THE ZERO-IN-THE-SPECTRUM QUESTION

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ABSTRACT. This is an expository article on the question of whether zero lies in the spectrum of the Laplace-Beltrami operator acting on differential forms on a manifold.

1. INTRODUCTION

Let M be a complete connected oriented Riemannian manifold. The Laplace-Beltrami operator $\triangle_p$ acts on the square-integrable p-forms on M. We asked the following question in 1991 :

Zero-in-the-Spectrum Question : Is zero always in the spectrum of $\triangle_p$ for some p?

To our knowledge, nobody has found a counterexample. The question was also raised by Gromov in the case of a contractible manifold with a discrete cocompact group of isometries [14, p. 21].

Being able to answer the above question is a first step toward understanding the spectrum of the Laplace-Beltrami operator. We would also like to be able to say whether or not zero is in the spectrum of $\triangle_p$ for a given p. This problem is partly topological in nature and partly geometric, in a sense which will be made precise later. In fact, it is equivalent to knowing the (unreduced) L^2-cohomology of M. The study of L^2-cohomology touches on many branches of mathematics, including combinatorial group theory, topology, differential geometry and algebraic geometry. It is most commonly considered when M is the universal cover of a compact manifold or when M is a finite-volume Hermitian locally symmetric space. We refer to [21, 25] and [29] for surveys of these two cases. In this article we will instead emphasize general complete Riemannian manifolds and give some partial positive answers to the zero-in-the-spectrum question.

The sections of the article are

Date: April 14, 1996.
Research supported by NSF grant DMS-9403652.

In what follows, all manifolds will be smooth, connected, oriented and of positive dimension. All maps between manifolds will be orientation-preserving. Unless otherwise indicated, all Riemannian manifolds will be complete.

We have tried to give as many complete proofs as reasonably possible. All unattributed results are of unknown origin or are due to the author. I thank Wolfgang Lück for conversations on some of the topics discussed herein. I thank Marie-Claude Vergne for making the figures. This article is based on lectures given at the Troisième Cycle Romand held at Les Diablerets, Switzerland, March, 1996. I warmly thank Alain Valette and the other organizers and participants of the meeting.

2. Definition of L^2-Cohomology

Let M be as above. Let $\Lambda^p(M)$ denote the Hilbert space of square-integrable p-forms on M. The completeness of M enters in one crucial way, in allowing us to integrate by parts on M in the sense of the following lemma.

Lemma 1. *(Gaffney [12]) Suppose that ω, η, $d\omega$ and $d\eta$ are smooth square-integrable differential forms on M. Then*

$$\int_M d\omega \wedge \eta + (-1)^{deg(\omega)} \int_M \omega \wedge d\eta = 0. \tag{2.1}$$

Proof. We claim that there is a sequence $\{\phi_i\}_{i=1}^{\infty}$ of compactly-supported functions on M with the properties that

1. There is a constant $C > 0$ such that for all i and almost all $m \in M$, $|\phi_i(m)| \le C$ and $|d\phi_i(m)| \le C$.
2. For almost all $m \in M$, $\lim_{i\to\infty} \phi_i(m) = 1$ and $\lim_{i\to\infty} |d\phi_i(m)| = 0$.

To construct the sequence $\{\phi_i\}_{i=1}^{\infty}$, let m_0 be a basepoint in M. Let $f \in C_0^\infty([0,\infty))$ be a nonincreasing function such that if $x \in [0, \frac{1}{2}]$ then $f(x) = 1$. Put $\phi_i(m) = f\left(\frac{1}{i} d(m_0, m)\right)$. This gives the desired sequence. The completeness of M ensures that ϕ_i is compactly-supported. Note that ϕ_i is *a priori* only a Lipschitz function, but this is good enough for our purposes.

Using Lebesgue Dominated Convergence and the fact that we can integrate by parts for compactly-supported forms, we have

$$\begin{aligned} \int_M d\omega \wedge \eta + (-1)^{deg(\omega)} \int_M \omega \wedge d\eta &= \int_M d(\omega \wedge \eta) = \lim_{i\to\infty} \int_M \phi_i \, d(\omega \wedge \eta) \\ &= - \lim_{i\to\infty} \int_M d\phi_i \wedge \omega \wedge \eta = 0. \end{aligned} \tag{2.2}$$

This proves the lemma. □

Let d^* be the formal adjoint to d. Using Lemma 1, one can construct a self-adjoint operator $\triangle = dd^* + d^*d$ acting on $\Lambda^*(M)$, with domain

$$\mathrm{Dom}(\triangle) = \{\omega \in \Lambda^*(M) : d\omega,\ d^*\omega,\ dd^*\omega \text{ and } d^*d\omega \text{ are square-integrable}\}. \tag{2.3}$$

Let $\triangle_p$ denote the restriction of $\triangle$ to $\Lambda^p(M)$. The spectrum $\sigma(\triangle_p)$ of $\triangle_p$ is a closed subset of $[0, \infty)$.

Lemma 2. *The kernel of* $\triangle_p$ *is* $\{\omega \in \Lambda^p(M) : d\omega = d^*\omega = 0\}$.

Proof. Clearly $\{\omega \in \Lambda^p(M) : d\omega = d^*\omega = 0\} \subseteq \mathrm{Ker}(\triangle_p)$. If $\omega \in \mathrm{Ker}(\triangle_p)$ then by elliptic regularity, ω is smooth. Using integration by parts, $0 = \langle \omega, \triangle_p \omega \rangle = \langle d\omega, d\omega \rangle + \langle d^*\omega, d^*\omega \rangle$, so $d\omega = d^*\omega = 0$. □

Warning : Unlike what happens with compact manifolds, it is possible that $\mathrm{Ker}(\triangle_p) = 0$ but nevertheless $0 \in \sigma(\triangle_p)$. The simplest example of this is when $M = \mathbb{R}$ and $p = 0$. By Lemma 2, $\mathrm{Ker}(\triangle_0)$ consists of square-integrable functions f on $\mathbb{R}$ such that $df = 0$. Clearly the only such function is the zero function. However, under Fourier transform, $\triangle_0$ is equivalent to the multiplication operator by k^2 on $L^2(\mathbb{R})$ and hence $\sigma(\triangle_0) = [0, \infty)$.

Examples : We now give $\sigma(\triangle_p)$ for simply-connected space forms.

1. M is the standard sphere S^n. From [13],

$$\sigma(\triangle_p) = \{(k+p)(k+n+1-p)\}_{k=0}^{\infty} \cup \{(k+p+1)(k+n-p)\}_{k=0}^{\infty}. \tag{2.4}$$

(See Fig. 1.) The details of the spectrum are not important for us. We only wish to note that $\sigma(\triangle_p)$ is discrete, and $0 \in \sigma(\triangle_p)$ if $p = 0$ or $p = n$. These statements are a consequence of the fact that M is closed. Namely, if M^n is any closed Riemannian manifold then $\sigma(\triangle_p)$ is discrete and $\mathrm{Ker}(\triangle_p) \cong \mathrm{H}^p(M; \mathbb{C})$. In particular, $\mathrm{Ker}(\triangle_0) \cong \mathrm{H}^0(M; \mathbb{C}) = \mathbb{C}$ consists of the constant functions and $\mathrm{Ker}(\triangle_n) \cong \mathrm{H}^n(M; \mathbb{C}) = \mathbb{C}$ consists of multiples of the volume form.

2. M is the standard Euclidean space $\mathbb{R}^n$. As the p-forms on $\mathbb{R}^n$ consist of $\binom{n}{p}$ copies of the functions, it is enough to consider $\sigma(\triangle_0)$. By Fourier analysis, $\sigma(\triangle_0) = [0, \infty)$. Thus $\sigma(\triangle_p) = [0, \infty)$ for all $0 \le p \le n$. (See Fig. 2.) Note that $\mathrm{Ker}(\triangle_p) = 0$ for all p.

3. M is the hyperbolic space H^{2n}. From [8],

$$\sigma(\triangle_p) = \begin{cases} \left[\frac{(2n-2p-1)^2}{4}, \infty\right) & \text{if } 0 \le p \le n-1, \\ \{0\} \cup \left[\frac{1}{4}, \infty\right) & \text{if } p = n, \\ \left[\frac{(2p-2n-1)^2}{4}, \infty\right) & \text{if } n+1 \le p \le 2n. \end{cases}$$

(See Fig. 3.) There is an infinite-dimensional kernel to $\triangle_n$. Otherwise, the spectrum is strictly bounded away from zero.

4. M is the hyperbolic space H^{2n+1}. From [8],

$$\sigma(\triangle_p) = \begin{cases} \left[\frac{(2n-2p)^2}{4}, \infty\right) & \text{if } 0 \le p \le n, \\ \left[\frac{(2p-2n-2)^2}{4}, \infty\right) & \text{if } n+1 \le p \le 2n+1. \end{cases}$$

(See Fig. 4.) For all p, $\mathrm{Ker}(\triangle_p) = 0$. The continuous spectrum extends down to zero in degrees n and $n+1$, and is strictly bounded away from zero in other degrees.
End of Examples

Comparing Figures 1-4, the spectra do not have much in common. However, one common feature is that zero lies in $\sigma(\triangle_p)$ for some p, although for different reasons in the different cases. In Figure 1, it is because $\triangle_0$ has a nonzero finite-dimensional kernel. In Figure 2, it is because zero lies in the continuous spectrum of $\triangle_p$ for all p. In Figure 3, it is because $\triangle_n$ has an infinite-dimensional kernel. And in Figure 4, it is because zero lies in the continuous spectrum of $\triangle_p$ for $p = n$ and $p = n+1$.

The above examples, along with others, motivate the zero-in-the-spectrum question. One can pose the question for various classes of manifolds, such as
1. Complete Riemannian manifolds.
2. Complete Riemannian manifolds of bounded geometry, meaning that the injectivity radius is positive and the sectional curvature K satisfies $|K| \le 1$.
3. Uniformly contractible Riemannian manifolds, meaning that for all $r > 0$, there is an $R(r) \ge r$ such that for all $m \in M$, the metric ball $B_r(m)$ can be contracted to a point within $B_{R(r)}(m)$.
4. Universal covers of closed Riemannian manifolds.
5. Universal covers of closed aspherical Riemannian manifolds.

