ON $\mathbb{C}$-FIBRATIONS OVER PROJECTIVE CURVES

TATIANA BANDMAN AND LEONID MAKAR-LIMANOV

1. Introduction

Rational affine surfaces, i.e., affine surfaces birationally equivalent to a plane is an interesting and rich class of surfaces worthy of investigation. One of the tools which was used for a classification of such surfaces is so called ML invariant of a surface which is a characteristic subring of the ring of regular functions of the surface. It consists of regular functions which are invariant under all possible $\mathbb{C}^+$-actions on the surface. Any $\mathbb{C}^+$-action on a surface induces a $\mathbb{C}$-fibration over an affine curve. The invariant answers to the question how many fibrations of this kind exist for the surface.

Naturally enough the fibrations over projective base are less studied (([DR], [GM] [Za] [KiKo] [GMMR]).

The goal of this paper is to present a modified version of ML invariant which should take into account projective rulings also and allow further stratification of the rational surfaces. Of course, it is much easier to introduce an invariant than to be able to compute it in a particular case. Say, we still do not know how to compute ML invariant for a given surface though some technique is available (see [KML1],[KML2]).

Unfortunately computation of the modified version of the invariant is even more involved. Nevertheless we present a non-trivial example where we were able to finish the computation. Hopefully further techniques will be developed in due course.

Let us recall the definition of the ML invariant. Let R be the ring of regular functions of an affine algebraic variety V. Let $\mathrm{LND}(R)$ be the set of all locally nilpotent derivations (lnd) of R. Then $ML(R) = ML(V) = \bigcap_{\partial\in \mathrm{LND}(R)} R^{\partial}$ where R^{∂} stands for the kernel of ∂.

2000 *Mathematics Subject Classification.* Primary 13B10, 14E09; Secondary 14J26, 14J50, 14L30, 14D25, 16W50.

Key words and phrases. Affine varieties, $\mathbb{C}^+$-actions, locally-nilpotent derivations.

The first author is partially supported by the Ministry of Absorption (Israel), the Israeli Science Foundation (Israeli Academy of Sciences, Center of Excellence Program), the Minerva Foundation (Emmy Noether Research Institute of Mathematics).

The second author is partially supported by an NSA grant, Max-Planck Institute of Mathematics, and Weizmann Institute of Science.

Here is the modified version. Let $F(R) = Frac(R)$ be the field of fractions of R. Take an element $f \in F(R)$ and consider the ring $R[f] \subset F(R)$, i.e., extension of R by the polynomial functions of f. Call $\partial \in \mathrm{DER}(F(R))$ a generalized locally nilpotent derivation (*glnd*) of R if it is locally nilpotent on $R[f]$ and $\partial(f) = 0$. Define $GML(R) = \bigcap_{\partial \in \mathrm{GLND}(R)} F(R)^\partial$ where $\mathrm{GLND}(R)$ is the set of all generalized locally nilpotent derivations of R.

If $R = \mathcal{O}(S)$, the ring of regular functions on a surface S, we will denote $F(R)$ by $F(S)$. Of course, $F(S)^\partial$ is the algebraic closure of $\mathbb{C}(f)$ in $F(S)$ when $\partial \in GLND(R)$.

Therefore $GML(R)$ is

-either $F(R)$ when the only element of $\mathrm{GLND}(R)$ is zero derivation,

-or a field of rational functions of a curve C when non-zero *lnd* are possible on $R[f]$ only for $f \in \mathbb{C}(u)$ where u is a fixed element of $F(R)$,

-or $\mathbb{C}$ when there are at least two substantially different possible choices of f.

If S is rational then $C \cong \mathbb{P}^1$.

Geometrically speaking if $R = O(S)$ where S is a surface, a non-zero *glnd* of R which is not equivalent to an *lnd* of R corresponds to $\mathbb{C}$-fibration of S over a projective curve. Therefore S contains a cylinder like subset. By a result of M. Miyanishi and T. Sugie ([MiSu]) it is equivalent to $\overline{k} = -\infty$ where $\overline{k}$ is the logarithmic Kodaira dimension of S. We can think about $\mathrm{LND}(R)$ as a subset of $\mathrm{GLND}(R)$ (just take $f = 1$). So in the case of surfaces the logarithmic Kodaira dimension of S is $-\infty$ if and only if $\mathrm{GLND}(R)$ contains a non-zero derivation.

In Section 2 we give some definitions and demonstrate the first properties of GML**.**

In Section 3 we compute invariant GML for a "rigid" surface: smooth affine rational surface admitting no $\mathbb{C}^+$-actions. In Section 4 we apply GML invariant to computing ML invariant of some threefolds.

It appears that GML invariant of a surface S is closely connected to ML invariant of the line bundles over S. Namely, let $\mathcal{L} = (L, \pi, S)$ be a line bundle over S and $\partial \in LND(\mathcal{O}(L))$. Then there exists $\partial' \in GML(S)$ such that $\partial f = 0$ for any $f \in \pi^*(F(S)^{\partial'})$ (see Proposition 1). On the other hand for any $\partial \in GML(S)$ there is a line bundle $\mathcal{L} = (L, \pi, S)$ and a *lnd* $\partial' \in LND(\mathcal{O}(L))$ such that $\partial' f = 0$ for any $f \in \pi^*(F(S)^\partial)$ (Lemma 10).

This is why the GML invariant is useful for understanding whether ML-invariant of a surface is stable under reasonable geometric constructions. In our previous work the cylinder over a surface played the role of a "reasonable" geometric construction. Here we are replacing the cylinder by an algebraic line bundle.

It is not always possible to generalize the results known for the cylinders to this setting. E. g. for "rigid" surfaces

$$ML(S \times \mathbb{C}) = \mathcal{O}(S),$$

but it is not valid for some non-trivial line bundles, because $GML(S)$ is not trivial.

In Section 4 , Corollary 3 we describe the line bundles for which the equality nevertheless is true.

Below we denote by $\mathbb{C}^n_{x_1,\dots,x_n}$ the n - dimensional complex affine space with coordinates $x_1, \dots, x_n$, and for an irreducible subvariety C of codimension 1 we denote by $[C]$ the effective divisor with this support and coefficient 1; $\mathrm{supp}(G)$ and $Cl(G)$ stand for support and class of divisor G respectively. $(C_1, C_2) = ([C_1], [C_2])$ is the intersection number of two curves (resp. divisors). $\overline{A}$ stands for a closure of A. For a rational function f we denote by $(f), (f)_0, (f)_\infty$ divisors of f, of its zeros and of its poles respectively. If $\mathcal{L} = (L, \pi, S)$ is a line bundle over a smooth surface S, then D_L stands for the Weil divisor (since S is smooth we do not distinguish between Weil and Cartier divisors) on S, associated to $\mathcal{L}$. Two $\mathbb{C}^+$-actions are equivalent if they have the same generic orbit.

For a ring R we denote by $\mathrm{DER}(R)$ the set of derivations on R, by $\mathrm{LND}(R) \subset \mathrm{DER}(R)$ the set of locally nilpotent derivations, by $F(R)$ the field of fractions of R. For a derivation $\partial \in \mathrm{DER}(R)$ we denote by R^∂ and $F(R)^\partial$ the kernel of ∂ in R and $F(R)$ respectively.

The main information on properties of $\mathrm{LND}(R)$ may be found in [KML2]. Our Encyclopedia on affine surfaces with fibrations is the book of M. Miyanishi [Mi2].

2. Properties of GML

Let S be a smooth affine complex surface, $R = \mathcal{O}(S)$ be the ring of regular functions on S, and $F(S) = \mathrm{Frac}(\mathcal{O}(S))$ stand for the field of fractions of $\mathcal{O}(S)$.

Definition 1. The derivation $\partial \in \mathrm{DER}(R)$ is a generalized locally nilpotent derivation (*glnd*) if there is $f \in F(S)$, such that $\partial \in \mathrm{LND}(R[f])$ and $\partial(f) = 0$. The set of all *glnd*'s for the ring R is denoted by $\mathrm{GLND}(R)$

Definition 2. Two elements ∂_1 and ∂_2 in $\mathrm{GLND}(R)$ are equivalent, if $F(R)^{\partial_1} = F(R)^{\partial_2}$.

