Comparison of Auslander-Reiten theory and Gabriel-Roiter measure approach to the module categories of tame hereditary algebras

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Abstract. Let Λ be a tame hereditary algebra over an algebraically closed field, i.e. $\Lambda = kQ$ with Q a quiver of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, or $\widetilde{\mathbb{E}}_8$. Two different kinds of partitions of the module category can be obtained by using Auslander-Reiten theory, and on the other hand, Gabriel-Roiter measure approach. We compare these two kinds of partitions and see how the modules are rearranged according to Gabriel-Roiter measure. We also show that the Gabriel-Roiter submodules can be used to build orthogonal exceptional pairs for indecomposable preprojective Λ-modules when Λ is of type $\widetilde{\mathbb{A}}_{n,n\geq 2}$ and $\widetilde{\mathbb{D}}_n$.

Keywords. Tame hereditary algebras, Gabriel-Roiter measure, irreducible maps, defect function.

Mathematics Subject Classification (2000). 16G70

1 Introduction

The Gabriel-Roiter measure has been introduced by Gabriel (under the name 'Roiter measure', [7]) in 1973, in order to clarify the induction scheme used by Roiter in his proof of the first Brauer-Thrall conjecture. Ringel used the Gabriel-Roiter measure as a foundation tool for representation theory of artin algebras ([12],[13]). So-called Gabriel-Roiter submodules of an indecomposable module are indecomposable submodules with a certain maximality condition. Gabriel-Roiter submodules of an indecomposable module Y always exist in case Y is not simple. One of the most interesting properties is that if Y is an indecomposable non-simple module and X is a Gabriel-Roiter submodule of Y, then Y/X, the Gabriel-Roiter factor module, is indecomposable ([12],[13]). Therefore, any indecomposable non-simple module Y is an extension of indecomposable modules.

By using Gabriel-Roiter measure, Ringel obtained a partition of the module category of a representation-infinite algebra ([12]): The module category consists of take-off part, central part and landing part. Moreover, he showed that all modules lying in the landing part are preinjective modules in the sense of Auslander and Smalø ([2]). This naturally leads us to compare the different kinds of partitions of the module category.

Throughout the paper, we assume that k is an algebraically closed field and Λ is a finite dimensional basic connected k-algebra. We denote by $\operatorname{mod}\Lambda$ the category of finite dimensional left Λ-modules and by $\operatorname{ind}\Lambda$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of indecomposable Λ-modules. Let

$\operatorname{ind}\mathcal{X} = \operatorname{ind}\Lambda \cap \mathcal{X}$ for a full subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$. We denote by $|M|$ the length of a Λ-module M. We use the symbol $\subset$ to denote proper inclusion. For any two Λ-modules X and Y, $\operatorname{Ext}^i(X,Y)$ always stands for $\operatorname{Ext}^i_\Lambda(X,Y)$ for any $i \geq 0$.

If Λ is a representation-infinite hereditary k-algebra, the module category contains preprojective modules, regular modules and preinjective modules. Ringel's result implies all landing modules are preinjective modules. In this note, we shall compare these two kinds of partitions of the module category of a tame hereditary algebra obtained by using Auslander-Reiten theory and Gabriel-Roiter measure approach, respectively. We show that all preprojective modules lies in the take-off part(Theorem 4.4). A direct consequence of this theorem is that a Gabriel-Roiter submodule of a homogeneous regular module, which is not regular simple, is always given by an irreducible monomorphism (Corollary 4.5). However, we will see a stronger result which says that for a Gabriel-Roiter inclusion of homogeneous regular modules $H \subset H'$, the measure for H' is a direct successor of the measure for H, i.e. there does not exist indecomposable module with Gabriel-Roiter measure lying in between (Theorem 4.6).

A known interesting application of Gabriel-Roiter submodules is that they can be used to construct orthogonal exceptional pairs for indecomposable modules over representation-directed algebras ([4],[5],[14]). We will extend this result to indecomposable preprojective modules over tame hereditary algebras of type $\widetilde{\mathbb{A}}_{n,n\geq 2}$ and $\widetilde{\mathbb{D}}_n$ (Theorem 5.1).

We recall some preliminaries of tame hereditary algebras in section 2. Some definitions and properties of Gabriel-Roiter measure are presented in section 3. Section 4 is devoted to a discussion of these two kinds of partitions. In section 5, we will discuss the orthogonal property of a Gabriel-Roiter inclusion of preprojective modules.

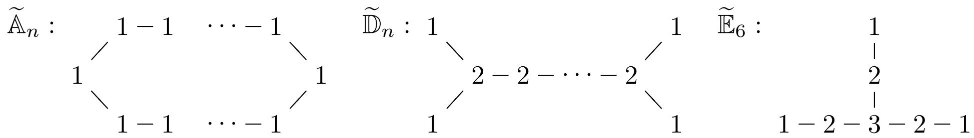
2 Preliminaries

In this section, we present some preliminaries which will be used later on. For details, we refer to [1], [6], [10]. Let $\Lambda = kQ$ be a path algebra with the underlying graph of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, or $\widetilde{\mathbb{E}}_{6,7,8}$. The dimension vector for a Λ-module M is denoted by $\underline{\dim} M$. We call a module M **sincere** if $(\underline{\dim} M)_i \geq 1$ for each i, and **thin** if $(\underline{\dim} M)_i \leq 1$ for each i.

We have a bilinear form $\langle a, b\rangle = aC^{-t}b^t$ for all $a, b \in \mathbb{Z}^n$ where C is the Cartan matrix and t denotes the transpose of a matrix. Then given two modules $X, Y \in \operatorname{mod}\Lambda$, we have

$$\langle \underline{\dim} X, \underline{\dim} Y\rangle = \dim \operatorname{Hom}(X,Y) - \dim \operatorname{Ext}^1(X,Y)$$

We denote by q the quadratic form on $\mathbb{Z}$ defined by $q(a) = \langle a, a\rangle$. Then q is positive semi-definite with **radical** $\mathbb{Z}\delta$, that is, $q(\delta) = 0$ and $h = r\delta$ for some $r \in \mathbb{Z}$ whenever $q(h) = 0$. We list the underlying graphs of the quivers of tame hereditary algebras and indicate δ for each case.



