SPECTRUM OF THE RUELLE OPERATOR AND EXPONENTIAL DECAY OF CORRELATIONS FOR OPEN BILLIARD FLOWS

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Abstract. The paper deals with the billiard flow in the exterior of several strictly convex disjoint domains in the plane with smooth boundaries satisfying an additional (visibility) condition. Using a modification of the technique of Dolgopayt [D], we get spectral estimates for the Ruelle operator related to a Markov family for the non-wandering (trapping) set of the flow similar to these in [D] in the case of transitive Anosov flows on compact manifolds with smooth jointly non-integrable horocycle foliations. As a consequence, we get exponential decay of correlation for Hölder continuous potentials on the non-wandering set. Combining the spectral estimate for the Ruelle operator with an argument of Pollicott and Sharp [PoS], one also derives the existence of a meromorphic continuation of the dynamical zeta function of the billiard flow to a half-plane $\mathrm{Re}(s) < h_T - \epsilon$, where h_T is the topological entropy of the billiard flow, and an asymptotic formula with an error term for the number $\pi(\lambda)$ of closed orbits of least period $\lambda > 0$.

1 Introduction

Let K be a subset of $\mathbb{R}^2$ of the form $K = K_1 \cup K_2 \cup \ldots \cup K_{k_0}$, where K_i are compact strictly convex disjoint domains in $\mathbb{R}^2$ with C^p $(p \geq 5)$ *boundaries* ∂K_i and $k_0 \geq 3$. Set $\Omega = \overline{\mathbb{R}^2 \setminus K}$.

We will assume that K satisfies the following condition: for every pair K_i, K_j of different connected components of K the convex hull of $K_i \cup K_j$ has no common points with any other connected component of K. With this condition, the billiard generated in the exterior of K is sometimes called an *open billiard*.

As usual, we will denote by $T^*(\Omega)$ the *cotangent bundle* of Ω and by $S^*(\Omega)$ its *cosphere bundle*. The *billiard flow* ϕ_t (also known as the *broken geodesic flow*) is defined on $S^*(\Omega)$ (and more generally on $T^*(\Omega) \setminus \{0\}$) in the standard way. Clearly this flow has singularities, however its restriction to the *non-wandering set* Λ only has simple discontinuities at reflection points. Moreover, Λ is compact, ϕ_t is hyperbolic and transitive on Λ, and it follows from [Mor] that ϕ_t is non-lattice and therefore by a result of Bowen [B1], it is topologically weak-mixing on Λ.

Our main result in this paper is a spectral estimate for the Ruelle operator similar to that of Dolgopayt [D] in the case of transitive Anosov flows on compact manifolds with C^1 jointly non-integrable horocycle foliations. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for ϕ_t near Λ consisting of rectangles $R_i = [U_i \cap \Lambda, S_i \cap \Lambda]$, where U_i (resp. S_i) are local unstable (resp. stable) manifolds near Λ (cf. Sect.2 for details). The *first return time function* $\tau : R = \cup_{i=1}^k R_i \longrightarrow [0, \infty)$ and the standard Poincaré map $\mathcal{P} : R \longrightarrow R$ have natural C^1 extensions on a neighbourhood of R in the union of the corresponding cross-sections containing the rectangles R_i. Setting $U = \cup_{i=1}^k U_i$, the *shift map* $\sigma : U \cap \Lambda \longrightarrow U \cap \Lambda$ is defined by $\sigma = p \circ \mathcal{P}$, where $p : R \longrightarrow U \cap \Lambda$ is the projection along the leaves of local stable manifolds. It follows from the existence of tubular families for Λ (cf. Robinson [Ro]) that σ has a C^1 extension on a neighbourhood of $U \cap \Lambda$ in U (cf. Sect. 2 for details), and choosing the size of the rectangles R_i sufficiently small, we may assume that σ has a C^1 extension

$\sigma : U \longrightarrow U$. Given a C^1 real-valued function f on U, set $g = f - P\tau$, where $P \in \mathbb{R}$ is the unique number such that the topological pressure of g with respect to σ is zero (cf. e.g. [PP2]). For $a, b \in \mathbb{R}$, one defines the *Ruelle operator* $L_{g-(a+\mathbf{i}b)\tau} : C^1(U) \longrightarrow C^1(U)$ in the usual way (cf. Sect.2). Here $C^1(U)$ is the space of continuously differentiable complex-valued functions h with $\|dh\| = \sup_{u\in U} \|dh(u)\| < \infty$. We will denote by $\|h\|$ the *standard* sup *norm* of h.

THEOREM 1.1. *For every $\epsilon > 0$ there exist constants $\rho \in (0,1)$, $a_0 \in (0,1)$, $b_0 \geq 1$ and $C > 0$ such that for every integer $n > 0$, every $\beta > 0$ and every $h \in C^1(U)$, if $a, b \in \mathbb{R}$, are such that $|a| \leq a_0$, $|b| \geq b_0$, then*

$$|L^n_{f-(P+a+\mathbf{i}b)\tau} h\|_{1,b} \leq C\ \rho^n\ |b|^\epsilon\ \|h\|_{1,b}\ ,$$

where the norm $\|.\|_{1,b}$ on $C^1(U)$ is defined by $\|h\|_{1,b} = \|h\| + \frac{\|dh\|}{|b|}$ for $|b| \geq 1$. In particular, for the spectral radius $\rho(L_{f-(P+a+\mathbf{i}b)\tau})$ of $L_{f-(P+a+\mathbf{i}b)\tau}$ in $C^1(U)$ we have $\rho(L_{f-(P+a+\mathbf{i}b)\tau}) \leq \rho < 1$.

Similar estimate holds for the Ruelle operator acting on the space $\mathcal{F}_\beta(U \cap \Lambda)$ of Hölder continuous functions with respect to an appropriate norm $\|\cdot\|_{\beta,b,U\cap\Lambda}$; see Corollary 3.3 below for details.

Theorem 1.1 is obtained as a consequence of the central Theorem 3.1 which in turn is analogous to Lemma 4 in [D]. As another consequence of Theorem 3.1 and the procedure described in [D] (based on previous arguments of Pollicott [Po] and Ruelle [R2]) one gets exponential decay of correlations for open billiard flows in the plane.

Throughout, given $\alpha > 0$, we denote by $\mathcal{F}_\alpha(\Lambda)$ the set of *Hölder continuous functions* with Hölder exponent α and by $\|h\|_\alpha$ the *Hölder constant* of $h \in \mathcal{F}_\alpha(\Lambda)$.

THEOREM 1.2. *Let F be a Hölder continuous function on Λ in $S^*(\Omega)$ and let ν_F be the Gibbs measure determined by F on Λ. For every $\alpha > 0$ there exist constants $C = C(\alpha) > 0$ and $c = c(\alpha) > 0$ such that*

$$\left|\int_\Lambda A(x)B(\phi_t(x))\ d\nu_F(x) - \left(\int_\Lambda A(x)\ d\nu_F(x)\right)\left(\int_\Lambda B(x)\ d\nu_F(x)\right)\right| \leq Ce^{-ct}\|A\|_\alpha\ \|B\|_\alpha$$

for any two functions $A, B \in \mathcal{F}_\alpha(\Lambda)$.

Using Theorem 1.1 and applying the argument of Pollicott and Sharp [PoS] in the case of geodesic flows on compact surfaces of negative curvature, one derives that the *zeta function*

$$\zeta(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-1}$$

of the billiard flow ϕ_t has an analytic continuation in a half-plane $\mathrm{Re}(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$. Here h_T is the *topological entropy* of the flow ϕ_t, γ denotes a closed orbit of ϕ_t and $\ell(\gamma)$ is the least period of γ. Moreover, folowing [PoS] again (cf. also [DP]), one derives that there exists $c \in (0, h_T)$ such that

$$\pi(\lambda) = \#\{\gamma : \ell(\gamma) \leq \lambda\} = \mathrm{li}(e^{h_T\lambda}) + O(e^{c\lambda})$$

as $\lambda \to \infty$, where $\mathrm{li}(x) = \int_2^x du/\log u$. The latter is a much stronger result than the standard Prime Orbit Theorem for open planar billiards in [Mor] (cf. [St1] for the higher dimensional case) derived by means of a result of Parry and Pollicott [PP1].