There are obvious inclusions
$$\begin{array}{ccccc} 5 & \subset & 4 & \subset & 2 \\ \cap & & & & \cap \\ 3 & & \subset & & 1. \end{array}$$
As we shall discuss, there are some reasons to believe that the answer to the zero-in-the-spectrum question is "yes" in class 5, but the evidence for a "yes" answer in class 1 consists mainly of a lack of counterexamples.

In order to make the study of the spectrum of $\triangle_p$ more precise, the Hodge decomposition

$$\Lambda^p(M) = \mathrm{Ker}(\triangle_p) \oplus \overline{\mathrm{Im}(d)} \oplus \Lambda^p(M)/\mathrm{Ker}(d) \tag{2.5}$$

is useful. The operator $\triangle_p$ decomposes with respect to (2.5) as a direct sum of three operators. If we know the spectrum of the Laplace-Beltrami operator on all forms of degree less than p then the new information in degree p consists of $\mathrm{Ker}(\triangle_p)$ and the spectrum of $\triangle_p$ on $\Lambda^p(M)/\mathrm{Ker}(d)$. So we can ask the more precise questions :
1. What is $\dim(\mathrm{Ker}(\triangle_p))$?
2. Is zero in $\sigma\,(\triangle_p$ on $\Lambda^p(M)/\mathrm{Ker}(d))$?

By its definition, $\triangle_p$ involves the first derivatives of the metric tensor. We now show that the answer to the zero-in-the-spectrum question only depends on the C^0-properties of the metric tensor. To do so, we reformulate the question in terms of L^2-cohomology. Define a subspace $\Omega^p(M)$ of $\Lambda^p(M)$ by

$$\Omega^p(M) = \{\omega \in \Lambda^p(M) : d\omega \text{ is square-integrable}\}, \tag{2.6}$$

where $d\omega$ is initially interpreted in a distributional sense. The subspace $\Omega^p(M)$ is cooked up so that we have a cochain complex

$$\dots \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \Omega^{p+1}(M) \xrightarrow{d_{p+1}} \dots \tag{2.7}$$

Lemma 3. $\mathrm{Ker}(d_p)$ *is a closed subspace of* $\Lambda^p(M)$.

Proof. Suppose that $\{\eta_i\}_{i=1}^{\infty}$ is a sequence in $\mathrm{Ker}(d_p)$ which converges to $\omega \in \Lambda^p(M)$ in an L^2-sense. We must show that the distributional form $d\omega$ vanishes. Given a smooth compactly-supported $(p+1)$-form ρ, we have

$$\langle d\omega, \rho\rangle = \langle \omega, d^*\rho\rangle = \lim_{i\to\infty}\langle \eta_i, d^*\rho\rangle = \lim_{i\to\infty}\langle d\eta_i, \rho\rangle = 0. \tag{2.8}$$

The lemma follows. □

Definition 1. *The p-th unreduced L^2-cohomology group of M is* $\mathrm{H}^p_{(2)}(M) = \mathrm{Ker}(d_p)/\mathrm{Im}(d_{p-1})$. *The p-th reduced L^2-cohomology group of M is* $\overline{\mathrm{H}}^p_{(2)}(M) = \mathrm{Ker}(d_p)/\overline{\mathrm{Im}(d_{p-1})}$, *a Hilbert space.*

The square-integrability condition on the forms should be thought of as a global decay condition, not as a local regularity condition. One can also compute $\mathrm{H}^*_{(2)}(M)$ using a complex as in (2.7) where the forms are additionally required to be smooth [19, Prop. 9].

There is an obvious surjection $i_p : \mathrm{H}^p_{(2)}(M) \to \overline{\mathrm{H}}^p_{(2)}(M)$. Clearly i_p is an isomorphism if and only if d_{p-1} has closed image.

Proposition 1. *1.* $\mathrm{Ker}(\Delta_p) \cong \overline{\mathrm{H}}^p_{(2)}(M)$.
2. $0 \notin \sigma\,(\Delta_p$ *on* $\Lambda^p(M)/\mathrm{Ker}(d))$ *if and only if* i_{p+1} *is an isomorphism.*

Proof. 1. Using Lemma 2, we have

$$\mathrm{Ker}(\Delta_p) = \{\omega \in \Lambda^p(M) : d\omega = d^*\omega = 0\} = \mathrm{Ker}(d_p) \cap \overline{\mathrm{Im}(d_{p-1})}^{\perp} \cong \overline{\mathrm{H}}^p_{(2)}(M). \tag{2.9}$$

The first part of the proposition follows.
2. Suppose first that Δ_p has a bounded inverse on $\Lambda^p(M)/\mathrm{Ker}(d)$. Given $\mu \in \Lambda^p(M)$, let $\overline{\mu}$ denote its class in $\Lambda^p(M)/\mathrm{Ker}(d)$. Define an operator S on smooth compactly-supported $(p+1)$-forms by $S(\omega) = d\Delta_p^{-1}\overline{d^*\omega}$. Then S extends to a bounded operator on $\Lambda^{p+1}(M)$. Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence in $\Omega^p(M)$ such that $\lim_{i\to\infty} d\eta_i = \omega$ for some $\omega \in \Lambda^{p+1}(M)$. Then for each i, we have $d\eta_i = S(d\eta_i)$ and so $\omega = S(\omega)$. Thus $\omega \in \mathrm{Im}(d)$ and so $\mathrm{Im}(d)$ is closed.

Now suppose that Δ_p does not have a bounded inverse on $\Lambda^p(M)/\mathrm{Ker}(d)$. Then there is a sequence of positive numbers $r_1 > s_1 > r_2 > s_2 > \ldots$ tending towards zero and an orthonormal sequence $\{\eta_i\}_{i=1}^{\infty}$ in $\Lambda^p(M)/\mathrm{Ker}(d)$ such that with respect to the spectral projection P of Δ_p (acting on $\Lambda^p(M)/\mathrm{Ker}(d)$), $\eta_i \in \mathrm{Im}(P([s_i, r_i]))$. Put $\lambda_i = \|d\eta_i\|$. Then $\lim_{i\to\infty}\lambda_i = 0$. Let $\{c_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{R}^+$ such that $\sum_{i=1}^{\infty} c_i^2 = \infty$ and $\sum_{i=1}^{\infty} c_i\lambda_i < \infty$. Put $\omega = \sum_{i=1}^{\infty} c_i d\eta_i$. Then $\omega \in \overline{\mathrm{Im}(d)}$. Suppose that $\omega = d\mu$ for some $\mu \in \Omega^p(M)$. By the spectral theorem, we must have $\overline{\mu} = \sum_{i=1}^{\infty} c_i\eta_i$. However, this is not square-integrable. Thus $\mathrm{Im}(d)$ is not closed. The proposition follows. □

Corollary 1. *Zero does not lie in* $\sigma(\Delta_p)$ *for any p if and only if* $\mathrm{H}^p_{(2)}(M) = 0$ *for all p, i.e. if the complex (2.7) is contractible.*

So a counterexample to the zero-in-the-spectrum question would consist of a manifold M whose complex (2.7) is contractible. By way of comparison, recall that the compactly-supported complex-valued cohomology of M is computed by a cochain complex similar to (2.7), except using compactly-supported smooth forms. As $\mathrm{H}_c^{dim(M)}(M;\mathbb{C}) \neq 0$, this latter complex is never contractible. And the ordinary complex-valued cohomology of M is computed by a cochain complex similar to (2.7), except using smooth forms without any decay conditions. Again, as $\mathrm{H}^0(M;\mathbb{C}) \neq 0$, this latter complex is never contractible.

If M is closed then $\overline{\mathrm{H}}^*_{(2)}(M)$ is independent of the Riemannian metric on M. This is no longer true if M is not closed - consider $\mathbb{R}^2$ and H^2. However, the L^2-cohomology groups of M do have some invariance properties which we now discuss.

Definition 2. *Riemannian manifolds M and M' are biLipschitz diffeomorphic if there is a diffeomorphism $F : M \to M'$ and a constant $K > 0$ such that the Riemannian metrics g and g' satisfy the pointwise inequality*

$$K^{-1} g \leq F^* g' \leq K g. \tag{2.10}$$

If M and M' are biLipschitz diffeomorphic then their reduced and unreduced L^2-cohomology groups are isomorphic, as the Riemannian metric only enters in the complex (2.7) in determining which forms are square-integrable. Thus the answer to the zero-in-the-spectrum question only depends on the biLipschitz diffeomorphism class of M. More generally, we can consider a category whose objects are Lipschitz Riemannian manifolds and whose morphisms are Lipschitz maps. Then the reduced and unreduced L^2-cohomology groups are Lipschitz-homotopy-invariants.

Note that L^2-cohomology groups are not coarse quasi-isometry invariants. For example, any closed manifold is coarsely quasi-isometric to a point, but its L^2-cohomology is the same as its ordinary complex-valued cohomology, which may not be that of a point. However, some aspects of L^2-cohomology only depend on the large-scale geometry of the manifold.

Proposition 2. [19, Prop. 12] *If M and M' are isometric outside of compact sets then*
1. $\mathrm{Ker}(\triangle_p)$ is finite-dimensional on M if and only if it is finite-dimensional on M'.
2. Zero is in $\sigma\left(\triangle_p \text{ on } \Lambda^p/\mathrm{Ker}(d)\right)$ on M if and only if the same statement is true on M'.

Consider uniformly contractible Riemannian manifolds of bounded geometry. If two such manifolds are coarsely quasi-isometric then they are Lipschitz-homotopy-equivalent and hence their L^2-cohomology groups are isomorphic [14, p. 219]. The next proposition gives an extension of this result in which uniform contractibility is replaced by uniform vanishing of cohomology, the latter being defined as follows.