$\widetilde{\mathbb{E}}_7$:

$$\begin{array}{ccccccccccccc} & & & & & & 2 & & & & & & \\ & & & & & & | & & & & & & \\ 1 & - & 2 & - & 3 & - & 4 & - & 3 & - & 2 & - & 1 \end{array}$$

$\widetilde{\mathbb{E}}_8$:

$$\begin{array}{ccccccccccccccc} & & & & 3 & & & & & & & & & & \\ & & & & | & & & & & & & & & & \\ 2 & - & 4 & - & 6 & - & 5 & - & 4 & - & 3 & - & 2 & - & 1 \end{array}$$

We have a decomposition of the Auslander-Reiten quiver Γ_Λ, into the preprojective part $\mathcal{P}$, the preinjective part $\mathcal{I}$ and the regular one $\mathcal{R}$, where $\mathcal{R}$ is a sum of stable tubes $\mathcal{T}_\lambda$ of ranks $r_\lambda \geq 1$, for $\lambda \in \mathbb{P}^1(k) = k \cup \{\infty\}$. A tube of rank 1 is called **homogeneous** and the ones of rank greater than 1 are called **exceptional**. Note that $\mathcal{T}_\lambda$ is exceptional for at most three $\lambda \in \mathbb{P}^1(k)$. For indecomposable Λ-modules X, Y, if $\mathrm{Hom}(X, Y) \neq 0$ and X and Y do not belong to the same connected component of Γ_Λ, then X is preprojective or Y is preinjective.

The following proposition is very useful.

Proposition 2.1 ([8]). *Assume that Λ is a hereditary algebra and X, Y are indecomposable Λ-modules with $\mathrm{Ext}^1(Y, X) = 0$. Then any non-zero map from X to Y is either injective or surjective.*

For each regular component $\mathcal{T}_\lambda$ with rank r_λ, let $E_1, \cdots, E_{r_\lambda}$ be the indecomposable modules on the mouth. We call them **regular simple** modules. Note that $\delta = \sum_{i=1}^{r_\lambda} \underline{\dim}\, E_i$. For each indecomposable regular module $X \in \mathcal{T}_\lambda$ with $X \not\cong E_i$, the middle term of the Auslander-Reiten sequence $0 \to \tau X \to M \to X \to 0$ has exactly two indecomposable summands such that one irreducible map to X is surjective and the other one is injective. Recall that a **sectional path** in the Γ_Λ is a sequence of irreducible maps $X_1 \to X_2 \to \cdots \to X_m \to \cdots$ with $X_i \not\cong \tau(X_{i+2})$ for each i. For any indecomposable regular module X, we may write $X = E_i[r]$ since there is a unique sectional path of irreducible monomorphisms $E_i = E_i[1] \to E_i[2] \to \cdots \to E_i[r] = X \to \cdots$ starting with a regular simple module E_i for some $1 \leq i \leq s$. Recall that $\mathrm{add}\, \mathcal{T}_\lambda$ is a serial abelian category and closed under extension.

The **defect** of a Λ-module X is defined to be $\langle \delta, \underline{\dim}\, X \rangle = -\langle \underline{\dim}\, X, \delta \rangle$. We thus get a **defect function** which is also denoted by $\delta : \delta(X) = \langle \delta, \underline{\dim}\, X \rangle$. It is well-known that an indecomposable Λ-module X is preprojective, (resp. regular, preinjective) if and only if $\delta(X)$ is negative (resp. zero, positive).

Lemma 2.2 ([3]). *Assume that X and Y are indecomposable preprojective modules such that the defect, $\delta(X)$, of X is -1. If $0 \neq f \in \mathrm{Hom}(X, Y)$, then f is injective.*

Proof. Since $\mathrm{Im}\, f$ is a submodule of Y, it is a preprojective module. We thus have $-1 = \delta(X) = \delta(\mathrm{Im}\, f) + \delta(\mathrm{Ker}\, f)$. It follows that either $\delta(\mathrm{Im}\, f) = 0$ or $\delta(\mathrm{Ker}\, f) = 0$. But $f \neq 0$ implies $\delta(\mathrm{Ker}\, f) = 0$. Therefore, $\mathrm{Ker}\, f = 0$ and f is injective. □

Corollary 2.3. *Assume that Λ is of type $\widetilde{\mathbb{A}}_n$.*
(1) *All non-zero maps between indecomposable preprojective modules are injective and the corresponding factors are regular modules. In particular, all irreducible maps between indecomposable preprojective modules are monomorphisms.*
(2) *All non-zero maps between indecomposable preinjective modules are surjective and the corresponding kernels are regular modules. In particular, all irreducible maps between indecomposable preinjective modules are epimorphisms.*

Proof. Note that in this case, all indecomposable preprojective modules are of defect -1. (2) is dual to (1). □

We now introduce some notations. Assume that Λ is of type $\widetilde{\mathbb{D}}_n$ or $\widetilde{\mathbb{E}}_{6,7,8}$. Let $Q = (Q_0, Q_1)$ be the ordinary quiver of Λ with Q_1 the set of arrows and Q_0 the one of vertices, and Γ_Λ be the Auslander-Reiten quiver. Fix a vertex $j \in Q_0$, a **sectional sequence** to j is a sequence of vertices $i_1, i_2, \cdots, i_s = j$ such that i_k and i_{k+1} are neighbors each other, i.e. they are connected by an edge. A sectional sequence to j is **complete** if i_1 is an ending vertex of Q, i.e. a vertex with only one neighbor in Q. Let P_i be the indecomposable projective module corresponding to i for each $i \in Q_0$. If $M = \tau^{-r}P_j$ is an indecomposable preprojective module, then we say a sectional path $X_{i_1} \to X_{i_2} \to \cdots \to X_{i_s} = M$ in Γ_Λ is complete if $\tau^{r_k}X_{i_k} = P_{i_k}$ and $i_i, i_2, \cdots, i_s = j$ is a complete sectional sequence to j. A sectional path $X_1 \to X_2 \to \cdots X_s = M$ to M in Γ_Λ is said to be **maximal** if any path $Y \to X_1 \to \cdots X_s = M$ of irreducible maps is not a sectional path for any Y. Then, a maximal sectional path $X_1 \to X_2 \to \cdots \to X_s = M$ being not complete implies that X_1 is projective. For each indecomposable preprojective module M, we denote by $(\to M)$ the subquiver of the Auslander-Reiten quiver consisting of all maximal sectional paths to M. We say $(\to M)$ is complete if each maximal sectional path to M is complete. We may also define $(M \to)$ for a preprojective module M. Similarly, we have $(N \to)$ and $(\to N)$ for an indecomposable preinjective module.

3 The Gabriel-Roiter measure

In this section, we assume that Λ is a fixed artin algebra. We first recall some definitions. Let $\mathbb{N}_1 = \{1, 2, \cdots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N}_1)$ the set of all subsets of $\mathbb{N}_1$. We consider the set $\mathcal{P}(\mathbb{N}_1)$ as a totally ordered set as follows: If I,J are two different subsets of $\mathbb{N}_1$, write $I < J$ provided the smallest element in $(I \backslash J) \cup (J \backslash I)$ belongs to J. Also we write $I \ll J$ provided $I \subset J$ and for all elements $a \in I$, $b \in J \backslash I$, we have $a < b$. We say that J **starts with** I provided $I = J$ or $I \ll J$. It is easy to check that:
(1) If $I \subseteq J \subseteq \mathbb{N}_1$, then $I \leq J$.
(2) If $I_1 \leq I_2 \leq I_3$, and I_3 starts with I_1, then I_2 starts with I_1.

