# Traces of intertwiners for quantum groups and Difference Equations II

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**Abstract**

In this paper we study twisted traces of products of intertwining operators for quantum affine algebras. They are interesting special functions, depending on two weights $\lambda, \mu$, three scalar parameters $q, \omega, k$, and spectral parameters $z_1, ..., z_N$, which may be regarded as q-analogs of conformal blocks of the Wess-Zumino-Witten model on an elliptic curve. It is expected that in the rank 1 case they essentially coincide with the elliptic hypergeometric functions defined in [FV2]. Our main result is that after a suitable renormalization the traces satisfy four systems of difference equations – the Macdonald-Ruijsenaars equation, the q-Knizhnik-Zamolodchikov-Bernard equation, and their dual versions. We also show that in the case when the twisting automorphism is trivial, the trace functions are symmetric under the permutation $\lambda \leftrightarrow \mu$, $k \leftrightarrow \omega$. Thus, our results generalize those of [ES1], dealing with the case $q = 1$, and [EV, ES2], dealing with the finite dimensional case.

## 1 Introduction

Let us recall the results of [EV, ES1, ES2]. In [EV], the main objects of study are suitably normalized traces of products of intertwining operators for the quantum group $U_q(\mathfrak{g})$ corresponding to a finite dimensional simple Lie algebra $\mathfrak{g}$. These traces are functions of two weights of $\mathfrak{g}$, which take values in the endomorphism algebra of the zero weight space of a tensor product of finite dimensional $U_q(\mathfrak{g})$-modules. It is shown in [EV] that these traces satisfy four systems of difference equations – the Macdonald-Ruijsenaars (MR), dual MR, quantum Knizhnik-Zamolodchikov-Bernard (qKZB), and dual qKZB equations. It is also shown that the traces are symmetric with respect to the two weights on which they depend.

In [ES2], the results of [EV] are generalized to the "twisted" case, i.e. with traces involving an arbitrary diagram automorphism $T$ of $\mathfrak{g}$. In this case, the four systems of difference equations can still be derived, but the symmetry no longer holds (unless $T = 1$).

It is an interesting problem to prove similar results in the case when $\mathfrak{g}$ is replaced by the affine Lie algebra $\widetilde{\mathfrak{g}}$. For the classical case $q = 1$, this was done earlier. Namely, the untwisted case $T = 1$ goes back to Bernard [Be], while the twisted case was first considered in [E1] in a special case, and then in [ES1] in general. In the quantum case, an attempt to attack this problem was made in [E2], but only partial results were obtained.

The goal of this paper is to give a complete solution of this problem. More specifically, we derive the Macdonald-Ruijsenaars and qKZB equations and their dual versions for traces of products of intertwining operators for a quantum affine algebra, twisted by a diagram automorphism $T$ of this algebra, and show that the trace functions are symmetric if $T = 1$. In particular, this gives a representation-theortic proof of the result from [FTV] that the dual difference equation to qKZB equation for $\mathfrak{g} = \mathfrak{sl}_2$ is also the qKZB equation, with the step and elliptic modulus interchanged, and allows us to generalize this result to other simple Lie algebras.

It turns out that the methods of [EV] and [ES2] apply with only minor modifications, related to the fact that certain completions need to be taken in the affine case. Therefore, we will omit proofs of the statements, referring the reader to appropriate statements in [EV, ES2], and explaining only the required modifications.

To simplify notation, we will work with simple Lie algebras of type A,D,E. The results can be generalized to other types.

The structure of the paper is as follows. Section 2 serves to make the main definitions and set notations. In Section 3, we remind the basic facts about dynamical twists and R-matrices, which are essential ingredients in the paper. In Section 4 we introduce the trace functions – the main characters of the paper. In Section 5 we discuss the MR and dual MR equations, in Section 6 – the qKZB and dual qKZB equations. The untwisted case and the symmetry identity are discussed in Section 7.

The content of this paper has the following connection with other literature. For $T = 1$, the trace functions are q-analogs of conformal blocks on an elliptic curve, which are elliptic deformations of the q-conformal blocks on $\mathbb{P}^1$, defined by Frenkel and Reshetikhin [FR]. If $\mathfrak{g} = \mathfrak{sl}(2)$, it is expected that the trace function $F_T^{V_1,...,V_N}(z_1, ..., z_n, \lambda, \omega, \mu, k)$ introduced here is (up to simple renomalizations) the elliptic hypergeometric function $u(z_1, ..., z_n, \lambda, \tau, \mu, p)$ introduced in [FV1, FV2, FTV] by an explicit integral. This conjecture is motivated by the fact that both functions satisfy qKZB and dual qKZB equations and the symmetry relations. The conjecture has been checked in the trigonometric limit ([EV], section 8), and in the classical limit $q \to 1$, where it turns into the integral formula for conformal blocks on the elliptic curve. We plan to discuss the general case in a subsequent paper.

Another interesting case is $\mathfrak{g} = \mathfrak{sl}_n$, and $T$ is the rotation of the affine Dynkin diagram (which is an n-gon) by $r/n$ of a full circle, where $(r, n) = 1$. This is the only case when the trace function does not depend on the dynamical parameters $\lambda$ and $\mu$, and is a function of $z_1, ..., z_n$ and two "elliptic moduli" $\omega$ and $k$ (more precisely, $\mu$ takes a finite set of values). The qKZB equation is in this case the elliptic qKZ equation considered in [JM, JMN], which involves Belavin's elliptic R-matrix. Thus we show that the dual difference equation to this equation (i.e., its "monodromy") is the difference equation involving special values of Felder's $R$-matrix at finitely many points.

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# 2 Notations

## 2.1 Simple Lie algebras

We will use the following notations:

$\mathfrak{g}$: a simple complex Lie algebra of type A,D,E.

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$: its Cartan decomposition.

$\Gamma = \{\alpha_1, \ldots, \alpha_r\}$: the Dynkin diagram.

$\Delta \subset \mathfrak{h}^*$: the root system.

$\Delta_+$: the set of positive roots.

$( \, , \, )$: the nondegenerate invariant bilinear form on $\mathfrak{g}$, normalized in such a way that the induced form on $\mathfrak{h}^*$ (also denoted by $( \, , \, )$) satisfies the equation $(\alpha, \alpha) = 2$ for roots $\alpha$.

$\theta \in \Delta^+$: the maximal root.

$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$: the half sum of positive roots.

h: the Coxeter (=dual Coxeter) number of $\mathfrak{g}$.

$e_i, f_i, h_i$, $i = 1, ..., r$: the Chevalley-Serre generators of $\mathfrak{g}$.

$\Omega_{\mathfrak{h}} \in \mathfrak{h} \otimes \mathfrak{h}$: the inverse element to the form $( \, , \, )$ on $\mathfrak{h}$.

