On frequencies of small oscillations of some dynamical systems associated with root systems

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Abstract

In the paper by Calogero and author [CP 1978] the formula for frequencies of small oscillations of Sutherland system (A_l case) was found. In present note the generalization of this formula for the case of arbitrary root system is given.

1. Introduction. In this note, we consider the classical Hamiltonian systems associated to the root systems, with the potential $v(q) = g^2 \sin^{-2} q$. Such systems were introduced in [OP 1976] as the generalization of the Calogero-Sutherland systems [Ca 1971], [Su 1972].

It is known that these systems have the equilibrium configurations which are distinguished by many remarkable features (see papers [Ca 1977ab], [Pe 1978], [CP 1978], [CP 1979], [ABCOP 1979], [MOP 1981], [OP 1981], [KPS 1992], [BMS 2000], [CS 2002] and books [Pe 1990], [Ca 2001]). In the simplest case (A_l case with the potential $v(q) = g^2 \sin^{-2} q$), the explicit expression for frequencies of small oscillations near equilibrium was found in the paper [CP 1978]. For the case of the potential $v(q) = q^2 + g^2 q^{-2}$, these frequencies are known for the case of arbitrary root system and they are proportional to the degrees of basic invariants of the Weyl group of the corresponding root system [Pe 1978].

The main result of this note is the explicit formula for the frequencies of such oscillations for trigonometric case. Namely, these frequencies are

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proportional to the coefficients $r_j(g)$ in the expansion of the sum of positive roots in the simple roots $\{\alpha_j\}$:

$$2\,\rho(g) = \sum_{\alpha\in R^+} g_\alpha\,\alpha = \sum_{j=1}^{l} r_j(g)\,\alpha_j. \tag{1}$$

This is the generalization of the old result by Calogero and author [CP 1978]. [1]

As a byproduct at $g_\alpha = 1$, we discovered the identity for the root systems which we cannot find in the literature, namely,

$$\prod_{j=1}^{l}(d_j - 1)d_j = z \cdot \prod_{k=1}^{l} r_k. \tag{2}$$

Here $\{d_j\}$ are the degrees of the basic invariants of the root systems, and $z = z(G)$ is the dimension of the center of the simple compact simply-connected Lie group G corresponding to the root system R.

2. General description. The systems under consideration were introduced in [OP 1976] and they are described by the Hamiltonian (for more details, see [OP 1981] and [Pe 1990])

$$H = \frac{1}{2}\,p^2 + U(q), \qquad p^2 = (p,p) = \sum_{j=1}^{l} p_j^2, \tag{3}$$

where $p = (p_1, ..., p_l)$, and $q = (q_1, ..., q_l)$ are the momentum vector and the coordinate one in the l-dimensional vector space $V \sim \mathbb{R}^l$ with the standard scalar product $(\cdot,\cdot)$. The potential $U(q)$ is constructed by means of the certain system of vectors $R = \{\alpha\} = R^+ \cup R^-$ in V, so-called root system.

$$U = \sum_{\alpha\in R^+} g_\alpha^2\, v(q_\alpha), \qquad q_\alpha = (\alpha, q). \tag{4}$$

The constants g_α satisfy the condition $g_\alpha = g_\beta$ if $(\alpha,\alpha) = (\beta,\beta)$. Such systems are completely integrable for $v(q)$ of five types. Here we consider only the case $v(q) = \sin^{-2} q$.

Note that these systems are the generalization of the Calogero-Sutherland systems [Ca 1971] and [Su 1972], for which $R^+ = \{e_i - e_j,\ i < j\}$ and $\{e_j\}$, $j = 1, ..., l+1$ is the standard basis in $\mathbb{R}^{l+1}$.

[1] For numerical calculations of such type quantities see [CS 2002].

