textstyle\int_0^\delta uvP - u(\delta)V(\delta)P(\delta) - \int_\delta^1 V(u'P+uP'). \end{aligned} \tag{48}$$

The second term in (48) is $O(1/t)$ and, since $|1-z| \le 2|1-sz|$ for $|z|<1$,

$$(1-z)^{3/2}\int_\delta^1 Vu'P \ll |1-z|\int_\delta^1 |V||u'| \ll \frac{1}{t}\int_\delta^1 (1-s)^{1/2}s^{-3/2} \ll \frac{1}{t},$$

$$(1-z)^{3/2}\int_\delta^1 VuP' \ll \frac{1}{t}\int_\delta^1 (1-s)^{1/2}s^{-1/2} \ll \frac{1}{t}.$$

Consequently

$$(1-z)^{3/2}\int_0^1 uvP = (1-z)^{3/2}\int_0^\delta uvP + O\left(\frac{1}{t}\right).$$

We study the integral on the right-hand side of the previous equation:

$$\begin{aligned}\int_0^\delta uvP &= \int_0^1 uv - \int_\delta^1 uv + \int_0^\delta uv(P-1)\\ &= \frac{\Gamma(1/2)\Gamma(1/2-it)}{\Gamma(1-it)} - \int_\delta^1 uv + \int_0^\delta uv(P-1),\end{aligned} \tag{49}$$

where we used the beta integral to evaluate $\int_0^1 uv$. The second integral in (49) is

$$\int_\delta^1 uv = u(\delta)V(\delta) - \int_\delta^1 u'V = O\left(\frac{1}{t}\right).$$

For the third term in (49) we have

$$\int_0^\delta uv(P-1) = u(\delta)V(\delta)(P(\delta)-1) - \int_0^\delta V[u'(P-1)+uP'], \tag{50}$$

since $\lim_{s\to 0}\frac{P(s)-1}{s} = P'(0)$, which is bounded. The first term on the right of Eq.(50) is $O(1/t)$. Moreover, since the function $(P(s)-1)/s$ is bounded on $[0,\delta]$ and $s^{-1/2}$ is integrable on $[0,\delta]$,

$$\int_0^\delta Vu'(P-1) = -\frac{1}{2}\int_0^\delta Vu(P-1)/s \ll \frac{1}{t}.$$

Since $(1-z)^{3/2}P'$ is bounded on $[0,\delta]$,

$$(1-z)^{3/2}\int_0^\delta VuP' \ll \frac{1}{t}.$$

As a result, when multiplied by $(1-z)^{3/2}$, all terms are $O(1/t)$, apart from $\Gamma(1/2)\Gamma(1/2-it)\Gamma(1-it)^{-1}$. In the end we multiply the hypergeometric integral by $\frac{\Gamma(1-it)}{\Gamma(1/2)\Gamma(1/2-it)}$, which, due to (24), is asymptotic to $t^{1/2}$. The result now follows. □

Remark 3. The estimate (16) in lemma 3 is the best possible as far as the bahavior of the hypergeometric function as $j\to\infty$ is concerned. This can be seen by setting $z=0$. Since

$$F(a,b,c,z) = (1-z)^{-b}F(c-a,b,c,\frac{z}{z-1})$$

[2, 2.10.(6), pp. 109] and

$$F(a,b,c,z) = 1 + O(|c|^{-1})$$

as $|c|\to\infty$ ([2, 2.3.2 (10), pp. 76]) we see that

$$F(1/2+i\lambda+j,1/2+i\lambda,1+j,z) = (1-z)^{-1/2-i\lambda}[1+O(1/j)]$$

for $|z|<|z-1|$, so the estimate cannot be improved in general. We see also that the estimate (17) is best possible.

8. Appendix B

In this section we prove the claim made at the beginning of Sect. 6. By Eq.(25) there exist $\epsilon_0 > 0$ and $t_0 > 0$ such that $|h_t(t)| > 2\epsilon_0$ for all $t > t_0$. If we prove that

$$\left|\frac{dh_t(s)}{ds}\right| \leq K$$

for $|s-t| < \epsilon_1$, t sufficiently large and K independent of t, then the mean value theorem allows to deduce $|h_t(s)| > \epsilon_0$ for $|s-t| < \epsilon = \min(\epsilon_1, \epsilon_0/K)$, t sufficiently large. Using (9) we get

$$\frac{dh_t(s)}{ds} = 2\pi \int_0^\infty \frac{d}{ds} P_{-1/2+is}(\cosh r) \frac{t P_{-1/2+it}(\cosh r)\sinh^3 r}{\cosh^8 r} dr.$$