Definition 3. The invariant $GML(R)$ (or $GML(S)$ if $R = \mathcal{O}(S)$) is the field $\bigcap\limits_{\partial \in \mathrm{GLND}(R)} F^\partial$.

Definition 4. A smooth affine rational surface S is rigid, if log-Kodaira dimension $\overline{k}(S) = -\infty$ and $ML(S) = \mathcal{O}(S)$.

The invariant $GML(S)$ has the following properties:

Property 1. *$\overline{k}(S) = -\infty$ if and only if $GML(S) \neq F(S)$.*

Proof. Indeed, by definition, $GML(S) \neq F(S)$ is equivalent to the existence of a cylinder-like subset in S, which is equivalent ([MiSu]) to $\overline{k}(S) = -\infty$. $\square$

Property 2. *If there exists a Zariski open affine subset $U \subseteq S$ such that $ML(U) = \mathbb{C}$, then $GML(S) = \mathbb{C}$.*

Proof. Let $\varphi_1 : U \to \mathbb{C}$ and $\varphi_2 : U \to \mathbb{C}$ be two $\mathbb{C}$ fibrations on U. Let $\overline{S}$ be a closure of S, such that the rational extensions $\overline{\varphi}_1 : \overline{S} \to \mathbb{P}^1$ and $\overline{\varphi}_2 : \overline{S} \to \mathbb{P}^1$ of φ_1 and φ_2, respectively, are regular. Let $S = \overline{S} - D$, $U = \overline{S} - (D \cup D')$, $D = \bigcup_{i=1}^{n} C_i$, $D' = \bigcup_{i=1}^{n} B_i$, where C_i and B_i are reduced components of D and D' respectively. All of them are smooth and rational (see [Mi2], ch. III, Lemma 1.4.1).

We denote by D_1^∞, D_2^∞ such components in $D \cup D'$ that $\overline{\varphi}_i : D_i^\infty \to \mathbb{P}^1$ is an isomorphism, $i = 1, 2$. If $D_i^\infty \subset D'$ for some i then for a generic $a \in \mathbb{P}^1$ the intersection $\overline{\varphi}_i^{-1}(a) \cap (D \cup D') \in D' - D$, since $\overline{\varphi}_i^{-1}(a)$ has only a single point in $\overline{S} - U$. It follows that a compact curve $\overline{\varphi}_i^{-1}(a) \subset S$, thus S is not affine. Hence $D_i^\infty \subset D$ for $i = 1, 2$ and $\overline{\varphi}_i \big|_{B_j} =$ const for every $j = 1, \dots, n$. Thus $\overline{\varphi}_i \big|_S : S \to \mathbb{P}^1$ are nonequivalent $\mathbb{C}$-fibrations, and $GML(S) = \mathbb{C}$. $\square$

Property 3. *(See* [GMMR], [Za]*.) For a $\mathbb{Q}$- homology plane S*

$$GML(S) = Frac(ML(S)).$$

Property 4. *(see* [GM]*, Th. 4.1) If there exist a $\mathbb{C}$-fibration $f : S \to B$ and the curve $B \cong \mathbb{C}$ ($B \cong \mathbb{P}^1$), all the fibers of f are irreducible and there are at least two (resp. three) multiple fibers, then $GML(S) = \mathbb{C}(f)$.*

The next Lemma is a simple fact about locally-nilpotent derivations, which was proved in another form in [BML1]. We will need it further.

Lemma 1. *Let R be a finitely generated ring and $r \in R$. Assume that there is a non-zero lnd ∂ on $R[r^{-1}]$. Then there is a non-zero lnd on R.*

Proof. Indeed, let $r_1, \dots, r_n$ be a generating set of R. Then $\partial(r_i) = p_i r^{-d_i}$ where $p_i \in R$ and d_i is a natural number. It is clear that $\partial(r) = 0$ since both r and r^{-1} are in $R[r^{-1}]$.

Take m which is larger then all d_i. Then $\epsilon = r^m \partial$ is also an *lnd* on $R[r^{-1}]$. Since $\epsilon(r_i) \in R$ for all i the derivation ϵ is a derivation of R. So it is an *lnd* of R. $\square$

Remark 1. Same consideration works if there is a non-zero lnd ∂ on $R(r)$. Again $\partial(r)=0$ and instead of r^m take a common denominator of all $\partial(r_i)$ which is a polynomial in r.

3. Example

In this section we compute $GML(S)$ for a surface $S\subset\mathbb{C}^7$, introduced in [BML2] (example 3) and defined by

$$uv = z(z-1) \tag{1}$$

$$v^2z = uw \tag{2}$$

$$z^2(w-1) = xu^2 \tag{3}$$

$$u^2(z-1) = tv \tag{4}$$

$$(z-1)^2(t-1) = yv^2 \tag{5}$$

$$u^2v^2 = wt \tag{6}$$

$$yz^2 = u^2(t-1) \tag{7}$$

$$x(z-1)^2 = v^2(w-1) \tag{8}$$

$$v^4x = w^2(w-1) \tag{9}$$

$$u^4y = t^2(t-1) \tag{10}$$

$$v^3 = (z-1)w \tag{11}$$

$$u^3 = tz \tag{12}$$

$$xy = (w-1)(t-1). \tag{13}$$

Equations (6)-(13) are the consequences of the equations (1)-(5).

The surface is smooth, because the rank of the Jacobi matrix of equations (1)-(13) is maximal everywhere.

The surface S has the following properties

Property 5. (1) $\overline{\kappa}(S)=-\infty$
(2) $R=ML(S)=\mathcal{O}(S)$
(3) $\pi_1(S)=\mathbb{Z}/2\mathbb{Z}$
(4) $Pic(S)=\mathbb{Z}+\mathbb{Z}/2\mathbb{Z}$
(5) *It admits an automorphism* $a:(u,v,z,t,w,x,y)\to(-v,-u,1-z,w,t,y,x)$;

(6) *Morphism* $b : S \to \mathbb{P}^1$, *defined as* $b(s) = \frac{z}{u}$ *for a point* $s \in S$, *is a* $\mathbb{C}$*-fibration. All the fibers of this fibration are isomorphic to* $\mathbb{C}^1$. *The fibers* $B_0 = b^{-1}(0)$ *and* $B_\infty = b^{-1}(\infty)$ *have multiplicity* 2.

(7) *The following relations are valid:*

$$z = ub,\ v = b(ub-1),\ w = b^3(ub-1)^2,$$
$$x = b^2(b^3(ub-1)^2 - 1),\ t = \frac{u^2}{b},\ y = \frac{u^2-b}{b^3}.$$

(8) *The surfaces* $S_0 = S - B_\infty$ *and* $S_\infty = S - B_0$ *are isomorphic to the hypersurface* $S' = \{\beta^3\gamma = \alpha^2 - \beta\}$. *Isomorphisms* $\tau_0 : S_0 \to S'$ *and* $\tau_\infty : S_\infty \to S'$ *are defined respectively by* $\beta = b, \alpha = u, \gamma = y$ *and* $\beta = 1/b, \alpha = v, \gamma = x$. *Indeed,* $R[b] = \mathbb{C}[u, b, \frac{u^2-b}{b^3}]$ *and* $R[1/b] = \mathbb{C}[b(ub-1), 1/b, b^2(b^3(ub-1)^2-1)]$.

Theorem 1. $GML(S) = \mathbb{C}(b)$.

Proof of Theorem 1. The proof is rather long but the main idea is as follows: if $GML(S) \neq \mathbb{C}(b)$ then there exists a $\mathbb{C}$-fibration $\varphi \in F(S)$, such that $\mathrm{LND}(R[\varphi]) \neq \{0\}$, where $R = \mathcal{O}(S)$ and φ is algebraically independent with b. We introduce some weights for the generators u, v, z, t, w, x, y and consider the corresponding graded algebra $\widehat{R[\varphi]}$ (since φ is a rational function the weight of φ will be also defined). Then we will show that for these weights $\mathrm{LND}(\hat{R}) = \{0\}$, where $\hat{R}$ is a corresponding to R graded algebra. Then we will show that the leading forms of the numerator and the denominator of φ are algebraically dependent, and finally that $\mathrm{LND}(\widehat{R[\varphi]}) = \{0\}$. This will bring us to a contradiction because (as it was shown in [KML1], see also [KML2]) $\mathrm{LND}(R[\varphi]) \neq \{0\}$ implies $\mathrm{LND}(\widehat{R[\varphi]}) \neq \{0\}$.