The present work was motivated by Ikawa's approach ([I2]) to the study of the so called Modified Lax-Phillips Conjecture (cf. the Epilogue in [LP] or Sect. 4 in [Z]) concerning the distribution of poles of the scattering matrix in obstacle scattering for the wave equation with Dirichlet boundary condition. In the case when the obstacle K in $\mathbb{R}^n$ ($n \geq 3$, n odd) is a finite disjoint union of strictly convex bodies Ikawa showed that the MLPC is equivalent to the existence of analytic singularities of a specially defined zeta function $\zeta_D(s)$. As for the classical dynamical zeta function $\zeta(s)$, the Ruelle operator related to the billiard flow on $S^*(\Omega)$ turns out to be an useful tool for the study of the scattering zeta function $\zeta_D(s)$ as well (cf. [I2], [P]). Using such a technique, Ikawa proved the MLPC for special classes of obstacles[1] in $\mathbb{R}^3$. It is natural to expect that estimates as that in Theorem 1.1 in the higher dimensional case would lead to the proof of the MLPC for more general classes of obstacles.

What concerns results on various types of decay of correlations, the reader is referred to the survey article of Baladi [Ba] (cf. also the Addendum by Dolgopyat and Pollicott [DP]) for general information and historical remarks. Amongst the most recent achievements one should mention the important articles of Liverani [L], Young [Y], Chernov [Ch1] and Dolgopyat [D] answering long standing questions. It apppears that for billiards the only results of this type that have been known so far concern the corresponding discrete dynamical system (generated by the billiard ball map from boundary to boundary). To my knowledge, these results are: the subexponential decay of correlations established for a very large class of dispersing billiards by Bunimovich, Sinai and Chernov [BSC] and the exponential decay of correlations for some classes of dispersing billiards in the plane and on the two-dimensional torus established by Young [Y] and Chernov [Ch2] as consequences of their more general arguments. Theorem 1.2 above describes a non-trivial class of billiard flows with exponential decay of correlations for any Hölder continuous potential. The main feature of these flows is that they have a unique maximal compact invariant set which is hyperbolic and has a zero-dimensional intersection with any (un)stable manifold. However they have other special properties that have been essentially used below. It is plausible that Dolgopyat's method could be used in more general situations than the one considered here.

The paper is organized as follows. Sect. 2 contains some basic definitions and notation along with some preliminary information. In Sect. 3 we state the central result (Theorem 3.1) and the main lemma (Lemma 3.4) needed for its proof. Theorem 1.1 is then derived in the same way as in [D]. Sections 4-8 are devoted to the proof of Lemma 3.4. We follow the method of Dolgopayt in Sects. 6-8 of [D], however essential new difficulties have to be overcome. These are due to the non-compactness of the phase space and the singularity of the Gibbs measures[2]. As remarked in [DP], one of the main features of the flow required in order to apply Dolgopyat's method is the Federer property of the corresponding Gibbs measures. Dislike the case of compact Anosov manifolds, this property is non-trivial in the

[1] E.g. when K is a finite union of disjoint balls of the same radius the distaces between them being large compared to the radius.

[2] $U' \cap \Lambda$ has Lebesgue measure zero in U' for any local unstable manifold U'.

situation considered here. It is proved in Sect. 5 using the distortion bounds in Sect. 4 concerning the discretization map $\sigma : U \longrightarrow U$. These distortion bounds and the uniform non-integrability of the so called temporal distance function established in Sect. 6 (by adapting a result of Katok and Burns [KB]) are later used in Sect. 7. There we construct for every $\epsilon > 0$ a partition of U into closed intervals of length $\leq \epsilon$ such that the ratio of the lengths of any two of these intervals is bounded from above and below by constants independent of ϵ, and such that the so called triple intersection property with respect to Λ is satisfied[3]: every interval I from the partition that contains a point of Λ is a member of a family of three consecutive intervals from the partition having common points with Λ and lying in the same connected component of U. This construction enables us to exploit the smoothness of σ and use the mean value theorem and similar "infinitesimal" tools on (or rather near) the Cantor set $U \cap \Lambda$. This is done in Sect. 7, where the main part of Dolgopyat's method is described and used. In Sect. 8 we very briefly outline how Theorem 1.2 is obtained from the main estimate (more precisely from Corollary 3.3 below) – now the procedure used in [D] applies automatically.

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2 Preliminaries

The *non-wandering set* Λ for the flow ϕ_t is the set of those $x \in S^*(\Omega)$ such that the trajectory $\{\phi_t(x) : t \in \mathbb{R}\}$ is bounded. Notice that the natural projection of ϕ_t on the quotient space $S^*(\Omega)/\sim$, where $\sim$ is the equivalence relation $(q, v) \sim (p, w)$ iff $q = p$ and $v = w$ or $q = p \in \partial K$ and v and w are symmetric with respect to $T_q(\partial K)$, is continuous. Moreover whenever both x and $\phi_t(x)$ are in the interior of $S^*(\Omega)$ and sufficiently close to Λ, the map $y \mapsto \phi_t(y)$ is smooth on a neighbourhood of x. It follows from results of Sinai ([Si1]) that Λ is a hyperbolic set for ϕ_t, and it is easily seen that there exists an open neighbourhood U of Λ in $S^*(\Omega)$ such that $\Lambda = \cap_{t\in\mathbf{R}}\phi_t(U)$. In fact, Λ is clearly the maximal compact ϕ_t-invariant subset of $S^*(\Omega)$. Moreover, it follows from the natural symbolic coding for the natural section of the flow (the so called billiard ball map; cf. Sect. 4) that the periodic points are dense in Λ (cf. e.g. [Mor]), and ϕ_t is transitive on λ. Thus, Λ is a basic set for ϕ_t and the classical theory of hyperbolic flows applies in the case under consideration (see e.g. Part 4 in the monograph of Katok and Hasselblatt [KH]).

Consider the *metric* ρ on $S^*(\mathbb{R}^2)$ given by $\rho((q, v), (p, w)) = \|q - p\| + \|v - w\|$. It is easy to see that there exists a metric $\tilde{\rho}$ on $S^*(\Omega)/\sim$ which generates the topology of the quotient space and coincides with ρ for pairs of points that are not very close to $S^*_{\partial K}(\Omega)$. The particular choice of this metric will not be important below (in fact we will only need it sifficiently far away from $S^*_{\partial K}(\Omega)$), so we are not going to specify it.

For $x \in \Lambda \setminus S^*_{\partial K}(\Omega)$ and a sufficiently small $\epsilon > 0$ let

$$W^s_\epsilon(x) = \{y \in S^*(\Omega) : \rho(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0 \ , \ \rho(\phi_t(x), \phi_t(y)) \to_{t\to\infty} 0 \} \ ,$$

[3]It is enough to know that there exists $m \geq 3$, possibly depending on the choice of the partition, such that for any choice of m consecutive intervals of the partition lying in the same connected component of U whose union has a common point with Λ, at least 3 of these intervals have common points with Λ.

$$W_\epsilon^u(x) = \{y \in S^*(\Omega) : \rho(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0 \ , \ \rho(\phi_t(x), \phi_t(y)) \to_{t \to -\infty} 0 \}$$

be the (strong) *stable* and *unstable sets* of size ϵ. It is easy to show (either by using arguments from [Si2] or directly) that for every $x \in \Lambda \setminus S^*_{\partial K}(\Omega)$ and every sufficiently small $\epsilon > 0$, $W_\epsilon^s(x)$ and $W_\epsilon^u(x)$ are 1-dimensional submanifolds of $S^*(\Omega)$ such that $E^s(x) = T_x(W_\epsilon^s(x)) \neq E^u(x) = T_x(W_\epsilon^u(x))$. Notice that both $W_\epsilon^s(x) \cap \Lambda$ and $W_\epsilon^u(x) \cap \Lambda$ are Cantor sets, i.e. they are infinite compact totally disconnected sets without isolated points.

For $A \subset S^*(\Omega)$ and a subset I of $\mathbb{R}$, following [B2] denote $\phi_I(A) = \{ \ \phi_t(y) \ : \ y \in A, t \in I \ \}$. Given $\epsilon > 0$, set

$$S^*_\epsilon(\Omega) = \{x = (q, v) \in S^*(\Omega) : \mathrm{dist}(q, \partial K) > \epsilon\} \quad , \quad \Lambda_\epsilon = \Lambda \cap S^*_\epsilon(\Omega) \ .$$

In what follows in order to avoid ambiguity and unnecessary complications we will consider stable and unstable manifolds only for points x in $S^*_\epsilon(\Omega)$ or Λ_ϵ; this will be enough for our purposes.

It follows from the genaral theory of horocycle foliations (cf. [S] or [KH]) that if $\epsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $\rho(x, y) < \delta$, then $W_\epsilon^s(x) \cap \Lambda$ and $\phi_{[-\epsilon,\epsilon]}(W_\epsilon^u(y) \cap \Lambda)$ intersect at exactly one point $[x, y]$. That is, there exists a unique $t \in [-\epsilon, \epsilon]$ such that $\phi_t([x, y]) \in W_\epsilon^u(x)$. Setting $\Delta(x, y) = t$, defines the so called *temporary distance function* Δ ([Ch1]) which will be used in Sect. 6 below. If $z \in \Lambda_\epsilon$ and $U \subset W_\epsilon^u(z)$ and $S \subset W_\epsilon^s(z)$ are closed (i.e. containing their end points) curves containing z and such that $U \cap \Lambda$ and $S \cap \Lambda$ have no isolated points, then

$$\Pi = [U \cap \Lambda, S \cap \Lambda] = \{[x, y] : x \in U \cap \Lambda \ , \ y \in S \cap \Lambda \ \}$$

is called a *rectangle* in Λ_ϵ. Notice that Π is "foliated" by leaves $[x, S \cap \Lambda]$ of stable manifolds.

Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a family of rectangles such that each R_i is contained in a C^1 cross-section $D_i \subset S^*_\epsilon(\Omega)$ to the flow ϕ_t. Thus, for each i, $R_i = [U_i \cap \Lambda, S_i \cap \Lambda]$, where U_i and S_i are closed curves in $W_\epsilon^u(z_i)$ and $W_\epsilon^s(z_i)$, respectively, for some $z_i \in \Lambda_\epsilon$. Set

$$R = \bigcup_{i=1}^k R_i \ .$$

The family $\mathcal{R}$ is called *complete* if there exists $T > 0$ such that for every $x \in \Lambda$ there exist $t_1 \in [-T, 0)$ and $t_2 \in (0, T]$ with $\phi_{t_1}(x) \in R$ and $\phi_{t_2}(x) \in R$. Thus, $\tau(x) = t_2(x) > 0$ is the smallest positive time with $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, and $\mathcal{P} : R \longrightarrow R$ is the *Poincaré map* related to the family $\mathcal{R}$. The function $\tau : R \longrightarrow [0, \infty)$ is called the *root function* (or the *first return time*) associated with $\mathcal{R}$.

Following [B2] and [Ra] we will say that a complete family $\mathcal{R} = \{R_i\}_{i=1}^k$ of rectangles in $S^*_\epsilon(\Omega)$ is a *Markov family* of size $\chi \in (0, \epsilon/2)$ for the billiard flow ϕ_t if:

(a) $R_i \cap R_j = \emptyset$ for $i \neq j$ and for each i the sets $U_i \cap \Lambda$ and $S_i \cap \Lambda$ are contained in the interior of the curves U_i and S_i, respectively;

(b) $\mathrm{diam}(R_i) \leq \chi$;

(c) For any $i \neq j$ and any $x \in R_i \cap \mathcal{P}^{-1}(R_j)$ we have $\mathcal{P}([x, S_i \cap \Lambda]) \subset [\mathcal{P}(x), S_j]$ and $\mathcal{P}([U_i \cap \Lambda, x]) \supset [U_j, \mathcal{P}(x)]$;

(d) For any $i = 1, \ldots, k$ and $x \in R_i$ the function τ is constant on the set $[x, S_i \cap \Lambda]$.

The existence of a Markov family $\mathcal{R}$ of an arbitrarily small size $\chi > 0$ for ϕ_t follows from the construction of Bowen [B2] (cf. also Ratner [Ra]). Apart from that, one can easily arrange one extra property of $\mathcal{R}$. Namely, for $i \neq j$, $(1 \leq i, j \leq k_0)$, denote by $\Pi_{i,j}$ *the set of those* $x = (q, v) \in \Lambda$ *such that* $q \in \partial K_i$, $\langle \nu(q), v \rangle > 0$, where $\nu(q)$ is the *outward unit normal* to ∂K at q, and the first reflection point of the billiard trajectory $\{\mathrm{pr}_1(\phi_t(x)) : t > 0\}$ belongs to ∂K_j. Define $\omega : \cup_{i=1}^k R_i \longrightarrow M$ by $\omega(q, v) = \phi_{-t}(q, v)$, where $t = t(q, v) > 0$ is the minimal number with $\mathrm{pr}_1(\phi_{-t}(q, v)) \in \partial K$. Using the construction of Bowen starting with appropriate cross-sections, one gets a Markov family $\mathcal{R}$ which satisfies the following extra condition:

(e) For every $i = 1, \ldots, k$ there exist $s(i), t(i) \in \{1, \ldots, k_0\}$ such that $\omega(R_i) \subset \Pi_{s(i)t(i)}$ and for any $x \in R_i$ the trajectory $\phi_{[0,\tau(x))}(x)$ has at most one reflection point.

It follows from a result of C. Robinson [Ro] that there exists an open neighbourhood $\mathcal{V}$ of Λ_ϵ in $S^*(\Omega)$ and C^1 *transverse foliations* $\mathcal{F}^u$ and $\mathcal{F}^s$ on $\mathcal{V}$ such that $W_\epsilon^u(x) \cap \mathcal{V} = \mathcal{F}^u(x)$ and $W_\epsilon^s(x) \cap \mathcal{V} = \mathcal{F}^s(x)$ for any $x \in \Lambda \cap \mathcal{V}$. Fix a neighbourhood $\mathcal{V}$ and C^1 foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ with these properties.

Choosing $\chi > 0$ sufficiently small, we may assume that our Markov family $\mathcal{R}$ satisfies the following additional condition:

(f) For each i the cross-section D_i (and therefore the rectangle R_i) is contained in $\mathcal{V}$.

¿From now on we will assume that $\mathcal{R} = \{R_i\}_{i=1}^k$ is a fixed Markov family for ϕ_t satisfying the extra conditions (e) and (f). Then

$$U = \bigcup_{i=1}^k U_i \, ,$$

is a finite disjoint union of compact curves in $\mathcal{V}$.

Using the foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ in $\mathcal{V}$, assuming again that $\mathcal{V}$ is sufficiently small, we can extend the product $[x, y]$ over the whole $U_i \times S_i$ for any i as follows. Given $x \in U_i \subset W_\epsilon^u(z_i) = \mathcal{F}^u(z_i) \cap \mathcal{V}$ and $y \in S_i \subset W_\epsilon^s(z_i) = \mathcal{F}^s(x_i) \cap \mathcal{V}$, the sets $\mathcal{F}^s(x)$ and $\phi_{[-\epsilon,\epsilon]}(\mathcal{F}^u(y))$ intersect at exactly one point $[x, y]$. The image of the C^1 map $U_i \times S_i \ni (x, y) \mapsto [x, y]$ is a 2-dimensional sumanifold of $S_\epsilon^*(\Omega)$ which will be denoted by $\tilde{R}_i = [U_i, S_i]$. Changing appropriately the cross-section D_i, we may assume[4] that $\tilde{R}_i \subset D_i$ for each i. The *projection* $p : \tilde{R} = \cup_{i=1}^k \tilde{R}_i \longrightarrow U$ along the leaves of $\mathcal{F}^s$ is C^1. Then the Poincaré map $\mathcal{P} : \tilde{R} \longrightarrow \cup_{i=1}^k D_i$ and the corresponding *root function* $\tau : \tilde{R} \longrightarrow [0, \infty)$ are well-defined and C^1. Thus, σ can be extended to a C^1 map $\sigma : U \longrightarrow U$ by the same formula: $\sigma = p \circ \mathcal{P}$. In the same way one observes that $\Delta(x, y)$ is well-defined and C^1 for $(x, y) \in \cup_{i=1}^k \tilde{R}_i \times \tilde{R}_i$.

Convention. Given a subset A of U and an integer $n > 0$ we will denote by $\sigma^n(A)$ the set of all well-defined $\sigma^n(x)$ with $x \in A$. That is, $\sigma^n(A) = \sigma^n(A \cap \sigma^{-n}(U))$.

A conjugacy between $\Lambda \cap R$ and the Bernoulli shift on a symbolic space Σ_A is now defined in the usual way. Let $A = (A_{ij})_{i,j=1}^k$ be the matrix given by $A_{ij} = 1$ if $\mathcal{P}(R_i) \cap R_j \neq \emptyset$ and $A_{ij} = 0$ otherwise. Then

$$\Sigma_A = \{(i_j)_{j=-\infty}^\infty : 1 \leq i_j \leq k, A_{i_j \, i_{j+1}} = 1 \quad \text{for all} \;\; j \;\},$$

[4]This is just for convenience; practically one does not need it.

and the *Bernoulli shift* $\tilde{\sigma} : \Sigma_A \longrightarrow \Sigma_A$ is given by $\tilde{\sigma}((i_j)) = ((i'_j))$, where $i'_j = i_{j+1}$ for all j. Define $\Phi : R \longrightarrow \Sigma_A$ by $\Phi(x) = (i_j)_{j=-\infty}^{\infty}$, where $\mathcal{P}^j(x) \in R_{i_j}$ for all $j \in \mathbf{Z}$. Notice that $\mathcal{P} : R \longrightarrow R$ is invertible, so $\mathcal{P}^j$ is well-defined for all integers j. The map Φ is a homeomorphism when Σ_A is considered with the product topology, and $\tilde{\sigma} \circ \Phi = \Phi \circ \mathcal{P}$. The *projection* r of τ on Σ_A is given by $r \circ \Phi = \tau$.