## 2.2 Affine Lie algebras

Let $\widetilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\partial \oplus \mathbb{C}c$ be the affine algebra Lie associated to $\mathfrak{g}$, where $c$ is the central element and $\partial$ is the grading element. That is, for Laurent polynomials $a(t), b(t)$ one has

$$[a(t), b(t)] = [ab](t) + \mathrm{Res}_0(a'(t), b(t))c,$$

and $[\partial, a(t)] = ta'(t)$, $[\partial, c] = 0$. Let $\widetilde{\mathfrak{b}}_\pm$ be the Borel subalgebras of $\widetilde{\mathfrak{g}}$ and let $\widetilde{\mathfrak{n}}_\pm$ be their nilpotent radicals. The Cartan subalgebra of $\widetilde{\mathfrak{g}}$ is $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}\partial$. The dual of the Cartan subalgebra is given by $\widetilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$ where $\Lambda_0, \delta$ are defined by the relations $\langle \Lambda_0, \mathfrak{h} \rangle = \langle \Lambda_0, \partial \rangle = \langle \delta, \mathfrak{h} \rangle = \langle \delta, c \rangle = 0$ and $\langle \Lambda_0, c \rangle = \langle \delta, \partial \rangle = 1$.

Set $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$. The Lie algebra $\widehat{\mathfrak{g}}$ is a Kac-Moody Lie algebra. Let $\widetilde{\Gamma}$ and $\widetilde{\Delta}$ be the Dynkin diagram and root system of this algebra, respectively. Thus $\widetilde{\Delta} = (\Delta + \mathbb{Z}\delta) \cup (\mathbb{Z} \backslash \{0\})\delta$ and $\widetilde{\Gamma} = \Gamma \cup \{\alpha_0\}$ where $\alpha_0 = \delta - \theta$ is the affine simple root. Let $(a_{ij})$ be the Cartan matrix of $\widetilde{\mathfrak{g}}$. Let $e_i, f_i, h_i$, $i = 0, ..., r$, be the standard Kac-Moody generators of $\widehat{\mathfrak{g}}$. The span of $h_i$ (i.e. the space $\mathfrak{h} \oplus \mathbb{C}c = \widehat{\mathfrak{g}} \cap \widetilde{\mathfrak{h}}$) will be denoted by $\widehat{\mathfrak{h}}$.

Let $D \in \mathfrak{h} \oplus \mathbb{C}\partial$ be the principal gradation element: $[D, e_i] = e_i$ and $[D, f_i] = -f_i$ for all $i$.

The Lie algebra $\widetilde{\mathfrak{g}}$ carries a nondegenerate invariant form $(,)$, which restricts to the form on $\mathfrak{g}$ defined above. Such a form is not unique: it depends on one free parameter. We will uniquely determine the form by the condition that $(D, D) = 0$. (We note that this form is somewhat different from the commonly used form, which satisfies $(\partial, \partial) = 0$.) The restriction of this form to $\widetilde{\mathfrak{h}}$ is nondegenerate, so it defines a form on the dual space.

Let $\mathfrak{h}'$ be the orthogonal complement to $\mathbb{C}c \oplus \mathbb{C}D$ in $\widetilde{\mathfrak{h}}$. It is easy to see that the map $x \to x - \frac{(\rho, x)}{\mathrm{h}}c$ defines an orthogonal isomorphism $\mathfrak{h} \to \mathfrak{h}'$.

Let $\widetilde{\rho} = \rho + \mathrm{h}\Lambda_0$. Then $(\widetilde{\rho}, \alpha_i) = 1$ for $i = 0, \dots, r$. Thus, $\widetilde{\rho}$ corresponds to $D$ under the identification $\widetilde{\mathfrak{h}}^* \to \widetilde{\mathfrak{h}}$ defined by the inner product.

Finally, for any $z \in \mathbb{C}^*$, we define an automorphism $D_z \in \mathrm{Aut}\, U(\widehat{\mathfrak{g}})$ by $D_z(e_i) = ze_i$ and $D_z(f_i) = z^{-1}f_i$ for any $i \in \widetilde{\Gamma}$, and $D_z c = c$. In other words, we have $D_z(x) = z^D x z^{-D}$.

## 2.3 Quantum groups

Let $q$ be a nonzero complex number, such that $|q| < 1$. We fix $\hbar$ such that $q = e^\hbar$. For any operator $A$, the expression $q^A$ will stand for $e^{\hbar A}$.

Let $U_q(\widetilde{\mathfrak{g}})$ be the quantized affine algebra corresponding to $\widetilde{\mathfrak{g}}$. It is a Hopf algebra over $\mathbb{C}$ with generators $E_i,\ F_i,\ i=0,1,...,r$ and $q^h,\ h\in\widetilde{\mathfrak{h}}$ and with relations

$$q^{x+y}=q^xq^y,\ x,y\in\widetilde{\mathfrak{h}},\quad q^0=1$$

$$q^hE_iq^{-h}=q^{\alpha_i(h)}E_i,\quad q^hF_iq^{-h}=q^{-\alpha_i(h)}F_i$$

$$E_iF_j-F_jE_i=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_q E_i^{1-a_{ij}-k}E_jE_i^k=0,\quad i\neq j,$$

and

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_q F_i^{1-a_{ij}-k}F_jF_i^k=0,\quad i\neq j.$$

$$\begin{bmatrix}n\\ k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!},\quad [n]_q!=[1]_q\cdot[2]_q\cdot\dots\cdot[n]_q,\quad [n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$$

The comultiplication $\Delta$, antipode $S$ and counit $\epsilon$ in $U_q(\widetilde{\mathfrak{g}})$ are given by

$$\Delta(E_i)=E_i\otimes q^{h_i}+1\otimes E_i,\quad \Delta(F_i)=F_i\otimes 1+q^{-h_i}\otimes F_i,$$

$$\Delta(q^h)=q^h\otimes q^h,$$

$$S(E_i)=-E_iq^{-h_i},\quad S(F_i)=-q^{h_i}F_i,\quad S(q^h)=q^{-h},$$

$$\epsilon(E_i)=\epsilon(F_i)=0,\quad \epsilon(q^h)=1.$$

The quantum group $U_q(\mathfrak{g})$ is the Hopf subalgebra of $U_q(\widetilde{\mathfrak{g}})$ generated by $E_i,F_i,\ i\geq 1$, and $q^h, h\in\mathfrak{h}$.

If $\widetilde{\lambda}\in\widetilde{\mathfrak{h}}^*$, then we will write $q^{\widetilde{\lambda}}$ for the element $q^{h_{\widetilde{\lambda}}}$, where $h_{\widetilde{\lambda}}$ is the image of $\widetilde{\lambda}$ under the identification $\widetilde{\mathfrak{h}}^*\to\widetilde{\mathfrak{h}}$.

Let $U_q(\widehat{\mathfrak{g}})$ be the subalgebra of $U_q(\widetilde{\mathfrak{g}})$ generated by $E_i,\ F_i,\ \alpha_i\in\widetilde{\Gamma}$ and $q^h,\ h\in\mathfrak{h}\oplus\mathbb{C}c$. Let $U_q(\widehat{\mathfrak{n}}_\pm)$ be the subalgebra generated by $(E_i)_{\alpha_i\in\widetilde{\Gamma}}$ and $(F_i)_{\alpha_i\in\widetilde{\Gamma}}$ respectively, and $U_q(\widehat{\mathfrak{b}}_\pm)$ be generated by $U_q(\widehat{\mathfrak{n}}_\pm)$ and elements $q^h$, $h\in\mathfrak{h}\oplus\mathbb{C}c$.