Let us specify the weights (ω) by $\omega(u) = 4$ and $\omega(b) = -1+\rho$ where $\rho << 1$ is an irrational number. Then $\omega(z) = 3+\rho$, $\omega(v) = 2+2\rho$, $\omega(w) = 3+5\rho$, $\omega(x) = 1+7\rho$, $\omega(t) = 9-\rho$, $\omega(y) = 11-3\rho$.

Lemma 2. *LND*$(\hat{R}) = \{0\}$**.**

Proof of Lemma 2. Let $\partial \in \mathrm{LND}(\hat{R})$ be a non-zero derivation.

The system

$$uv = z^2,\ v^2z = uw,\ z^2w = xu^2,\ u^2z = tv \tag{14}$$

$$z^2t = yv^2,\ u^2v^2 = wt,\ yz^2 = u^2t,\ xz^2 = v^2w \tag{15}$$

$$v^4x = w^3,\ u^4y = t^3,\ v^3 = zw,\ u^3 = tz,\ xy = wt. \tag{16}$$

defines a reduced (the rank of Jacobian matrix is maximal in Zariski open subset $\{uvz \neq 0\}$) and irreducible surface. The last follows from the fact, that each fiber of a rational function $k = \frac{u}{z} = \frac{z}{v}$ is irreducible.

According to [KML2] (Lemma 6.2) the system (14),(15)(16) defines $\hat{R}$.

So

$$\hat{R} = \mathbb{C}[u, z, u^{-1}z^2, u^{-3}z^5, u^{-5}z^7, u^3z^{-1}, u^5z^{-3}].$$

We want to show that this ring does not have a non-zero locally nilpotent derivation. With our choice of weights we will get that the induced non-zero locally nilpotent derivation $\hat{\partial}$ also belongs to $\mathrm{LND}(\hat{R})$ since $\hat{R}$ is a graded algebra relative to this weights. The weights are not comesurable, that is why both $\hat{\partial}(u)$ and $\hat{\partial}(z)$ are monomials and $\hat{\partial}$ of any monomial is a monomial.

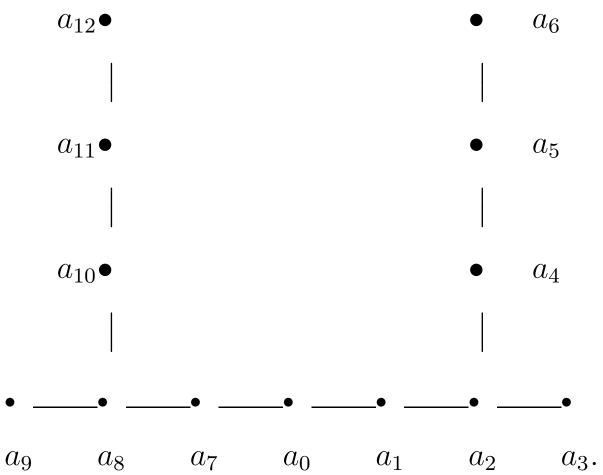
We present monomials in u, z of $\hat{R}$ by points of the two-dimensional integer lattice. The set A of points $(1, 0), (0, 1), (-1, 2), (-3, 5), (-5, 7), (3, -1), (5, -3)$ which correspond to the generating set of $\hat{R}$ is located on a plane with coordinates (r, s) inside the angle between lines $L_1 = \{7r + 5s = 0\}$ and $L_1 = \{5s + 3r = 0\}$ containing the first quadrant. The points $(-5, 7), (5, -3)$ belong to L_1, L_2 respectively. There is an involution $\hat{a} : (u, z) \to (u^{-1}z^2, z)$ of the ring $\hat{R}$, which changes roles of lines L_1 and L_2.

Since $\hat{\partial}$ is locally nilpotent and non-zero it implies that there is a monomial $f \in ker(\hat{\partial})\setminus\mathbb{C}$. This means that the $ker(\hat{\partial})$ is generated by a monomial, say f. Now, let us take a monomial g for which $\hat{\partial}(g) \neq 0$ and $\hat{\partial}^2(g) = 0$. It is known (see [KML2]), that $\hat{R} \subset \mathbb{C}(f)[g]$. The action of $\hat{\partial}$ is represented on the plane (r, s) as the translation by the vector, corresponding to g toward the line passing through the point corresponding to f. Since both $f, g \in \hat{R}$ it implies that f must be represented by a point of the boundary line, i.e. L_1 or L_2. Because of involution we may assume that $\hat{\partial}(u^5z^{-3}) = 0$. Let deg be the degree function induced by $\hat{\partial}$ ([FLN]). Then $5 \deg u - 3 \deg z = 0$, i.e. $\deg u = 3n, \deg z = 5n$ for some $n \in \mathbb{N}$. Since $\hat{\partial}(u)$ is a monomial, n should divide $3n - 1$, so $n = 1$.

Now, $\deg(g) = 1$. So one of the monomials in $\hat{R}$ has degree 1. But $\deg u = 3, \deg z = 5, \deg u^{-1}z^2 = 7, \deg u^{-3}z^5 = 16, \deg u^{-5}z^7 = 20, \deg u^3z^{-1} = 4, \deg u^5z^{-3} = 0$ and since g is a product of these monomials it cannot have degree equal to 1. □

The next step is computation of the leading form $\hat{\varphi}$ of the function φ. We need several Lemmas. We will denote by the same letter the function u on S, its extension to $\overline{S}$ and its lift to any blow-up $\tilde{S}$ of S.

Lemma 3. *The map b can be extended to a morphism $\overline{b} : \overline{S} \to \mathbb{P}^1$ to the closure $\overline{S}$ such that the divisor $D = \overline{S} - S$ has the following graph:*



where vertex a_i, $0 \le i \le 12$ represent a component A_i of divisor D. Moreover, they enjoy

Property 6. (1) $A_i^2 = -2$ *for* $i > 0$

(2) $A_6 \cap B_0 \neq \emptyset, \quad A_{12} \cap B_\infty \neq \emptyset$

(3) $F_0 = \overline{b}^{-1}(0) = A_1 + A_3 + 2A_2 + 2A_4 + 2A_5 + 2A_6 + 2B_0, \quad F_i = \overline{b}^{-1}(\infty) = A_7 + A_9 + $
$+ 2A_8 + 2A_{10} + 2A_{12} + 2A_{11} + 2B_\infty.$

(4) $u \,\Big|_{\bigcup_2^6 A_i} = u \,\Big|_{B_0} = u \,\Big|_{B_\infty} = 0,\ u \,\Big|_{A_1}$ *is linear;*
$v \,\Big|_{\bigcup_8^{12} A_i} = v \,\Big|_{B_0} = v \,\Big|_{B_\infty} = 0,\ v \,\Big|_{A_7}$ *is linear.*

Proof of Lemma 3. Due to Property 5(5) and Property 5(8) it is sufficient to analyze the structure of the closure of the surface $S_0 = S - B_\infty$ and to proof only Property 6, (1)-(4). The detailed description of the graph of the divisor $D_0 = \overline{S_0} - S_0$ is given in [MiMa] and [TtD], together with the proof of Property 6, (1)-(3).