3. Root systems. We give here the basic definitions. For more details, see [Hu 1972], [Ma 1987] and [OV 1990]. Let V be the l-dimensional real vector space with a standard scalar product $(\cdot,\cdot)$, $(\alpha,\beta)=\sum \alpha_j\beta_j$, and let s_α be the reflection in the hyperplane through the origin orthogonal to the vector α

$$s_\alpha\, q = q - (q,\alpha^\vee)\,\alpha, \qquad \alpha^\vee = \frac{2}{(\alpha,\alpha)}\,\alpha. \tag{5}$$

Consider a finite set of nonzero vectors $R=\{\alpha\}$ generating V and satisfying the following conditions:

1. For any $\alpha\in R$, the reflection s_α conserves R : $s_\alpha R = R$.

2. For all $\alpha,\beta\in R$, we have $(\alpha^\vee,\beta)\in\mathbb{Z}$.

The set $\{s_\alpha\}$ generates the finite group $W(R)$ – the Weyl group of R. The root system R is called *reduced system* if only vectors in R collinear to α are $\pm\alpha$. Let us choose the hyperplane which does not contain the root. Then we have $R=R^+\bigcup R^-$, and R^+ is the set of positive roots. In R^+ there is the basis $\{\alpha_1,\ldots,\alpha_l\}$ (the set of simple roots) such that $\alpha=\sum_j n_j\alpha_j$, $n_j\geq 0$ for any $\alpha\in R^+$. The root system R is called *irreducible system* if it can not be the union of two non-empty subsets R_1 and R_2 which are orthogonal each other.

Let $\{\alpha_1,\ldots,\alpha_l\}$ be the set of simple roots in R, R^+ be the set of positive roots, and $\{\lambda_j\}$ be a dual basis or the weight basis: $(\lambda_j,\alpha_k)=\delta_{jk}$.

Let Q be the root lattice, and Q^+ be the cone of positive roots

$$Q=\left\{\beta{:}\,\beta=\sum_{j=1}^{l} m_j\alpha_j,\ m_j\in\mathbb{Z}\right\};\quad Q^+=\left\{\gamma{:}\,\gamma=\sum_{j=1}^{l} n_j\alpha_j,\ n_j\in\mathbb{N}\right\}. \tag{6}$$

Let P be the weight lattice, and P^+ be the cone of dominant weights

$$P=\left\{\lambda{:}\,\lambda=\sum_{j=1}^{l} m_j\lambda_j,\ m_j\in\mathbb{Z}\right\};\quad P^+=\left\{\mu{:}\,\mu=\sum_{j=1}^{l} n_j\lambda_j,\ n_j\in\mathbb{N}\right\}. \tag{7}$$

Following [Ma 1987], we define a partial order on P as follows: $\lambda\geq\mu$ if and only if $\lambda-\mu\in Q^+$ (or $(\lambda,\,\lambda_j)\geq(\mu,\,\lambda_j)$ for all $j=1,\ldots,l$). The set of linear combinations over $\mathbb{R}$ of the functions $f_\lambda(q)=\exp\{2i(\lambda,q)\}$, $\lambda\in P$, $q\in V$ may be considered as the group algebra A over $\mathbb{R}$ of the free Abelian group P. For any $\lambda\in P$, let us denote $e^\lambda\sim f_\lambda(q)$ as the corresponding element of A, so that $e^{\lambda}e^{\mu} = e^{\lambda+\mu}$, $(e^{\lambda})^{-1} = e^{-\lambda}$, and $e^0 = 1$, the identity element of A. Then e^{λ}, $\lambda \in P$ form the $\mathbb{R}$-basis of A.

The Weyl group $W(R)$ acts on P and hence also on A : $s(e^{\lambda}) = e^{s\lambda}$ for $s \in W$ and $\lambda \in P$. We denote as A^W the subalgebra of W-invariant elements of A. It is known that A^W is the algebra free generated by basic invariants of degrees $d_1, ..., d_l$. We introduce also a g-deformed vector

$$\rho(g) = \frac{1}{2}\sum_{\alpha\in R^+} g_\alpha\,\alpha.$$

Let δ be the highest root, i.e. the positive root such that $\delta \geq \alpha$ for all $\alpha \in R^+$.