For each Λ-module M, we denote by $|M|$ the length of M. Let $\mu(M)$ be the maximum of the sets $\{|M_1|, |M_2|, \cdots, |M_t|\}$ where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of indecomposable submodules of M. We call $\mu(M)$ the **Gabriel-Roiter measure** (briefly **GR measure**) of M. If M is an indecomposable Λ-module, then a chain of indecomposable submodules $M_1 \subset M_2 \subset \cdots \subset M_t = M$ with $\mu(M) = \{|M_1|, |M_2|, \cdots |M_t|\}$ is called a **Gabriel-Roiter filtration** (briefly **GR filtration**) of M. We call an inclusion $T \subset M$ of indecomposable Λ-modules a **Gabriel-Roiter inclusion** (briefly **GR inclusion**) provided $\mu(M) = \mu(T) \cup \{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(T)$. In this case, we call T a **Gabriel-Roiter submodule** (briefly, **GR submodule**) of M. Note that a chain $M_1 \subset M_2 \subset \cdots \subset M_t = M$ is a GR filtration if and only if all the inclusions $M_i \subset M_{i+1}$ are GR inclusions. The factor module of a GR inclusion is called a **Gabriel-Roiter factor** (briefly **GR factor**). A short exact sequence $0 \longrightarrow T \xrightarrow{f} M \xrightarrow{g} X \longrightarrow 0$ is called a **GR sequence** provided f is a GR inclusion.

We obtain the following conclusion from the above concept, which is useful in what will follow:

Lemma 3.1. *Let X, Y and Z be Λ-modules.*
(1) *If X is a proper submodule of Y, then $\mu(X) \leq \mu(Y)$.*
(2) *Assume that X, Y, Z are indecomposable. If $\mu(X) < \mu(Y) < \mu(Z)$ and X is a GR submodule of Z, then $|Y| > |Z|$.*

The following Main Property of Gabriel-Roiter measure is essentially due to Gabriel ([7]), and proved by Ringel ([12]) for arbitrary modules.

Main Property (Gabriel). *Let X, $Y_1, \cdots, Y_t$ be indecomposable modules and assume that there is a monomorphism $f : X \longrightarrow \oplus_{i=1}^{t} Y_i$. Then*
(1) $\mu(X) \leq \max\{\mu(Y_i)\}$.
(2) *If $\mu(X) = \max\{\mu(Y_i)\}$, then f splits.*
(3) *If $\max\{\mu(Y_i)\}$ starts with $\mu(X)$, then there is some j such that $\pi_j f$ is injective, where $\pi_j : \oplus_{i=1}^{t} Y_i \longrightarrow Y_j$ is the canonical projection.*

In the following proposition, we collect some basic properties of the GR inclusions which will be needed in the sequel. We refer to [12] and [5] for a proof.

Proposition 3.2. *Let $\epsilon : 0 \longrightarrow T \stackrel{l}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} M/T \longrightarrow 0$ be a GR sequence. Then the following statements hold:*
(1) *T is a direct summand of all proper submodules of M containing T.*
(2) *M/T is indecomposable.*
(3) *Any map to M/T which is not an epimorphism factors through π.*
(4) *All irreducible maps to M/T are epimorphisms.*
(5) *If all irreducible maps to M are monomorphisms, then l is an irreducible map.*
(6) *M/T is a factor module of $\tau^{-1}T$ and $M/T \cong \tau^{-1}T$ if and only if ϵ is an Auslander-Reiten sequence.*

The following proposition will be quite often used in our discussion.

Proposition 3.3. *Assume that T is a GR submodule of M. Then there is an irreducible monomorphism $T \to X$ with X indecomposable and an epimorphism $X \to M$.*

Proof. Assume that $l : T \to M$ is the inclusion map and $T \stackrel{f=(f_i)}{\longrightarrow} \oplus_{i=1}^{r} X_i$ is a minimal left almost split map. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{f=(f_i)} & \oplus X_i \\ \downarrow{\scriptstyle l} & \swarrow{\scriptstyle g=(g_i)} & \\ M & & \end{array}$$

Assume that g_i is not an epimorphism for any i. The induced monomorphism $(g_i f_i) : T \to \oplus X_i \to \oplus_i \operatorname{Im} g_i$ implies $\mu(T) \leq \max\{\mu(\operatorname{Im} g_i)\} \leq \mu(T)$ since T is a GR submodule of Y and $\operatorname{Im} g_i$ is a proper submodule of M. By the Main Property, we obtain that the map $(g_i f_i)$ splits, thus (f_i) splits. But $f = (f_i)$ is an almost split map. This contradiction implies that there is an index j such that g_j is an epimorphism. Since $|X| \geq |M| > |T|$, we obtain that f_j is a monomorphism. Now we take $X = X_j$. □

Remark. We may also require that the composition of the maps $T \to X \to M$ obtained in the proposition is a monomorphism. This follows from the fact that the subset consisting of all non-monomorphisms in $\operatorname{Hom}(T, M)$ is a subgroup, see [15] for details.

Now we recall the partition obtained by using Gabriel-Roiter measure approach. As in [12],[13], we say $I \in \mathcal{P}(\mathbb{N}_1)$ is a Gabriel-Roiter measure for Λ if there exists an indecomposable Λ-module M

with $\mu(M) = I$. A measure I is said to be of **finite type** if there are only finitely many isomorphism classes of indecomposable modules with measure I. Let I and J be two measures for Λ, we say J is a **direct successor** of I if there is no measure J' with $I < J' < J$.

Theorem 3.4 ([12]). *Let Λ be a representation infinite artin algebra. Then there are Gabriel-Roiter measures I_t, I^t for Λ such that*

$$I_1 < I_2 < I_3 < \quad \cdots \quad < I^3 < I^2 < I^1$$

and such that any other measure J satisfies $I_t < J < I^t$ for all t. Moreover, all these measures I_t and I^t are of finite type.

The measures $I_t(I^t)$ are called **take-off (landing)** measures and any other measure is called a **central** measure. An indecomposable module M with GR measure I is called a take-off (resp. central, landing) module if I is a take-off (resp. central, landing) measure. It is easy to see that if J is the direct successor of I, then I is a take-off (resp. central, landing) measure if and only if so is J.

In [12], Ringel showed the following proposition:

Proposition 3.5 ([12]). *Let Λ be a representation infinite artin algebra. Then all landing modules are preinjective (in the sense of Auslander and Smalø [2]).*

The following is the *Successor Lemma* in [13]. Note that a GR measure different from I_1 may not have direct predecessor.