In order to obtain S_0 you have to consider the open set $U \cong \mathbb{C}^2_{b,u}$ of a Hirzebruch surface and to blow-up several times the point $b = 0, u = 0$ of the fiber $B = \{b = 0\}$. That is why Property 6(4) is valid: $u = 0$ on all exceptional components $A_i, 1 \le i \le 6$ of this process and

u is linear along the proper transform A_1 of B. The equality $u\,\big|_{B_\infty} = 0$ follows from equation (1) in the definition of the surface. $\square$

Any non-equivalent to b fibration $\varphi : S \to \mathbb{P}^1, \varphi \in F(S)$ has the following

Property 7. (1) *every fiber* $\Phi_q = \overset{-1}{\varphi}(q)$ *is isomorphic to* $\mathbb{C}$ *(since* rank $Pic_{\mathbb{Q}}(S) = 1$*)* *(*[Mi2]*, Ch.3, 2.4.3.1 p).*

(2) *There are precisely two values* $q_0, q_1 \in \mathbb{P}^1$*, such that the fibers* Φ_{q_0}*,* Φ_{q_1} *have multiplicities* 2 *; all other fibers are of multiplicities* 1 *(*[Fu]*, 4.19. 4.20, 5.9);*

(3) φ *is not a function of* b *(since they define non-equivalent fibrations);*

(4) *there is* $\partial \in LND(R[\varphi])$ *such that* $\partial \neq \{0\}$ *and* $\partial(\varphi) = 0$*.*

Lemma 4. *There is no* $p \in \mathbb{P}^1$ *such that* φ *is constant along the fiber* $B_p = \overset{-1}{b}(p)$*.*

Proof of Lemma 4 . If such p exists, then the affine surface $S'' = S - B_p$ admits two non-equivalent $\mathbb{C}$ fibrations over $\mathbb{C}$, i.e. $ML(S'') = \mathbb{C}$.

If $p \neq 0, \infty$, then $b\,\big|_{S''}$ has two singular fibers, thus $ML(S'') \neq \mathbb{C}$ ([Giz],[Ber]).

If $p = 0$ or ∞, then $S'' \cong S'$ (see Property 5(8)). But $ML(S'') = \mathbb{C}[\beta] \neq \mathbb{C}$ as well ([MiMa], Theorem 2.3).

Thus, both cases are impossible. $\square$

Lemma 5. *The extension* $\overline{\varphi}$ *of rational function* φ *to* $\overline{S}$ *is not regular and has only one singular point.*

Proof of Lemma 5. Assume first that $\overline{\varphi}$ is morphism of $\overline{S}$ onto $\mathbb{P}^1$.

Then for one of the components A_i of divisor $D = \overline{S} - S$ (see Lemma 3) the restriction $\varphi\,\big|_{A_i}: A_i \to \mathbb{P}^1$ is an isomorphism. Due to the existence of the automorphism a of S (see Property 5(5)), we may assume that $0 \leq i \leq 6$.

Case 1. $i = 0$. Then the generic fiber $\overline{\Phi}_q = \overline{\varphi}^{-1}(q) = \overline{\varphi^{-1}(q)} \cong \mathbb{P}^1$ of $\overline{\varphi}$ intersects A_0 transversally. Since the function u is linear along the generic fiber ([BML2], Ex. 3) it has a simple pole along A_0. Since this is the only puncture of Φ_q and $u \in \mathcal{O}(S)$, the restriction $u\,\big|_{\overline{\Phi}_q}$ has the only simple pole at the point $A_0 \cup \overline{\Phi}_q$. But it has zero at every point of intersections $\overline{\Phi}_q \cap B_0 \neq \emptyset$ and $\overline{\Phi}_q \cap B_\infty \neq \emptyset$. Hence the number of zeros is at least two. The contradiction shows that $i \neq 0$.

Case 2. $0 < i \leq 6$. In this case $\overline{\Phi}_q$ intersects D at a point of A_i only and for a general fiber u is finite at the intersection point (see Property 6(4)). Thus it is finite everywhere in $\overline{\Phi}_q$, hence constant. Since the curve $\{u = const\} \not\cong \mathbb{C}$ in S it is impossible.

Thus $\overline{\varphi}$ is regular on S but is not a morphism of $\overline{S}$, i.e. the singular point of $\overline{\varphi}$ is at the puncture of the generic fiber Φ_q (or, the same, at the intersection of generic fibers $\overline{\Phi}_q$). Since Φ_q has only one puncture, there is only one singular point $s \in \overline{S}$. $\square$

Let $\overline{b}(s) = p_0$. We may assume that $p_0 \neq 0$ (due to the involution a we may always change the roles of 0 and ∞.)

Let $\pi : \tilde{S} \to \overline{S}$ be a resolution of $\overline{\varphi}$, i.e. π is an isomorphism outside $\overset{-1}{\pi}(s)$. Let $\overset{-1}{\pi}(s) = \bigcup_0^k E_j$, where E_j are exceptional components in $\tilde{D} = \tilde{S} - S$, let $\tilde{\varphi} = \overline{\varphi} \circ \pi$, $\tilde{b} = \overline{b} \circ \pi$, let $\tilde{A}_i$ be proper transforms of A_i and let $\tilde{\varphi} \mid_{E_0}$ be an isomorphism.Then $\tilde{\varphi}$ has to be constant along each connected component of $\tilde{D} - E_0$.

Let $\tilde{\Phi}_q = \tilde{\varphi}^{-1}(q)$ and $\tilde{B}_q = \tilde{b}^{-1}(q)$ for a point $q \in \mathbb{P}^1$. As above, $\overline{\Phi}_q = \overline{\varphi^{-1}(q)}$ and $\tilde{\Phi}_q = \overline{\Phi}_q$ for the generic q.

Consider the connected component R of $\tilde{D} - E_0$ containing the proper transform $\tilde{A}_0$ of A_0.

If $\tilde{\varphi} \mid_R = \kappa \in \mathbb{P}^1$, then $\tilde{\Phi}_\kappa = \tilde{\varphi}^{-1}(\kappa) = R \cup C$, where $C = \overline{\Phi}_\kappa$ is the closure of Φ_κ (this means that C is the only component of $\tilde{\Phi}$ that intersects S).

Lemma 6. *$\tilde{b}(s_1) \neq 0$, where $s_1 = R \cap C$.*

Proof of Lemma 6. Assume that $s_1 \in \tilde{b}^{-1}(0)$. We remind that π is isomorphism in the neighborhood of $\tilde{b}^{-1}(0)$. The point s_1 cannot be the intersection point of $\tilde{A}_0$ and $\tilde{b}^{-1}(0)$, since three components $(C, \tilde{A}_0, \tilde{A}_1)$ of the fiber of $\tilde{\varphi}$ cannot intersect at a point ([Mi2], Ch.3 1.4.1). Thus, $s_1 \in (\bigcup_1^6 \tilde{A}_i) - (\tilde{A}_0 \cap \tilde{A}_1)$ and $u(s_1)$ is finite. But then u is finite at every point of C, which is impossible. $\square$

Lemma 7. *The fiber Φ_κ has multiplicity 2 in fibration φ.*

Proof of Lemma 7. Let $\tilde{\Phi}_\kappa = \bigcup_0^6 \tilde{A}_i \cup C \cup R_1$, where R_1 is the union of other components of R, and let the corresponding divisor G of the fiber $\tilde{\Phi}_\kappa = \tilde{\varphi}^{-1}(\kappa)$ be $G = \sum_0^6 k_i \tilde{A}_i + \epsilon C + H$, $(\operatorname{supp} H = R_1)$.

We want to prove that $\epsilon \neq 1$. We have

$(\tilde{A}_6, G) = 0$ implies $-2k_6 + k_5 = 0$,

$(\tilde{A}_5, G) = 0$ implies $-2k_5 + k_6 + k_4 = 0$,

$(\tilde{A}_4, G) = 0$ implies $-2k_4 + k_5 + k_2 = 0$,

$(\tilde{A}_3, G) = 0$ implies $-2k_3 + k_2 = 0$,

$(\tilde{A}_2, G) = 0$ implies $-2k_2 + k_3 + k_4 + k_1 = 0$,

$(\tilde{A}_1, G) = 0$ implies $-2k_1 + k_2 + k_0 = 0$,

$(\tilde{A}_0, G) = 0$ implies $k_0(A_0^2) + k_1 + (A_0, \epsilon C + H) = 0$.

It follows that $k_1 = \frac{3}{2}k_0$ and $k_0(A_0^2 + \frac{3}{2}) + (A_0, \epsilon C + H) = 0$. Since $(A_0, \epsilon C + H) > 0$, and $k_0 > 0$ we have $A_0^2 \neq -1$. Along all components of G except A_0 and C the map $\tilde{b}$ is constant. If any of them were a (-1) curve, it would be possible to contract it. The new divisor still would have normal crossings, because it was obtained by blow-up process from the normal crossing divisor. Hence we may assume that the only (-1) curve in G is C. But then it cannot be of multiplicity 1 ([Mi2], Ch.3 1.4.1). According to Property 7 (2), multiplicity should be 2. □

Lemma 8. *$\hat{\varphi} = \hat{y}^k$ for some $k \in \mathbb{Z}$.*

Proof of Lemma 8. By bilinear transformation of φ we may always achieve that $q_0 = \kappa = 0, q_1 = \infty$ (see Property 7 (2)).