It is known that $U_q(\widetilde{\mathfrak{g}})$ has a (generalized) quasitriangular structure, given by the R-matrix

$$\mathcal{R}=\mathcal{R}_0q^{(c\otimes D+D\otimes c)/\mathrm{h}+\Omega_{\mathfrak{h}'}},$$

where $\mathcal{R}_0$ belongs to an appropriate completion of $U_q(\widehat{\mathfrak{b}}_+)\otimes U_q(\widehat{\mathfrak{b}}_-)$, and its degree zero part (under the principal gradation of the first component of the tensor product) is 1. (Here $\Omega_{\mathfrak{h}'}$ denotes the inverse to the form on $\mathfrak{h}'$).

The automorphism $D_z$ of $U_q(\widetilde{\mathfrak{g}})$ is defined as in the classical case. Given a finite dimensional representation $V$ of $U_q(\widehat{\mathfrak{g}})$, and any $z \in \mathbb{C}^*$, define the shifted representation $V(z)$ to be the vector space $V$ with the new action of $U_q(\widehat{\mathfrak{g}})$ defined by $\pi_{V(z)}(a) = \pi_V(D_z(a))$.

## 2.4 Dynamical notation

We will use the following notations throughout the paper. Let $V_1, \dots, V_N$ be $U_q(\widehat{\mathfrak{g}})$-modules. For an element $A$ of $U_q(\widehat{\mathfrak{g}})$ and $i \in \{1, \dots, N\}$, we denote by $A^i$ the action of $A$ on the $i$th component of the tensor product $V_1 \otimes \cdots \otimes V_N$. Now let $\mathfrak{l} \subset \widehat{\mathfrak{h}}$ and suppose that $V_1, \dots, V_N$ are $\mathfrak{l}$-semisimple. Let $S(\lambda) : \mathfrak{l}^* \to U_q(\widehat{\mathfrak{g}})$ be a function. Then for any $i, j \in \{1, \dots, N\}$ we denote by $S^i(\lambda + h^{(j)})$ the endomorphism of $V_1 \otimes ... \otimes V_N$ defined by

$$S^i(\lambda + h^{(j)})(v_1 \otimes \cdots \otimes v_N) = S_i(\lambda + \mu_j)(v_1 \otimes \cdots \otimes v_N)$$

if $v_j$ is of weight $\mu_j \in \mathfrak{l}^*$.

# 3 Dynamical twists and quantum dynamical R-matrices

In this section we construct dynamical twists and R-matrices arising from a Dynkin diagram authomorphism ([JKOS, ESS, ES2]).

Let $T \in$ Aut $(\widetilde{\Gamma})$ be an automorphism of order $N$. Let us denote by $B_T : U_q(\widetilde{\mathfrak{g}}) \to U_q(\widetilde{\mathfrak{g}})$ the automorphism defined by

$$B_T(E_i) = E_{T(i)}, \quad B_T(F_i) = F_{T(i)}, \quad B_T(h_i) = h_{T(i)},$$

and $B_T(D) = D$. For brevity, we will denote $B_T$ simply by $B$, assuming that $T$ has been fixed.

It is easy to see that $B$ is an orthogonal automorphism of $\widetilde{\mathfrak{h}}$, which preserves $\mathfrak{h}'$, $c$ and $D$. In particular, it preserves the principal grading, i.e $D_z B = B D_z$ for any $z \in \mathbb{C}^*$.

Let $\widetilde{\mathfrak{l}} = \left(\sum_{\alpha \in \widetilde{\Gamma}} \mathbb{C}(\alpha - T\alpha)\right)^\perp = \widetilde{\mathfrak{h}}^B \subset \widetilde{\mathfrak{h}}$. Set $\mathfrak{l} = \widetilde{\mathfrak{l}} \cap \mathfrak{h}'$. Since $B$ is orthogonal, the form is nondegenerate on $\widetilde{\mathfrak{l}}$ and on $\mathfrak{l}$. We identify $\mathfrak{l}$ (resp. $\widetilde{\mathfrak{l}}$) with their duals using $( \, , \, )$. Let $\mathfrak{h}_0 \subset \mathfrak{h}'$ be the orthogonal complement of $\widetilde{\mathfrak{l}}$ in $\widetilde{\mathfrak{h}}$, and let $\Omega_{\mathfrak{h}_0} \in \mathfrak{h}_0 \otimes \mathfrak{h}_0$ and $\Omega_{\widetilde{\mathfrak{l}}} \in \widetilde{\mathfrak{l}} \otimes \widetilde{\mathfrak{l}}$ be the inverse elements to the restriction of $( \, , \, )$ to $\mathfrak{h}_0$ and $\widetilde{\mathfrak{l}}$ respectively.

Define the Cayley transform $C_T : \mathfrak{h}_0 \to \mathfrak{h}_0$ of $T$ by $C_T = \frac{B+1}{B-1}$. This is a skew-symmetric operator on $\mathfrak{h}_0$.

Let $I_\pm \subset U_q(\widetilde{\mathfrak{b}}_\pm)$ be the kernel of the projection $U_q(\widetilde{\mathfrak{b}}_\pm) \to U_q(\widetilde{\mathfrak{h}})$ on the elements of zero degree. Set

$$\mathcal{A}_\lambda = \mathbb{C}[q^{\pm 2(\lambda,\alpha_1)}, \dots, q^{\pm 2(\lambda,\alpha_r)}].$$

We define an automorphism $B \in \text{Aut}\,(U_q(\widetilde{\mathfrak{g}}))$ in the same fashion as in the classical case. Then $BD_z = D_z B$ for any $z \in \mathbb{C}^*$. Let

$$Z = -\frac{1}{2}((1 + C_T) \otimes 1)\Omega_{\mathfrak{h}_0}.$$

Let $q^{-2\omega}$ be a formal parameter. More precisely, we will work over the ring of power series in the expression $q^{-2\omega}$.

Let $\widetilde{\lambda} = \lambda + \omega\widetilde{\rho}/\mathrm{h}$, $\lambda \in \mathfrak{l}^*$. (This makes sense, since $\widetilde{\lambda}$ will occur only in the expression $q^{-2(\lambda,\alpha_i)}$, which expresses via the formal parameter $q^{-2\omega}$).

**Theorem 3.1 ([ESS]).** *There exists a unique $\widetilde{\mathfrak{l}}$-invariant element $\mathcal{J}_T(\widetilde{\lambda}) = \sum_{m\geq 0} \mathcal{J}_{T,m}(\widetilde{\lambda})$, such that*

$$\mathcal{J}_{T,0} = q^Z, \mathcal{J}_{T,m} \in ((I_-[-m] \otimes I_+[m])^{\widetilde{\mathfrak{l}}} \otimes \mathcal{A}_\lambda)[[q^{-2\omega}]]$$

*(where $I_\pm[m] = \{x \in I_\pm, [\rho, x] = mx\}$), satisfying the ABRR equation ([ABRR])*

$$\mathcal{R}^{21}\big(\mathrm{Ad}(q^{2\widetilde{\lambda}}B) \otimes 1\big)\mathcal{J}_T(\widetilde{\lambda}) = \mathcal{J}_T(\widetilde{\lambda})q^{\Omega_{\widetilde{\mathfrak{l}}}}. \tag{3.1}$$

*Moreover, $\mathcal{J}_T(\widetilde{\lambda})$ satisfies the following shifted 2-cocycle condition :*

$$\mathcal{J}_T^{12,3}(\widetilde{\lambda})\mathcal{J}_T^{12}(\widetilde{\lambda} + \frac{1}{2}h^{(3)}) = \mathcal{J}_T^{1,23}(\widetilde{\lambda})\mathcal{J}_T^{23}(\widetilde{\lambda} - \frac{1}{2}h^{(1)}). \tag{3.2}$$