Definition 3. *We say that $\mathrm{H}^j(M;\mathbb{C})$ vanishes uniformly if for all $r > 0$, there is an $R(r) \geq r$ such that for all $m \in M$,*

$$\mathrm{Im}\left(H^j(B_{R(r)}(m);\mathbb{C}) \to H^j(B_r(m);\mathbb{C})\right) = 0. \tag{2.11}$$

Proposition 3. *(Pansu [24]) Consider a Riemannian manifold M of bounded geometry such that for some $k > 0$, $\mathrm{H}^j(M;\mathbb{C})$ vanishes uniformly for $1 \leq j \leq k$. Then within the class of such manifolds,*
1. $\overline{\mathrm{H}}^p_{(2)}(M)$ and $\mathrm{H}^p_{(2)}(M)$ are coarse quasi-isometry invariants for $0 \leq p \leq k$.
2. $\mathrm{Ker}(\overline{\mathrm{H}}^{k+1}_{(2)}(M) \to \mathrm{H}^{k+1}(M;\mathbb{C}))$ and $\mathrm{Ker}(\mathrm{H}^{k+1}_{(2)}(M) \to \mathrm{H}^{k+1}(M;\mathbb{C}))$ are coarse quasi-isometry invariants.

3. General Properties of L^2-Cohomology

In this section we give some general results about the L^2-cohomology of complete Riemannian manifolds. First, we give a useful sufficient condition for the reduced L^2-cohomology to be nonzero.

Proposition 4. *For all* p, $\mathrm{Im}\,(\mathrm{H}^p_c(M;\mathbb{C}) \to \mathrm{H}^p(M;\mathbb{C}))$ *injects into* $\overline{\mathrm{H}}^p_{(2)}(M)$.

Proof. Suppose that ω is a smooth compactly-supported closed p-form which represents a nonzero class in $\mathrm{H}^p(M;\mathbb{C})$. By Poincaré duality, there is a smooth compactly-supported closed $(\dim(M)-p)$-form ρ such that $\int_M \omega \wedge \rho \neq 0$.

As ω is compactly-supported, it is square-integrable and so represents an element $[\omega]$ of $\overline{\mathrm{H}}^p_{(2)}(M)$. Suppose that $[\omega] = 0$. Then there is a sequence $\{\eta_i\}_{i=1}^{\infty}$ in $\Omega^{p-1}(M)$ such that $\omega = \lim_{i\to\infty} d\eta_i$, where the limit is in an L^2-sense. It follows that

$$\int_M \omega \wedge \rho = \lim_{i\to\infty} \int_M d\eta_i \wedge \rho = \lim_{i\to\infty} \int_M d(\eta_i \wedge \rho) = 0, \tag{3.1}$$

which is a contradiction. Thus $[\omega] \neq 0$. □

Corollary 2. *Let* N^{4k} *be a compact manifold-with-boundary with nonzero signature. Then if* M *is any complete Riemannian manifold which is diffeomorphic to* $\mathrm{int}(N)$, $\overline{\mathrm{H}}^{2k}_{(2)}(M) \neq 0$.

Proof. By definition, the signature of N is the signature of the intersection form on

$$\mathrm{Im}\left(\mathrm{H}^{2k}(N,\partial N;\mathbb{C}) \to \mathrm{H}^{2k}(N;\mathbb{C})\right) \cong \mathrm{Im}\left(\mathrm{H}^{2k}_c(M;\mathbb{C}) \to \mathrm{H}^{2k}(M;\mathbb{C})\right). \tag{3.2}$$

If the signature of N is nonzero then $\mathrm{Im}\left(\mathrm{H}^{2k}_c(M;\mathbb{C}) \to \mathrm{H}^{2k}(M;\mathbb{C})\right)$ must be nonzero. The corollary follows from Proposition 4. □

Example : Let N be $\mathbb{C}P^2$ with a small 4-ball removed. Then N satisfies the hypothesis of Corollary 2.

We now show that the middle-dimensional reduced L^2-cohomology is a conformal invariant of M.

Proposition 5. *If* M^{2k} *is even-dimensional then* $\mathrm{Ker}(\triangle_k)$ *is conformally-invariant.*

Proof. Suppose that g and $e^{\phi}g$ are conformally equivalent Riemannian metrics on M, with $\phi \in C^{\infty}(M)$. We use the fact that the action of the Hodge duality operator $*$ on $\Lambda^k(M)$ is independent of ϕ. If ω is a k-form on M, its L^2-norm $\int_M \omega \wedge *\omega$ is independent of ϕ. Thus the Hilbert space $\Lambda^k(M)$ is independent of ϕ. Furthermore,

$$\begin{aligned}\mathrm{Ker}(\triangle_k) &= \{\omega \in \Lambda^k(M) : d\omega = d^*\omega = 0\} &(3.3)\\ &= \{\omega \in \Lambda^k(M) : d\omega = \pm * d * (\omega) = 0\} \\ &= \{\omega \in \Lambda^k(M) : d\omega = d * (\omega) = 0\} &(3.4)\end{aligned}$$

is independent of ϕ. □

Example : Take $M = H^2$. Then M is conformally equivalent to a Euclidean disk D. The harmonic square-integrable 1-forms on D are of the form $f_1(x,y)dx + f_2(x,y)dy$, where f_1 and f_2 are square-integrable harmonic functions on D. There is clearly an infinite number of such functions, and so $\dim(\overline{\mathrm{H}}^1_{(2)}(H^2)) = \infty$. The same argument applies to H^{2k}, to give $\dim(\overline{\mathrm{H}}^k_{(2)}(H^{2k})) = \infty$.

In the case of functions, one has a good control of when zero is in the spectrum of the Laplacian.

Lemma 4. $\mathrm{Ker}(\Delta_0) \neq 0$ *if and only if* $\mathrm{vol}(M) < \infty$.

Proof. If $\mathrm{vol}(M) < \infty$ then the constant functions on M are square-integrable and harmonic. Conversely, if $f \in \mathrm{Ker}(\Delta_0)$ then by Lemma 2, f is constant. If f is nonzero and square-integrable then $\mathrm{vol}(M) < \infty$. □

Definition 4. *M is open at infinity if there is a constant $C > 0$ such that for all domains D in M with smooth compact closure, $\frac{\mathrm{area}(\partial D)}{\mathrm{vol}(D)} \geq C$.*

Examples :
1. $\mathbb{R}^n$ is not open at infinity, as can be seen by taking large balls for D.
2. H^n is open at infinity.

Proposition 6. *(Buser [3]) Let M have infinite volume. Suppose that there is a constant $c \geq 0$ such that* $\mathrm{Ricci}_M \geq -c^2$. *Then $0 \notin \sigma(\Delta_0)$ if and only if M is open at infinity.*

Proof. 1. Suppose that M is open at infinity. By Cheeger's inequality,

$$\inf(\sigma(\Delta_0)) \geq \inf_D \frac{1}{4}\left(\frac{\mathrm{area}(\partial D)}{\mathrm{vol}(D)}\right)^2 > 0. \tag{3.5}$$

2. Suppose that M is not open at infinity. The bottom of the spectrum of Δ_0 is given in terms of Rayleigh quotients by

$$\inf(\sigma(\Delta_0)) = \inf_{f \neq 0} \frac{\int_M |df|^2}{\int_M f^2}, \tag{3.6}$$

where f ranges over compactly-supported Lipschitz functions on M. We want to find compactly-supported Lipschitz functions on M of arbitrarily small Rayleigh quotient. By assumption, for all $\epsilon > 0$ there is a domain D such that $\frac{\mathrm{area}(\partial D)}{\mathrm{vol}(D)} \leq \epsilon$. Put

$$N_1(\partial D) = \{m \in M : m \notin D \text{ and } d(m, \partial D) \leq 1\}. \tag{3.7}$$

Define a function f, which approximates the characteristic function of D, by

$$f(m) = \begin{cases} 1 & \text{if } m \in D \\ 1 - d(m, \partial D) & \text{if } m \in N_1(\partial D) \\ 0 & \text{if } m \notin D \text{ and } m \notin N_1(\partial D). \end{cases} \tag{3.8}$$

Clearly $\int_M f^2 \geq \mathrm{vol}(D)$. As f has nonzero gradient only in $N_1(\partial D)$, where $|df| = 1$ almost everywhere, we have $\int_M |df|^2 = \mathrm{vol}(N_1(\partial D))$. If D is nice and round then we expect that

$$\mathrm{vol}(N_1(\partial D)) \sim \mathrm{area}(\partial D) \tag{3.9}$$

and the Rayleigh quotient $\frac{\int_M |df|^2}{\int_M f^2}$ will be comparable to ϵ.

The only problem with this argument is that D may not be nice and round, but may have long thin legs coming out of it, like an octupus. Then (3.9) may not be valid. The content of [3] is that if this is the case, we can cut the legs off of D to come up with a new domain for which the above heuristic argument is valid. It is in this step that the lower bound on the Ricci curvature is used. We refer to [3] for details. □

Corollary 3. *(Brooks [2]) Let M be a normal covering of a compact manifold X with covering group Γ. Then $0 \in \sigma(\Delta_0)$ on M if and only if Γ is amenable.*

Proof. If Γ is finite then $0 \in \sigma(\triangle_0)$ and Γ is amenable. If Γ is infinite then by Proposition 6, $0 \in \sigma(\triangle_0)$ if and only if M is not open at infinity. Let S be a finite set of generators of Γ. Let G be the Cayley graph of Γ, constructed using S. There is a notion of G being open at infinity which is similar to Definition 4. As M is coarsely quasi-isometric to G, M is not open at infinity if and only if G is not open at infinity. However, this is one of the characterizations of amenability of Γ. □

We now prove a result about manifolds which, roughly speaking, are at least as large as Euclidean space.

Definition 5. *M is hyperEuclidean if there is a proper distance-nonincreasing map $F : M \to \mathbb{R}^{\dim(M)}$ of nonzero degree.*

Remarks :
1. A map is proper if preimages of compact sets are compact. Instead of requiring that F be distance-nonincreasing, we could require that F have a finite Lipschitz constant. By postcomposing F with a dilatation of $\mathbb{R}^{\dim(M)}$, the two conditions are equivalent.
2. If M is hyperEuclidean then a compactly-supported modification of M is also hyperEuclidean.
3. Examples of hyperEuclidean manifolds are given by simply-connected nonpositively-curved manifolds M. Namely, fix $m_0 \in M$ and put $F = \exp_{m_0}^{-1}$.
4. There was once a conjecture that all uniformly contractible manifolds are hyperEuclidean (with a degree-one map to $\mathbb{R}^{\dim(M)}$), but this turns out to be wrong [10]. There is still an open conjecture that a uniformly contractible manifold of bounded geometry is hyperEuclidean, and in particular, that the universal cover of an aspherical closed manifold is hyperEuclidean.