According to Lemma 3 and Property 5 (8)

$$S - B_\infty = S_0 = \{b^3 y = u^2 - b\}. \tag{17}$$

Since $\mathrm{Pic}(S_0) = \mathbb{Z}/2\mathbb{Z}$, divisors $2[\Phi_0 \cap S_0] \cong 0$ and $2[\Phi_\infty \cap S_0] \cong 0$. This implies that there exist polynomials $P(u, b, y)$, and $Q(u, b, y)$, such that

$$2[\Phi_0 \cap S_0] = (P(u, b, y))_0 \cap S_0,$$

$$2[\Phi_\infty \cap S_0] = (Q(u, b, y))_0 \cap S_0$$

and

$$\varphi \Big|_{S_0} = \frac{P(u, b, y)}{Q(u, b, y)}.$$

On the other hand, $\varphi \Big|_{B_\infty} \neq const$. It follows, that in S

$$\varphi = \frac{P(u, b, y)}{Q(u, b, y)}. \tag{18}$$

We may substitute u^2 by $b^3 y - b$ into polynomials P and Q and obtain

$$P(u,b,y) = P_1(y) + uP_2(y,b) + bP_3(y,b), \tag{19}$$

$$Q(u,b,y) = Q_1(y) + uQ_2(y,b) + bQ_3(y,b), \tag{20}$$

Along B_0 function y is linear, $u = b = 0$, along the generic fiber B_p we have $b = p$, u is linear, $y = \frac{u^2-p}{p^3}$.

For two generic fibers $B_p = b^{-1}(p) \subset S$ and $\Phi_q = \varphi^{-1}(p) \subset S$ we denote by $|B_p, \Phi_q|$ the number of points in their intersection $B_p \cap \Phi_q$ counted with multiplicities. We consider B_p and Φ_q as reduced curves isomorphic to $\mathbb{C}$. Recall that B_p and $\overline{\Phi}_q$ are the closures in $\tilde{S}$ of B_p and Φ_q respectively.

Let for two generic points $p, q \in \mathbb{P}^1$

$$|B_p, \Phi_q| = (\overline{B}_p, \overline{\Phi}_q) = N. \tag{21}$$

For $q \neq 0, \infty$

$$|B_0, \Phi_q| = (\overline{B}_0, \overline{\Phi}_q) = N/2. \tag{22}$$

Let r be the multiplicity of zero of function $\tilde{\varphi}$ along $\tilde{A}_0$. For a generic p

$$|B_p, \Phi_0| = (\overline{B}_p, \overline{\Phi}_0) = (N-r)/2 \tag{23}$$

$$|B_p, \Phi_\infty| = (\overline{B}_p, \overline{\Phi}_\infty) = N/2 \tag{24}$$

and

$$|B_0, \Phi_\infty| = (\overline{B}_0, \overline{\Phi}_\infty) = N/4. \tag{25}$$

In order to compute $|B_0, \Phi_0|$ we denote by $B = 2B_0 + \sum_1^6 n_i\tilde{A}_i$ the divisor of zero fiber $\tilde{b}^{-1}(0)$. Due to Lemma 6 $\tilde{B}_0$ intersects $\overline{\Phi}_0$ only inside the surface S, thus for the generic p

$$|B_0, \Phi_0| = (\overline{B}_0, \overline{\Phi}_0) = \frac{1}{2}(B, \overline{\Phi}_0) = \frac{1}{2}(\overline{B}_p, \overline{\Phi}_0) = (N-r)/4. \tag{26}$$

Combining (23), (26), (19), we get

$$degP_1(y) = |B_0, \Phi_0| = (N-r)/4 \tag{27}$$

$$2deg_yP + deg_uP = |B_z, \Phi_0| = (N-r)/2 \tag{28}$$

Here deg_sH stand for degree of polynomial H relative to indefinite s.

Combining (24), (25), (20), we get

$$degQ_1(y) = |B_0, \Phi_\infty| = N/4 \tag{29}$$

$$2deg_yQ + deg_uQ = |B_p, \Phi_\infty| = N/2 \tag{30}$$

For our weights $\omega(u) = 1, \omega(b) = -1 + \rho, \omega(y) = 11 - 3\rho$, it gives
$\hat{P} = \hat{P}_1 = \hat{y}^{\frac{N-r}{4}}, \ \hat{Q} = \hat{Q}_1 = \hat{y}^{\frac{N}{4}}, \ \hat{\varphi} = \hat{y}^{-\frac{r}{4}}$ □

Now we can prove the Theorem. Were there a fibration φ, there would be a non-zero locally nilpotent derivation on $\widehat{R[\varphi]}$. Since the system which defines $\hat{R}$ and $\hat{\varphi} y^{\frac{r}{4}} = 1$ again defines a reduced irreducible surface we can conclude that $\widehat{R[\varphi]} = \hat{R}[\hat{\varphi}]$. But that is impossible due to Lemma 2 and Lemma 1. □

Remark 2. The curve $\{y = 0\} \subset S$ contains two rational curves. As it was proven, none of them may be included into a $\mathbb{C}$ fibration (compare with [GMMR], where such curves are called anomalous).

Conjecture 1. *Let S be a rigid surface which admits a morphism $b : S \to \mathbb{P}^1$ such that the divisor at infinity built as in Lemma 3 has the graph which is different from the graph in the Lemma only by the number of vertices in the vertical components of the graph. We would like to conjecture that then Theorem 1 remains valid.*

4. ML invariant of a line bundle over a rigid surface

In this section we establish a connection between the GML-invariant of a surface and the ML-invariant of the total space of a line bundle over the surface. The computation of ML-invariant is often a very involved matter even for surfaces and cylinders over surfaces. That is why we find it interesting to compute the invariant for threefolds of another type. We consider line bundles over rigid surfaces. The information on $GML(S)$ appears to be very helpful.

Let us remind some notions and notations which we use in this section.

The triple $\mathcal{L} = (L, \pi, X)$, where L, X are affine varieties and $\pi : L \to X$ is a morphism defines a line bundle if there is a covering of X by Zariski open affine subsets U_α such that $L_\alpha = \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}_{t_\alpha}$ and in the intersection $L_\alpha \cap L_\beta$ the function $g_{\alpha\beta} = \frac{t_\alpha}{t_\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ and does not vanish.

Assume that there are functions $h_\alpha \in F(U_\alpha)$ (i.e.rational in U_α) such that $g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$. Let the divisor D_L be such, that $D_L \cap U_\alpha = (h_\alpha) \cap U_\alpha$ (recall that (h_α) is the divisor of h_α). We say that the divisor D_L and its class $[D_L]$ are associated to the line bundle $\mathcal{L}$ and vice versa (since the surface is smooth, we do not differ between Cartier and Weil divisors). If $h_\alpha \in \mathcal{O}(U_\alpha)$, divisor D_L is effective.

The set of functions $f_\alpha \in \mathcal{O}(U_\alpha)$ such that $\frac{f_\alpha}{h_\alpha} = \frac{f_\beta}{h_\beta}$ is a globally defined rational function $f \in F(X)$ defines the section of $\mathcal{L}$ by $t_\alpha(u) = f_\alpha(u)$ for a point $u \in U_\alpha$. If D_L is effective, it has a section $t_\alpha(u) = h_\alpha(u)$ which vanishes (intersects a zero section) at D_L. The quotient of two sections is a rational function on X. The sheaf $\mathcal{F}$ of germs of sections of the line bundle is coherent, and the global sections $\Gamma(\mathcal{F})$ form a projective module over $\mathcal{O}(X)$, which generates $\mathcal{F}_x$ at every point $x \in X$ as $\mathcal{O}_{x,X}$-module ([S], Theorem 2,§45, Prop. 5, 41).