Below we will need a slightly modified version of $\mathcal{J}$. Namely, we set

$$\mathbb{J}_T(\widetilde{\lambda}) = \mathcal{J}_T(\widetilde{\lambda} + \frac{1}{2}(h^{(1)} + h^{(2)})).$$

**Remark.** At $q^{-2\omega} = 0$, both $\mathcal{J}_T$ and $\mathbb{J}_T$ specialize to $\mathcal{R}_0^{21}$.

Now define the exchange operator

$$\mathbb{R}_T(\widetilde{\lambda}) = \mathbb{J}_T(\widetilde{\lambda})^{-1}\mathcal{R}^{21}\mathbb{J}_T^{21}(\widetilde{\lambda}), \tag{3.3}$$

**Proposition 3.1.** *The element $\mathbb{R}_T(z, \widetilde{\lambda})$ satisfies the quantum dynamical Yang-Baxter equation :*

$$\begin{aligned}&\mathbb{R}_T^{12}(\widetilde{\lambda})\mathbb{R}_T^{13}(\widetilde{\lambda} + h^{(2)})\mathbb{R}_T^{23}(\widetilde{\lambda})\\ &\quad= \mathbb{R}_T^{23}(\widetilde{\lambda} + h^{(1)})\mathbb{R}_T^{13}(\widetilde{\lambda})\mathbb{R}_T^{12}(\widetilde{\lambda} + h^{(3)}).\end{aligned} \tag{3.4}$$

*Proof.* By definition, $\mathbb{R}_T(\widetilde{\lambda})^{21}$ is the twist of the quantum R-matrix $\mathcal{R}$ by the noncommutative dynamical 2-cocycle $\mathbb{J}_T(\widetilde{\lambda})$. This implies the statement.

### 3.1 Spectral parameter

Now let us introduce a spectral parameter. Consider the expression

$$\mathbb{J}_T(z,\widetilde{\lambda}) = (D_z \otimes 1)\mathbb{J}_T(\widetilde{\lambda}).$$

It is clear that we have

$$\mathbb{J}_T(z,\widetilde{\lambda}) \in ((U_q(\widetilde{\mathfrak{b}}_-) \otimes U_q(\widetilde{\mathfrak{b}}_+))^{\mathfrak{l}}[[z^{-1}]] \otimes \mathcal{A}_\lambda)[[q^{-2\omega}]].$$

Since $BD_z = D_z B$, we have

$$(D_{z_1} \otimes D_{z_2})\mathbb{J}_T(\widetilde{\lambda}) = (D_{z_1/z_2} \otimes 1)\mathbb{J}_T(\widetilde{\lambda}) = \mathbb{J}_T(z_1/z_2,\widetilde{\lambda}). \tag{3.5}$$

Now set

$$\mathbb{R}_T(z,\widetilde{\lambda}) = (D_z \otimes 1)\mathbb{R}_T(\widetilde{\lambda}) \tag{3.6}$$

Note that $\mathbb{R}_T(z,\widetilde{\lambda})$ belongs to the space [1]

$$(U_q(\widetilde{\mathfrak{g}})\otimes U_q(\widetilde{\mathfrak{g}})((z^{-1}))\widehat{\otimes}\mathcal{A}_\lambda)[[q^{-2\omega}]].$$

**Proposition 3.2.** *The element $\mathbb{R}_T(z,\widetilde{\lambda})$ satisfies the quantum dynamical Yang-Baxter equation with spectral parameters :*

$$\begin{aligned}&\mathbb{R}_T^{12}(z_1/z_2,\widetilde{\lambda})\mathbb{R}_T^{13}(z_1/z_3,\widetilde{\lambda}+h^{(2)})\mathbb{R}_T^{23}(z_2/z_3,\widetilde{\lambda})\\ &\quad= \mathbb{R}_T^{23}(z_2/z_3,\widetilde{\lambda}+h^{(1)})\mathbb{R}_T^{13}(z_1/z_3,\widetilde{\lambda})\mathbb{R}_T^{12}(z_1/z_2,\widetilde{\lambda}+h^{(3)}).\end{aligned} \tag{3.7}$$

*Proof.* Equation (3.7) is obtained by applying $D_{z_1} \otimes D_{z_2} \otimes D_{z_3}$ to (3.4) and using (3.5). ■

Now let $V$ be a finite dimensional representation of $U_q(\widehat{\mathfrak{g}})$. Since the level of such a representation is always zero (see e.g. [CP]), we can evaluate the element $\mathcal{R}$ and hence $\mathbb{R}_T(z,\widetilde{\lambda})$ in the tensor product $V \otimes V$. The obtained operator $\mathbb{R}_T^{VV}(z,\lambda,\omega)$ is a finite dimensional solution of the quantum dynamical Yang-Baxter equation with a spectral parameter $z$ and dynamical parameter $\lambda$, which belongs to $(\mathrm{End}(V \otimes V)((z^{-1})) \otimes \mathcal{A}_\lambda)[[q^{-2\omega}]]$.

In fact, the series defining $\mathbb{R}_T^{VV}$ has a nonempty region of convergence. More precisely, the coefficients to all powers of $q^{-2\omega}$ are convergent series in $z^{-1}$ outside a common disk, and for any point of this disk the resulting series in $q^{-2\omega}$ (with numerical coefficients) is convergent if the real part of $\omega$ is small enough.

[1]Here and below, the sign $\widehat{\otimes}$ will denote a completed tensor product. The nature of completion in each situation should be clear from the context.

To show this, for any two finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules $V$ and $W$ we set

$$\mathcal{R}^{V,W}(z) = (\pi_V \otimes \pi_W)((D_z \otimes 1)\mathcal{R}).$$

This is a power series in $z$. Then we have the following result.

**Theorem 3.2 ([KS]).** *The power series $\mathcal{R}^{V,W}(z)$ converges to a holomorphic function in the neighborhood of $z = 0$, which extends to a meromorphic function on $\mathbb{C}$.*

Using this theorem and the Birkhoff's theory of q-difference equations (see e.g. [EFK], Chapter 10), we get from the ABRR equation that the series $\mathbb{R}_T^{VV}$ defines a meromorphic function of $z, \lambda, \omega$ in the region $z \neq 0$, $\mathrm{Re}(\omega) < 0$.

**Remark 1.** Another proof of Theorem 3.2 can be found in [EM].

**Remark 2.** We note that after evaluation in $V \otimes W$, in the case $T = 1$, the defining equation 3.1 for $\mathbb{J}$ becomes the quantum KZ equation of Frenkel and Reshetikhin [FR]. In the case $T \neq 1$, this is a twisted version of the quantum KZ equation.

**Remark 3.** If $\mathfrak{g} = \mathfrak{sl}_n$, $V$ is its vector representation, and $T = 1$, then the dynamical R-matrix $\mathbb{R}_T^{VV}(z, \lambda, \omega)$ is gauge equivalent to Felder's elliptic R-matrix (see [FR, Mo]). If $T$ is the rotation of the Dynkin diagram of $\widehat{\mathfrak{g}}$ by $r/n$ of a full circle, $(r, n) = 1$, then $R_T^{VV}$ does not depend on $\lambda$ (as $\mathfrak{l} = 0$), and is a usual R-matrix with a spectral parameter. Namely, as was shown in [T], it is the Belavin R-matrix.