The subset $\Lambda \cap U$ of R can be naturally identified with

$$\Sigma_A^+ = \{ \ (i_j)_{j=0}^{\infty} \ : \ 1 \leq i_j \leq k \ , \ A_{i_j \ i_{j+1}} = 1 \quad \text{for all} \ \ j \geq 0 \ \}.$$

Namely, if $\pi : \Sigma_A \longrightarrow \Sigma_A^+$ is the *natural projection*, then $\Phi^+ = \pi \circ \Phi_{|\Lambda \cap U} : \Lambda \cap U \longrightarrow \Sigma_A^+$ is a bijection with $\tilde{\sigma} \circ \Phi^+ = \Phi^+ \circ \sigma$, where $\tilde{\sigma}$ denotes the corresponding Bernoulli shift on Σ_A^+.

The *suspended flow* $\tilde{\sigma}^r$ over $\Sigma_A^r = \{(\xi, t) \ : \ \xi \in \Sigma_A, 0 \leq t \leq r(\xi) \ \}$ is defined by $\tilde{\sigma}^r_s(\xi, t) = (\xi, t+s)$, where we use the identification $(\xi, r(\xi)) = (\sigma(\xi), 0)$ in $\Sigma_A \times \mathbb{R}$. Then $\Sigma_A^{r,+} = \{(\xi, t) \in \Sigma_A^r : \xi \in \Sigma_A^+\}$ is a closed subset of Σ_A^r which is invariant under the semiflow $\tilde{\sigma}^r_t$, $t \geq 0$ (cf. [PP2] for details). A conjugacy between $\tilde{\sigma}^r_t$ and ϕ_t on Λ is defined by $\Phi^r(\phi_s(x)) = (\Phi(x), s)$ for $x \in R$ and $0 \leq s \leq \tau(x)$.

Given $\theta \in (0, 1)$, consider the *metric* d_θ on Σ_A given by $d_\theta(\xi, \eta) = 0$ if $\xi = \eta$ and $d_\theta(\xi, \eta) = \theta^n$ if $\xi_i = \eta_i$ for $|i| \leq n$ and n is maximal with this property. In a similar way one defines a metric d_θ on Σ_A^+. Using it, we get a metric on $\Sigma_A^{r,+}$ by setting $d^r_\theta((\xi, t), (\eta, s)) = d_\theta(\xi, \eta) + |t - s|^\theta$. The *spaces of Lipschitz functions* on Σ_A^+ and $\Sigma_A^{r,+}$ with respect to the metrics just defined will be denoted by $\mathcal{F}_\theta(\Sigma_A^+)$ and $\mathcal{F}_\theta(\Sigma_A^{r,+})$, respectively.

Let $F \in \mathcal{F}_\alpha(\Lambda)$ for some $\alpha > 0$. Denote by $\tilde{F}$ the function on Σ_A^r such that $\tilde{F} \circ \Phi^r = \Phi^r \circ F$. There exists $\theta = \theta(\alpha) \in (0, 1)$ such that $\tilde{F} \in \mathcal{F}_\theta(\Sigma_A^r)$. Let $\tilde{\mu}$ be the *Gibbs measure* related to $\tilde{F}$ with respect to the suspended flow $\tilde{\sigma}^r$, and let $P = \Pr_{\tilde{\sigma}^r}(\tilde{F})$ be the *topological pressure* of $\tilde{F}$ with respect to $\tilde{\sigma}^r$. A function $\tilde{f} \in \mathcal{F}_\theta(\Sigma_A^+)$ related to $\tilde{F}$ is defined by

$$\tilde{f}(\xi) = \int_0^{r(\xi)} \tilde{F}(\xi, s) \ ds \ .$$

Let $\tilde{\nu}$ be the *Gibbs measure* on Σ_A determined by the function $\tilde{f} - Pr$. Then (cf. [PP2])

$$d\tilde{\mu}(\xi, s) = \frac{1}{\nu(r)} d\tilde{\nu}(\xi) ds \ , \ \text{where} \quad \nu(\tilde{g}) = \int_{\Sigma_A} \tilde{g}(\xi) \ d\tilde{\nu}(\xi) \ .$$

Moreover, we have $\Pr_{\tilde{\sigma}}(\tilde{f} - Pr) = 0$. The Gibbs measures $\tilde{\mu}$ and $\tilde{\nu}$ give rise to measures μ and ν on R and $\Lambda \cap U$, respectively, via the conjugacies Φ^r and Φ. If f is the function on $\Lambda \cap U$ such that $\tilde{f} \circ \Phi = \Phi \circ f$, then $f(x) = \int_0^{\tau(x)} F(\phi_s(x)) \ ds$, so $f \in \mathcal{F}_\alpha(\Lambda \cap U)$. Thus, according to Kirszbraun's theorem (cf. Ch. 3 in [WW]), f has an extension to a function $f \in \mathcal{F}_\alpha(U)$ with the same *Hölder constant* $\|f\|_\alpha = \|f_{|\Lambda \cap U}\|_\alpha$. The measures μ and ν are called the *Gibbs measures* related to F and $f - P\tau$, respectively. It follows from above that $\Pr_{\phi_t}(F) = P$ and $\Pr_\sigma(f - P\tau) = 0$.

Let $C(U)$ be the space of *bounded continuous functions* on U. For $g \in C(U)$ denote $\|g\| = \|g\|_0 = \sup_{x \in U} |g(x)|$. Given g, the *Ruelle operator* $L_g : C(U) \longrightarrow C(U)$ is defined in the usual way:

$$(L_g h)(u) = \sum_{\sigma(v) = u} e^{g(v)} h(v).$$

If $g \in C^1(U)$, then clearly L_g preserves the space $C^1(U)$. Here $C^1(U)$ denotes the *space of differentiable function* with bounded derivatives on U. For $g \in C^1(U)$ we will denote by $\|dg(u)\|$ the norm of the linear map $dg(u) : T_uU \longrightarrow \mathbb{R}$ with respect to the Euclidean norm on $T_uU \subset \mathbb{R}^4$.

Given $f \in \mathcal{F}_\alpha(U)$ and $b \in \mathbb{R}$ with $|b|$ sufficiently large, following [D], denote by $f^{(b)}$ the average of f over subintervals of U of length $1/\sqrt{|b|}$. To be more precise, one has to first fix an arbitrary extension $f \in \mathcal{F}_\alpha(S^*(\Omega))$ (with the same Hölder constant), then extend each connected component U_i of U appropriately (so every point of U_i is far enough from the ends of the extension) and then take the averages in question. Then $f^{(b)} \in C^1(U)$, and as mentioned in [D], it has the following properties: (a) $\|f - f^{(b)}\|_0 \leq \|f\|_\alpha/|b|^{\alpha/2}$; (b) $\|df^{(b)}\| \leq \text{const } \|f\|_\alpha \, |b|^{(1-\alpha)/2} \leq \text{const } \|f\|_\alpha \sqrt{|b|}$; (c) $\|f - f^{(b)}\|_\beta \to 0$ as $|b| \to \infty$ for any $\beta < \alpha$. In the special case $f \in C^1(U)$ we set $f^{(b)} = f$ for all b.

It follows from Theorem 3.6 in [R2] that if $g \in C^1(U)$, then $L_g : C^1(U) \longrightarrow C^1(U)$ has an eigenvalue λ_g equal to the spectral radius of L_g and a corresponding (positive) eigenfunction. Given real numbers a and b (with $|a| + \frac{1}{|b|}$ small), denote by λ_{ab} the *largest eigenvalue* of $L_{f^{(b)}-(P+a)\tau}$ and by h_{ab} the corresponding (positive) eigenfunction such that $\sup_{u\in U} h_{ab}(u) = 1$. The above remark implies that $h_{ab} \in C^1(U)$. As in [D], one shows that for $|a| + \frac{1}{|b|}$ sufficiently small we have $\|d_u \ln h_{ab}(u)\| \leq \text{const } \sqrt{|b|}$ for all $u \in U$. Since $\Pr(f - P\tau) = 0$, it follows from the main properties of pressure (cf. e.g. Ch. 3 in [PP2]) and the properties of the approximation $f^{(b)}$ mentioned above that $|\Pr(f^{(b)} - (P+a)\tau)| \leq \text{const } (|a| + \frac{1}{|b|^{\alpha/2}})$, where const > 0 is a constant depeding on f. Thus, for small $|a|$ and large $|b|$, the maximal eigenvalues λ_{ab} and the eigenfunctions h_{ab} are uniformly close to 1. In particular, for small $|a|$ and large $|b|$ we have $h_{ab}(u) \geq \frac{1}{2}$ for all $u \in U$.