We will prove that, for $|s-t| < \epsilon_1$, the integrand is bounded by a function of r which is integrable on $[0,\infty)$ (independent of s and t). We review some facts about the spherical function on the symmetric space $\mathbb{H}$ (see [4, p. 144, 150–152]). The spherical function $P_{-1/2+i\lambda}(\cosh r)$ can be split as

$$P_{-1/2+i\lambda}(\cosh r) = \varphi_\lambda(r) = c(\lambda)\Phi_\lambda(r) + c(-\lambda)\Phi_{-\lambda}(r), \tag{51}$$

where $c(\lambda)$ is the Harish-Chandra c function and $\Phi_\lambda(r)$ is the unique solution of the equation

$$\frac{\partial^2\varphi}{\partial r^2} + \coth r \frac{\partial \varphi}{\partial r} + (\varrho^2 + \lambda^2)\varphi = 0$$

satisfying $\Phi_\lambda(r) = e^{(i\lambda-\varrho)r}(1+o(1))$ as $r \to \infty$. Here ϱ is the half sum of the roots, in our case $\varrho = 1/2$. We have

$$c(\lambda) = \frac{\Gamma(i\lambda)}{\Gamma(1/2+i\lambda)\sqrt{\pi}}, \tag{52}$$

whose absolute value is asymptotic to $\pi^{-1/2}\lambda^{-1/2}$ as $\lambda \to \infty$, by (24). Moreover,

$$\Phi_\lambda(r) = e^{(i\lambda-\varrho)r}\sum_{m=0}^{\infty} \Gamma_m(\lambda) e^{-mr}, \tag{53}$$

where the $\Gamma_m(\lambda)$ satisfy the following recursion formula

$$4n(n-i\lambda)\Gamma_{2n} = \sum_{k=0}^{n-1}(2k - i\lambda + \varrho)2\Gamma_{2k} \tag{54}$$

with $\Gamma_0 = 1$ and $\Gamma_{2n-1} = 0$. The convergence of (53) is uniform on $[c,\infty)$ for any $c > 0$ by the estimate $|\Gamma_m(\lambda)| \leq K(1+m)^d$, for some K, $d > 0$. This is explained in [4, Lemma 7] or [5, p. 57]. We need more precise information. We have the following two lemmas:

Lemma 4. *There is a constant $K > 0$ such that $|\Gamma_m(\lambda)| \leq K$ for all $\lambda > 0$ and $m \in \mathbb{N}$.*

Proof. The proof is essentially in [4]. We set $a_n(\lambda) = \Gamma_{2n}(\lambda)$ and assume that $|a_k(\lambda)| \leq K$ for $k < n$. Then, since $|4k - 2i\lambda + 1|^2 \leq 16|n - i\lambda|^2$, Eq.(54) gives

$$|a_n(\lambda)| \leq \sum_{k=0}^{n-1} \frac{|4k - 2i\lambda + 1|}{4n|n-i\lambda|} K \leq K.$$

□

Lemma 5. *For all $d > 0$, there exists a $K_1 > 0$ such that for all $m \in \mathbb{N}$, $\lambda > 0$*

$$\left|\frac{d}{d\lambda}\Gamma_m(\lambda)\right| \le K_1 m^d.$$

Proof. We differentiate (54) to get

$$-4ina_n(\lambda) + 4n(n - i\lambda)\frac{da_n(\lambda)}{d\lambda} = \sum_{k=0}^{n-1}\left(-2ia_k(\lambda) + (4k - 2i\lambda + 1)\frac{da_k(\lambda)}{d\lambda}\right)$$

and $a_0'(\lambda) = 0$. Therefore

(55)

$$\left|\frac{da_n(\lambda)}{d\lambda}\right| \le \sum_{k=0}^{n-1}\frac{|a_k(\lambda)|}{2n|n-i\lambda|} + \sum_{k=0}^{n-1}\frac{|4k - 2i\lambda + 1|}{4n|n - i\lambda|}\left|\frac{da_k(\lambda)}{d\lambda}\right| + \frac{|a_n(\lambda)|}{|n - i\lambda|}.$$

If we assume that

$$\left|\frac{da_k(\lambda)}{d\lambda}\right| \le K_1 k^d$$

for $k < n$, we get using the previous lemma and (55)

$$\left|\frac{da_n(\lambda)}{d\lambda}\right| \le \frac{3K}{2n} + K_1\sum_{k=0}^{n-1}\frac{k^d}{n}.$$

For n sufficiently large, say $n > N_0$, $3K/(2n) + K_1\sum_{k=0}^{n-1}k^d/n \le K_1 n^d$, since $\sum_{k=0}^{n-1}(k/n)^d/n \to \int_0^1 x^d dx = 1/(d+1) < 1$. Eq.(55) shows that we can bound $\frac{da_n(\lambda)}{d\lambda}$ for all $n \le N_0$ independently from λ, so we can start the induction and the inductive step is complete. □