Therefore the ring $\mathcal{O}(L) = \mathcal{O}(X)[tr_0, tr_1, ..., tr_K]$, where $r_i \in \mathcal{O}(X)$ and t is rational on L.

This ring naturally admits an *lnd* $\partial_\pi \in \mathrm{LND}(\mathcal{O}(L))$ such that $\partial_\pi f = 0$ if $f \in \mathcal{O}(X)$ and $\partial_\pi t = 1$. The $\mathbb{C}^+$-action ψ_π corresponding to ∂_π acts along the fibers of π.

Lemma 9. *Let X be a smooth affine variety admitting a $\mathbb{C}^+$-action $\phi : \mathbb{C} \times X \to X$. Let $\mathcal{L} = (L, \pi, X)$ be an algebraic line bundle over X. Then the total space L of $\mathcal{L}$ admits a $\mathbb{C}^+$-action $\phi' : \mathbb{C} \times L \to L$ such that the image $\pi(\Phi')$ of a general orbit Φ' of the action ϕ' is a generic orbit of the action ϕ.*

Proof of Lemma 9. Since the action ϕ corresponds to a $\partial \in lnd(R)$ which is non-zero we can find an element $r \in R = \mathcal{O}(X)$ such that $\partial(r) = p \neq 0$ and $\partial(p) = 0$. Put $A = R^\partial[r]$ and $B = \mathrm{Frac}(R^\partial)[r] = \mathrm{Frac}(R)^\partial[r]$ ([ML], Lemma 1 of O. Hadas).

As we know, $r_i \in B$. Consider the ideal generated by $r_i, i = 1, ..., K$ in B. Since B is a principal ideal domain this ideal is generated by some element q. So we can write $r_i = q\rho_i$ ($\rho_i \in B$) and polynomials $\rho_0, \ldots, \rho_K$ are relatively prime. Thus we can find $\varsigma_0, \ldots, \varsigma_K \in B$ for which $\sum_i \rho_i \varsigma_i = 1$. Since all elements in B are elements of A divided by elements from R^∂ it means that we can find elements $\widetilde{\varsigma}_i, i = 1, ...K$ in A such that $\sum_i r_i \widetilde{\varsigma}_i = q\Delta$ where $\Delta \in R^\partial$. Therefore $tq\Delta \in \mathcal{O}(L)$. Next, $tr_i = tq\rho_i$. Let $\delta \in R^\partial$ be a common denominator for the coefficients of all ρ_i. We can define now $\widehat{\partial}$ by $\widehat{\partial}(tq) = \delta$, $\widehat{\partial}(r) = \delta\Delta$, $\widehat{\partial}(r') = 0$ for every function $r' \in R^\partial$. $\square$

Corollary 1. *If $ML(X) = \mathbb{C}$ and (L, π, X) is an algebraic line bundle $\mathcal{L}$ over X then $ML(L) = \mathbb{C}$.*

Our main object of interest is rigid surfaces.

Definition 5. If the generic orbit of a $\mathbb{C}^+$-action $\varphi : \mathbb{C}_\lambda \times L \to L$ on the total space of a line bundle $\mathcal{L} = (L, \pi, S)$ over a smooth affine surface S is not contained in a fiber of π we will call φ a skew $\mathbb{C}^+$-action.

Example 1. Define the projection $\pi : \mathbb{C}^9 \to \mathbb{C}^7$ by

$$\pi(u, v, z, w, x, t, y, s, r) = (u, v, z, w, x, t, y) \tag{31}$$

and define the affine variety $L \subset \mathbb{C}^9$ by equations (1)–(13) and the following ones:

$$su = rz \tag{32}$$

$$s(z - 1) = rv. \tag{33}$$

Then $\mathcal{L} = (L, \pi, S)$ is a line bundle over the surface S defined in Section 3 by (1)–(13).

Indeed in notations of Section 3 $S = S_0 \cup S_\infty$ and

$$\pi^{-1}(S_0) \cong S_0 \times \mathbb{C}^1_r,\ s = rb;$$

$$\pi^{-1}(S_\infty) \cong S_\infty \times \mathbb{C}^1_s,\ r = s/b.$$

There is $\partial \in \mathrm{LND}(L)$ that is defined as

$$\partial s = \partial r = 0, \partial b = 0, \partial u = s^m r^{n-m}, \partial z = s^{m+1} r^{n-m-1},$$

$$\partial v = s^{m+2} r^{n-m-2}, \partial w = 2vs^{m+3} r^{n-m-3}, \partial x = 2vs^{m+5} r^{n-m-5},$$

$$\partial t = 2us^{m-1} r^{n-m+1}, \partial y = 2us^{m-3} r^{n-m+3}.$$

For any $m \geq 3$ and $n \geq m+5$ this *lnd* is well defined and provides a skew $\mathbb{C}^+$-action. Note that this line bundle has a section $Z = \{r = u, s = z\} \subset L$. The divisor D of intersection Z with the zero section Z_0 is associated to $\mathcal{L}$ divisor. Let $C = \{u = 0,\ b \neq 0,\ b \neq \infty\}$ and let F be a fiber $b = const \neq 0, \infty$. Then $D = C + B_0$ and since $(u)_0 = C + B_0 + 2B_\infty \sim 0$, we have $D \sim -2B_\infty \sim -F$.

Similar example may be constructed over any rigid surface S.

Lemma 10. *Let S be a rigid surface and $\partial' \in GML(S)$. There exists a line bundle $(\mathcal{L}, \pi, S)$ and $\partial \in LND(L)$ such that $\partial f = 0$ for any $f \in \pi^*(F^\partial)$ (as we mentioned in Introduction).*

Proof of Lemma 10. Consider a $\mathbb{C}$-fibration $f : S \to \mathbb{P}^1$ on S induced by ∂' and a non-singular fiber $F = f^{-1}(\infty)$. Consider the line bundle $(\mathcal{L}, \pi, S)$ associated to the divisor $-mF$. Let $U_1 = S - F$ and $U_2 = S - F'$, where $F' = \{f = 0\}$ is another non-singular fiber. (We may always assume that fibers F and F' are nonsingular). Then $L = L_1 \cup L_2$ where $L_1 = \pi^{-1}(U_1) \cong U_1 \times \mathbb{C}_{t_1}$ and $L_2 = \pi^{-1}(U_2) \cong U_2 \times \mathbb{C}_{t_2}$ and $t_2 = f^m t_1$. The function $\tau = t_1 = f^{-m} t_2 \in \mathcal{O}(L)$: it has zero of order m along F because f has a simple pole there. The divisor $(\tau) = Z_0 + m\pi^* F$. Thus, $\mathcal{O}(L) = \mathcal{O}(S)[\tau, \tau\omega_1^*, ...\tau\omega_n^*]$, where ω_i are rational functions on S, such that $(\omega_i) \geq -mF$. Since $f(U_1)$ is an affine curve, there exists an *lnd* $\partial_1 \in LND(\mathcal{O}(U_1))$ such that $\partial_1 f = 0$. Let N be bigger than the order of poles of $\partial_1 \omega_i$ along F for all $i = 1, ..., n$. One can define an *lnd* $\partial \in \mathcal{O}(L)$ by $\partial\tau = \partial f = 0, \partial u = \tau^N \partial_1 u$ for $u \in \mathcal{O}(S)$. □

Take now any morphism $f : S \to \mathbb{P}^1$ of a rigid surface S onto $\mathbb{P}^1$ such that the general fiber of f is isomorphic to $\mathbb{C}$. Picard group of S is generated by divisor $[F]$ of the generic fiber F and the divisors $[E_{i,j}]$ of the irreducible components $E_{i,j}$ of the singular fibers F_i, $[F_i] = \sum\limits_1^{n_i} \alpha_{i,j}[E_{i,j}], i = 1, .., n$ with relations reflecting that all the fibers are equivalent.

The group $\mathrm{Pic}(S) \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus N}$, where $N = (\sum n_i) - n + 1$, and is generated by $[F]$ and $[E_{i,j}], j > 1$. ([Mi2], Ch.3, Lemma 2.4.3.1).