Given real numbers a and b, as in [D], consider the function

$$f^{(ab)}(u) = f^{(b)}(u) - (P+a)\tau(u) + \ln h_{ab}(u) - \ln h_{ab}(\sigma(u)) - \ln \lambda_{ab}$$

and the operators

$$L_{ab} = L_{f^{(ab)}} : C(U) \longrightarrow C(U) \quad , \quad \mathcal{M}_{ab} = L_{f^{(ab)}} : C(U) \longrightarrow C(U).$$

One checks that $\mathcal{M}_{ab}\, 1 = 1$.

For a connected curve I in $S^*(\Omega)$ we will denote by $|I|$ the *arc length* of I with respect to the standard metric in $S^*(\Omega) \subset \mathbb{R}^2 \times S^1$. Given $x \in U_i$ for some i and $r > 0$ we will denote by $I(x,r)$ the set of all $y \in U_i$ with $\text{dist}_U(x,y) \leq r$, where $\text{dist}_U(x,y)$ is the *arc length* of the connected subcurve of U_i with end points x and y.

It is well known (see also Sect. 5 below) that σ^n is an expanding map (on its domain $\sigma^{-n}(U)$) for sufficiently large integers n. Since σ is C^1, shrinking slightly R if necessary (and therefore U), it is uniformly Lipschitz on U. Thus, there exist constants $c_0 > 0$ and $\gamma_1 > \gamma > 1$ such that

$$c_0\gamma^n \,\text{dist}_U(u_1,u_2) \leq \text{dist}_U(\sigma^n(u_1),\sigma^n(u_2)) \leq \frac{\gamma_1^n}{c_0}\text{dist}(u_1,u_2) \tag{2.1}$$

whenever u_1 and u_2 belong to the same connected component of U and $\sigma^n(u_i)$, $i = 1,2$, is well-defined. Set $C_0 = \frac{1}{c_0(\gamma-1)}$. ¿From now on **we will assume that** c_0**,** C_0**,** γ **and** γ_1

are fixed constants with the above properties.

LEMMA 2.1. (Lemma 3 in [D]) *Let $h \in C^1(U)$ and let $v(u)$ be a C^1 map defined on some open subset V of U and such that $\sigma^n(v(u)) = u$ for all $u \in V$. Then*

$$\|dv(u)\| \leq \frac{1}{c_0\ \gamma^n} \quad \text{and} \quad \|d_u h_n(v(u))\| \leq C_0 \sup_{v \in U} \|dh(v)\| \ .$$

If $h \in \mathcal{F}_\alpha(U)$ for some $\alpha \in (0,1]$, then $\|h_n \circ v\|_\alpha \leq C_0 \|h\|_\alpha$.

Set $\tilde{\tau} = \sup_{u \in U} \tau(u)$. Given a sufficiently small number $a_0' > 0$, there exists $T = T(a_0') > 0$ such that

(2.2) $$T \geq \sup_{u \in U} |f^{(ab)}(u)| \quad , \quad T|b| \geq \sup_{u \in U} \|df^{(ab)}(u)\|$$

for all $|a| \leq a_0'$, $1/|b| \leq a_0'$. Fix $a_0' > 0$ and $T > 0$ and with these properties.

LEMMA 2.2. (Lemma 2 in [D]) *There exists a constants $A_0 > 0$ such that for all $a \in \mathbb{R}$ with $|a| \leq a_0'$ and all positive integers n we have:*

(a) $|(L^n_{ab} h)(u)| \leq (\mathcal{M}^n_{ab}|h|)(u)$ *for all $u \in U$ and $h \in C(U)$;*

(b) $\|d_u(L^n_{ab} h)(u)\| \leq A_0[\ |b|(\mathcal{M}^n_{ab}|h|)(u) + \dfrac{1}{\gamma^n}(\mathcal{M}^n_{ab}\|dh\|)(u)\ ]$ *for all $u \in U$ and $h \in C^1(U)$;*

(c) $\|d_u(\mathcal{M}^n_{ab} h)(u)\| \leq A_0[\ |b|(\mathcal{M}^n_{ab}|h|)(u) + \frac{1}{\gamma^n}(\mathcal{M}^n_{ab}\|dh\|)(u)\ ]$ *for all $u \in U$ and $h \in C^1(U)$;*

(d) $\|L^n_{ab} h\|_\beta \leq A_0[\ |b|\ \|h\|_0 + \frac{1}{\gamma^{n\beta}}\|h\|_\beta\ ]$ *for all $u \in U$ and $h \in \mathcal{F}_\beta(U)$, $\beta > 0$.*

3 The main estimate

The central point in this paper is to prove the L^1_ν-contraction property of Dolgopyat's normalized operator L_{ab} with respect to the norm $\|h\|_{1,b}$ defined by $\|h\|_{1,b} = \|h\|_0 + \frac{1}{|b|}\|dh\|_0$ for $|b| \geq 1$. The main results stated in Sect. 1 are consequences of this estimate. We will also use the notation $\|h\|_{0,U\cap\Lambda} = \sup_{u \in U\cap\Lambda} |h(u)|$ and $\|h\|_{1,b,U\cap\Lambda}$ defined as above replacing h by $h_{|U\cap\Lambda}$.

THEOREM 3.1. *There exist a positive integer N and constants $\rho_1 \in (0,1)$ and $a_0 > 0$ such that for any $a, b \in \mathbb{R}$ with $|a| \leq a_0$ and $|b| \geq 1/a_0$ and every $h \in C^1(U)$ with $\|h\|_{1,b} \leq 1$ we have*

$$\int_U |L^{Nn}_{ab} h|^2\ d\nu \leq \rho_1^n$$

for every positive integer n, where ν is the Gibbs measure determined by $f - P\tau$ on U.

Remark 3.2. Since ν is a probability measure on U, combining the above with the Cauchy-Schwartz inequality, gives $\int_U |L^{Nn}_{ab} h|\ d\nu \leq \rho_1^{n/2}$ for $\|h\|_{1,b} \leq 1$ and $n > 0$. Hence $\int_U |L^{Nn}_{ab} h|\ d\nu \leq \rho_1^{n/2} \|h\|_{1,b}$ for any positive integer n and any $h \in C^1(U)$. On the other hand, it follows from Whitney's extension theorem (cf. Ch. 1 in [Mal]) that there exists a constant $D > 0$ such that for every $h \in C^1(U)$ there is $\tilde{h} \in C^1(U)$ with $\tilde{h}_{|U\cap\Lambda} = h_{|U\cap\Lambda}$, $(d\tilde{h})_{|U\cap\Lambda} = (dh)_{|U\cap\Lambda}$ and $\|\tilde{h}\|_0 \leq D \sup_{u\in U\cap\Lambda} |h(u)|$ and $\|d\tilde{h}\|_0 \leq D \sup_{u\in U\cap\Lambda} \|dh(u)\|$. Hence in the above statement the norm $\|.\|_{1,b}$ in $C^1(U)$ can be replaced by the seminorm (for $|b| \geq 1$)

$$\|h\|_{1,b,U\cap\Lambda} = \|h\|_{0,U\cap\Lambda} + \frac{1}{|b|}\|dh\|_{0,U\cap\Lambda} .$$

That is, we have $\int_U |L_{ab}^{Nn} h|\, d\nu \leq D\rho_1^{n/2}$ for all $n = 1, 2, \ldots$ and $h \in C^1(U)$ with $\|h\|_{1,b,U\cap\Lambda} \leq 1$. In the case when $h \in \mathcal{F}_\beta(U\cap\Lambda)$ for some $\beta > 0$ we have a similar inequality involving the norm

$$\|h\|_{\beta,b,U\cap\Lambda} = \|h\|_{0,U\cap\Lambda} + \frac{1}{|b|}\|h\|_{\beta,U\cap\Lambda} .$$

Indeed, according to Kirszbraun's theorem (cf. Ch. 3 in [WW]), such h can always be extended to a function $h \in \mathcal{F}_\beta(U)$ with the same Hölder constant $\|h\|_\beta = \|h\|_{\beta,U\cap\Lambda} = \|h_{|U\cap\Lambda}\|_\beta$. Approximating h by functions $h^{(p)}$ $(p \to \infty)$ from $C^1(U)$ (cf. Sect. 2), one gets

$$\text{(3.1)} \qquad \int_U |L_{ab}^{Nn} h|\, d\nu \leq D\rho_1^{n/2}\, \|h\|_{\beta,b,U\cap\Lambda} \qquad (n = 1, 2, \ldots\, , \quad h \in \mathcal{F}_\beta(U\cap\Lambda)) .$$

Theorem 1.1 follows immediately from part (a) of the following consequence of Theorem 3.1.