4. Results. Let us consider two classical systems characterized by the Hamiltonians

$$H_m = \frac{1}{2}\,p^2 + U_m(q), \qquad m = 1, 2 \tag{8}$$

with

$$\begin{aligned} U_1(q) &= -\sum_{\alpha\in R^+} g_\alpha \log|\sin(q_\alpha)|, & (9)\\ U_2(q) &= \sum_{\alpha\in R^+} g_\alpha^2 \sin^{-2}(q_\alpha),\ q_\alpha = (\alpha, q),\ g_\alpha = g_\beta \,\mathrm{if}\, (\alpha,\alpha) = (\beta,\beta). & (10) \end{aligned}$$

The configuration space of such system is the Weyl alcove

$$\{q\,|\,(q,\alpha_j) > 0,\ j = 1,\ldots,l,\ (q,\delta) < \pi\}, \tag{11}$$

where $\{\alpha_j\}$ is the set of the simple roots, δ is the highest root, and as it well known the potential energies U_1 and U_2 have an isolated minimum at the point $\bar{q}$ inside the alcove [2]. Let us denote this minimum as $U_1^{(0)}$ and $U_2^{(0)}$, $U_m^{(0)} = U_m(\bar{q})$. Then near this point we have

$$U_m(q) \sim U_m^{(0)} + \frac{1}{2}\,a_{jk}^{(m)}\,\xi_j\xi_k, \qquad \xi_j = q_j - \bar{q}_j. \tag{12}$$

The matrices $a^{(m)}$ are positive definite and as it is known ([Pe 1978]),

$$a^{(2)} = c\left(a^{(1)}\right)^2, \qquad c = \mathrm{const}. \tag{13}$$

[2] Let us note that the first system is the generalization of the so-called "Dyson system", [Dy 1962a], [Dy 1962b].

So, we need to know just one of them. The frequencies of small oscillations $\omega_j^{(m)}$ near the equilibrium configuration are square roots of eigenvalues of these matrices and we have $\omega_j^{(2)} = c\left(\omega_j^{(1)}\right)^2$. The eigenvalues of these matrices were known before only for A_l case (see [CP 1978]). Here we give the explicit expression for them for general case.

Theorem. *Eigenvalues of the matrix $a^{(1)}$ are equal to the quantities $\{2\,r_j(g)\}$, $j=1,\ldots,l$, where $\{r_j(g)\}$ are the coefficients in the expansion*

$$2\,\rho(g) = \sum_{\alpha\in R^+} g_\alpha\,\alpha = \sum_{j=1}^{l} r_j(g)\,\alpha_j. \tag{14}$$

Proof. Note that the function $U_1(q)$ is equal to $-\log|\Psi_0^\kappa|$, where Ψ_0^κ is the solution of the Schrödinger equation ([OP 1983])

$$\left(\frac{1}{2}\,\hat{p}^2 + U_2(q)\right)\Psi_0^\kappa = E_0(\kappa)\,\Psi_0^\kappa, \qquad g_\alpha^2 = \kappa_\alpha(\kappa_\alpha - 1),\; \hat{p}_j = -i\,\frac{\partial}{\partial q_j}, \tag{15}$$

$$\Psi_0^\kappa = \prod_{\alpha\in R^+} (\sin q_\alpha)^{\kappa_\alpha}. \tag{16}$$