Proposition 7. *(Gromov [14, p. 238]) If M is hyperEuclidean then $0 \in \sigma(\triangle_p)$ for some p.*

Proof. Put $n = \dim(M)$. First, suppose that n is even. We will construct a vector bundle E with connection on $\mathbb{R}^n$ which is topologically nontrivial but analytically trivial, in a sense which will be made precise. Then assuming that zero is not in the spectrum of M, we will apply the relative index theorem to F^*E in order to get a contradiction.

Recall that $\mathrm{K}^0(S^n) = \mathbb{Z} \oplus \mathbb{Z}$. If $\mathcal{E}$ is a (virtual) vector bundle on S^n, the two $\mathbb{Z}$ factors correspond to $\mathrm{rk}(\mathcal{E})$ and $\int_{S^n} \mathrm{ch}(\mathcal{E})$, respectively. This means that for some $N > 0$, there is a complex $\mathbb{C}^N$-bundle $\mathcal{E}$ on S^n with $\int_{S^n} \mathrm{ch}(\mathcal{E}) \neq 0$. Fixing a point $\infty \in S^n$, we can trivialize $\mathcal{E}$ in a neighborhood of ∞. Furthermore, we can put a Hermitian metric and Hermitian connection on $\mathcal{E}$ so that the connection is flat in a neighborhood of ∞.

Let E be the restriction of $\mathcal{E}$ to $\mathbb{R}^n = S^n - \{\infty\}$. Let ∇ be the restriction of the Hermitian connection on $\mathcal{E}$ to $\mathbb{R}^n$. Then E is trivialized outside of a compact set $K \subset \mathbb{R}^n$ and ∇ is flat outside of K.

As $\mathbb{R}^n$ is contractible, there is an isomorphism of Hermitian vector bundles $i : \mathbb{R}^n \times \mathbb{C}^N \to E$. Then $i^*\nabla$ can be considered to be a $u(N)$-valued 1-form ω on $\mathbb{R}^n$. The curvature of ω is the $u(N)$-valued 2-form $\Omega = d\omega + \omega^2$. The nontriviality of $\mathcal{E}$ translates to the facts that
1. Ω vanishes outside of K and
2. The de Rham cohomology class of the compactly-supported form $\mathrm{Tr}\left(e^{-\frac{\Omega}{2\pi i}}\right) - N$ is a nonzero multiple of the fundamental class $[\mathbb{R}^n] \in \mathrm{H}^n_c(\mathbb{R}^n; \mathbb{R})$.

In fact, we can take ω to have a finite L^∞-norm $\| \omega \|_\infty$. For example, if $n = 2$, take $N = 1$. Let $f \in C_0^\infty([0,\infty))$ be a nonincreasing function such that if $x \in [0, \frac{1}{2}]$ then $f(x) = 1$. Put $\omega = -i(1 - f(r))d\theta$. Then

$$\Omega = d\omega = if'(r)dr \wedge d\theta. \tag{3.10}$$

We have $\| \omega \|_\infty = \sup_{r \geq 0} \frac{1-f(r)}{r}$ and $\int_{\mathbb{R}^2} \left[\mathrm{Tr} \left(e^{-\frac{\Omega}{2\pi i}} \right) - 1 \right] = 1$.

Returning to the case of general even n, for $\epsilon > 0$, let $\Phi_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ be the map $\Phi_\epsilon(\mathbf{x}) = \epsilon \mathbf{x}$. Put $\omega_\epsilon = \Phi_\epsilon^* \omega$ and $\Omega_\epsilon = d\omega_\epsilon + \omega_\epsilon^2$. Then

$$\| \omega_\epsilon \|_\infty = \epsilon \| \omega \|_\infty \text{ and } \int_{\mathbf{R}^n} \left[\mathrm{Tr} \left(e^{-\frac{\Omega_\epsilon}{2\pi i}} \right) - N \right] = \int_{\mathbf{R}^n} \left[\mathrm{Tr} \left(e^{-\frac{\Omega}{2\pi i}} \right) - N \right] \neq 0. \tag{3.11}$$

We now turn our attention to M. Suppose that $0 \notin \sigma(\triangle_p)$ for all p. Consider the self-adjoint operator $d + d^*$ on $\Lambda^*(M)$. As $(d+d^*)^2 = \triangle$, it follows that $0 \notin \sigma(d + d^*)$. In other words, $d + d^*$ is L^2-invertible. Define an operator μ on $\Lambda^*(M)$ by saying that if $\omega \in \Lambda^p(M)$ then

$$\mu(\omega) = i^{\frac{n(n-1)}{2}} (-1)^{\frac{p(p-1)}{2}} * (\omega). \tag{3.12}$$

One can check that $\mu^2 = 1$ and $\mu(d + d^*) + (d + d^*)\mu = 0$.

Clearly the operator $(d + d^*) \otimes \mathrm{Id}_N$, acting on $\Lambda^*(M) \otimes \mathbb{C}^N$, is also invertible. Consider the $u(N)$-valued 1-form $F^*\omega_\epsilon$ on M. As F is distance-nonincreasing,

$$\| F^*\omega_\epsilon \|_\infty \leq \| \omega_\epsilon \|_\infty = \epsilon \| \omega \|_\infty . \tag{3.13}$$

Let $e(F^*\omega_\epsilon)$ denote exterior multiplication by $F^*\omega_\epsilon$, acting on $\Lambda^*(M) \otimes \mathbb{C}^N$ and let $i(F^*\omega_\epsilon)$ denote interior multiplication by $F^*\omega_\epsilon$. By making ϵ small enough, the operator $e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon)$ has arbitrarily small norm and so the operator $((d + d^*) \otimes \mathrm{Id}_N) + e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon)$ is also invertible.

Put $D = (d \otimes \mathrm{Id}_N) + e(F^*\omega_\epsilon)$. Then D is exterior differentiation; using the connection $F^*\omega_\epsilon$, and

$$D + D^* = ((d + d^*) \otimes \mathrm{Id}_N) + e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon). \tag{3.14}$$

As $(d + d^*) \otimes \mathrm{Id}_N$ and $D + D^*$ anticommute with $\mu \otimes \mathrm{Id}_N$, they have well-defined indices which happen to vanish, as the operators are invertible. On the other hand, let $L(M)$ be the Hirzebruch L-form. The relative index theorem of Gromov and Lawson [9, 15] says that

$$\mathrm{ind}(D + D^*) - \mathrm{ind}((d + d^*) \otimes \mathrm{Id}_N) = \int_M L(M) \wedge \left[\mathrm{Tr} \left(e^{-\frac{F^*\Omega_\epsilon}{2\pi i}} \right) - N \right]. \tag{3.15}$$

As F is proper, the de Rham cohomology class of $\mathrm{Tr} \left(e^{-\frac{F^*\Omega_\epsilon}{2\pi i}} \right) - N = F^* \left[\mathrm{Tr} \left(e^{-\frac{\Omega_\epsilon}{2\pi i}} \right) - N \right]$ is well-defined as a multiple of the fundamental class $[M] \in \mathrm{H}_c^n(M; \mathbb{R})$. As the series for

$L(M)$ starts off as $L(M) = 1 + \ldots$, we obtain

$$\begin{aligned} \operatorname{ind}(D + D^*) - \operatorname{ind}((d + d^*) \otimes \operatorname{Id}_N) &= \int_M \left[\operatorname{Tr}\left(e^{-\frac{F^*\Omega_c}{2\pi i}}\right) - N \right] \\ &= \int_M F^* \left[\operatorname{Tr}\left(e^{-\frac{\Omega_c}{2\pi i}}\right) - N \right] \\ &= \deg(F) \int_{\mathbf{R}^n} \left[\operatorname{Tr}\left(e^{-\frac{\Omega_c}{2\pi i}}\right) - N \right] \neq 0. \end{aligned} \tag{3.16}$$

This contradicts the vanishing of $\operatorname{ind}(D + D^*)$ and $\operatorname{ind}((d + d^*) \otimes \operatorname{Id}_N)$. Thus zero must be in the spectrum of M after all.

Now suppose that n is odd. As M is hyperEuclidean, so is $\mathbb{R} \times M$. With respect to the decomposition $\Lambda^*(\mathbb{R} \times M) = \Lambda^*(\mathbb{R}) \otimes \Lambda^*(M)$, the Laplace-Beltrami operator on $\mathbb{R} \times M$ decomposes as

$$\triangle_{\mathbf{R}\times M} = (\triangle_{\mathbb{R}} \otimes I) + (I \otimes \triangle_M). \tag{3.17}$$

Then

$$\sigma(\triangle_{\mathbf{R}\times M}) = \{\lambda_1 + \lambda_2 : \lambda_1 \in [0, \infty) \text{ and } \lambda_2 \in \sigma(\triangle_M)\}. \tag{3.18}$$

From what has already been proved, $0 \in \sigma(\triangle_{\mathbf{R}\times M})$. It follows that $0 \in \sigma(\triangle_M)$. □

Remarks :

1. We have shown that if M is hyperEuclidean then $0 \in \sigma(\triangle_p)$ for some p. One can ask whether the number p can be pinned down. In general, when computing the index of the operator $d + d^*$, the differential forms outside of the middle dimensions do not contribute. This is a reflection of the fact that the signature of a closed manifold can be computed using only the middle-dimensional cohomology. So this gives some reason to think that if $\dim(M)$ is even then $0 \in \sigma\left(\triangle_{\frac{dim(M)}{2}}\right)$.

Unfortunately, the operator $(D + D^*)^2$ does not preserve the degree of a differential form and so we cannot use the above proof to reach the desired conclusion. However, with a more refined index theorem [27, Theorem 6.24], one can indeed conclude that $0 \in \sigma\left(\triangle_{\frac{dim(M)}{2}}\right)$ if $\dim(M)$ is even and that $0 \in \sigma\left(\triangle_{\frac{dim(M)\pm 1}{2}}\right)$ if $\dim(M)$ is odd.