Any element $l \in \mathrm{Pic}(S)$ may be represented uniquely as

$$l = m[F] + \sum_1^n \sum_1^{n_i} m_{i,j}[E_{i,j}],$$

where

(1) $m_{i,j} < \alpha_{i,j}$ for any $i, 1 \leq i \leq n$ and any $j, 1 \leq j \leq n_i$;
(2) $m_{i,j} \geq 0$ for at least one of $j, 1 \leq j \leq n_i$ for any $i, 1 \leq i \leq n$;

Definition 6. We will call the representation with properties (1)-(2) standard for fibration f. We will call the element $l \in \mathrm{Pic}(S)$ positive relative to fibration f, if in the standard representation $m \geq 0$.

The crucial fact for Lemma 10 and Example 1 is that the line bundles are associated with the non-positive (relative to a given fibration) element of the Picard group. The following example presents the line bundle associated to a positive divisor.

Example 2. Define the same projection $\pi : \mathbb{C}^9 \to \mathbb{C}^7$ by

$$\pi(u, v, z, w, x, t, y, s, r) = (u, v, z, w, x, t, y) \tag{34}$$

and the affine variety $L \subset \mathbb{C}^9$ by equations (1)–(13) and the following ones:

$$su = rv \tag{35}$$

$$st = ru(z-1), \tag{36}$$

$$svz = rw. \tag{37}$$

Then $\mathcal{L} = (L, \pi, S)$ is a line bundle over the surface S, defined in Section 3 by (1)–(13).

In notations of Section 3

$$\pi^{-1}(S_0 - C) \cong (S_0 - C) \times \mathbb{C}^1_r,\ s = rv/u;$$

$$\pi^{-1}(S_\infty - C_1) \cong (S_\infty - C_1) \times \mathbb{C}^1_s,\ r = su/v.$$

Here $C_1 = \{v = 0,\ b \neq 0,\ b \neq \infty\}$ does not intersect $C = \{u = 0,\ b \neq 0,\ b \neq \infty\}$.

The associated to $\mathcal{L}$ divisor being the intersection divisor of the section $Z_1 = \{r = u, s = v\} \subset L$ and the zero section Z_0 is $B_0 + B_\infty$. Therefore $\mathcal{L}$ is associated to a positive (relative to the fibration) divisor. We will show that there is no skew actions on L.

Proposition 1. *Let $\mathcal{L} = (L, \pi, S)$ be an algebraic line bundle over a rigid surface S. Assume that L admits a skew $\mathbb{C}^+$-action $\alpha : \mathbb{C} \times L \to L$. Then there exists a skew $\mathbb{C}^+$-action $\beta : \mathbb{C} \times L \to L$, $\partial' \in GML(S)$ and an induced by ∂' morphism $g : S \to \mathbb{P}^1$ of S such that*

(1) *the generic fiber of g is $\mathbb{C}$;*
(2) *$g(\pi(O))$ is a point for a general orbit O of β;*
(3) *there is no non-zero section Z of $\mathcal{L}$ over an open subset $U \subset S$, such that*
    (a) *$g(U) = \mathbb{P}^1$;*
    (b) *the components of $g^{-1}(p) \cap U$ are isomorphic to $\mathbb{C}$ for each $p \in \mathbb{P}^1$;*
    (c) *$g(Z \cap Z_0)$ is a finite set in $\mathbb{P}^1$.*

Proof of Proposition 1.

Lemma 11. *Let R be an affine ring and $Q = R[t, t^{r_1}\omega_1, \ldots, t^{r_k}\omega_k]$ where t is a variable and $\omega_i \in Frac(R)$. Let $\partial \in LND(Q)$ which is not identically zero on R. Then there exists a locally nilpotent derivation on Q which is t-homogeneous and is not identically zero on R.*

Proof of Lemma 11. Let us introduce a weight function on Q by $w(t) = 1$, $w(r) = 0$ for $r \in R^*$, and $w(0) = -\infty$. Consider a (non-zero) locally nilpotent derivation $\overline{\partial}$ which corresponds to this weight function ([KML2]). Clearly $\overline{\partial} \in \mathrm{LND}(Q)$ since Q is a graded algebra relative to the introduced weight function. Then $\overline{\partial}(t) = t^{k+1}\epsilon(t)$, $\overline{\partial}(r) = t^k\epsilon(r)$ where $\epsilon \in \mathrm{DER}(Q)$ such that $\epsilon(t), \epsilon(r) \in \mathrm{Frac}(R)$ if $r \in R$. Since our goal is to produce a locally nilpotent derivation on R we may assume that $k > 0$ (otherwise ∂ can be restricted on R). It remains to show that $\overline{\partial}$ is not identically zero on R. So assume that $\overline{\partial}$ is identically zero on R. Then $\overline{\partial}(t) = t^{k+1}\epsilon(t)$ implies that $\overline{\partial}(t) = 0$, so $\overline{\partial}$ would be identically zero contrary to the facts. Indeed, if deg is the degree function induced on Q by $\overline{\partial}$ we have $\deg(t) - 1 = (k+1)\deg(t) + \deg(\epsilon(t))$. But since we assumed that $\overline{\partial}$ is identically zero on R we have $\deg(\epsilon(t)) = 0$ if $\epsilon(t) \neq 0$. (If $\epsilon(t) = 0$ then $\deg(\epsilon(t)) = -\infty$.) So if $\epsilon(t) \neq 0$ then $\deg(t) - 1 = (k+1)\deg(t)$. Since $k > 0$ we see that then $\deg(t) < 0$ which is impossible. So the lemma is proved. □

Corollary 2. *If* $\dim(R) > 1$ *then* $\mathit{Frac}(Q)^{\overline{\partial}}$ *contains a non-constant rational function from* $\mathit{Frac}(R)$.

Proof of Corollary 2. Since $\overline{\partial}$ is t-homogeneous the ring of $\overline{\partial}$-constants is generated by t-homogeneous elements. Since $\dim(Q) > 2$ there are two algebraically independent homogeneous $\overline{\partial}$-constants, say $f_1 = t^m\omega_1$ and $f_2 = t^n\omega_2$. Then $f_1^n f_2^{-m} \in \mathrm{Frac}(R)$. □

We apply the Corollary 2 assuming $R = \mathcal{O}(S)$ and $Q = \mathcal{O}(L)$, and $t \in \mathcal{O}(L)$ is any regular function on L that is linear along the generic fiber and vanishing at zero section. Let β be the $\mathbb{C}^+$-action defined by locally nilpotent derivation $\overline{\partial}$. By construction, all the points of zero section $Z_0 \subset L$ are fixed for β and there exists β-invariant function $f = \pi^*g \in \mathrm{Frac}(\mathcal{O}(L))$ with $g \in \mathrm{Frac}(\mathcal{O}(S))$. Using Stein factorization we may assume that the generic fiber of $g^{-1}(p), p \in \mathbb{P}^1$ is connected (and irreducible).

Lemma 12. $g : S \to \mathbb{P}^1$ *is morphism.*

Proof of Lemma 12. We will identify S with the zero section Z_0, i.e. $S \subset L$. By construction it is β-invariant. The function f is the composition of rational maps: $L \xrightarrow{\pi} S \xrightarrow{g} \mathbb{P}^1$. Let p be a point in $\mathbb{P}^1$. Let $C_p = g^{-1}(p) \subset S$ and let $T_p = \pi^{-1}(C_p) = f^{-1}(p)$. Since f is β-invariant, T_p is β-invariant as well, thus consists β-orbits. Since β is a skew action, these orbits are not mapped to a point by π. Hence, $C_p = \pi(T_p) = T_p \cap Z_0 \cong \mathbb{C}$. By construction T_p is the restriction of our line bundle $\mathcal{L}$ over C_p, thus $T_p \cong \mathbb{C}^2$.

If g were not a morphism there would be a point $s \in S$ contained in every fiber $C_p = \{g = p\}$. Then for every p the set T_p would contain two β-invariant intersecting curves: C_p and $A_s = \pi^{-1}(s)$. But then all the points of T_p for all p would be fixed by β. The contradiction shows that such point s does not exist and g is morphism. □

Items (1),(2) of Proposition 1 are proved in Lemma 12. Assume now that there exists a section Z as in item (3).