COROLLARY 3.3. (a) *For every $\epsilon > 0$ there exist constants $\rho \in (0, 1)$, $a_0 \in (0, 1)$ and $C > 0$ such that for every integer $n > 0$, every $\beta > 0$ and every $h \in \mathcal{F}_\beta(U\cap\Lambda)$, if $a, b \in \mathbb{R}$, are such that $|a| \leq a_0$, $1/|b| \leq a_0$, then:*

$$\|L_{ab}^n h\|_{\beta,b,U\cap\Lambda} \leq C\, \rho^n\, |b|^\epsilon\, \|h\|_{\beta,b,U\cap\Lambda} \quad \textit{and} \quad |L_{f^{(b)}-(P+a+\mathbf{i}b)\tau}^n h\|_{1,b,U\cap\Lambda} \leq C\, \rho^n\, |b|^{1/2}\, \|h\|_{\beta,b,U\cap\Lambda} .$$

When $f \in C^1(U)$ and $\beta \in (0, 1]$, under the same assumptions we have

$$|L_{f-(P+a+\mathbf{i}b)\tau}^n h\|_{\beta,b,U\cap\Lambda} \leq C\, \rho^n\, |b|^\epsilon\, \|h\|_{\beta,b,U\cap\Lambda} .$$

(b) *If $\beta \leq \alpha$, there exist constants $\rho \in (0, 1)$, $a_0 \in (0, 1)$, $\epsilon' > 0$, $B > 0$ and $C > 0$ such that for $h \in \mathcal{F}_\beta(U\cap\Lambda)$ and $a, b \in \mathbb{R}$ with $|a| \leq a_0$ and $1/|b| \leq a_0$ we have*

$$\|L_{f-(P+a+\mathbf{i}b)\tau}^{2[B\ln|b|]} h\|_{\beta,b,U\cap\Lambda} \leq \frac{C}{|b|^{\epsilon'}}\, \|h\|_{\beta,b,U\cap\Lambda} .$$

The derivation of Corollary 3.3 from Theorem 3.1 essentially repeats the proofs of Corollaries 2 and 3 in [D] (cf. also Sect. 6 in [PoS]).

Proof of Corollary 3.3. (a) Let $\rho_1 \in (0, 1)$ and N be as in Theorem 3.1, and let $\epsilon > 0$. We will assume that $\rho_1 \geq \frac{1}{\gamma}$, where $\gamma > 1$ is determined in Sect. 2. Again we will denote by const a global constant (independent of a, b, n). Let $a, b \in \mathbb{R}$ be such that $|a| \leq a_0$

and $|b|^{-\alpha/2} \leq a_0$. As in Sect. 2, $\alpha > 0$ denotes the Hölder exponent of f, i.e. we have $f \in \mathcal{F}_\alpha(U)$. Without loss of generality we may assume that $\epsilon < 1$. Set $q = 1/\epsilon$ and let $p > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

Assume that $h \in \mathcal{F}_\beta(U \cap \Lambda)$ for some $\beta > 0$ and $\|h\|_{\beta,b,U\cap\Lambda} \leq 1$. According to Remark 3.2, we may assume that $h \in \mathcal{F}_\beta(U)$ with the same $\|h\|_0$ and $\|h\|_\beta$. Since ν is the Gibbs measure for $g = f - P\tau$ on U and $\Pr(g) = 0$, it folows from the Perron-Ruelle-Frobenius theorem that if we take $\rho_1 \in (0,1)$ from Theorem 3.1 sufficiently close to 1 and const > 0 sufficiently large, then

$$\left| L_g^m h(u) - \int_{U\cap\Lambda} h \, d\nu \right| \leq \text{const } \rho_1^m \, \|h\|_\beta \qquad u \in U \cap \Lambda \ , \ m = 1, 2, \dots \ . \tag{3.2}$$

Lemma 2.2 and Hölder's inequality imply

$$\begin{aligned} \left| L_{ab}^{2nN} h(u) \right| &= \left| L_{ab}^{nN} \left(L_{ab}^{nN} \right) (u) \right| \leq \mathcal{M}_{ab}^{nN} \left(|L_{ab}^{nN} h| \right) (u) \\ &= L_{f^{(ab)}}^{nN} \left(\left| L_{ab}^{nN} h \right| \right) (u) = L_g^{nN} \left(e^{(f^{(ab)} - g)_{nN}} \, |L_{ab}^{nN} h| \right) (u) \\ &\leq \left[L_g^{nN} \left(e^{p(f^{(ab)} - g)_{nN}} \right) (u) \right]^{1/p} \, \left[L_g^{nN} \left(|L_{ab}^{nN} h|^q \right) (u) \right]^{1/q} \ . \end{aligned} \tag{3.3}$$

To estimate the first term, write $L_g^{nN} \left(e^{p(f^{(ab)} - g)_{nN}} \right) (u) = \left[L_{g+p(f^{(ab)}-g)}^{nN} 1 \right] (u)$. Let $\hat{h} = \hat{h}_{a,b}$ be an eigenfunction of $L_{g+p(f^{(ab)}-g)}$ with $\sup \hat{h} = 1$. Since

$$|\Pr(g + p(f^{(ab)} - g))| = |\Pr([f - P\tau] + p[f^{(b)} - f] - pa\tau) - p \ln \lambda_{ab}| \leq A_1 \left(|a| + \frac{1}{|b|^{\alpha/2}} \right)$$

for some constant $A_1 > 0$ (depending on ϵ), it follows that for small $|a|$ and large $|b|$ we have $\hat{h}(u) \geq 1/2$ for all $u \in U \cap \Lambda$. Moreover, the Perron-Ruelle-Frobenius theorem implies

$$\left[L_{g+p(f^{(ab)}-g)}^{nN} 1 \right] (u) \leq 2 \left[L_{g+p(f^{(ab)}-g)}^{nN} \hat{h} \right] (u) = 2 e^{nN \Pr(g+p(f^{(ab)}-g))} \hat{h}(u) \leq 2 e^{2A_1 a_0 nN}.$$

This estimate, $q > 1$, $\|h\|_0 \leq 1$, (3.1) and (3.2) imply that for $u \in U \cap \Lambda$ we have

$$\begin{aligned} L_g^{nN} \left(|L_{ab}^{nN} h|^q \right) (u) &\leq L_g^{nN} \left(|L_{ab}^{nN} h| \right) (u) \leq \int_{U\cap\Lambda} |L_{ab}^{nN} h| \, d\nu + \text{const } \ \rho_1^{nN} \|L_{ab}^{nN} h\|_\beta \\ &\leq \text{const } \ \rho_1^{n/2} + \text{const } \ \rho_1^{nN} \|L_{ab}^{nN} h\|_\beta \ . \end{aligned}$$

Assuming that $1/\gamma \leq \rho_1$ and using Lemma 2.2(d), one gets $\|L_{ab}^{nN} h\|_{\beta,U\cap\Lambda} \leq A_0 |b| \, \|h\|_{\beta,b} \leq$ const $|b|$. Thus, $L_g^{nN} \left(|L_{ab}^{nN} h|^q \right) (u) \leq \text{const } \rho_1^{n/2} \, |b|$ for all $u \in U \cap \Lambda$. Combining this with (3.3), one gets

$$\|L_{ab}^{2nN} h\|_{0,U\cap\Lambda} \leq \text{const } e^{2A_1 nN a_0/p} \rho_1^{n/2q} \, |b|^{1/q} \leq \text{const } \rho_2^n \, |b|^\epsilon \ , \tag{3.4}$$

where $\rho_2 = e^{2A_1 N a_0/p} \rho_1^{1/2q} < 1$ provided $a_0 > 0$ is sufficiently small.

Next, applying Lemma 2.2(d) again and using (3.4) with $2nN$ replaced by nN, we get

$$\begin{aligned} \|L_{ab}^{2nN} h\|_{\beta,U\cap\Lambda} &= \|L_{ab}^{nN} (L_{ab}^{nN} h)\|_{\beta,U\cap\Lambda} \leq A_0 \left(|b| \, \|L_{ab}^{nN} h\|_{0,U\cap\Lambda} + \rho_1^{nN\beta} \|L_{ab}^{nN} h\|_{\beta,U\cap\Lambda} \right) \\ &\leq \text{const } |b|^{1+\epsilon} \, \rho_2^{n/2} + \text{const } \rho_1^{nN\beta} \left[|b| \, \|h\|_0 + \rho_1^{nN\beta} \|h\|_\beta \right] \\ &\leq \text{const } |b|^{1+\epsilon} \, \rho_2^{n/2} + \text{const } \rho_1^{nN\beta} \, |b| \, \|h\|_{1,b} \leq \text{const } \ |b|^{1+\epsilon} \rho_3^n \ , \end{aligned}$$

where $\rho_3 = \max\{\rho_2^{1/2}, \rho_1^\beta\}$. Thus, $\frac{1}{|b|}\|L_{ab}^{2nN} h\|_{\beta,U\cap\Lambda} \leq \text{const}\ \ |b|^\epsilon\ \rho_3^n$. This and (3.4) yield

$$\|L_{ab}^{2nN} h\|_{\beta,b,U\cap\Lambda} \leq \text{const}\ \ |b|^\epsilon\ \rho_3^n \tag{3.5}$$

for all positive integers n and all $h \in \mathcal{F}_\beta(U\cap\Lambda)$ with $\|h\|_{\beta,b,U\cap\Lambda} \leq 1$. Taking $\rho = \rho_3$, proves the first part of (a).