# 4 Twisted traces of intertwiners

## 4.1 Intertwiners

If $W$ is a graded space, we will denote by $W[\nu]$ the weight subspace of weight $\nu$ in $W$.

For any $\mu \in \mathfrak{h}'^*$ and $k \in \mathbb{C}$, let $\widetilde{\mu} = \mu + k\widetilde{\rho}/\mathrm{h}$. Let $M_{\widetilde{\mu}}$ be the $U_q(\widetilde{\mathfrak{g}})$-Verma module of weight $\widetilde{\mu}$ (it has level $k$). Let $x_{\widetilde{\mu}}$ be a highest weight vector in this module. Let $M^*_{\widetilde{\mu}}$ be the restricted dual module of $M_{\widetilde{\mu}}$, and $x^*_{\widetilde{\mu}} \in M^*_{\widetilde{\mu}}[-\widetilde{\mu}]$ be such that $\langle x^*_{\widetilde{\mu}}, x_{\widetilde{\mu}} \rangle = 1$.

Let $V$ be a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module. Let $\beta \in \widehat{\mathfrak{h}}^*$.

**Lemma 4.1.** *[FR] Assume that $M^*_{\widetilde{\mu}+\beta}$ is irreducible. Then*

$$\mathrm{Hom}_{U_q(\widehat{\mathfrak{g}})}(M_{\widetilde{\mu}}, M_{\widetilde{\mu}+\beta}\widehat{\otimes}V) \simeq V[\beta]$$

*where the isomorphism is as follows : to $\Phi : M_{\widetilde{\mu}} \to M_{\widetilde{\mu}+\beta}\widehat{\otimes}V$ we associate the element $\langle\Phi\rangle = \langle x^*_{\widetilde{\mu}+\beta}, \Phi(x_{\widetilde{\mu}})\rangle$.*

We will denote the operator corresponding to $v \in V[\beta]$ under the correspondence of Lemma 4.1 by $\Phi^v_{\widetilde{\mu}}$. Define also the intertwining operator $\Phi^v_{\widetilde{\mu}}(z) : M_{\widetilde{\mu}} \to M_{\widetilde{\mu}+\beta}\widehat{\otimes}V(z)$. This operator can be regarded as an infinite in both directions Laurent series in $z$ with operator coefficients (for the coefficients, the completion of the tensor product is unnecessary).

Let $V_1, \ldots, V_N$ be finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules and let $v_1 \in V_1[\nu_1], \ldots, v_N \in V_N[\nu_N]$. For generic values of $\mu \in \mathfrak{h}'^*$ and $k$ we can consider the composition

$$\Phi^{v_1,\ldots v_N}_{\widetilde{\mu}}(z_1, \ldots, z_N) = \Phi^{v_1}_{\widetilde{\mu}-\nu_2-\ldots-\nu_N}(z_1)\cdots\Phi^{v_N}_{\widetilde{\mu}}(z_N) :$$

$$M_{\widetilde{\mu}} \to M_{\widetilde{\mu}-\nu_1-\ldots-\nu_N}\widehat{\otimes}V_1\widehat{\otimes}\cdots\widehat{\otimes}V_N,$$

It is easy to see that, for a given $\tilde{\mu}$, the matrix coefficients of $\Phi^{v_1,\ldots,v_N}_{\widetilde{\mu}}(z_1, \ldots, z_N)$ belong to the space $z_1^{l_1}\cdots z_N^{l_N}\mathbb{C}[[\frac{z_2}{z_1}, \ldots, \frac{z_N}{z_{N-1}}]]$ for some $l_1, \ldots, l_n \in \mathbb{Z}$.

**Theorem 4.1 ([FR],[EFK]).** *The matrix elements of the operator valued series $\Phi^{v_1,\ldots,v_N}_{\widetilde{\mu}}(z_1, \ldots, z_N)$ converge in the the region $z_1 \gg z_2 \gg \cdots \gg z_N$ and extend to meromorphic functions on $(\mathbb{C}^*)^N$.*

## 4.2 Trace functions

For any $\mu \in \mathfrak{h}'^*$ we define the linear operator $B : M_{\widetilde{\mu}} \to M_{B\widetilde{\mu}}$ by setting $B(u \cdot x_{\widetilde{\mu}}) = B(u)\cdot x_{B\widetilde{\mu}}$ for any $u \in U_q(\widetilde{\mathfrak{n}}_-)$.

Consider the following *trace function*, which will be the main object of study of this paper.

$$\Psi^{v_1,\ldots,v_N}_T(z_1, \ldots, z_N, \lambda, \omega, \mu, k) = \mathrm{Tr}_{|M_{\widetilde{\mu}}}(\Phi^{v_1,\ldots,v_N}_{B\widetilde{\mu}}(z_1, \ldots, z_N)Bq^{2\widetilde{\lambda}}),$$

where $\widetilde{\lambda} = \lambda + \omega\widetilde{\rho}/\mathrm{h}$, and $\lambda \in \mathfrak{l}^*$. Note that $\Psi^{v_1,\ldots,v_N}_T(z_1, \ldots, z_N, \lambda, \omega, \mu, k) = 0$ unless $\sum_i \nu_i = B\mu - \mu$, and that $\Psi$ is independent of $z_i$ if $N = 1$. Moreover, given $\nu_1, \ldots, \nu_N$, the set of all $\mu \in \mathfrak{h}'^*$ satisfying this condition is an $\mathfrak{l}^*$-homogeneous space. By Theorem 4.1, for (almost) every such fixed value of $\mu$ and $k$, the series $\Psi_T^{v_1,\dots,v_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k)$ converges in the region $z_1 \gg \cdots \gg z_N$ and extends to a meromorphic function on $(\mathbb{C}^*)^N$ with values in $q^{2(\lambda,\mu)}\mathcal{A}_\lambda[[q^{-2\omega}]]$.

Now we will consider $k$ as a formal parameter. Namely, we set

$$\mathcal{A}_\mu^- = \mathbb{C}[q^{-2(\mu,\alpha_1)},\dots,q^{-2(\mu,\alpha_r)}], \qquad \mathcal{A}_{\mu,k} = \mathcal{A}_\mu^-[[q^{-2k}]].$$

The following lemma is easy to prove.

**Lemma 4.2.** *The function $\Psi_T^{v_1,\dots,v_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k)$ has a series expansion, which lies in $(V_1\otimes\cdots\otimes V_N)^{\mathfrak{l}}\otimes q^{2(\lambda,\mu)}\mathcal{A}_{\lambda,\omega}\widehat{\otimes}\mathcal{A}_{\mu,k}$, where $\widetilde{\mu} = \mu + k\widetilde{\rho}/\mathrm{h}$ and $\widetilde{\lambda} = \lambda + \omega\widetilde{\rho}/\mathrm{h}$.*

We also define the following universal trace function, which takes values in $(V_1\otimes\cdots\otimes V_N)^{\mathfrak{l}}\otimes(V_N^*\otimes\cdots\otimes V_1^*)^{\mathfrak{l}}\otimes q^{2(\lambda,\mu)}\mathcal{A}_{\lambda,\omega}\widehat{\otimes}\mathcal{A}_{\mu,k}$ :

$$\begin{aligned}\Psi_T^{V_1,\dots,V_N}&(z_1,\dots,z_N,\lambda,\omega,\mu,k)\\ &= \sum_{v_i\in\mathcal{B}_i} \Psi_T^{v_1,\dots,v_N}(z_N,\dots,z_1,\lambda,\omega,\mu,k)\otimes v_N^*\otimes\cdots\otimes v_1^*\end{aligned}$$

where $\mathcal{B}_i$ is a homogeneous basis of $V_i$ and $\{v_i^*\}_{v_i\in\mathcal{B}_i}$ is the dual basis of $V_i^*$. This function is defined for $\mu\in\mathfrak{h}'^*$, but has support on finitely many shifted copies of $\mathfrak{l}^*$.