2. If M is an irreducible noncompact globally symmetric space G/K, with $G = \operatorname{Isom}(M)$ and K a maximal compact subgroup, then one can say more about the bottom of the spectrum. If $\operatorname{rk}(G) = \operatorname{rk}(K)$ then $\operatorname{Ker}\left(\triangle_{\frac{dim(M)}{2}}\right)$ is infinite-dimensional and the spectrum of $\triangle$ is bounded away from zero otherwise. If $\operatorname{rk}(G) > \operatorname{rk}(K)$ then $\operatorname{Ker}(\triangle) = 0$ and $0 \in \sigma(\triangle_p)$ if and only if $p \in \left[\frac{dim(M)}{2} - \frac{rk(G)-rk(K)}{2}, \frac{dim(M)}{2} + \frac{rk(G)-rk(K)}{2}\right]$ [18, Section VIIB].

Finally, we state a result about uniformly contractible Riemannian manifolds.

Definition 6. [14, p. 29] *A metric space Z has finite asymptotic dimension if there is an integer n such that for any $r > 0$, there is a covering $Z = \bigcup_{i\in I} C_i$ of Z by subsets of uniformly bounded diameter so that no metric ball of radius r in Z intersects more than $n + 1$ elements of $\{C_i\}_{i\in I}$. The smallest such integer n is called the asymptotic dimension* $\operatorname{asdim}_+(Z)$ *of Z.*

Proposition 8. *(Yu [32]) If M is a uniformly contractible Riemannian manifold with finite asymptotic dimension then $0 \in \sigma(\Delta_p)$ for some p.*

The proof of Proposition 8 uses methods of coarse index theory [27].

4. Very Low Dimensions

In this section we show that the answer to the zero-in-the-spectrum question is "yes" for one-dimensional simplicial complexes and two-dimensional Riemannian manifolds.

4.1. One Dimension.

As a one-dimensional manifold is either S^1 or $\mathbb{R}$, zero is clearly in the spectrum.

A more interesting problem is to consider a connected one-dimensional simplicial complex K. Let V be the set of vertices of K and let E be the set of oriented edges of K. That is, an element e of E consists of an edge of K and an ordering (s_e, t_e) of ∂e. We let $-e$ denote the same edge with the reverse ordering of ∂e. For $x \in V$, let m_x denote the number of unoriented edges of which x is a boundary. We assume that $m_x < \infty$ for all x. Put

$$C^0(K) = \{f : V \to \mathbb{C} \text{ such that } \sum_{x \in V} m_x \, |f(x)|^2 < \infty\}, \tag{4.1}$$

$$C^1(K) = \{F : E \to \mathbb{C} \text{ such that } F(-e) = -F(e) \text{ and } \frac{1}{2} \sum_{e \in E} |F(e)|^2 < \infty\}.$$

Then $C^0(K)$ and $C^1(K)$ are Hilbert spaces. The edges e such that $s_e = t_e$ do not enter in $C^1(K)$ and can be deleted for our purposes. The weighting used to define $C^0(K)$ is natural in certain respects [7].

There is a bounded operator $d : C^0(K) \to C^1(K)$ given by $(df)(e) = f(t_e) - f(s_e)$. Define the Laplace-Beltrami operators by $\Delta_0 = d^*d$ and $\Delta_1 = dd^*$. An element of $\mathrm{Ker}(\Delta_1)$ is an $F \in C^1(K)$ such that for each vertex x the total current flowing into x vanishes, i.e. $\sum_{e \in E : t_e = x} F(e) = 0$.

The next proposition is essentially due to Gromov [14, p. 236], who proved it in the case when $\{m_x\}_{x \in V}$ is bounded.

Proposition 9. *$0 \in \sigma(\Delta_0)$ or $0 \in \sigma(\Delta_1)$.*

Proof. As the nonzero spectra of d^*d and dd^* are the same, for our purposes it suffices to consider $\sigma(\Delta_0)$ and $\mathrm{Ker}(\Delta_1)$. We argue by contradiction. Suppose that $0 \notin \sigma(\Delta_0)$ and $\mathrm{Ker}(\Delta_1) = 0$. First, K must be infinite, as otherwise $\mathrm{Ker}(\Delta_0) \neq 0$. Second, K must be a tree, as if K had a loop then we could create a nonzero element of $\mathrm{Ker}(\Delta_1)$ by letting a current of unit strength flow around the loop.

We now show that K has lots of branching. For $x, y \in V$, let $[x, y]$ be the geodesic arc from x to y and let (x, y) be its interior. Let $d(x, y)$ be the number of edges in $[x, y]$.

Lemma 5. *There is a constant $L > 0$ such that if $d(x, y) > L$ then there is an infinite subtree of K which intersects (x, y) but does not contain x or y.*

Proof. If the lemma is not true then for all $N > 1$, there are vertices x and y such that $d(x, y) > N$ but there are no infinite subtrees as in the statement of the lemma. In other words, the connected component C of $K - \{x\} - \{y\}$ which contains (x, y) is finite. As K

is a tree, x is only connected to the vertices in C by a single edge, and similarly for y (see Fig. 5). Define $f \in C^0(K)$ by

$$f(v) = \begin{cases} 1 & \text{if } v \in C, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Then

$$\frac{\langle df, df \rangle}{\langle f, f \rangle} \leq \frac{2}{2(d(x,y)-1)} \leq \frac{1}{N}. \tag{4.3}$$

As N can be taken arbitrarily large, this contradicts the assumption that $0 \notin \sigma(\triangle_0)$. □

It follows that K contains a subtree K' which is topologically equivalent to an infinite triadic tree, with the distances between branchings at most L (see Fig. 6). We can create a nonzero square-integrable harmonic 1-cochain F' on K' by letting a unit current flow through it, as in Fig. 6. Let $F \in C^1(K)$ be the extension of F' by zero to K. If x is a vertex of K' then the total current flowing into x is still zero, as no new current comes in along the edges of $K - K'$. Thus $\mathrm{Ker}(\triangle_1) \neq 0$, which is a contradiction. □

4.2. Two Dimensions.

Proposition 10. *(Lott, Dodziuk) The answer to the zero-in-the-spectrum question is "yes" if M is a two-dimensional manifold.*

Proof. The Hodge decomposition gives

$$\Lambda^0(M) = \mathrm{Ker}(\triangle_0) \oplus \Lambda^0(M)/\mathrm{Ker}(d), \tag{4.4}$$

$$\Lambda^1(M) = \mathrm{Ker}(\triangle_1) \oplus \overline{d\Lambda^0(M)} \oplus \overline{*d\Lambda^0(M)}, \tag{4.5}$$

$$\Lambda^2(M) = *\mathrm{Ker}(\triangle_0) \oplus *(\Lambda^0(M)/\mathrm{Ker}(d)). \tag{4.6}$$

Thus it is enough to look at $\mathrm{Ker}(\triangle_0)$, $\mathrm{Ker}(\triangle_1)$ and $\sigma\,(\triangle \text{ on } \Lambda^0(M)/\mathrm{Ker}(d))$.

We argue by contradiction. Assume that zero is not in the spectrum. By Proposition 4, $\mathrm{Im}(\mathrm{H}^1_c(M) \to \mathrm{H}^1(M)) = 0$. Thus M must be planar, in the sense of either of the following two equivalent conditions :

1. Any simple closed curve in M separates it into two pieces.
2. M is diffeomorphic to the complement of a closed subset of S^2.

As $\mathrm{Ker}(\triangle_0) = 0$, M cannot be S^2. By Proposition 5, the possible existence of nonzero square-integrable harmonic 1-forms on M only depends on the underlying Riemann surface coming from the Riemannian metric on M.

We recall some notions from Riemann surface theory [1]. A function $f \in C^\infty(M)$ is *superharmonic* if $\triangle_0 f > 0$. (This is a conformally-invariant statement.) The Riemann surface underlying M is *hyperbolic* if it has a positive superharmonic function and *parabolic* otherwise. If M is planar and hyperbolic then there is a nonconstant harmonic function $f \in C^\infty(M)$ such that $\int_M df \wedge *df < \infty$ [1, p. 208]. Then df would be a nonzero element of $\mathrm{Ker}(\triangle_1)$. Thus M must be parabolic.

Put $\lambda_0 = \inf(\sigma(\triangle_0))$. Choose some λ such that $0 < \lambda < \lambda_0$. Then there is a positive $f \in C^\infty(M)$ (not square-integrable!) such that $\triangle_0 f = \lambda f$ [30, Theorem 2.1]. However, this contradicts the parabolicity of M. □

We do not know of any result analogous to Proposition 10 for general two-dimensional simplicial complexes, say uniformly finite. See, however, Subsection 5.2.

5. Universal Covers

Suppose that M is the universal cover of a compact Riemannian manifold X. We give M the pulled-back Riemannian metric and consider the Laplace-Beltrami operator $\triangle_p$ on M. There are numerical invariants which measure the density of $\sigma(\triangle_p)$ near zero, the so-called L^2-Betti numbers $\{b_p^{(2)}(X)\}_{p\geq 0}$ and Novikov-Shubin invariants $\{\alpha_{p+1}(X)\}_{p\geq 0}$. We refer to [20, 21, 25] for the definitions of these invariants. We will only need the following properties :

Properties : 1. $b_p^{(2)}(X) = 0 \iff \mathrm{Ker}(\triangle_p) = 0$.
2. $0 \notin \sigma(\triangle_p \text{ on } \Lambda^p(M)/\mathrm{Ker}(d)) \iff \alpha_{p+1} = \infty^+$.
3. $b_p^{(2)}(X)$ and $\alpha_p(X)$ are homotopy-invariants of X.
4. $b_0^{(2)}(X)$, $b_1^{(2)}(X)$, $\alpha_1(X)$ and $\alpha_2(X)$ only depend on $\pi_1(X)$.
5. $b_0^{(2)}(X) = 0$ if and only if $\pi_1(X)$ is infinite.
6. $\alpha_1(X) = \infty^+$ if and only if $\pi_1(X)$ is finite or nonamenable.
7. The Euler characteristic of X satisfies

$$\chi(X) = \sum_p (-1)^p\, b_p^{(2)}(X) \tag{5.1}$$

8. If X^n is closed then $b_{n-p}^{(2)}(X) = b_p^{(2)}(X)$.
9. If X^{4k} is closed then there are nonnegative numbers $b_{2k,\pm}^{(2)}(X)$ such that $b_{2k}^{(2)}(X) = b_{2k,+}^{(2)}(X) + b_{2k,-}^{(2)}(X)$ and the signature of X satisfies

$$\tau(X) = b_{2k,+}^{(2)}(X) - b_{2k,-}^{(2)}(X). \tag{5.2}$$

One can extend properties 1-7 from compact Riemannian manifolds X to finite CW-complexes K.