Proposition 3.6 ([13]). *Any Gabriel-Roiter measure I different from I^1 has a direct successor.*

4 Comparison of two kinds of partitions for tame hereditary algebras

Let Λ be a representation infinite hereditary k-algebra. We denote by $\mathcal{T}_\Lambda$, $\mathcal{C}_\Lambda$, $\mathcal{L}_\Lambda$ the full subcategory of take-off modules, central modules and landing modules, respectively. Note that under our convention, they are all collections of indecomposable modules. We denote by $\mathcal{P}_\Lambda$, $\mathcal{R}_\Lambda$, $\mathcal{I}_\Lambda$ preprojective modules, regular modules and preinjective modules, respectively. Then $\operatorname{ind}\mathcal{P}_\Lambda$ (resp. $\operatorname{ind}\mathcal{R}_\Lambda, \operatorname{ind}\mathcal{I}_\Lambda$) is the collection of all indecomposable preprojective (resp. regular, preinjective) modules. We shall always omit the subscript Λ in the sequel since only one algebra is involved.

Proposition 4.1. *Let Λ be a representation infinite hereditary algebra and M be an indecomposable Λ-module.*
(1) *If M is a preprojective take-off module, then there exists a natural number n such that $\mu(\tau^{-i}M) > \mu(M)$ for all $i > n$.*
(2) *If M is a preinjective landing module, then there exists a natural number n such that $\mu(\tau^{i}M) < \mu(M)$ for all $i > n$.*

Proof. We show (1), then (2) is the dual. Assume not, then since M is preprojective, there are infinitely many indecomposable modules of the form $\tau^{-i}M$ such that $\mu(\tau^{-i}M) \leq \mu(M)$. But M is a take-off module implies that only finitely many measures are smaller than $\mu(M)$. It follows that there exist a measure $I \leq \mu(M)$, which is thus a take-off measure, such that there are infinitely many indecomposable modules with measure I. This is a contradiction since any take-off measure is of finite type. $\square$

Remark. Part (2) of this proposition may not hold if M is preinjective but not a landing module. For example, we consider the quiver $\widetilde{\mathbb{A}}_2$ (for details see [12],[13]). The landing part consists of half of the indecomposable preinjective modules, namely, the modules with length $3m+1(m > 0)$. The indecomposable preinjective modules of length $3m + 2(m > 0)$ are central modules. For each preinjective central module M, there are infinitely many indecomposable preinjective modules of the form $\tau^i M$ such that $\mu(\tau^i M) > \mu(M)$.

Lemma 4.2 ([9]). *Let Λ be a representation infinite hereditary algebra. If X, Y are nonzero preprojective Λ-modules, then X is cogenerated by $\tau^{-m} Y$ for $m \gg 0$.*

Proposition 4.3. *Let Λ be a representation infinite hereditary algebra and X be an indecomposable preprojective Λ-module. Then*

$$|\{Y \in \operatorname{ind} \mathcal{P} : \mu(Y) \leq \mu(X)\}| < \infty.$$

Proof. Assume that $P_i(i \in Q_0)$ are indecomposable projective Λ-modules. Then for each i, there exists a natural number m_0^i such that X is cogenerated by $\tau^{-m} P_i$ for all $m \geq m_0^i$ (Lemma 4.2). Since there are only finitely many indecomposable projective modules, we may choose $m_0 = \max\{m_0^i\}$. Then X is cogenerated by $\tau^{-m} P$, thus $\mu(X) < \mu(\tau^{-m} P)$, for each indecomposable projective module P and $m \geq m_0$. Therefore, an indecomposable module Y with $\mu(Y) \leq \mu(X)$ satisfies $Y = \tau^{-t} P$ for some indecomposable projective module P and some $t \leq m_0$. Thus, there are only finitely many indecomposable preprojective modules with measures smaller than $\mu(X)$. $\square$

Let Λ be a representation infinite hereditary algebra and P be an indecomposable projective Λ-module. We call $\mathcal{O}(P) = \{\tau^{-i} P | i \geq 0\}$ the τ-orbit of P. Baer, see [3], also [9], has called the orbit $\mathcal{O}(P)$ a **mono orbit**, provided

- If $X \in \mathcal{O}(P)$ is indecomposable and Y is preprojective , then any nonzero homomorphism $f : X \to Y$ is injective.
- If X and Y in $\mathcal{O}(P)$ are indecomposable and $f : X \to Y$ is a nonzero (hence a monomorphism), then the cokernel C of f is regular.

Now assume that $\Lambda = kQ$ is a tame hereditary algebra. If P_i is an indecomposable projective module with $\delta(P_i) = -1$, then $\mathcal{O}(P_i)$ is a mono-orbit. Namely, assume tht $X = \tau^{-t} P_i$ and f is a nonzero map from X to a preprojective module Y. Thus $\operatorname{Im} f$ is a preprojective module. But $\delta(X) = \delta(\operatorname{Im} f) + \delta(\operatorname{Ker} f)$ and $\delta(X) = -1$ implies $\operatorname{Im} f = 0$ or $\operatorname{Ker} f = 0$. Thus f is a monomorphism. If $Y \in \mathcal{O}(P_i)$, then $\delta(X) = \delta(Y) = -1$ and $\delta(\operatorname{Coker} f) = 0$. In particular, X has no preprojective factor module. The same argument shows that any nonzero map from an indecomposable preprojective module with defect -1 to a regular simple module is either injective or surjective.

Dually, we have τ-orbits $\mathcal{O}(I)$ for indecomposable injective modules I and can define **epi orbits** in a similar way. If Y is an indecomposable preinjective module with defect $\delta(Y) = 1$, then any nonzero map from a regular simple module to Y is either injective or surjective.

Note that for a tame hereditary algebras kQ, $\mathcal{O}(P_i)$ ($\mathcal{O}(I_i)$) is a mono (an epi) orbit if and only if $-\langle\delta, P_i\rangle = \delta_i = \langle\delta, I_i\rangle = 1$.

Theorem 4.4. *Let $\Lambda = kQ$ be a tame hereditary algebra. Then $\operatorname{ind}\mathcal{P} \subset \mathcal{T}$, i.e. every indecomposable preprojective module lies in the take-off part.*

Proof. Since an indecomposable Λ-module X is a take-off module if and only if there are only finitely many indecomposable modules with GR measure smaller than $\mu(X)$, the theorem is a direct consequence of Proposition 4.3 and the following three statements:

(1) Let X be an indecomposable preprojective module. Then

$$|\{Y \in \operatorname{ind}\mathcal{R} : Y \text{ is an exceptional regular module}, \quad \mu(Y) \le \mu(X)\}| < \infty.$$

(2) Let H_1 be a homogeneous regular simple module. Then

$$|\{X \in \operatorname{ind}\mathcal{I} : X \quad \text{is indecomposable}, \quad \mu(X) < \mu(H_1)\}| < \infty.$$

(3) Let H be a homogeneous regular module. Then $\mu(H) > \mu(X)$ for all $X \in \operatorname{ind}\mathcal{P}$.