It is more convenient to renormalize the function $\Psi_T^{V_1,\dots,V_N}$ as follows. Define

$$\mathbb{J}_T^{1\dots N}(\widetilde{\lambda}) = \mathbb{J}_T^{1,2\dots N}(\widetilde{\lambda})\cdots\mathbb{J}_T^{N-1,N}(\widetilde{\lambda}),$$
$$\mathbb{J}_T^{1\dots N}(z_1,\dots,z_N,\widetilde{\lambda}) = (D_{z_1}\otimes\cdots\otimes D_{z_N})\mathbb{J}_T^{1\dots N}(\widetilde{\lambda}).$$

It is easy to check that

$$\mathbb{J}_T^{1\dots N}(z_1,\dots,z_N,\tilde{\lambda}) \in U_q(\widetilde{\mathfrak{g}})^{\otimes N}[[\frac{z_2}{z_1},\dots,\frac{z_N}{z_{N-1}}]]\widehat{\otimes}\mathcal{A}_{\lambda,\omega}.$$

Further, put $\mathbb{Q}_T(\widetilde{\lambda}) = m_{21}((1\otimes S)[(\mathcal{R}^{21})^{-1}\mathbb{J}_T(\widetilde{\lambda})])$ (here $m_{21}$ is the multiplication of components, i.e. $m_{21}(a\otimes b) := ba$). This is a formal power series lying in $U_q(\widetilde{\mathfrak{g}})\widehat{\otimes}\mathcal{A}_{\lambda,\omega}$. Note that $\mathbb{Q}_T(\widetilde{\lambda})$ is $B$-invariant and $D_z$-invariant. If $T=1$, we will drop the subscript $T$ and denote the corrseponding elements $\mathbb{J}_T,\mathbb{Q}_T$, $\mathbb{R}_T$ simply by $\mathbb{J}$, $\mathbb{R}$, $\mathbb{Q}$.

Finally, let

$$\delta_q^T(\lambda,\omega) = \left(\mathrm{Tr}_{|M_{-\widetilde{\rho}}}(Bq^{2(\lambda+\omega\widetilde{\rho}/\mathrm{h})})\right)^{-1}$$

be the twisted Weyl denominator. It can be written as an infinite product of binomial factors (see [ES1]).

Define the renormalized trace function by

$$\begin{aligned}&F_T^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k)\\&=[\mathbb{Q}^{-1}(\mu+h^{(*1\cdots*N)},k)^{(*N)}\otimes\cdots\otimes\mathbb{Q}^{-1}(\mu+h^{(*1)},k)^{(*1)}]\\&\qquad\qquad\varphi_T^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,-\mu-\rho,-k-\mathrm{h})\end{aligned}$$

where

$$\begin{aligned}&\varphi_T^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k)\\&\qquad=\mathbb{J}_T^{1\cdots N}(z_1,\dots,z_N,\lambda,\omega)^{-1}\Psi_T^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k)\delta_q^T(\lambda,\omega).\end{aligned}$$

(here $\mathbb{Q}(\lambda,\omega):=\mathbb{Q}(\widetilde{\lambda})$, $\mathbb{J}(\lambda,\omega):=\mathbb{J}(\widetilde{\lambda})))$. Note that $F_T^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k)$ is in the space

$$\mathcal{E}=(V_1\otimes\cdots\otimes V_N)^{\mathfrak{l}}\otimes(V_N^*\otimes\cdots\otimes V_1^*)^{\mathfrak{l}}((\frac{z_2}{z_1},\dots,\frac{z_N}{z_{N-1}}))\widehat{\otimes}q^{-2(\lambda,\mu)}\mathcal{A}_{\lambda,\omega}\widehat{\otimes}\mathcal{A}_{\mu,k}.$$

**Remark 1.** Here we regard the function $F_T$ as a formal power series. It is expected, but, as far as we know, not proved in general, that this series is actually convergent in some region, defining an analytic function. In the case $T=1$, it is expected that $F_T$ has an integral representation coming from the free bosonic realization, which could be used for a proof of analyticity. Such a presentation has not been worked out in general. However, in the case of $\mathfrak{g}=\mathfrak{sl}_2$, such presentation is known ([Ko1, Ko2]), which means that in this case the function $F_T$ for $T=1$ is analytic.

**Remark 2.** It is important to note that the element $\mathbb{Q}_T(\widetilde{\lambda})$ defined here is somewhat different from the one used in [EV, ES2]. This is done because the formula $m_{21}((1\otimes S^{-1})(\mathbb{J}_T(\lambda)))$ used in [EV, ES2] does not make sense for affine Lie algebras due to appearance of infinite summations. Indeed, at $q^{-2\omega}=0$ we have $\mathbb{J}=\mathcal{R}_0^{21}$, so already the computation of the leading term of $\mathbb{Q}_T$ as in [EV, ES2] in a finite dimensional representation $V$ would involve evaluating the R-matrix of $V,V^*$ at $z=1$ and multiplying the dual to the second component by the first component. However, this operation contains infinite summations, which are manifestly illegal, since the R-matrix $\mathcal{R}_{V,V^*}(z)$ has a pole at $z=1$. On the other hand, the definition of $\mathbb{Q}_T(\lambda)$ given here makes sense, because the expression $(\mathcal{R}_0^{21})^{-1}\mathbb{J}$ is a Taylor series in $q^{-2\omega}$, whose coefficients involve only finite summations.

Now we explain the connection between our definition and that of [ES2] in more detail.

In the finite dimensional case, both definitions make sense, and the relationship is given by the formula $\mathbb{Q}_T(\lambda)=S(u)^{-1}\mathbb{Q}_T^{ES2}(\lambda)$, where $u$ is the

Drinfeld element [D], $u = m_{21}((1\otimes S)(\mathcal{R}))$, The trace functions $F_T^{V_1,\ldots,V_N}$ can be defined using either definition: the difference would be just a constant factor.

In the affine case, neither of the factors $S(u)^{-1}$ and $\mathbb{Q}_T^{ES2}(\widetilde{\lambda})$ makes sense because of infinite summations, but the product $S(u)^{-1}\mathbb{Q}_T(\tilde{\lambda})$ does. So we use this product as a substitute for the non-existenct $\mathbb{Q}_T^{ES2}(\widetilde{\lambda})$.

**Remark 3.** One may also consider traces over integrable modules rather than Verma modules. In the particular case $\mathfrak{g} = \mathfrak{sl}_n$, $N = 1$, $V_1 = S^{kn}\mathbb{C}^n$ they are affine analogues of Macdonald's polynomials and were considered in [EK1].