In what follows, Γ will denote a finitely-presented group. Given a presentation of Γ, there is an associated 2-dimensional CW-complex K which we call the *presentation complex*. To form it, make a bouquet of circles indexed by the generators of Γ. Attach 2-cells based on the relations of Γ. (We allow trivial or repeated relations in the presentation.) This is the presentation complex.

Definition 7. *Put* $b_0^{(2)}(\Gamma) = b_0^{(2)}(K)$, $b_1^{(2)}(\Gamma) = b_1^{(2)}(K)$, $\alpha_1(\Gamma) = \alpha_1(K)$ *and* $\alpha_2(\Gamma) = \alpha_2(K)$.

By Property 4 above, Definition 7 makes sense in that the choice of presentation of Γ does not matter.

5.1. Big and Small Groups.

Definition 8. *The group* Γ *is* big *if it is nonamenable,* $b_1^{(2)}(\Gamma) = 0$ *and* $\alpha_2(\Gamma) = \infty^+$. *Otherwise,* Γ *is* small.

Proposition 11. *Let* X *and* M *be as above. The group* $\pi_1(X)$ *is small if and only if* $0 \in \sigma(\triangle_0)$ *or* $0 \in \sigma(\triangle_1)$.

Proof. This follows immediately from Properties 1, 2, 4, 5 and 6 above. □

The question arises as to which groups are big and which are small. Clearly any amenable group is small.

Proposition 12. *Fundamental groups of compact surfaces are small.*

Proof. Suppose that Σ is a compact surface and $\Gamma = \pi_1(\Sigma)$. If Σ has boundary then Γ is a free group F_j on some number j of generators. If $j = 0$ or $j = 1$ then Γ is amenable. If $j > 1$ then $b_1^{(2)}(\Gamma) = j - 1 > 0$.

Suppose now that Σ is closed. If $\chi(\Sigma) \geq 0$ then Γ is amenable. If $\chi(\Sigma) < 0$ then $b_1^{(2)}(\Gamma) = -\chi(\Sigma) > 0$. □

We now extend Proposition 12 to 3-manifold groups. We use some facts about compact connected 3-manifolds Y, possibly with boundary. (See, for example, [20, Section 6]). Again, all of our manifolds are assumed to be oriented. First, Y has a decomposition as a connected sum $Y = Y_1 \# Y_2 \# \ldots \# Y_r$ of *prime* 3-manifolds. A prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Seifert, Haken or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known and it is likely that there are none.

Proposition 13. *(Lott-Lück) Suppose that Y is a compact connected oriented 3-manifold, possibly with boundary, none of whose prime factors are exceptional. Then $\pi_1(Y)$ is small.*

Proof. We argue by contradiction. Suppose that $\pi_1(Y)$ is big. First, $\pi_1(Y)$ must be infinite. If ∂Y has any connected components which are 2-spheres then we can cap them off with 3-balls without changing $\pi_1(Y)$. So we can assume that ∂Y does not have any 2-sphere components. In particular, $\chi(Y) = \frac{1}{2}\chi(\partial Y) \leq 0$. From [20, Theorem 0.1.1],

$$b_1^{(2)}(Y) = (r-1) - \sum_{i=1}^{r} \frac{1}{|\pi_1(Y_i)|} - \chi(Y). \tag{5.3}$$

As this must vanish, we have $\chi(Y) = 0$ and either
1. $\{|\pi_1(Y_i)|\}_{i=1}^{r} = \{2, 2, 1, \ldots, 1\}$ or
2. $\{|\pi_1(Y_i)|\}_{i=1}^{r} = \{\infty, 1, \ldots, 1\}$.

It follows that ∂Y is empty or a disjoint union of 2-tori. As there are no 2-spheres in ∂Y, if $|\pi_1(Y_i)| = 1$ then Y_i is a homotopy 3-sphere. Thus Y is homotopy-equivalent to either
1. $\mathbb{R}P^3 \# \mathbb{R}P^3$ or
2. A prime 3-manifold Y' with infinite fundamental group whose boundary is empty or a disjoint union of 2-tori.

If Y is homotopy-equivalent to $\mathbb{R}P^3 \# \mathbb{R}P^3$ then $\pi_1(Y)$ is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that $Y = Y'$. Then as Y is prime, it follows from [23, Chapter 1] that either $Y = S^1 \times D^2$ or Y has incompressible (or empty) boundary. If $Y = S^1 \times D^2$ then $\pi_1(Y)$ is amenable. If Y has incompressible (or empty) boundary then from [20, Theorem 0.1.5], $\alpha_2(Y) \leq 2$ unless Y is a closed 3-manifold with an $\mathbb{R}^3$, $\mathbb{R} \times S^2$ or *Sol* geometric structure. In the latter cases, Γ is amenable. Thus in any case, we get a contradiction. □

The next proposition gives examples of big groups.

Proposition 14. *1. A product of two nonamenable groups is big.*
2. If Y is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in $\widetilde{Y}$, then $\pi_1(Y)$ is big.

Proof. 1. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 and Γ_2 nonamenable. Then Γ is nonamenable. Let K_1 and K_2 be presentation complexes with fundamental groups Γ_1 and Γ_2, respectively. Put $K = K_1 \times K_2$. Then $\Gamma = \pi_1(K)$. Let $\Delta_p(\widetilde{K})$, $\Delta_p(\widetilde{K_1})$ and $\Delta_p(\widetilde{K_2})$ denote the Laplace-Beltrami operator on p-cochains on $\widetilde{K}$, $\widetilde{K_1}$ and $\widetilde{K_2}$, respectively, as defined in Subsection 5.2 below. Then

$$\inf(\sigma(\Delta_1(\widetilde{K}))) = \min\Big(\inf(\sigma(\Delta_1(\widetilde{K_1}))) + \inf(\sigma(\Delta_0(\widetilde{K_2}))), \quad \inf(\sigma(\Delta_0(\widetilde{K_1}))) + \inf(\sigma(\Delta_1(\widetilde{K_2})))\Big) > 0. \tag{5.4}$$

Using Proposition 11, the first part of the proposition follows.
2. If $\widetilde{Y}$ is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If $\widetilde{Y}$ is reducible then we can use an argument similar to (5.4). □

5.2. **Two and Three Dimensions.** In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let K be a finite connected 2-dimensional CW-complex. Let $\widetilde{K}$ be its universal cover. Let $C^*(\widetilde{K})$ denote the Hilbert space of square-integrable cellular cochains on $\widetilde{K}$. There is a cochain complex

$$0 \longrightarrow C^0(\widetilde{K}) \xrightarrow{d_0} C^1(\widetilde{K}) \xrightarrow{d_1} C^2(\widetilde{K}) \longrightarrow 0. \tag{5.5}$$

Define the Laplace-Beltrami operators by $\Delta_0 = d_0^* d_0$, $\Delta_1 = d_0 d_0^* + d_1^* d_1$ and $\Delta_2 = d_1 d_1^*$. These are bounded self-adjoint operators and so we can talk about zero being in the spectrum of $\widetilde{K}$.

Proposition 15. *Zero is not in the spectrum of $\widetilde{K}$ if and only if $\pi_1(K)$ is big and $\chi(K) = 0$.*

Proof. Suppose that zero is not in the spectrum of $\widetilde{K}$. From the analog of Proposition 11, Γ must be big. Furthermore, from Properties 1 and 7, $\chi(K) = 0$.

Now suppose that $\pi_1(K)$ is big and $\chi(K) = 0$. From the analog of Proposition 11, $0 \notin \sigma(\Delta_0)$ and $0 \notin \sigma(\Delta_1)$. In particular, $\mathrm{Ker}(\Delta_0) = \mathrm{Ker}(\Delta_1) = 0$. From Properties 1 and 7, $\mathrm{Ker}(\Delta_2) = 0$. As $C^2(\widetilde{K}) = \mathrm{Ker}(\Delta_2) \oplus \overline{d_1 C^1(\widetilde{K})}$, we conclude that $0 \notin \sigma(\Delta_2)$. □

Let Γ be a finitely-presented group. Consider a fixed presentation of Γ consisting of g generators and r relations. Let K be the corresponding presentation complex. Then $\chi(K) = 1 - g + r$. Thus zero is not in the spectrum of $\widetilde{K}$ if and only if $\pi_1(K)$ is big and $g - r = 1$.

Recall that the *deficiency* $\mathrm{def}(\Gamma)$ is defined to be the maximum, over all finite presentations of Γ, of $g - r$. If $b_1^{(2)}(\Gamma) = 0$ then from the equation

$$\chi(K) = 1 - g + r = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(K), \tag{5.6}$$

we obtain $\mathrm{def}(\Gamma) \le 1$. This is the case, for example, when Γ is big or when Γ is amenable [5].

As any finite connected 2-dimensional CW-complex is homotopy-equivalent to a presentation complex, it follows from Proposition 15 that the answer to the zero-in-the-spectrum question is "yes" for universal covers of such complexes if and only if the following conjecture is true.

Conjecture 1. *If* Γ *is a big group then* $\mathrm{def}(\Gamma) \leq 0$.

Remark : If $\pi_1(K)$ has property T then the ordinary first Betti number of K vanishes, and so $\chi(K) = 1 + b_2(K) > 0$. Thus zero lies in the spectrum of $\widetilde{K}$.