For the proof of (1), we consider the sectional path $Y_1 \to Y_2 \to \cdots \to Y_n \to \cdots$ with Y_1 an exceptional regular simple module in an exceptional tube of rank r. Note that $\underline{\dim}\, Y_r = \delta$ and Y_n is sincere for each $n \ge r$. It is known that for each $n \ge r$, we may get indecomposable preprojective module P_n which is a proper submodule of Y_n such that $\lim_{n\to\infty} |P_n| = \infty$

For the given X, we may, by Lemma 4.2, obtain a natural number m_0 such that X is cogenerated by the indecomposable modules $\tau^{-m}P$ for all indecomposable projective modules P and all $m \ge m_0$. Since $\lim_{n\to\infty} |P_n| = \infty$, we have X is cogenerated by P_n for n large enough. In particular, we have $\mu(X) < \mu(P_n) < \mu(Y_n)$ for $n \gg 0$. Since there are at most 3 exceptional tubes, only finitely many indecomposable exceptional regular modules have measures smaller than $\mu(X)$ for the given X.

To prove (2), we assume that Y is an indecomposable preinjective module with $\underline{\dim}\, Y > \delta$. If $\delta(Y) = 1$, then any nonzero map from H_1 to Y is either injective or surjective. If $\delta(Y) \ge 2$ (this occurs when Λ is of type $\widetilde{\mathbb{D}}_n$, or $\widetilde{\mathbb{E}}_{6,7,8}$), we may find an indecomposable module X with defect $\delta(X) = 1$ (using Proposition 2.1) and there is an injective map from X to Y. In fact we consider $(\to Y)$ (see section 2 for definition) which is complete since Y is sincere. Take any module X with defect $\delta(X) = 1$ in $(\to Y)$. It is easy to see that the composition of the sectional path $X \to \cdots \to Y$ is injective. As upshot, if $Y = \tau^m I$ is indecomposable with I an injective module and $m \gg 0$, that is, $|Y| \gg |H_1|$, then we may get an indecomposable module X ($= Y$ if $\delta(Y) = 1$) with $\delta(X) = 1$, such that $|X| > |H_1|$ and, both $\operatorname{Hom}(H_1, X)$ and $\operatorname{Hom}(X, Y)$ contain monomorphisms. It follows that $\mu(H_1) < \mu(X) \le \mu(Y)$. Note that only finitely many indecomposable preinjective modules have dimension vectors smaller than δ. Therefore, only finitely many indecomposable preinjective modules have measures smaller than $\mu(H_1)$.

Finally, we prove (3) by showing that $\mu(X) < \mu(H_1)$ for any indecomposable preprojective module X and any homogeneous regular simple module H_1. We first note that $\mathrm{Hom}(X, H_1) \neq 0$ for any indecomposable preprojective module X and that no indecomposable preprojective (preinjective) module has length $|H_i| = i|\delta|$ where H_i is a homogeneous regular module. In particular, $\mu(M) \neq \mu(H_i)$ for any indecomposable module $M \in \mathrm{ind}\,\mathcal{P}(\mathrm{ind}\,\mathcal{I})$. Assume that X is an indecomposable preprojective module with $\mu(X) > \mu(H_1)$ such that $|X|$ is minimal. Since X is not simple, we take a GR submodule Y of X which is again preprojective. Thus, $\mu(Y) < \mu(H_1) < \mu(X)$ by the minimality property of X. We thus have $|H_1| > |X|$ (Lemma 3.1). Assume $K = \cap_{f:X\to H_1}\mathrm{Ker}\, f$ and consider the short exact sequence $0\to K\to X \xrightarrow{\pi} C\to 0$. By construction, $\mathrm{Hom}(\pi, H_1)$ is an isomorphism and C is cogenerated by H_1. Since $|H_1| > |X| \geq |C|$, we obtain that C is preprojective and thus $\mathrm{Ext}^1(C, H_1) = 0$. Applying the functor $\mathrm{Hom}(-, H_1)$, we obtain

$$0\to\mathrm{Hom}(C, H_1) \overset{\mathrm{Hom}(\pi,H_1)}{\longrightarrow} \mathrm{Hom}(X, H_1)\longrightarrow\mathrm{Hom}(K, H_1)\longrightarrow\mathrm{Ext}^1(C, H_1) = 0$$

Thus we get $\mathrm{Hom}(K, H_1) = 0$. Therefore $K = 0$ and X is cogenerated by H_1, and $\mu(X) < \mu(H_1)$. This is a contradiction. □

Remark. (1) Note that there may exist infinitely many exceptional regular take-off modules. See $\widetilde{\mathbb{A}}_2$, for details we refer to [12],[13].
(2) The last statement in the above theorem implies that the homogeneous modules over tame hereditary algebras are always central modules.
(3) A length category is said to be of infinite type if there are, up to isomorphism, infinitely many indecomposable objects. Ringel showed that any cogeneration closed length category which is of infinite type contains a minimal infinite cogeneration closed subcategory ([16]). Thus the preprojective component of a tame hereditary (concealed) algebras is a minimal cogeneration closed subcategory. And he shows that any infinite cogeneration closed subcategory of the module category of a tame hereditary (concealed) algebra always contains all preprojective modules. This yields an alternative proof of the theorem. But our proof here is direct and use only basic representation theory methods.

Corollary 4.5. *Let $\Lambda = kQ$ be a tame hereditary algebra and $|\delta| = \sum_j \delta_j$ where δ is the minimal radical vector. Let H_1 be a homogeneous simple module. Then for each $i \geq 2$, the homogeneous module H_i contains, up to isomorphism, H_{i-1} as the unique GR submodule. Therefore, $\mu(H_i) = \mu(H_1) \cup \{2|\delta|, 3|\delta|, \cdots, i|\delta|\}$.*

Proof. Note that a GR submodule of H_i with $i \geq 2$ is either H_{i-1} or a preprojective module. Then statement (3) in Theorem 4.4 implies the conclusion. □

Corollary 4.5 implies that the homogeneous modules behave very well according to Gabriel-Roiter inclusions, i.e. the GR submodule of a homogeneous module (not regular simple) is always given by an irreducible monomorphism. Next we will see a stronger consequence which claims that we can not insert any measure between those of a GR inclusion of homogeneous modules.