# 5 The twisted Macdonald-Ruijsenaars equations

## 5.1 The equations

Let $W$ be an integrable lowest weight $U_q(\widetilde{\mathfrak{g}})$-module of level $\eta$ (so $\eta$ is a nonpositive integer). Consider the following difference operator with respect to the variable $\lambda$ acting on functions in the space $\mathcal{E}$:

$$\mathcal{D}_W^T = \sum_{\nu\in\mathfrak{l}^*,a\in\mathbb{C}} \mathrm{Tr}_{|W[\nu+a\delta+\eta\widetilde{\rho}/\mathrm{h}]}\big(\mathbb{R}_T^{WV_1}(z_1^{-1},\lambda+h^{(2\cdots N)},\omega)\cdots\mathbb{R}_T^{WV_N}(z_N^{-1},\lambda,\omega)\big)q^{-2ka}\mathbb{T}_\nu \tag{5.1}$$

where $\mathbb{T}_\nu f(\lambda) = f(\lambda+\nu)$, and $\mathbb{R}(z,\lambda,\omega) := \mathbb{R}(z,\widetilde{\lambda})$. In the above, we only consider the trace of the "diagonal block" of the product $\mathbb{R}_T^{WV_1}(z_1^{-1},\lambda+h^{(2\cdots N)},\omega)\cdots\mathbb{R}_T^{WV_N}(z_N^{-1},\lambda,\omega)$, i.e the part that preserves $W[\nu+a\delta+\eta\widetilde{\rho}/\mathrm{h}]$.

**Theorem 5.1 (Macdonald-Ruijsenaars equations).**

$$\begin{aligned}\mathcal{D}_W^T F_T^{V_1,\ldots,V_N}&(z_1,\ldots z_N,\lambda,\omega+\eta,\mu,k)\\ &= \chi_W(q^{-2\widetilde{\mu}})F_T^{V_1,\ldots,V_N}(z_1,\ldots,z_N,\lambda,\omega,\mu,k),\end{aligned} \tag{5.2}$$

*where $\chi_W(x) = \sum \dim W[\nu]x^\nu$ is the character of $W$ and where $\mathcal{D}_W^T$ acts on the variable $\lambda$.*

**Remark 1.** Note that equation (5.2) is an infinite difference equation. That is, the difference operator on the left hand side involves an infinite linear combination of shifts by different weights. Nevertheless, it makes sense as an operator on power series in the formal parameters $q^{-2\omega}$ and $q^{-2k}$, since the coefficient of any fixed power of $q^{-2k}$ is an honest (finite) difference operator. Such operators were previously considered in [E3]. Abusing terminology, we will call them "difference operators", without specifying that they are infinite.

**Remark 2.** We do not know a simple explicit formula for the operator 5.1 in any nontrivial case.

Let us denote by $W^B$ the twist of $W$ by $B$: as a vector space $W = W^B$ and the $U_q(\widetilde{\mathfrak{g}})$-action is given by $\pi_{W^B}(u) = \pi_W(B^{-1}(u))$. Now suppose that $W \simeq W^B$ as $U_q(\widetilde{\mathfrak{g}})$-modules and let us consider the intertwiner $W \to W^B$ which preserves the lowest weight vector (abusing notation, we will denote it by the same letter $B$). This endows $W$ with the structure of a module over $\mathbb{C}[\langle B\rangle] \ltimes U_q(\widetilde{\mathfrak{g}})$, where $\langle B\rangle$ is the cyclic group generated by $B$.

Consider the following difference operator with respect to the variable $\widetilde{\mu}$ acting on functions in the space $\mathcal{E}$:

$$\mathcal{D}_W^{\vee,T} = \sum_{\nu,a} \mathrm{Tr}_{|W[\nu+a\delta+\eta\widetilde{\rho}/\mathrm{h}]}\big(\mathbb{R}^{WV_N^*}(z_N^{-1}, \mu + h^{(*1\cdots *N-1)}, k)\cdots \mathbb{R}^{WV_1^*}(z_1^{-1}, \mu, k) B_W\big) q^{-2\omega a}\mathbb{T}_\nu^\vee \tag{5.3}$$

where the sign $\vee$ means that the operators act on functions of $\mu$ rather than $\lambda$.

**Theorem 5.2 (Dual Macdonald-Ruijsenaars equations).**

$$\begin{aligned}\mathcal{D}_W^{\vee,T} F_T^{V_1,\dots,V_N}(z_1,\dots,z_n,\lambda,\omega,\mu,k+\eta) \\ = \mathrm{Tr}_{|W}(q^{-2\widetilde{\lambda}}B) F_T^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k).\end{aligned} \tag{5.4}$$

**Remark.** Any $B$-invariant integrable lowest weight $U_q(\widetilde{\mathfrak{g}})$-module is a direct sum of modules $\overline{V}^-_{\widetilde{\nu}_0} := \bigoplus_{\widetilde{\nu}\in\langle B\rangle\widetilde{\nu}_0} V^-_{\widetilde{\nu}}$, where $V^-_{\widetilde{\nu}}$ is the irreducible lowest weight module of lowest weight $-\widetilde{\nu}$, and $\widetilde{\nu}_0$ is a dominant integral weight. It is easy to see that both sides of (5.4) identically vanish when $W = \overline{V}_{\widetilde{\nu}_0}$ and $\widetilde{\nu}_0 \notin \mathfrak{l}^*$ (i.e when $B(\widetilde{\nu}_0) \neq \widetilde{\nu}_0$).

## 5.2 The proofs

The proof of Theorem 5.1 is an extension of the proofs of Theorem 2.1 of [ES2] and Theorem 1.1 of [EV]. We first introduce the notion of (twisted) radial part.

**Proposition 5.1.** *(see [E3]) Let $V$ be a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module. For any element $X \in U_q(\widehat{\mathfrak{g}})$ there exists a unique difference operator (with respect to the variable $\lambda$) $\mathcal{D}_X^T$ acting on power series in the space $V^{\mathfrak{l}} \otimes q^{2(\lambda,\mu)}\mathcal{A}_{\lambda,\omega}$ such that we have*

$$\mathrm{Tr}_{|M_\mu}(\Phi_{B\mu}^V X q^{2\widetilde{\lambda}} B) = \mathcal{D}_X^T \mathrm{Tr}_{|M_\mu}(\Phi_{B\mu}^V q^{2\widetilde{\lambda}} B).$$

The operator $\mathcal{D}_X^T$ is called the twisted radial part of $X$.

Let $W$ be any lowest-weight integrable $U_q(\widetilde{\mathfrak{g}})$-module. Consider the expression

$$C_W = \mathrm{Tr}_{|W}(1 \otimes \pi_W)(\mathcal{R}^{21}\mathcal{R}(1 \otimes q^{2\widetilde{\rho}})).$$

Set $\mathcal{M}_W^T = \mathcal{D}_{C_W}^T$.

**Proposition 5.2.** *We have*

1. $\mathcal{M}_W^T \mathcal{M}_{W'}^T = \mathcal{M}_{W'}^T \mathcal{M}_W^T$ *for any two integrable modules $W, W'$,*
2. *For any finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module $V$ we have $\mathcal{M}_W^T \Psi_T^V(z, \lambda, \omega, \mu, k) = \chi_W(q^{2(\widetilde{\mu}+\widetilde{\rho})})\Psi_T^V(\lambda, \omega, \mu, k)$ where $\chi_W(x) = \sum_\nu \dim W[\nu] x^\nu$ is the character of $W$.*

*Proof.* See [EK], [EV].