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \emptyset$, we define $\triangle_p$ on $\widetilde{Y}$ using absolute boundary conditions on $\partial \widetilde{Y}$.

Proposition 16. *Zero lies in the spectrum of* $\widetilde{Y}$.

Proof. This is a corollary of Propositions 11 and 13. □

5.3. **Four Dimensions.** In the subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

$$\begin{aligned}
\Lambda^0(M) &= \mathrm{Ker}(\triangle_0) \oplus \Lambda^0(M)/\mathrm{Ker}(d), \qquad (5.7)\\
\Lambda^1(M) &= \mathrm{Ker}(\triangle_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M)/\mathrm{Ker}(d),\\
\Lambda^2(M) &= \mathrm{Ker}(\triangle_2) \oplus \overline{d\Lambda^1(M)} \oplus \overline{*d\Lambda^1(M)},\\
\Lambda^3(M) &= *\mathrm{Ker}(\triangle_1) \oplus *\overline{d\Lambda^0(M)} \oplus *(\Lambda^1(M)/\mathrm{Ker}(d)),\\
\Lambda^4(M) &= *\mathrm{Ker}(\triangle_0) \oplus *(\Lambda^0(M)/\mathrm{Ker}(d)).
\end{aligned}$$

Thus for the zero-in-the-spectrum question, it is enough to consider $\mathrm{Ker}(\triangle_0)$, $\mathrm{Ker}(\triangle_1)$, $\sigma(\triangle_0 \text{ on } \Lambda^0/\mathrm{Ker}(d))$, $\sigma(\triangle_1 \text{ on } \Lambda^1/\mathrm{Ker}(d))$ and $\mathrm{Ker}(\triangle_2)$.

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ. Embed the resulting presentation complex in $\mathbb{R}^5$ and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ. Given X, we have $\chi(X \# \mathbb{C}P^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$\{\chi(X) : X \text{ is a closed connected oriented 4-manifold with } \pi_1(X) = \Gamma\} = \{n \in \mathbb{Z} : n \geq q(\Gamma)\} \qquad (5.8)$$

for some $q(\Gamma)$. *A priori* $q(\Gamma) \in \mathbb{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbb{Z}$ [16, Théorème 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

$$\chi(X) = 2\, b_0^{(2)}(\Gamma) - 2\, b_1^{(2)}(\Gamma) + b_2^{(2)}(X). \qquad (5.9)$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \geq 0$ and so $q(\Gamma) \geq 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].

Proposition 17. *Let* X *be a closed* 4*-manifold. Then zero is not in the spectrum of* $\widetilde{X}$ *if and only if* $\pi_1(X)$ *is big and* $\chi(X) = 0$.

Proof. Suppose that zero is not in the spectrum of $\widetilde{X}$. Then from Proposition 11, $\pi_1(X)$ must be big. Furthermore, $\mathrm{Ker}(\Delta_2) = 0$. From Property 1 and (5.9), $\chi(X) = 0$.

Now suppose that $\pi_1(X)$ is big and $\chi(X) = 0$. From Proposition 11, $0 \notin \sigma(\Delta_0)$ and $0 \notin \sigma(\Delta_1)$. From Property 1 and (5.9), $\mathrm{Ker}(\Delta_2) = 0$. Then from (5.7), zero is not in the spectrum of $\widetilde{X}$. □

Remark : If zero is not in the spectrum of $\widetilde{X}$ then it follows from Property 9 that in addition, $\tau(X) = 0$. Also, if $\pi_1(X)$ satisfies the Strong Novikov Conjecture then it will follow from Corollary 4 that $\nu_*([X])$ vanishes in $\mathrm{H}_4(B\pi_1(X); \mathbb{C})$.

In summary, we have shown that the answer to the zero-in-the-spectrum question is "yes" for universal covers of closed 4-manifolds if and only if the following conjecture is true.

Conjecture 2. *If Γ is a big group then $q(\Gamma) > 0$.*

We now give some partial positive results on the zero-in-the-spectrum question for universal covers of closed 4-manifolds. Recall that there is a notion, due to Thurston, of a manifold having a geometric structure. This is especially important for 3-manifolds. The 4-manifolds with geometric structures have been studied by Wall [31].

Proposition 18. *Let X be a closed 4-manifold. Then zero is in the spectrum of $\widetilde{X}$ if*
1. $\pi_1(X)$ has Property T or
2. X has a geometric structure (and an arbitrary Riemannian metric) or
3. X has a complex structure (and an arbitrary Riemannian metric).

Proof. 1. If X has Property T then the ordinary first Betti number of X vanishes. Thus $\chi(X) = 2 + b_2(X) > 0$. Part 1. of the proposition follows.
2. The geometries of [31] all fall into at least one of the following classes :
a. $b_0^{(2)} \neq 0$: S^4, $S^2 \times S^2$, $\mathbb{C}P^2$.
b. $0 \in \sigma(\Delta_0$ on $\Lambda^0/\mathrm{Ker}(d))$: $\mathbb{R}^4$, $S^3 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$, $Nil^3 \times \mathbb{R}$, Nil^4, Sol_0^4, Sol_1^4, $Sol_{m,n}^4$.
c. $b_1^{(2)} \neq 0$: $S^2 \times H^2$.
d. $0 \in \sigma(\Delta_1$ on $\Lambda^1/\mathrm{Ker}(d))$: $H^3 \times \mathbb{R}$, $\widetilde{SL_2} \times \mathbb{R}$, $H^2 \times \mathbb{R}^2$.
e. $\chi > 0$: H^4, $H^2 \times H^2$, $\mathbb{C}H^2$.
Part 2. of the proposition follows.
3. Suppose that zero is not in the spectrum of $\widetilde{X}$. From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [31, p. 148-149]. This contradicts part 2. of the proposition. □

5.4. More Dimensions.

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [17]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the $\mathbb{Z}_2$-grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d + d^*$. To say this in a more complicated way, the operator $d + d^*$ defines a element $[d + d^*]$ of the K-homology group $K_0(X)$. Let $\nu : X \to \mathrm{pt.}$ be the (only) map from X to a point. Then $\nu_*([d + d^*]) \in K_0(\mathrm{pt.})$. There is a map $A : K_0(\mathrm{pt.}) \to K_0(\mathbb{C})$ which is the identity, as both sides are $\mathbb{Z}$. So we can say that $\tau(X) = A(\nu([d + d^*])) \in K_0(\mathbb{C})$.

Now let M be a normal cover of X with covering group Γ. The fiber bundle $M \to X$ is classified by a map $\nu : X \to B\Gamma$, defined up to homotopy. Let $\widetilde{d}$ be exterior differentiation on M. Consider the operator $\widetilde{d} + \widetilde{d}^*$. Taking into account the action of Γ on M, one can define a refined index $\mathrm{ind}(\widetilde{d} + \widetilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^*-algebra of Γ.

We recall the statement of the Strong Novikov Conjecture (SNC). This is a conjecture about a countable discrete group Γ, namely that the assembly map $A : K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.

Proposition 19. *Let X be a closed Riemannian manifold with a surjective homomorphism $\pi_1(X) \to \Gamma$. Let M be the induced normal Γ-cover of X. Suppose that Γ satisfies SNC. Let $L(X) \in \mathrm{H}^*(X;\mathbb{C})$ be the Hirzebruch L-class of X and let $*L(X) \in \mathrm{H}_*(X;\mathbb{C})$ be its Poincaré dual. Then if $\nu_*(*L(X)) \neq 0$ in $H_*(B\Gamma;\mathbb{C})$, zero lies in the spectrum of M. In fact, $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\triangle_{\frac{dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. Suppose first that $\dim(X)$ is even. Suppose that zero does not lie in the spectrum of M. Then the operator $\widetilde{d} + \widetilde{d}^*$ is invertible. (More precisely, it is invertible as an operator on a Hilbert $C_r^*\Gamma$-module of differential forms on M.) This implies that $\mathrm{ind}(\widetilde{d} + \widetilde{d}^*)$ vanishes in $K_0(C_r^*\Gamma)$.

The higher index theorem says that

$$\mathrm{ind}(\widetilde{d} + \widetilde{d}^*) = A(\nu_*([d + d^*])). \tag{5.10}$$

Let $A_{\mathbb{C}} : K_0(B\Gamma) \otimes \mathbb{C} \to K_0(C_r^*\Gamma) \otimes \mathbb{C}$ be the complexified assembly map. Using the isomorphism $K_0(B\Gamma) \otimes \mathbb{C} \cong \mathrm{H}_{even}(B\Gamma;\mathbb{C})$, the higher index theorem implies that in $K_0(C_r^*\Gamma) \otimes \mathbb{C}$,

$$\mathrm{ind}(\widetilde{d} + \widetilde{d}^*)_{\mathbb{C}} = A_{\mathbb{C}}(\nu_*(*L(X))). \tag{5.11}$$

By assumption, $A_{\mathbb{C}}$ is injective. This gives a contradiction.

Let T be the operator obtained by restricting $\widetilde{d} + \widetilde{d}^*$ to

$$\Lambda^{\frac{dim(X)}{2}}(M) \oplus \overline{\widetilde{d}\Lambda^{\frac{dim(X)}{2}}(M)} \oplus \overline{*\widetilde{d}\Lambda^{\frac{dim(X)}{2}}(M)}.$$

One can show that the other differential forms on M cancel out when computing the rational index of $\widetilde{d} + \widetilde{d}^*$, so T will have the same index as $\widetilde{d} + \widetilde{d}^*$. Then the same arguments apply to T to give $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$.

