On primitive elements in the bialgebra of chord diagrams

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The bialgebra of chord diagrams arises naturally in the study of Vassiliev knot invariants (see [1, 2]). The bialgebra of chord diagrams is both commutative and cocommutative. It follows then from Milnor-Moore structure theorem [4] that considered over a field of characteristic zero it is a symmetric algebra in its primitive elements.

Over a field of characteristic zero investigating bialgebra of Vassiliev knot invariants is essentially equivalent to investigation of the bialgebra of chord diagrams. However, over an arbitrary ring both the structure of the bialgebra and the question, whether it determines Vassiliev invariants remain open. For example, it is unknown whether the bialgebra has torsion. In the present text we prove that the bialgebra of chord diagrams is generated by its primitive elements over **Z**.

The proof is obtained by presenting an explicit formula for a projection onto the submodule of primitive elements. In fact, the proof presented is valid in a more general situation, namely for a class of bialgebras where a multiplication and comultiplication may be presented as "disjoint union" and "partition" correspondently.

In the first section we give all the definitions concerning the algebra of chord diagrams and present the formula for a projection on the submodule of primitive elements. In the second section we introduce the bialgebra of partitions and prove the formula for the projector in this bialgebra. In the third section we apply this formula for bialgerba of chord diagrams as well as for bialgebra of weighted graphs.

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1 Bialgebra of chord diagrams

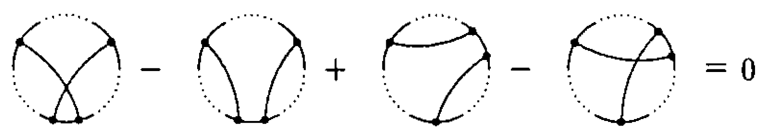
1.1 Notions

Definition 1.1 *A chord diagram of order n* is a circle with a distinguished set of n unordered pairs of points, regarded up to an orientation preserving diffeomorphism of the circle.

A chord diagram may be depicted as a circle with a set of n chords all of whose endpoints are distinct. Of course, these chords can be drawn as lines or curves whose actual shape is irrelevant; what matters is the way they bind their endpoints into pairs.

Consider a free $\mathbf{Z}$-module freely generated by all chord diagrams of order n and introduce a set of linear relations in this module:

(four-term relation)



for an arbitrary fixed position of $(n-2)$ chords (which are not drawn here) and the two additional chords positioned as shown in the picture. Here dotted arcs suggest that there might be further chords attached to their points, while on the solid portions of the circle all the endpoints are explicitely shown.

Let $\mathcal{M}_n$ be the $\mathbf{Z}$-module of chord diagrams of order n modulo all four-term relations. Consider the graded module

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots$$

Module $\mathcal{M}$ carries the structure of a bialgebra. [1] Explicit description of this structure is given in the next subsection.

1.2 Bialgebra structure

Definition 1.2 Let D_1, D_2 be two chord diagrams of order n_1, n_2 respectively. The *product* $[D_1] \cdot [D_2]$ of classes $[D_1] \in M_{n_1}, [D_2] \in M_{n_2}$ is a class $[D] \in M_{n_1+n_2}$ defined as follows.

Break the circle of the diagram D_1 in a point x_1 different from all the ends of the chords of D_1, and break the circle of the diagram D_2 in a point x_2 different from all the ends of the chords of D_2. Gluing two broken circles in a new circle taking into account their orientations one obtains a new chord diagram D of order $n_1 + n_2$. We set $[D_1] \cdot [D_2] = [D]$.

Definition 1.3 Let D be a chord diagram of order n. Denote by $V = V(D)$ the set of its chords. Any subset $V_1 \subset V$ of chords of D determines a diagram (V_1) consisting only of the chords belonging to the set V_1, so, for example, $(V) = D$. We set *coproduct* $\mu((V)) = \sum_{V=V_1 \sqcup V_2} (V_1) \otimes (V_2)$, where the sum is taken over all disjoint partitions $V = V_1 \sqcup V_2$.

Theorem 1.4 (see [1]) *The operations of multiplication and comultiplication defined above are consistent operations on the module $\mathcal{M}$ of chord diagrams. The graded module $\mathcal{M}$ supplied with these two operations becomes a (graded, even commutative and cocommutative) bialgebra over* $\mathbf{Z}$.

(The definition of a bialgebra see in [4], for example.)