The computation of the operator $\mathcal{M}_W^T$ is now identical to that in [ES2], Section 3. We obtain

$$\begin{aligned}\mathcal{M}_W^T \Psi_V^T(\lambda, \omega, \mu, k) = \sum_{\nu, a} \Big\{ & q^{2(k+\mathrm{h})a} \frac{\delta_q^T(\lambda+\nu, \omega+\eta)}{\delta_q^T(\lambda, \omega)} \times \\ & \mathrm{Tr}_{|W[\nu+a\delta+\eta\widetilde{\rho}/\mathrm{h}]}(\mathbb{R}_T^{WV}(z^{-1}, \lambda, \omega))\Psi_T^V(z, \lambda+\nu, \omega+\eta, \mu, k)\Big\}.\end{aligned} \tag{5.5}$$

Theorem 5.1 now follows from (5.5) and from the fusion identity

$$\mathbb{J}_T^{1\cdots N}(\widetilde{\lambda})^{-1}\mathbb{R}_T^{0,1\cdots N}(\widetilde{\lambda}))\mathbb{J}_T^{1\cdots N}(\widetilde{\lambda}+h^{(0)} = \mathbb{R}_T^{01}(\widetilde{\lambda}+h^{(2\cdots N)})\cdots(\mathbb{R}_T^{0N}(\widetilde{\lambda})). \tag{5.6}$$

The proof of Theorem 5.2 is completely parallel to the proofs of Theorem 2.3 in [ES2] and Theorem 1.2 in [EV].

# 6 The twisted qKZB and dual qKZB equations

Let $V_1, \dots, V_N$ be irreducible finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules. For each $j \in \{1, \dots, N\}$ we define the following operators acting on the space $\mathcal{E}$ :

$$\begin{aligned} D_j^T =& q_{*j}^{-2\mu - C_{\mathfrak{h}'}} q_{*j,*1}^{-2\Omega_{\mathfrak{h}'}} \cdots q_{*j,*j-1}^{-2\Omega_{\mathfrak{h}'}}, \end{aligned} \tag{6.1}$$

$$\begin{aligned} K_j^T =& \mathbb{R}_T^{j+1,j}(\frac{z_{j+1}}{z_j}, \lambda + h^{(j+2,\dots,N)}, \omega)^{-1} \cdots \mathbb{R}_T^{Nj}(\frac{z_N}{z_j}, \lambda, \omega)^{-1} \Gamma_j \times \\ & \mathbb{R}_T^{j1}(\frac{z_j}{z_1}, \lambda + h^{(2\dots,j-1)} + h^{(j+1\dots,N)}, \omega) \times \cdots \mathbb{R}_T^{j,j-1}(\frac{z_j}{z_{j-1}}, \lambda + h^{(j+1\dots,N)}, \omega) \end{aligned} \tag{6.2}$$

where $C_{\mathfrak{h}'} = m_{12}(\Omega_{\mathfrak{h}'}) \in U(\mathfrak{h}')$ is the quadratic Casimir element for $\mathfrak{h}'$, and where $\Gamma_j f(\lambda) = f(\lambda + h^{(j)})$.

**Theorem 6.1 (qKZB equations).** *For all $j = 1, \dots, N$, we have :*

$$\begin{aligned} & F_T^{V_1,\dots,V_N}(z_1, \dots, pz_j, \dots, z_N, \lambda, \omega, \mu, k) \\ & \qquad = (D_j^T \otimes K_j^T) F_T^{V_1,\dots,V_N}(z_1, \dots, z_N, \lambda, \omega, \mu, k), \end{aligned} \tag{6.3}$$

*where $p = q^{2k}$.*

Similarly, consider the operators

$$\begin{aligned} D_j^{\vee,T} =& q_j^{-2\lambda - C_{\mathfrak{l}}} q_{j,j+1}^{-\Omega_{\mathfrak{l}}} \cdots q_{j,N}^{-\Omega_{\mathfrak{l}}}, \\ K_j^{\vee,T} =& \mathbb{R}^{*j-1,*j}(\frac{z_{j-1}}{z_j}, \mu + h^{(*1\cdots *j-2)}, k)^{-1} \cdots \mathbb{R}^{*1,*j}(\frac{z_1}{z_j}, \mu, k)^{-1} \Gamma^*_{B^{-1}(j)} \times \\ & \mathbb{R}^{*j,*N}(\frac{z_j}{z_N}, \mu + h^{(*j+1\cdots *N-1)} + h^{(*1\cdots *j-1)}, k) \times \cdots \times \\ & \mathbb{R}^{*j,*j+1}(\frac{z_j}{z_{j+1}}, \mu + h^{(*1\cdots *j-1)}, k), \end{aligned} \tag{6.4}$$

where $C_{\mathfrak{l}} = m_{12}(\Omega_{\mathfrak{l}}) \in U(\mathfrak{l})$ and where $(\Gamma^*_{B^{-1}(j)} f)(\mu) = f(\mu + B^{-1}(h^{(*j)}))$.

**Theorem 6.2 (Dual qKZB equations).** *For all $j = 1 \dots, N$, we have :*

$$\begin{aligned} & F_T^{V_1,\dots,V_j,\dots,V_N}(z_1, \dots, sz_j, \dots, z_N, \lambda, , \omega, \mu, k) \\ & = (D_j^{\vee,T} \otimes K_j^{\vee,T}) F_T^{V_1,\dots,V_j^B,\dots,V_N}(z_1, \dots, z_N, \lambda, \omega, \mu, k), \end{aligned} \tag{6.5}$$

*where $s = q^{2\omega}$, and we identify $V_j$ with $V_j^B$.*

The proof of Theorem 5.1 is very similar to the proof of Theorem 2.2 of [ES2] and Theorem 1.2 of [EV] (with modifications described in Remark 2 in Section 4). The proof of the dual qKZB equation runs parallel to [ES2], Section 6.

# 7 The symmetry identity

In this section we assume that $T = 1$. Thus $\mathfrak{l} = \mathfrak{h}'$ and the renormalized trace function $F^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k)$ takes values in the space

$$(V_1\otimes\cdots\otimes V_N)[0]\otimes(V_N^*\otimes\cdots\otimes V_1^*)[0]((\frac{z_2}{z_1},\dots,\frac{z_N}{z_{N-1}}))\widehat{\otimes}q^{-2(\lambda,\mu)}\mathcal{A}_{\lambda,\omega}\widehat{\otimes}\mathcal{A}_{\mu,k}.$$

As in the case of finite-dimensional simple Lie algebra $\mathfrak{g}$ (see [EV]), we have the following symmetry identity :

**Theorem 7.1.**

$$F^{V_1,\dots,V_N}(z_1,\dots,z_N,\lambda,\omega,\mu,k) = F_*^{V_N^*,\dots,V_1^*}(z_N,\dots,z_1,\mu,k,\lambda,\omega),$$

*where $F_*$ is the result of interchanging the two factors $(V_1\otimes\cdots\otimes V_N)[0]$ and $(V_N^*\otimes\cdots\otimes V_1^*)[0]$.*

This result follows from Theorem 4.1 and Theorem 4.2 and the arguments in [EV], Section 5.

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