Theorem 4.6. *Let $\Lambda = kQ$ be a tame hereditary algebra and $H_1\to H_2\to H_3\to\cdots$ be a sectional path of irreducible monomorphisms starting with a homogeneous regular simple module H_1. Then $\mu(H_i)$ is a direct successor of $\mu(H_{i-1})$ for all $i \geq 2$.*

Before we present the proof of the theorem, we first show some interesting lemmas which will be used in the proof of Theorem 4.6.

Lemma 4.7. *Let $\Lambda = kQ$ be a tame hereditary algebra. Assume that $X \in \operatorname{ind}\mathcal{I} \setminus \mathcal{T}$ is an indecomposable module and $Y \notin \mathcal{I}$ is a GR submodule of X with $Y = Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_n \rightarrow \cdots$ a sequence of irreducible monomorphisms. Then $\mu(X) > \mu(Y_i)$ for all i.*

Proof. If Y is preprojective, then all Y_i's are preprojective and $\mu(X) > \mu(Y_i)$ for all i since $\operatorname{ind}\mathcal{P} \subset \mathcal{T}$ and $X \notin \mathcal{T}$. Now we assume that Y is a regular module. Then all Y_i's are regular modules and the sectional path of irreducible monomorphisms starting with Y is unique. Since $Y_1 = Y$ is a GR submodule of X, there is an epimorphism from Y_2 to X (Proposition 3.3). It follows that $|Y_i| > |X|$ for all $i \geq 2$. If $Y_1 \rightarrow Y_2$ is a GR inclusion, then $\mu(Y_1) < \mu(X) < \mu(Y_2)$ implies $|X| > |Y_2|$ which is a contradiction. We thus have $\mu(X) > \mu(Y_2)$. If Y_1 is not a GR submodule of Y_2 (By Corollary 4.5, this only happens when Y is an exceptional regular module), then a GR submodule T of Y_2 is preprojective. Therefore, $\mu(T) < \mu(X)$ since $X \notin \mathcal{T}$. If $\mu(X) < \mu(Y_2)$, then $|X| > |Y_2|$, again a contradiction. Continuing the induction steps, we get $\mu(X) > \mu(Y_i)$ for all i. □

Lemma 4.8. *Let $\Lambda = kQ$ be a tame hereditary algebra. Assume $X_1 \rightarrow \cdots \rightarrow X_r \rightarrow \cdots$ is a sectional path with X_1 an exceptional regular simple module and $\underline{\dim}\, X_r = \delta$.*
(1) *For any $j \leq r$, if $\mu(X_j) < \mu(H_1)$, then $X_j \in \mathcal{T}$; if $\mu(X_j) > \mu(H_1)$, then $\mu(X_j) > \mu(H_i)$ for all i.*
(2) *If $\mu(X_r) \geq \mu(H_1)$, then $\mu(X_j) > \mu(H_i)$ for all $j > r, i \geq 1$. In this case, $X_r \rightarrow X_{r+1} \rightarrow X_{r+2} \rightarrow \cdots$ is a chain of GR inclusions.*
(3) *If $\mu(X_r) < \mu(H_1)$, then $\mu(X_j) < \mu(H_1)$ for all $j \geq 1$.*

Proof. For the proof of (1), we assume that Y is a GR submodule of H_1, then Y lies in $\operatorname{ind}\mathcal{P}$. Suppose $\mu(X_j) < \mu(H_1)$ with $j \leq r$. If $\mu(Y) < \mu(X_j) < \mu(H_1)$, then $|X_j| > |H_1|$. This contradiction shows $\mu(X_j) \leq \mu(Y)$. In particular, $X_j \in \mathcal{T}$ since $Y \in \mathcal{P} \subset \mathcal{T}$. If $\mu(H_1) < \mu(X_j) < \mu(H_s)$ for some $s > 1$, then $\mu(X_j)$ starts with $\mu(H_1)$. It follows that $|X_j| > |H_1|$, again a contradiction since $j \leq r$.
For statement (2), we assume that $\mu(X_r) \geq \mu(H_1)$. Since $|X_{r+1}| < |X_r| + |\delta| = 2|\delta|$, we have

$$\mu(X_{r+1}) \geq \mu(X_r) \cup \{|X_{r+1}|\} > \mu(X_r) \cup \{2|\delta|, \cdots, i|\delta|\} \geq \mu(H_1) \cup \{2|\delta|, \cdots, i|\delta|\} = \mu(H_i)$$

Therefore, $\mu(X_j) > \mu(X_{r+1}) > \mu(H_i)$ for all $j > r+1$ and $i \geq 1$.
Finally, we prove (3) by showing $\mu(X_{r+1}) < \mu(H_1)$. Then using the same argument, we may show that $\mu(X_j) < \mu(H_1)$ for all j. If X_r is a GR submodule of X_{r+1} and $\mu(X_{r+1}) > \mu(H_1)$, we have $|H_1| > |X_{r+1}|$ which contradicts $|X_{r+1}| > |X_r| = |H_1|$. Thus, $\mu(X_{r+1}) < \mu(H_1)$. If X_r is not a GR submodule of X_{r+1}, we choose a GR submodule T of X_{r+1} which will be a preprojective module. Thus, we have $\mu(X_{r+1}) = \mu(T) \cup \{|X_{r+1}|\} < \mu(H_1)$ since $\mu(T) < \mu(H_1)$ and $|X_{r+1}| > |H_1|$. □

Lemma 4.9. *Let $\Lambda = kQ$ be a tame hereditary algebra and $H_1 \rightarrow H_2 \rightarrow \cdots$ be the sectional path of irreducible monomorphisms with H_1 a homogeneous regular simple module. If $X \in \operatorname{ind}\mathcal{I}$ with $\mu(X) > \mu(H_1)$, then $\mu(X) > \mu(H_i)$ for all i.*

Proof. We assume for a contradiction that $\mu(X) < \mu(H_j)$ for some j. Since there is no indecomposable preinjective module with length $s|\delta|$ for all natural number s, there is an index i such that $\mu(H_i) < \mu(X) < \mu(H_{i+1})$. Since H_i is a GR submodule of H_{i+1}, we obtain that $|X| > |H_{i+1}|$.

Assume T is a GR submodule of X. If $\mu(T) < \mu(H_i) < \mu(X)$, then $|H_i| > |X|$ since T is a GR submodule of X. This contradicts $|X| > |H_{i+1}| > |H_i|$. Thus $\mu(H_i) \leq \mu(T) < \mu(X) < \mu(H_{i+1})$ holds and T is not preprojective. Furthermore, T can not be a homogeneous module, since otherwise, from $\mu(H_i) \leq \mu(T) < \mu(X) < \mu(H_{i+1})$ we deduce that $\mu(T) = \mu(H_i)$ and, by Lemma 4.7, $\mu(X) > \mu(H_t)$ for all $t \geq 0$ which contradicts $\mu(X) < \mu(H_{i+1})$. Thus T is either an exceptional regular module or a preinjective module.