The unit $e \in \mathcal{M}_0$ is represented by the chord diagram with an empty set of chords. The counit is a mapping $\epsilon : \mathcal{M} \to \mathbf{Z}$, such that $\epsilon(e) = 1$, and the restriction of ϵ on $\mathcal{M}_n$ for $n > 0$ vanishes.

[1] I prefer to use the term *bialgebra* instead of *Hopf algebra* because a natural antipode in this bialgebra appears only after the structure theorem is applied.

1.3 Primitive elements of $\mathcal{M}$

We will make no difference in notation between a chord diagram and an element of $\mathcal{M}$ represented by this diagram.

Definition 1.5 An element $p \in \mathcal{M}$ is called *primitive*, if $\mu(p) = e \otimes p + p \otimes e$.

Theorem 1.6 *Let D be a chord diagram of order n, $V = V(D)$ its set of chords. Then the element*

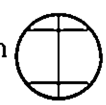
$$p(V) = (V) - 1! \sum_{V = V_1 \sqcup V_2} (V_1) \cdot (V_2) + 2! \sum_{V = V_1 \sqcup V_2 \sqcup V_3} (V_1) \cdot (V_2) \cdot (V_3) - \ldots, \quad (1)$$

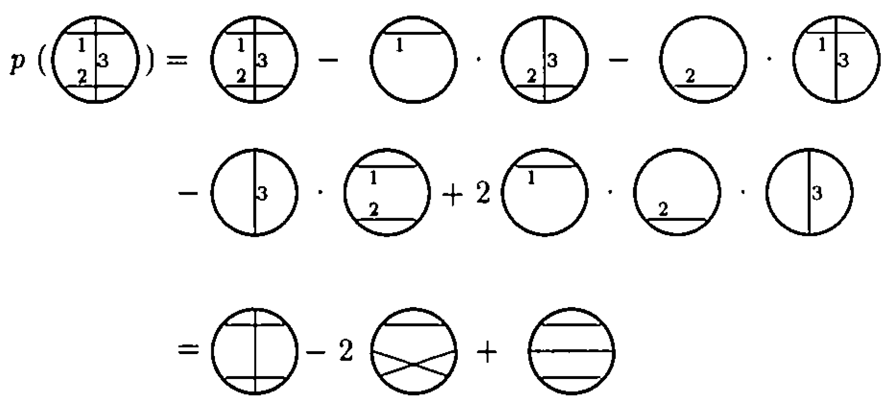
where sums are taken over all unordered disjoint partitions of V into non-empty *subsets, is a primitive element in $\mathcal{M}_n$.*

The theorem will be proved in section 3.

Remark 1.7 Changing unordered partitions for ordered ones one would rewrite the formula (1) as

$$p(V) = (V) - \frac{1}{2} \sum_{V = V_1 \sqcup V_2} (V_1) \cdot (V_2) + \frac{1}{3} \sum_{V = V_1 \sqcup V_2 \sqcup V_3} (V_1) \cdot (V_2) \cdot (V_3) - \ldots, \quad (2)$$

Example 1.8 For the chord diagram  we have



As an immediate corollary of the theorem 1.6 one obtains

Theorem 1.9 *Bialgebra of chord diagrams is generated by its primitive elements over* $\mathbf{Z}$.

Indeed, each chord diagram (V) may be represented as a sum of $p((V))$ and a polynomial in the diagrams of smaller order with integer coefficients.

2 Bialgebra of partitions

We associate with a positive integer n a bialgebra $\mathcal{B}(n)$, which we call *bialgebra of partitions.*

2.1 Algebra structure

The definition proceeds as follows. Let $N_n = \{1, 2, \ldots, n\}$. Consider all non-empty subsets $V \subset N_n$ as commuting variables. We prefer to denote such a subset as a sequence of elements in brackets: $(1), (12)$, and so on. As an algebra, $\mathcal{B}(n)$ is a polynomial algebra in these $2^n - 1$ variables.

The *order* of a variable $V \subset N_n$ is just the number of elements in V. This order makes the algebra $\mathcal{B}(n)$ a graded algebra $\mathcal{B}(n) = \mathcal{B}_0(n) \oplus \mathcal{B}_1(n) \oplus \ldots$, where $\mathcal{B}_0(n) \cong \mathbf{Z}$ (the generator of $\mathcal{B}_0(n)$ is the empty subset $()$), $\mathcal{B}_k(n)$ is a free $\mathbf{Z}$-module, freely generated over $\mathbf{Z}$ by all monomials of order k.