To prove the second inequality in (a), notice that

$$\begin{aligned}
[L_{ab}^n(h/h_{ab})](u) &= \sum_{\sigma^n v = u} e^{f_n^{(b)}(v) - (P+a+\mathbf{i}b)\tau_n(v) + (\ln h_{ab} - \ln(h_{ab}\circ\sigma))_n(v) - n\ln\lambda_{ab}} \frac{h(v)}{h_{ab}(v)} \\
&= \frac{1}{\lambda_a^n h_{ab}(u)} \sum_{\sigma^n v = u} e^{f_n^{(b)}(v) - (P+a+\mathbf{i}b)\tau_n(v)}\ h(v) = \frac{1}{\lambda_{ab}^n h_{ab}(u)} [L_{f^{(b)}-(P+a+\mathbf{i}b)\tau}^n h](u)\ .
\end{aligned}$$

Let $\beta \in (0,1]$ and $h \in \mathcal{F}_\beta(U\cap\Lambda)$ be such that $\|h\|_{\beta,b,U\cap\Lambda} \leq 1$. We will again consider an extension $h \in \mathcal{F}_\beta(U)$ with the same norms. Let $\tilde{h} = h/h_{ab}$. Since $\|d_u h_{ab}(u)\| \leq \text{const}\ \ \sqrt{b}$ as remarked in Sect. 2, we have $\|h_{a,b}\|_\beta \leq \text{const}\ \sqrt{b}$ which implies $\|\tilde{h}\|_{\beta,b} \leq \text{const}$. Using this, (3.5) with h replaced by $\tilde{h}/\text{const}$ and the fact that $\lambda_{ab} \leq e^{\text{const}\ (|a| + \frac{1}{|b|^{\alpha/2}})}$ (see above), it follows that

$$\|L_{f^{(b)}-(P+a+\mathbf{i}b)\tau}^n h\|_{\beta,b,U\cap\Lambda} \leq \text{const}\ |b|^{1/2}\ \rho_4^n\ , \tag{3.6}$$

where $\rho_4 = \rho_3\ e^{\text{const}\ a_0} < 1$, provided $a_0 > 0$ is sufficiently small. So, for the second inequality in (a) we can take $\rho = \rho_4$.

Finally, notice that when $f \in C^1(U)$ we have (by definition) $f^{(b)} = f$ for all b, so both λ_{ab} and h_{ab} do not depend on b. This implies $\|d_u h_{ab}(u)\| \leq \text{const}$, $\|h_{a,b}\|_\beta \leq \text{const}$ and $\lambda_{ab} \leq \text{const}$. Now the above argument and (3.5) give $\|L_{f-(P+a+\mathbf{i}b)\tau}^n h\|_{\beta,b,U\cap\Lambda} \leq \text{const}\ |b|^\epsilon\ \rho_4^n$ for all $h \in \mathcal{F}_\beta(U\cap\Lambda)$ with $\|h\|_{\beta,b,U\cap\Lambda} \leq 1$.

(b) This follows from (3.6) using the argument from Appendix 2 in [D]. We omit the details. ■

Given $A > 0$, define

$$K_A = K_A(U) = \Big\{ h \in C^1(V) : h > 0, \|d_u(\ln h(u))\|_V \leq A \ \text{for all} \ \ u \in U \Big\}\ .$$

Notice that if $h \in K_A$, then $|\ln h(u) - \ln h(v)| \leq A\ \text{dist}(u,v)$, which implies

$$e^{-A\ \text{dist}(u,v)} \leq \frac{h(u)}{h(v)} \leq e^{A\ \text{dist}(u,v)} \quad , \quad u, v \in U. \tag{3.7}$$

The following lemma is the analogue of Lemma 10″ in [D] [5].

Lemma 3.4. *There exist a positive integer N and constants $\rho \in (0,1)$, $a_0 > 0$ and $E > 0$ such that for every $a, b \in \mathbb{R}$ with $|a| \leq a_0$, $1/|b| \leq a_0$, there exist an operator $\mathcal{N} = \mathcal{N}(a,b) : C^1(U) \longrightarrow C^1(U)$ with the following properties:*

(a) *The operator $\mathcal{N}$ preserves the cone $K_{E|b|}$;*

[5] Lemma 10" in [D] is slightly more general.

(b) *For all* $H \in K_{E|b|}$ *we have*

$$\int_U |\mathcal{N} H|^2 \, d\nu \leq \rho \int_U H^2 \, d\nu \; .$$

(c) *If* $h, H \in C^1(U)$ *are such that* $|h(u)| \leq H(u)$ *and* $\|dh(u)\| \leq E|b|H(u)$ *for all* $u \in U$ *and* $H \in K_{E|b|}$, *then* $|L_{ab}^N h(u)| \leq (\mathcal{N} H)(u)$ *and* $\|d(L_{ab}^N h)(u)\| \leq E|b|(\mathcal{N} H)(u)$ *for all* $u \in U$.

Theorem 3.1 is an easy consequence of Lemma 3.4. The proof of the fact is exactly the same as in [D] and we omit it.

Sections 4–8 are devoted to the proof of Lemma 3.4.

4 Distortion bounds

In order to apply Dolgopyat's construction of the operators $\mathcal{N}(a, b)$ (cf. Lemma 3.4), we need some information roughly speaking about the distribution of points of $U' \cap \Lambda$ in U' for each (maximal in a sense) local unstable manifold U' of the flow ϕ_t. More precisely, given such U' with $U' \cap \Lambda \neq \emptyset$, it is clear that $U' \cap \Lambda$ is a Cantor set which is obtained by recursively applying the following procedure: given an nth generation interval (a connected subcurve of U'), divide it (in a special way) into $2k_0 - 1$ subintervals and throw away the odd numbered intervals. One of the aims of this section is to prove that the ratio of lengths of any two of these $2k_0 - 1$ subintervals is bounded by positive constants from below and above. This shows that the natural homeomorphism S between $U' \cap \Lambda$ and the corresponding (classical in a sense) Cantor subset $C(k_0)$ of $[0, 1]$ has some "Lipschitz-like" property. It should be mentioned however that in general S is not a Lipschitz equivalence[6].

Denote by M *the set of those* $(q, v) \in S^*_{\partial K}(\Omega)$ such that $\langle v, \nu(q) \rangle > 0$ whenever $q \in \partial\Omega = \partial K$. A natural cross-section map for ϕ_t is the *billiard ball map* B defined on the subset $M_0 = \Lambda \cap M$ of M. Clearly M_0 is the maximal invariant set for B.

First, we describe a standard symbolic representation of the billiard ball map B on M_0 which has some special features and in some cases is more convenient than a symbolic representation coming from an arbitrary Markov family.

Let $\mathcal{A}$ be the $k_0 \times k_0$ matrix with entries $\mathcal{A}(i, j) = 1$ if $i \neq j$ and $\mathcal{A}(i, i) = 0$ for all $i = 1, 2, \ldots, k_0$. Denote by $\Sigma_{\mathcal{A}}$ the *space of all infinite in both directions sequences*

$$\xi = (\ldots, \xi_{-i}, \xi_{-i+1}, \ldots, \xi_{-1}; \xi_0, \xi_1, \ldots, \xi_{i-1}, \xi_i, \ldots)$$

such that $\xi_i \in \{1, 2, \ldots, k_0\}$ and $\mathcal{A}(\xi_i, \xi_{i+1}) = 1$ for all i (the latter simply means $\xi_i \neq \xi_{i+1}$). Notice that ; separates the "negative" and "non-negative" parts of ξ. In a similar way one defines $\Sigma^+_{\mathcal{A}}$, the space of one-sided sequences $\xi = (\xi_0, \xi_1, \ldots)$ with the above properties.