If $\dim(X)$ is odd, consider the even-dimensional manifold $X' = X \times S^1$ and the group $\Gamma' = \Gamma \times \mathbb{Z}$. As the proposition holds for X', it must also hold for X. □

Corollary 4. *Let X be a closed Riemannian manifold. Let $[X] \in \mathrm{H}_{dim(X)}(X;\mathbb{C})$ be its fundamental class. Suppose that there is a surjective homomorphism $\pi_1(X) \to \Gamma$ such that Γ satisfies SNC and the composite map $X \to B\pi_1(X) \to B\Gamma$ sends $[X]$ to a nonzero element of $\mathrm{H}_{dim(X)}(B\Gamma;\mathbb{C})$. Let M be the induced normal Γ-cover of X. Then on M, $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\triangle_{\frac{dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. As the Hirzebruch L-class starts out as $L(X) = 1 + \ldots$, its Poincaré dual is of the form $*L(X) = \ldots + [X]$. The corollary follows from Proposition 19. □

Corollary 5. *Let X be a closed aspherical Riemannian manifold whose fundamental group satisfies SNC. Then on $\widetilde{X}$, $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\triangle_{\frac{dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. This follows from Corollary 4. □

Examples :
1. If $X = T^n$ then Corollary 5 is consistent with Example 2 of Section 2.
2. If X is a compact quotient of H^{2n} then Corollary 5 is consistent with Example 3 of Section 2.
3. If X is a compact quotient of H^{2n+1} then Corollary 5 is consistent with Example 4 of Section 2.
4. If X is a closed nonpositively-curved locally symmetric space then Corollary 5 is consistent with the second remark after Proposition 7.

If X is a closed aspherical manifold, it is known that SNC implies that the rational Pontryagin classes of X are homotopy-invariants [17] and that X does not admit a Riemannian metric of positive scalar curvature [28]. Thus we see that these three questions about aspherical manifolds, namely homotopy-invariance of rational Pontryagin classes, (non)existence of positive-scalar-curvature metrics and the zero-in-the-spectrum question, are roughly all on the same footing.

If X is a closed aspherical Riemannian manifold, one can ask for which p one has $0 \in \sigma(\triangle_p)$ on $\widetilde{X}$. The case of locally symmetric spaces is covered by the second remark after Proposition 7. Another interesting class of aspherical manifolds consists of those with amenable fundamental group. By [5], $\mathrm{Ker}(\triangle_p) = 0$ for all p. By Corollary 3, $0 \in \sigma(\triangle_0)$.

Proposition 20. *If X is a closed aspherical manifold such that $\pi_1(X)$ has a nilpotent subgroup of finite index then $0 \in \sigma(\triangle_p)$ for all $p \in [0, \dim(X)]$.*

Proof. First, X is homotopy-equivalent to an infranilmanifold, that is, a quotient of the form $\Gamma\backslash G/K$ where K is a finite group, G is the semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [11, Theorem 6.4]. We may as well assume that $X = \Gamma\backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [26, Corollary 7.28], $\mathrm{H}^p(X;\mathbb{C}) \cong \mathrm{H}^p(g,\mathbb{C})$, the Lie algebra cohomology of g. From [6], $\mathrm{H}^p(g,\mathbb{C}) \neq 0$ for all $p \in [0, \dim(X)]$. Thus for all $p \in [0, \dim(X)]$, $\mathrm{H}^p(X;\mathbb{C}) \neq 0$.

Now let ω be a nonzero harmonic p-form on X. Let $\pi^*\omega$ be its pullback to $\widetilde{X}$. The idea is to construct low-energy square-integrable p-forms on X by multiplying $\pi^*\omega$ by appropriate functions on X. We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of $\widetilde{X}$. If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \cup_{g \in E}\, g \cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty([0,\infty))$ which is identically one on $[0, \epsilon_1]$ and identically zero on $[\epsilon_2, \infty)$. Define a compactly-supported function f_E on $\widetilde{X}$

by $f_E(m) = \psi(d(m, H))$. Then there is a constant $C_1 > 0$, independent of E, such that

$$\int_{\widetilde{X}} |df|^2 \leq C_1 \operatorname{area}(\partial H). \tag{5.12}$$

Define $\rho_E \in \Lambda^p(\widetilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\,\pi^*\omega$. As f_E is identically one on H, it follows that there is a constant $C > 0$, independent of E, such that

$$\frac{\int_{\widetilde{X}} [|d\rho_E|^2 + |d^*\rho_E|^2]}{\int_{\widetilde{X}} |\rho_E|^2} \leq C\,\frac{\operatorname{area}(\partial H)}{\operatorname{vol}(H)}. \tag{5.13}$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\Delta_p)$. □

Question : Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable?

6. Topologically Tame Manifolds

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\operatorname{Ker}(\Delta_0) \neq 0$, so we restrict our attention to the infinite volume case. A limited result is given by Corollary 2.

An interesting subclass of topologically tame manifolds consists of those which are hyperbolic, that is, of constant sectional curvature -1. Complete hyperbolic manifolds are divided into those which are *geometrically finite* and those which are *geometrically infinite.* Roughly speaking, M is geometrically finite if its set of ends consists of a finite number of standard cusps and flares.

Proposition 21. *(Mazzeo-Phillips* [22, Theorem 1.11]*) Let M be an infinite-volume geometrically finite hyperbolic manifold. If* $\dim(M) = 2k$ *then* $\dim(\operatorname{Ker}(\Delta_k)) = \infty$. *If* $\dim(M) = 2k+1$ *then* $\sigma(\Delta_k) = \sigma(\Delta_{k+1}) = [0, \infty)$.

The paper [22] also computes $\dim(\operatorname{Ker}(\Delta_p))$ for such manifolds.

In general, geometrically infinite hyperbolic manifolds can have wild end behavior. However, in three dimensions one can show that the ends have a fairly nice structure. This is used to prove the next result.

Proposition 22. *(Canary* [4, Theorem A]*) If M is a geometrically infinite topologically tame hyperbolic 3-manifold then* $0 \in \sigma(\Delta_0)$.

Proof. The method of proof is to show that M is not open at infinity and then apply Theorem 6. See [4] for details. □

Thus zero lies in the spectrum of all topologically tame hyperbolic 3-manifolds. From Proposition 2, the same statement is true for compactly-supported modifications of such manifolds.

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Fig. 1

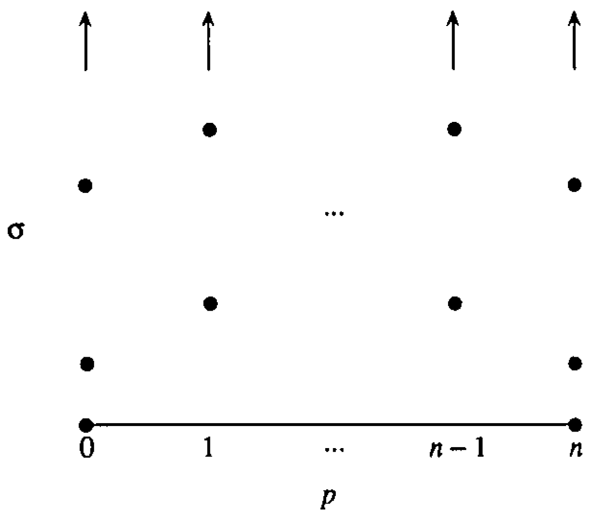

Fig. 2

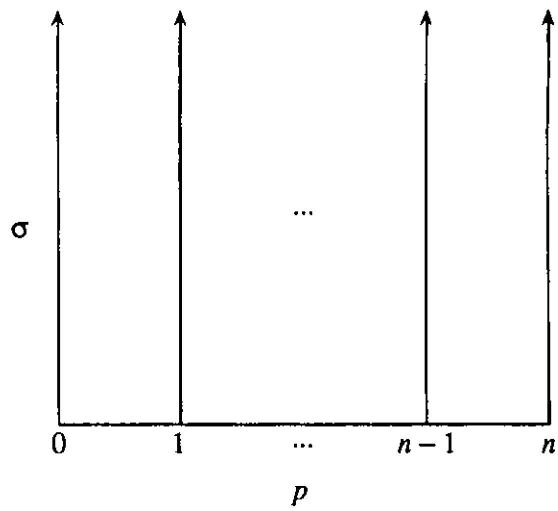

Fig. 3

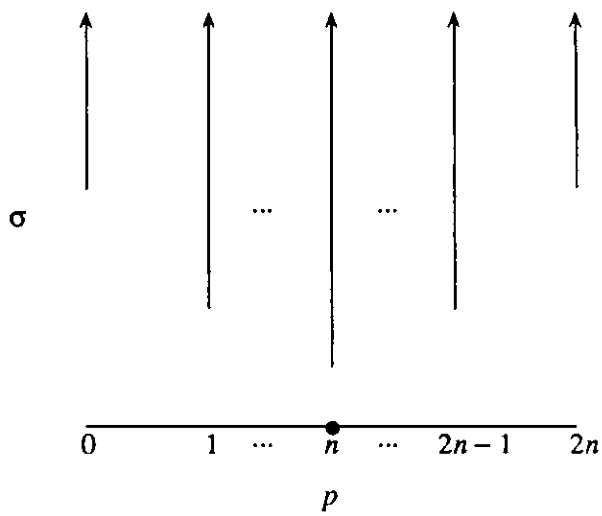

Fig. 4

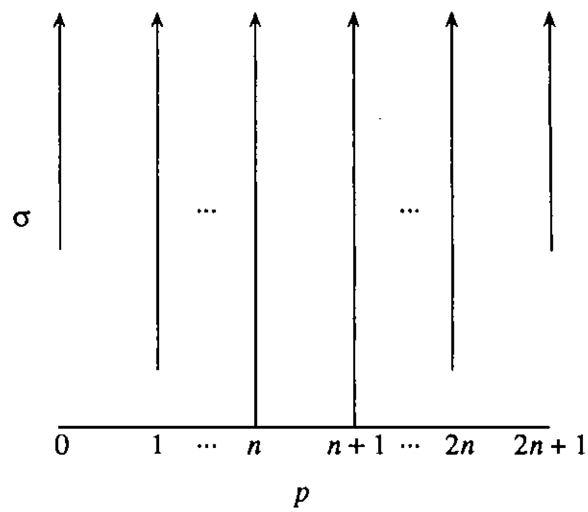

Fig. 5

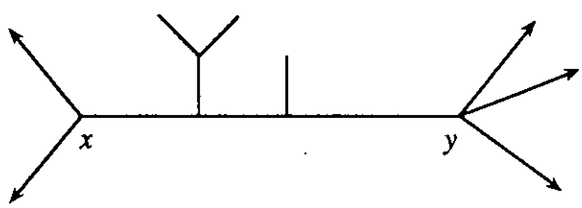

Fig. 6