For example, for $n = 3$

$$\begin{aligned}
\mathcal{B}_0(3) &= <()> \\
\mathcal{B}_1(3) &= <(1),(2),(3)> \\
\mathcal{B}_2(3) &= <(12),(13),(23),(1)(2),(1)(3),(2)(3),(1)(1),(2)(2),(3)(3)> \\
\mathcal{B}_3(3) &= <(123),(1)(12),(1)(13),(1)(23),\ldots,(1)(1)(1),(1)(1)(2),\ldots>
\end{aligned}$$

A monomial in $\mathcal{B}(n)$ will be called *a partition.*

The empty partition $e = ()$ forms the unit with respect to this multiplication.

2.2 Coalgebra structure

Let $V \subset N_n$, and (V) denote a partition of V, consisting of one set, namely V itself. We set

$$\mu((V)) = \sum_{V=V_1 \sqcup V_2} (V_1) \otimes (V_2),$$

where $V = V_1 \sqcup V_2$ is an arbitrary partition of V into a disjoint union of two subsets, and the sum is taken over all such partitions.

Example 2.1

$$\mu((13)) = () \otimes (13) + (1) \otimes (3) + (3) \otimes (1) + (13) \otimes ().$$

For other types of partitions the comultiplication $\mu: \mathcal{B}(n) \to \mathcal{B}(n) \otimes \mathcal{B}(n)$ is extended as an algebra homomorphism.

Statement 2.2 *Module $\mathcal{B}(n)$ with the multiplication and comultiplication thus introduced is a graded (even) commutative and cocommutative bialgebra.*

2.3 Primitive elements

Theorem 2.3 *For arbitrary subset $V \subset N_n$ the element*

$$p((V)) = (V) - 1! \sum (V_1)(V_2) + 2! \sum (V_1)(V_2)(V_3) - 3! \sum (V_1)(V_2)(V_3)(V_4) + \dots \quad (3)$$

where the sums are taken over all disjoint unordered partitions of V into nonempty subsets, is a primitive element in $\mathcal{B}(n)$.

Example 2.4 The element

$$p((123)) = (123) - (1)(23) - (2)(13) - (3)(12) + 2 \cdot (1)(2)(3) \in \mathcal{B}(n), n \geq 3$$

is primitive.

Corollary 2.5 *The submodule $\mathcal{P}(n) \subset \mathcal{B}(n)$ of primitive elements is generated by elements of type $p((V))$. The bialgebra $\mathcal{B}(n)$ is isomorphic to symmetric bialgebra of $\mathcal{P}(n)$.*

Remark 2.6 All primitive elements of $\mathcal{B}(n)$ are contained in the first n orders of grading.

Proof of theorem 2.3. It is apparently sufficient to prove the theorem for the case $V = N_n$.

The coefficient of the term

$$(V_1)\dots(V_k)\otimes(V_{k+1})\dots(V_{k+m})$$

in $\mu(p((N_n)))$ is uniquely determined by two numbers k and m. Denote this coefficient by $(-1)^{k+m-1}a_{km}$. We set $a_{00}=0$. One has

$$a_{km} = (k+m-1)! - \frac{(k+m-2)!km}{1!} + \frac{(k+m-3)!k(k-1)m(m-1)}{2!} - \dots$$

Indeed, the term we are considering may be obtained by comultiplying a term with $(k+m-j)$ subsets in partition for $j = 0, 1, \dots, \min(k, m)$. Such a term may be chosen by picking j subsets from $(V_1), \dots, (V_k)$ and j subsets from $(V_{k+1}), \dots, (V_{k+m})$ and taking their pairwise union. We divide by $j!$ as the partitions are unordered.

Multiplying and dividing by $k!m!$ we obtain the following formula for a_{km}:

$$\begin{aligned} a_{km} &= k!m!\left(\frac{(k+m-1)!}{0!k!m!} - \frac{(k+m-2)!}{1!(k-1)!(m-1)!} + \dots\right) \\ &= k!m!\sum_{j=0}^{\min(k,m)} \frac{(k+m-1-j)!}{j!(k-j)!(m-j)!} \end{aligned}$$