Items (3a),(3b),(3c) imply that $Z \cong U$ admits a $\mathbb{C}$-fibration over $\mathbb{P}^1$ such that $Z \cap Z_0$ is the union of finite set of fibers of this fibration. We want to show that Z is β-invariant and this fibration should be induced by the restriction of β on Z. It would lead to a contradiction, because a $\mathbb{C}^+$-action has an affine base ([MiMa1], Lemma 1.1).

In notations of Lemma 12 item (3c) means that for a generic $p \in \mathbb{P}^1$ the curve $B_p = Z \cap T_p$ does not intersect Z_0, in particular, the curves $B_p \subset T_p$ and $C_p = Z_0 \cap T_p \subset T_p$ do not intersect.

Since $C_p = \pi(T_p)$, and Z is a section, $B_p = Z \cap T_p$ is a section of the bundle over C_p and $\pi\big|_{B_p} : B_p \to C_p$ is an isomorphism. Hence $B_p \cong \mathbb{C}$. Thus, in β-invariant set $T_p \cong \mathbb{C}^2$ we have two rational disjoint curves. C_p is a β-orbit in T_p, therefore the same should be true for B_p. Therefore, Z is β-invariant, and the base of the restriction of the induced fibration should be affine. This contradicts to (3a). □

Corollary 3. *Let S be a rigid surface, let $GML(S) = \mathbb{C}(f)$ and let $f : S \to \mathbb{P}^1$ be the corresponding fibration. Let $\mathcal{L} = (L, \pi, S)$ be a line bundle over S. Then $ML(L) = \mathcal{O}(S)$ if $\mathcal{L}$ is associated to positive (relative to fibration f) element l of $Pic(S)$.*

Proof of Corollary 3. Let $\psi : \mathbb{C} \times L \to \mathbb{C}$ be a skew action on L. According to Proposition 1 it gives rise to a $\mathbb{C}$-fibration $g : S \to \mathbb{P}^1$. Since $GML(S) = \mathbb{C}(f)$, the fibrations g and f have to be equivalent. Let the associated to $\mathcal{L}$ element $l \in \operatorname{Pic}(S)$ have the standard representation

$$l = m[F] + \sum_1^n \sum_1^{n_i} m_{i,j}[E_{i,j}]$$

and let l_+ be a sum of summands with non-negative coefficient and l_- be a sum of summands with negative coefficient. Let D_+ and D_- be the union of components appearing in l_+ and l_- respectively.

Over $U = S - D_- \subset S$, the line bundle $\mathcal{L}$ is associated to the effective divisor, hence has a section Z_U such that intersection $Z_0 \cap Z_U \subset D_+$. Since $\operatorname{supp} D_+$ contains at least one component of every fiber of g, U enjoys all the properties of item (3) of Proposition 1 which is impossible if ψ is a skew $\mathbb{C}^+$-action. □

Corollary 3 provides the situation when similar to the case of trivial line bundle, the isomorphism $ML(S) \cong \mathcal{O}(S)$ implies $ML(L) \cong ML(S) \cong \mathcal{O}(S)$ ([BML2]). The following questions remain open.

Questions.

1. Let S be a rigid surface, $GML(S) = \mathbb{C}(f)$ and let $f : S \to \mathbb{P}^1$ be the corresponding fibration. Let $\mathcal{L} = (L, \pi, S)$ be a line bundle over S. Is it possible that $ML(L) = \mathcal{O}(S)$ if $\mathcal{L}$ is associated to non-positive relative to fibration f element l of $\operatorname{Pic}(S)$?

2. Assume that S is rigid and $GML(S) = \mathbb{C}$. When $ML(L) = \mathcal{O}(S)$?

Acknowledgments. We are grateful to Sh. Kaliman for his valuable advice and infinite help. This project was started while both authors were visiting Max Planck Institute of Mathematics and while the first author was visiting Wayne State University and University of California, San Diego.

References

[BML1] T. Bandman, L. Makar-Limanov, *Cylinders over affine surfaces*, Japan. J. Math., **26** (2000), 207-217.

[BML2] T. Bandman, L. Makar-Limanov, *Nonstability of AK invariant*, Michigan Journal of Mathematics, **53** (2005), 263-281.

[Ber] J.Bertin, *Pinceaux de droites et automorphismes des surfaces affines*, J. Reine und Angew. Math.,**341**(1983), 32-53.

[DR] D.Daigle, P. Russell, *Affine rulings of normal rational surfaces*, Osaka J. Math.,**38**(2001), 37-100.

[FLN] M. Ferrero, Y. Lequain, A. Nowicki, *A note on locally nilpotent derivations*, J. of Pure and Appl. Algebra, **79**(1992), 45–50.

[F] K.-H. Fieseler, *On complex affine surfaces with* $\mathbb{C}^+$*-action* , Comment. Math. Helvetici, **69**(1994), 5-27.

[Fu] T. Fujita *On the topology of non-complete algebraic surfaces* , J. Fac.Sci. Univ. Tokyo, Sect.IA **29**(1982), 503-566.

[GM] R.V. Gurjar, M. Miyanishi *Automorphisms of affine surfaces with* $\mathbb{A}^1$*-fibrations* , Mich. Math. J. **53**(2005), 33-55.

[GMMR] R.V. Gurjar, K. Masuda, M. Miyanishi P. Russell *Affine lines on affine surfaces and the Makar-Limanov invariant*, Can. J. Math, to appear.

[Giz] M.H.Gizatullin, *Quasihomogeneous affine surfaces*, Math. USSR Izvestiya, **5**(1971), 1057-81.

[Ha] R. Harstshorne, Algebraic Geometry, Springer-Verlag, (1977).

[KML1] S. Kaliman, L. Makar-Limanov, *On the Russell-Koras contractible threefolds* , Journ. of the Algebraic Geometry, **6(2)**(1997), 247–268.

[KML2] S. Kaliman, L. Makar-Limanov, *AK-invariant of affine domains* , Affine Algebraic Geometry, volume in honor of M. Miyanishi, to appear.

[KiKo] T. Kishimoto, H. Kojima Affine lines in $\mathbb{Q}$-homology planes with logarithmic Kodaira dimension $-\infty$ Transformation Groups, **11**(2006), 659-672.

[ML] L. Makar-Limanov, *AK-invariant, some conjectures, examples and counterexamples.* Annales Polonici Matematici **76**(2001), 139-145.

[Mi1] M. Miyanishi *On algebro-topological characterization of the affine space of dimension 3* , Amer. Math. Jour., **106**(1984), 1469-1485.

[Mi2] M.Miyanishi, Open algebraic surfaces, CRM Monograph series, **12**, AMS, RI, (2001).

[MiMa1] M. Miyanishi, K.Masuda, *The additive group action on* $\mathbb{Q}$- *homology planes*, Ann. de l'Inst. Fourier, **53**(2003), 429–464.

[MiMa] M. Miyanishi, K.Masuda, *Affine pseudoplanes and cancellation problem*, Transaction of AMS **357**(2005), 4867-4883.

[MiMa2] M. Miyanishi, K.Masuda, *Open algebraic surfaces with finite group actions*, Transformation groups, **7**(2002), 185-207.

[MiSu] M.Miyanishi, T. Sugie *Affine surfaces containing cylinderlike open set*, J. Math. Univ.Kyoto, **20**, (1980), 11-42.

[S] J.-P. Serre *Faiseaux algébriques cohérants*, Ann. of Math. **61**, (1955), 197-278.

[TtD] T. tom Dieck *Homology planes without cancellation property*, Arch. der Math. **59**, (1992), 105-115.

[Z] O. Zariski *Intrepretations algébrico-géomé triques du quatorzième probléme de Hilbert*, in "'O.Zariski: Collected papers"',**2**, The MIT Press, Cambridge (Massachusetts), (1973), 261-275.

[Za] M. Zaidenberg, *Affine lines on* $\mathbb{Q}$*-homology planes and group actions*, Transformation Groups, **11**(2006),725-735.

T. Bandman, Department of Mathematics, Bar-Ilan University, 52900, Ramat Gan, ISRAEL

E-mail address: `bandman@macs.biu.ac.il`

L. Makar-Limanov, Department of Mathematics, Wayne State University, Detroit, MI, 48202, USA

E-mail address: `lml@math.wayne.edu`