If T is an exceptional regular module, then $\mu(H_i) \leq \mu(T) < \mu(X) < \mu(H_{i+1})$ implies $\mu(T)$ starts with $\mu(H_i)$ and thus starts with $\mu(H_1)$. Therefore, T has a (regular) submodule Y with $\mu(Y) = \mu(H_1)$. It follows that there is a sectional path of irreducible monomorphisms

$$Y_1 \to Y_2 \to \cdots \to Y_r = Y \to \cdots \to Y_t = T \to Y_{t+1} \to \cdots$$

where Y_1 is an exceptional regular simple module, r is the rank of the exceptional tube and $t \geq r$. Then Lemma 4.8 shows $\mu(Y_{t+1}) > \mu(H_s)$ for all s and Lemma 4.7 shows $\mu(X) > \mu(Y_{t+1})$. This contradicts $\mu(X) < \mu(H_{i+1})$.

If T is a preinjective module, then $\mu(H_i) < \mu(T) < \mu(X) < \mu(H_{i+1})$ holds. Since $|T| < |X|$, we may use induction to obtain a contradiction.

As upshot, we have $\mu(X) > \mu(H_i)$ for all i. □

Proof of Theorem 4.6. Assume that X is an indecomposable Λ-module with $\mu(H_1) \leq \mu(H_i) < \mu(X) < \mu(H_{i+1})$. Then we may assume $X \notin \mathcal{I} \cup \mathcal{P}$ by Theorem 4.4 and Lemma 4.9. It follows that X is an exceptional regular module and is of the form $X = E[s]$ where E is the regular simple module on a tube of rank r. Now the theorem is a direct consequence of Lemma 4.8. □

5 Orthogonal exceptional pairs for type $\widetilde{\mathbb{A}}_{n,n\geq 2}$ and $\widetilde{\mathbb{D}}_n$

In this section, we shall give some interesting phenomena for tame hereditary algebra Λ of type $\widetilde{\mathbb{A}}_{n,n\geq 2}$ or $\widetilde{\mathbb{D}}_n$.

Let Λ be a finite dimensional k-algebra. An indecomposable Λ-module X is **exceptional** if $\mathrm{End}(X) = k$ and $\mathrm{Ext}^i(X, X) = 0$ for all $i > 0$. Two modules X, Y are **orthogonal** if $\mathrm{Hom}(X, Y) = \mathrm{Hom}(Y, X) = 0$. An **orthogonal exceptional pair** to an indecomposable module M is a pair of orthogonal exceptional modules (Y, X) such that there is a short exact sequence $0 \to X^u \to M \to Y^v \to 0$ with $\mathrm{Ext}^1(X, Y) = 0$. Schofield's Theorem ([11], [17]) tells us that if Λ is hereditary and M is an exceptional Λ-module, then there exist orthogonal exceptional pairs to M. But there does not yet exist a convenient procedure to determine the possible orthogonal exceptional pairs, when an exceptional module is given. In [4], we have shown that by taking GR submodules, we can construct orthogonal exceptional pairs for each indecomposable module over a representation-finite hereditary algebra. This was generalized to representation-directed algebra ([14]).

Our next proposition shows that for indecomposable preprojective modules over tame hereditary algebras of type $\widetilde{\mathbb{A}}_{n,n\geq 2}$ or $\widetilde{\mathbb{D}}_n$, we can construct orthogonal exceptional pairs by taking GR submodules.

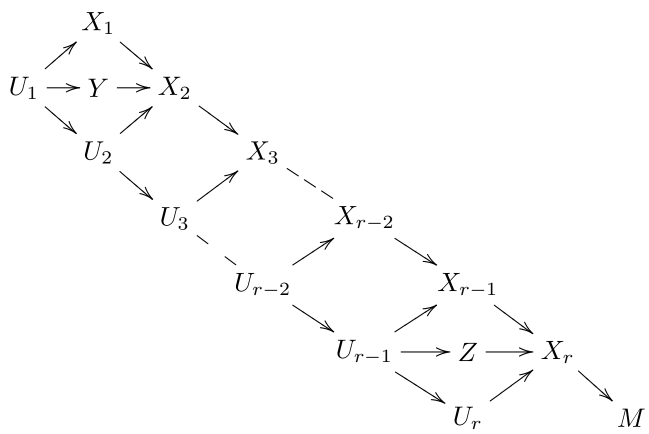
Proposition 5.1. *Let $\Lambda = kQ$ be a tame hereditary of type $\widetilde{\mathbb{A}}_{n,n\geq 2}$, or type $\widetilde{\mathbb{D}}_n$ and M be a non-simple indecomposable preprojective Λ-module. Then, for each GR submodule T of M, $(M/T, T)$ is* *an orthogonal exceptional pair to M.*

Proof. We first show that if T is a GR submodule of an indecomposable preprojective M, then $\dim \mathrm{Hom}(T, M) = 1$.

We suppose that Λ is of type $\widetilde{\mathbb{A}}_n$ with $n \geq 2$. In this case, there are no multiple edges in the Aulsander-Reiten quiver Γ_Λ. Note that any map between preprojective modules is a sum of compositions of irreducible maps, i.e. a sum of paths in Γ_Λ. Recall that the GR inclusions of preprojective modules are irreducible maps since all irreducible maps in preprojective component are monomorphisms. If $\dim \mathrm{Hom}(T, M) \geq 2$, then there exist another path $T \to \cdots \to N \to M$ of irreducible maps. But this implies there is a monomorphism from T to N. This contradiction shows $\dim \mathrm{Hom}(T, M) = 1$. Note that any indecomposable preprojective module has,up to isomorphism, at most 2 GR submodules.

Now we consider $\widetilde{\mathbb{D}}_n$ case. We assume that M is an indecomposable preprojective Λ-module with defect -2. We show that in this case any irreducible map to M is injective. Thus any GR submodule of M is given by irreducible maps. Therefore M has at most 4 GR submodules. (Note that 4 only occurs when Λ is of type $\widetilde{\mathbb{D}}_4$). Assume that $X \xrightarrow{f} M$ is an irreducible map. If $\delta(X) = -1$, then f is a monomorphism. Assume $\delta(X) = -2$ and f is an epimorphism. Then from the exact sequence $0 \to \mathrm{Ker}\, f \to X \to M \to 0$ we obtain $\delta(\mathrm{Ker}\, f) = \delta(X) - \delta(M) = -2 - (-2) = 0$. Thus, $\mathrm{Ker}\, f = 0$ and f is a monomorphism. Therefore, all irreducible maps to an indecomposable preprojective module of defect -2 are monomorphisms. Note that in this case $\dim \mathrm{Hom}(T, M) = 1$ and M/T is either a regular simple module or a preprojective module with defect -1.