The function $\Psi_0^\kappa(q)$ is positive inside the Weyl alcove and has a maximum at point $q=\bar{q}$. Let $\kappa_\alpha = s\,\tilde{\kappa}_\alpha$ and let us consider the limit at $s\to\infty$. Then the function

$$\Phi_0^{\tilde{\kappa}}(\xi) = \lim_{s\to\infty} \frac{\Psi_0^\kappa\,(\bar{q}+\xi/\sqrt{s})}{\Psi_0^\kappa(\bar{q})}, \qquad \kappa_\alpha = s\,\tilde{\kappa}_\alpha \tag{17}$$

takes the Gaussian form [3]

$$\Phi_0^{\tilde{\kappa}}(\xi) = \exp\left(-\frac{1}{2}\,a_{jk}\,\xi_j\xi_k\right). \tag{18}$$

[3] In other words at the limit $\kappa_\alpha \to \infty\,(s\to\infty)$ our quantum system become the oscillatory one for which the correspondence between classical and quantum systems is well known. Note that at this limit wave functions go to oscillatory ones. For example for the case of one degree of freedom we have

$$\lim_{s\to\infty} s^{(-n/2)} P_n^{(\alpha,\alpha)}(x/\sqrt{s}) = (2^n n!)^{-1} H_n(x),$$

where $P_n^{(\alpha,\beta)}(x)$ and $H_n(x)$ are Jacobi and Hermite polynomials correspondingly. For multidimensional case analogous formulae give the multivariable Hermite polynomials.

From other side, at this limit the Schrödinger equation

$$\left(\frac{1}{2}\,\hat{p}^2 + U_2(q)\right)\Psi_{\mathbf{m}} = E_{\mathbf{m}}\,\Psi_{\mathbf{m}}, \qquad \mathbf{m} = (m_1, \dots, m_l) \tag{19}$$

takes the form

$$\left(-\Delta_\xi + \frac{1}{2}\sum_{j,k=1}^{l} b_{j,k}\,\xi_j\xi_k \right)\Phi_{\mathbf{m}} = \tilde{E}_{\mathbf{m}}\,\Phi_{\mathbf{m}}, \tag{20}$$

where

$$b = a^2, \qquad \tilde{E}_{\mathbf{m}} = \tilde{E}_0 + \sum_{j=1}^{l}\omega_j\,m_j, \tag{21}$$

and ω_j are the eigenvalues of the matrix a. Note that the spectrum of the quantum problem

$$H\Psi_{\mathbf{m}} = E_{\mathbf{m}}\,\Psi_{\mathbf{m}}.$$

is given by the formula

$$E_{\mathbf{m}} = \sum_{i,j=1}^{l}(\lambda_i, \lambda_j)\,m_i\,m_j + 2\sum_{j=1}^{l}(\lambda_j, \rho(\kappa))\,m_j + E_0(\kappa). \tag{22}$$

And at $s \to \infty$ and we obtain

$$\tilde{E}_{\mathbf{m}} = \lim_{s\to\infty}\left(s^{-1}\,E_{\mathbf{m}}(\kappa)\right) = \tilde{E}_0 + 2\sum_{j=1}^{l}(\lambda_j, \rho(\tilde{\kappa}))\,m_j. \tag{23}$$

Comparing (21) and (23), we obtain the main formula

$$\left(\omega_j^{(1)}\right)^2 = 2\,r_j(\tilde{\kappa}) = 2\,(\lambda_j, \rho(\tilde{\kappa})). \tag{24}$$

So, the classical result is obtained from the quantum result at $s \to \infty$.

In conclusion, we give the explicit expression for the maximum value of the function $\Psi_0^{\kappa}(q)$. This formula was given by I. Macdonald as an conjecture [Ma 1982] that was proved later by E. Opdam [Op 1989]. For the simplest case ($\kappa_\alpha = 1$ for all α), it has the form

$$\left|\Psi_0^1(\bar{q})\right|^2 = \left\{\prod_{\alpha\in R^+}(\sin q_\alpha)\right\}^2 = \frac{|W|}{2^{|R|}}\prod_{j=1}^{l}\left(\frac{d_j}{d_j - 1}\right)^{d_j - 1}, \tag{25}$$

where $|W|$ is the order of W, and $|R|$ is the number of roots.

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