Consider the exponential generating function for the sequence of coefficients a_{km} and make the following transformations [2]

$$\begin{aligned} \sum_{k,m=0}^{\infty} \frac{a_{km}}{k!m!}x^k y^m &= \sum_{k,m=0}^{\infty} x^k y^m \sum_{j=0}^{\min(k,m)} (-1)^j \frac{(k+m-1-j)!}{j!(k-j)!(m-j)!} \\ &= \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!} \sum_{k,m=j}^{\infty} \frac{(k+m-1-j)!}{(k-j)!(m-j)!} x^k y^m \\ &= \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!} y^j \sum_{k=j}^{\infty} \frac{x^k}{(k-j)!} \sum_{m'=0}^{\infty} \frac{(k-1+m')!}{m'!} y^{m'} \end{aligned}$$

[2] This calculation is due to A. Levin.

$$
\begin{aligned}
&= \sum_{j=0}^{\infty}(-1)^j\frac{1}{j!}y^j\sum_{k=j}^{\infty}\frac{x^k}{(k-j)!}\left(\frac{1}{1-y}\right)^{(k-1)} \\
&= \sum_{j=0}^{\infty}(-1)^j\frac{1}{j!}y^jx^j\left(\frac{\partial}{\partial y}\right)^{j-1}\sum_{k'=0}^{\infty}\frac{\left(x\frac{\partial}{\partial y}\right)^{k'}}{k'!}\frac{1}{1-y} \\
&= \sum_{j=0}^{\infty}(-1)^j\frac{1}{j!}y^jx^j\left(\frac{\partial}{\partial y}\right)^{j-1}\frac{1}{1-y-x} \\
&= -\sum_{j=0}^{\infty}\frac{\left(-xy\frac{\partial}{\partial y}\right)^j}{j!}\log(1-x-y) \\
&= -\log(1-x-y+xy) \\
&= -\log(1-x)-\log(1-y)
\end{aligned}
$$

The upper index in parenthesis denotes as usual the corresponding derivative. Remark, that

$$\left(\frac{1}{1-y}\right)^{(-1)} = -\log(1-y).$$

In this calculation we used twice the fact, that the exponent of the derivative is a shift.

Thus, the coefficient a_{km} equals zero if both $k \neq 0, m \neq 0$, and $a_{0m} = a_{m0} = (m-1)!$. This means precisely, that $p((N_n))$ is primitive, and the theorem is thus proved.

Remark 2.7 Knowing primitive elements of the bialgebra $\mathcal{B}(n)$ we are able to make it Hopf algebra. The antipode just acts as reflection $p \mapsto -p$ on the submodule $\mathcal{P}(n)$ of primitive elements.

3 Proof of the main theorem

3.1 Proof of theorem 1.6

Let D be a chord diagram with n chords. Associate with D a homomorphism $m_D: \mathcal{B}(n) \to \mathcal{M}$ in the following way. Number the chords of D with elements of $\{1, \ldots, n\}$ arbitrarily, so that each chord has a unique number.

The partition (V) is taken to the chord diagram $m_D((V))$, consisting of the chords, numbered with elements of V. The partition $(V_1)\dots(V_k)$ is taken to the chord diagram $m_D((V_1))\dots m_D((V_k))$.

The mapping m_D obviously respects both multiplication and comultiplication and extends thus to a homomorphism of bialgebras. Primitive elements are taken to primitive elements, that proves the theorem.

Remark 3.1 It follows directly from [3] that image of the primitive element (3) under the mapping m_D is a non-zero primitive element, if the diagram D is not a product of two diagrams of smaller order.

3.2 Bialgebra of weighted graphs

The bialgebra $\mathcal{W}$ of weighted graphs has been introduced in [3]. It has been proven there that $\mathcal{W}$ is generated by its primitive elements over $\mathbf{Z}$. However, we may make use of the theorem 2.3 constructing a projection of the algebra onto the space of its primitive elements.

Theorem 3.2 *For arbitrary weighted graph G with weights of all vertices equal to* 1 *the element*

$$p(G) = G - 1!\sum G_1G_2 + 2!\sum G_1G_2G_3 - 3!\sum G_1G_2G_3G_4 + \dots$$

is a primitive element in $\mathcal{W}_n$.

Here G_k is an arbitrary complete subgraph of the graph G, and sums are taken over all possible disjoint unordered partitions of the set of vertices of G into non-empty subsets.

The proof is achieved by associating with the graph G a mapping w_G from a bialgebra $\mathcal{B}(n)$ into the bialgebra $\mathcal{W}$, similar to the mapping m_D from the previous subsection.

References

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