Now we assume that M is an indecomposable module with defect -1. Consider the following full subquiver of the preprojective component.



First note that if X_1 (or Y, Z) is nonzero, then the unique map from X_1 (or Y, Z) to M is a monomorphism since it has defect -1.

If $(\to M)$ is complete, then X_1, Y and Z are all non-zero. If T is a GR submodule of M and no sectional path goes from T to M, then the canonical inclusion factors through $X_1 \oplus Y \oplus Z$. Thus T is isomorphic to one of these summands by the Main Property. In particular, up to isomorphism, M has at most three GR submodules and for each GR submodule T, we have $\dim \mathrm{Hom}(T, M) = 1$ and M/T is regular simple module. If $(\to M)$ is not complete, then X_1, (or Y, Z) is zero. In case Z

is zero, we have X_r is projective. Then a GR submodule T of M is projective and on the sectional path $X_1(Y) \to X_2 \to \cdots \to X_r$. Thus $\dim \mathrm{Hom}(T, M) = 1$. Now assume $Z \neq 0$, but one of X_1 and Y, without loss of generality say X_1, is zero. Assume T is a GR submodule of M. If T is not on a sectional path to M, then the canonical inclusion factors through $Z \oplus Y$. Thus, $T \cong Z$ or Y, a contradiction. It follows that any GR submodule of M is on a sectional path to M. In particular, $\dim \mathrm{Hom}(T, M) = 1$ for any GR submodule T of M. Note that in each case, M has at most 3 GR submodules.

Now we show that for each GR submodule T of M, $(M/T, T)$ is an orthogonal exceptional pair. Note that an indecomposable preprojective module is an exceptional module and there is no map from regular (preinjective) modules to preprojective modules. Applying the functor $\mathrm{Hom}(T, -)$ to the GR sequence $0 \to T \to M \to M/T \to 0$, we obtain

$$0 \to \mathrm{Hom}(T, T) \to \mathrm{Hom}(T, M) \to \mathrm{Hom}(T, M/T) \to \mathrm{Ext}^1(T, T) = 0$$

It follows $\mathrm{Hom}(T, M/T) = 0$ since $\dim \mathrm{Hom}(T, T) = 1 = \dim \mathrm{Hom}(T, M)$. Applying the functor $\mathrm{Hom}(-, M/T)$ to the GR sequence, we obtain an exact sequence

$$0 = \mathrm{Hom}(T, M/T) \to \mathrm{Ext}^1(M/T, M/T) \to \mathrm{Ext}^1(M, M/T).$$

Then last term vanishes since there is a path from M to M/T in $\mathrm{mod}\,\Lambda$ and M is directing. Therefore, we have $\mathrm{Ext}^1(M/T, M/T) = 0$. This means M/T is an exceptional module, and $(M/T, T)$ is an orthogonal exceptional pair to M. We should note that M/T is not a homogeneous module. □

We call a GR inclusion a **preprojective pair** if the modules involved are both preprojective. The corresponding GR factor of a preprojective pair is called a **GR factor of preprojective pair**. The following is an immediate corollary of the proof of the above proposition.

Corollary 5.2. *Let Λ be a tame hereditary algebra of type $\widetilde{\mathbb{A}}_{n, n \geq 2}$ or $\widetilde{D}_n$.*
(1) There are only finitely many preinjective GR factors of preprojective pairs.
(2) Regular GR factors of preprojective pairs are exceptional.

Proof. For $\widetilde{\mathbb{A}}_n$ case, all GR factors of preprojective pairs are regular modules. In the proof of proposition 5.1, we have seen that, for $\widetilde{\mathbb{D}}_n$ type, A GR factor of a preprojective pair is preinjective if and only if $\delta(M) = -1$ and $\delta(T) = -2$. From the picture in the proof of the proposition, this only occurs when there is a monomorphism from X_i to M for some $2 \leq i \leq r$. But this can not happen if we take $M = \tau^{-t} P$ for some large enough natural number t. Thus, we have only finitely many choices of such M. This shows (1).

(2) is just the last sentence in the above proof of the theorem. □

Acknowledgments. I am grateful to the referee for spotting misprints and inaccuracies.

References

[1] M.Auslander; I.Reiten; S.O.Smalø, Representation Theory of Artin Algebras. Cambridge studies in advanced mathematics, **36** (Cambridge University Press, Cambridge, 1995).

[2] M.Auslander; S.O.Smalø, Preprojective modules over Artin algebras. *J. Algebra* **66**(1980), 61–122.

[3] D.Baer, Homological properties of wild hereditary Artin algebras. Lecture Notes in Mathematics **1177**(Springer-Verlag, 1986), 1–12.

[4] B.Chen, The Gabriel-Roiter measure for representation-finite hereditary algebras. Dissertation Bielefeld, 2006.

[5] B.Chen, The Gabriel-Roiter measure for representation-finite hereidtary algebras. *J. Algebra.* **309**(2007), 292–317.

[6] V.Dlab; C.M.Ringel, Indecomposable representations of graphs and algebras. *Mem. Amer. Math. Soc.* **6**(1976), no.173.

[7] P.Gabriel, Indecomposable representations II. *Symposia. Math. Ist. Naz. AltaMat.* **6**(1973), 81–104.

[8] D.Happel; C.M.Ringel, Tilted algebras. *Trans. Amer. Math. Soc.* **272**(1982), 399–443.

[9] O.Kerner; M.Takane, Mono orbits, epi orbits and elementary vertices of representation infinite quivers. *Comm. Alg.* **25**(1997), 51–77.

[10] C.M.Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Mathematics **1099**(Springer-Verlag, 1984).

[11] C.M.Ringel, Exceptional objects in hereditary categories. *An. St. Univ. Ovidius Constantza ser. Math* **4**(1998), no.2, 150–158.

[12] C.M.Ringel, The Gabriel-Roiter measure. *Bull. Sci. Math.* **129**(2005), no.9, 726-748.

[13] C.M.Ringel, Foundation of the representation theory of Artin algebras, Using the Gabriel-Roiter measure. *Trends in Representation Theory of Algebras and Related Topics. (Workshop Queretaro, Mexico, 2004). Contemporary Math.* **406**. *Amer. Math. Soc., Providence, RI,* (2006),105-135.

[14] C.M.Ringel, The theorem of Bo Chen and Hall polynomials. *Nagoya Math. J.* **183**(2006),143–160.

[15] C.M.Ringel, Gabriel-Roiter inclusions and irreducible monomorphisms. *preprint.*
http://www.math.uni-bielefeld.de/∼ringel

[16] C.M.Ringel, Take-off subcagtegories. *preprint.*
http://www.math.uni-bielefeld.de/∼ringel

[17] A.Schofield, The internal structure of real Schur representations. *Preprint.*

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