We are interested in the product

$$\frac{d\varphi_s(r)}{ds}\varphi_t(r) = \frac{d}{ds}\left\{c(s)\Phi_s(r) + c(-s)\Phi_{-s}(r)\right\}\cdot\left[c(t)\Phi_t(r) + c(-t)\Phi_{-t}(r)\right].$$

The products $|c(s)c(t)|$, $|c(-s)c(t)|$, $|c(s)c(-t)|$ and $|c(-s)c(-y)|$ are asymptotic to $\pi^{-1}s^{-1/2}t^{-1/2}$ as $t \to \infty$, $|s - t| < \epsilon_1$ $(s \to \infty)$. We study now

$$\begin{aligned} c'(s) &= i\frac{\Gamma'(is)\Gamma(1/2 + is) - \Gamma'(1/2 + is)\Gamma(is)}{\sqrt{\pi}\Gamma^2(1/2 + is)} \\ &= \frac{i\Gamma(is)}{\sqrt{\pi}\Gamma(1/2 + is)}\left[\psi(is) - \psi(1/2 + is)\right], \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the gamma function. We have the following asymptotics for $\psi(z)$

$$\psi(z) = \log z - \frac{1}{2z} + O(z^{-2})$$

as $|z| \to \infty$, see [2, 1.18 (7), p. 47]. We have

$$\log(1/2 + is) - \log(is) = \frac{1}{2}\log(1 + 1/(4s^2)) + i\left[\arg(1/2 + is) - \pi/2\right]$$

and

$$0 \le \log(1 + 1/(4s^2)) \le \frac{1}{4s^2}.$$

Moreover,

$$\arg(1/2+is)-\pi/2=O(1/s),$$

which follows by substituting $x=\arg(1/2+is)-\pi/2$, i.e. $2s=\tan(x+\pi/2)$, $x\to 0^-$, and noticing that $x=O(\cot(x+\pi/2))$. Also $-1/(2is)+1/(1+2is)=O(1/s^2)$. Finally $|c'(s)|=O(s^{-3/2})$ and the products $|c'(s)c(t)|$, $|c'(-s)c(t)|$, $|c'(s)c(-t)|$ and $|c'(-s)c(-t)|$ are $O(s^{-3/2}t^{-1/2})$. Equation (53) gives together with Lemma 4

$$|\Phi_s(r)|\le e^{-r/2}\sum_{m=0}^{\infty}|\Gamma_m(s)|e^{-mr}\le Ke^{r/2}/(e^r-1)$$

which blows like K/r as $r\to 0$. We take $d=1$ in Lemma 5. Then

$$\frac{d\Phi_s(r)}{ds}=e^{(is-\varrho)r}\left(\sum_{m=0}^{\infty}\left[ir\Gamma_m(s)+\frac{d\Gamma_m(s)}{ds}\right]e^{-mr}\right)$$

and

$$\left|\frac{d\Phi_s(r)}{ds}\right|\le Kre^{-r/2}/(1-e^{-r})+K_1e^{-r/2}\sum_{m=0}^{\infty}me^{-mr}$$

which behaves like K_1/r^2 as $r\to 0$. For $r\ge c>0$ we get for $\Phi_s(r)$ and $d\Phi_s(r)/ds$ bounds by exponentially decreasing functions of r with no dependence on s. The products $\Phi_s(r)\Phi_t(r)$, $\Phi_{-s}(r)\Phi_t(r)$, $\Phi_s(r)\Phi_{-t}(r)$ and $\Phi_{-s}(r)\Phi_{-t}(r)$ blow at most like c_1/r^2 as $r\to 0$ and the products $d\Phi_s/ds\cdot\Phi_t$, $d\Phi_{-s}/ds\cdot\Phi_t$, $d\Phi_s/ds\cdot\Phi_{-t}$ and $d\Phi_{-s}/ds\cdot\Phi_{-t}$ blow at most like c_2/r^3 ar $r\to 0$. Away from zero all these products can be bounded by a function of r that decreases exponentially as $r\to\infty$. Since $tO(s^{-1/2}t^{-1/2})=O(1)$ and $tO(s^{-3/2}t^{-1/2})=O(1/t)$ as $t\to\infty$, $|s-t|<\epsilon_1$, the function $d\varphi_s(r)/ds\cdot\varphi_t(r)t\sinh^3 r/\cosh^8 r$ can be bounded by an integrable function of r independently from s and t. This concludes the claim in Sect. 6.

Remark 4. One can actually use (53) and Lemma 4 to provide an alternate proof that $\liminf h_t(t)>0$. Eq.(20) corresponds to (51).

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