The natural *symbolic representation map* $Q : M_0 \longrightarrow \Sigma_{\mathcal{A}}$ is defined by $Q(x) = \xi$, where $\xi = (\xi_n)_n$ is such that $\mathrm{pr}_1(B^j(x)) \in \partial K_{\xi_j}$ for all $j \in \mathbf{Z}$. It is well-known that

[6]Indeed, otherwise the Hausdorff dimension $\dim_H(U' \cap \Lambda)$ would have been the same for all U', and so $\dim_H(\Lambda)$ would be a positive constant depending on the number of connected components of K only but not on K itself. However, this is not the case; one can see that preserving the shape of the components of K and letting the distance between them tend to ∞ (in an appropriate fashion) makes $\dim_H(\Lambda)$ tend to 0. This is established rigorously in [K], where one can also find some estimates of $\dim_H(\Lambda)$ from below and above.

Q is a homeomorphism and $Q \circ B = \sigma_{\mathcal{A}} \circ Q$, where $\sigma_{\mathcal{A}} : \Sigma_{\mathcal{A}} \longrightarrow \Sigma_{\mathcal{A}}$ is the *shift map*: $(\sigma_{\mathcal{A}}(\xi))_n = \xi_{n+1}$.

Given $i \neq j$, $(1 \leq i, j \leq k_0)$, denote by $\Pi_{i,j}$ *the set of those* $x = (q, v) \in M$ *such that* $q \in \partial K_i$, $B(x) = (p, w)$ is defined and $p \in \partial K_j$. Following an old idea in symbolic dynamics (apparently due to Smale) the sets $\Pi_{i,j}$ can be regarded as parallelograms of a Markov partition. Notice that $\widetilde{\Pi}_{i,j} = Q(M_0 \cap \Pi_{i,j}) = \{\xi \in \Sigma_{\mathcal{A}} : \xi_0 = i, \xi_1 = j\}$. Next, let $V_{i,j}$ be the set of those $(q, v) \in M$ such that $B^m(q, v) = (q_m, v_m)$ is defined for all $m \leq 1$, $q_0 = q \in \partial K_i$, $q_m \in \partial K_i$ for even $m \leq 0$ and $q_m \in \partial K_j$ for odd $m \leq 1$. Clearly $V_{i,j}$ is the *local unstable set* of the periodic point $x = (p, w)$ of B with period 2 which corresponds to the shortest segment connecting K_i and K_j. Notice that

$$\widetilde{V}_{i,j} = Q(M_0 \cap V_{i,j}) = \{\xi = (\ldots, i, j, i, j; i, j, \xi_2, \xi_3, \ldots) \in \Sigma_{\mathcal{A}}\}.$$

Let $\pi : \Sigma_{\mathcal{A}} \longrightarrow \Sigma_{\mathcal{A}}^+$ be the *natural projection*. We will denote by $\sigma_{\mathcal{A}}$ the *Bernoulli shift* on $\Sigma_{\mathcal{A}}^+$ and will identify $\widetilde{V}_{i,j}$ with its image $\pi(\widetilde{V}_{i,j})$ in $\Sigma_{\mathcal{A}}^+$. By $\mathrm{diam}(W)$ we denote the *diameter* of the subset W of $S^*(\Omega)$ with respect to the metric ρ (cf. Sect. 2).

Given a finite string $\eta = (\eta_0, \eta_1, \ldots, \eta_n)$ of integers $\eta_i = 1, 2, \ldots, k_0$, we will say that η is *permissible* if $\eta_i \neq \eta_{i+1}$ for $i = 0, 1, \ldots, n-1$. For such η one defines a *cylinder*

$$\widetilde{\mathcal{C}}[\eta] = \{\xi \in \Sigma_{\mathcal{A}}^+ : \xi_i = \eta_i \ , \ i = 0, 1, \ldots, n \ \} \subset \Sigma_{\mathcal{A}}^+ \ .$$

Fix for a moment integers $i_0 \neq j_0$ with $1 \leq i_0, j_0 \leq k_0$. It can be easily seen using a simple geometrical argument that the set $\widetilde{V}_{i_0,j_0} \subset \Sigma_{\mathcal{A}}^+$ is a cylinder, namely $\widetilde{V}_{i_0,j_0} = \widetilde{\mathcal{C}}[(i_0, j_0)]$. Given a permissible string $\eta = (\eta_1, \ldots, \eta_n)$ with $\eta_1 \neq j_0$, denote

$$\mathcal{C}_{i_0,j_0}[\eta] = \{x \in V_{i_0,j_0} : Q(x) \in \widetilde{\mathcal{C}}[(i_0, j_0, \eta_1, \ldots, \eta_n)] \ \},$$

and call this set a *cylinder* of length n in V_{i_0,j_0} (later on we will call these sets big cylinders). One can easily see that $\mathcal{C}_{i_0,j_0}[\eta]$ is a connected closed (i.e. containing its end points) curve in V_{i_0,j_0}, and it is well-known that there exist a global constant $\mu \in (0, 1)$ (i.e. independent of i_0, j_0, n and η) such that

$$\mathrm{diam}(\mathcal{C}_{i_0,j_0}[\eta]) \leq \mathrm{const}\ \mu^n \ ,$$

for any η of length n, where const denotes another global positive constant (cf. [Si2], [BSC] or [I1], [Sj]). For any $r = 1, \ldots, k_0$ with $r \neq \eta_n$ the cylinder $\mathcal{C}_{i_0,j_0}[(\eta, r)]$ of length $n+1$ is a closed subcurve of $\mathcal{C}_{i_0,j_0}[\eta]$. Consider an arbitrary orientation of the curve $\mathcal{C}_{i_0,j_0}[\eta]$. Then, using again a simple geometrical argument, one can find a permutation $\{r_1, \ldots, r_{k_0-1}\}$ of $\{1, \ldots, k_0\} \setminus \{\eta_n\}$ such that the curves $\mathcal{C}_{i_0,j_0}[(\eta, r_1)]$, $\mathcal{C}_{i_0,j_0}[(\eta, r_2)]$, …, $\mathcal{C}_{i_0,j_0}[(\eta, r_{k_0-1})]$ follow successively one after another with respect to the fixed orientation of $\mathcal{C}_{i_0,j_0}[\eta]$. Denote by $D_1, D_2, \ldots, D_{k_0}$ the closed intervals in $\mathcal{C}_{i_0,j_0}[\eta]$ that supplement the intervals $\mathcal{C}_{i_0,j_0}[(\eta, r_s)]$, i.e.

(4.1) $$D_1, \mathcal{C}_{i_0,j_0}[(\eta, r_1)], D_2, \mathcal{C}_{i_0,j_0}[(\eta, r_2)], \ldots, D_{k_0-1}, \mathcal{C}_{i_0,j_0}[(\eta, r_{k_0-1})], D_{k_0}$$

is an ordered sequence of intervals in $\mathcal{C}_{i_0,j_0}[\eta]$ with disjoint interiors whose union equals $\mathcal{C}_{i_0,j_0}[\eta]$. So, the sequence (4.1) consists of "gaps", i.e. intervals without common points with the Cantor set M_0, and cylinders of length $n+1$ contained in $\mathcal{C}_{i_0,j_0}[\eta]$.

Most likely the following lemma can be derived from [BSC]. Here we give an elementary proof of it using Sinai's estimates ([Si1], [Si2]) of the curvature of a convex wave front propagating in a dispersing billiard.

LEMMA 4.1. *There exists a global constant $\lambda \in (0,1)$ such that for any $i_0 \neq j_0$, n and η as above we have*

$$\lambda\, |\mathcal{C}_{i_0,j_0}[\eta]| \leq |\mathcal{C}_{i_0,j_0}[(\eta; r_i)]| \quad , \quad \lambda\, |\mathcal{C}_{i_0,j_0}[\eta]| \leq |D_j| \tag{4.2}$$

for all $i = 1, \ldots, k_0 - 1$ and $j = 1, \ldots, k_0$.

Proof. Take a small $\epsilon > 0$ and consider the curve $X = \{q + \epsilon v : (q,v) \in \mathcal{C}_{i_0,j_0}[\eta]\ \}$. Then (cf. [Si1],[Si2]) X is a strictly convex smooth curve in the exterior of K in $\mathbb{R}^2$ and for any $x = q + \epsilon v \in X$ with $(q,v) \in \mathcal{C}_{i_0,j_0}[\eta]$, the vector $\nu(x) = v$ coincides with the "exterior" *unit normal* to X at x. Moreover, the map $f : \mathcal{C}_{i_0,j_0}[\eta] \longrightarrow X$, $f(q,v) = q + \epsilon v$, is a diffeomorphism. Denote $X_i = f(\mathcal{C}_{i_0,j_0}[(\eta, r_i)])$, $Y_j = f(D_j)$. Then $Y_1, X_1, Y_2, X_2, \ldots, Y_{k_0-1}, X_{k_0-1}, Y_{k_0}$ is an ordered family of closed curves in X (with respect to the orientation on X induced by $\mathcal{C}_{i_0,j_0}[\eta]$) with disjoint interiors and whose union equals X. Since f is smooth and X is smooth and convex, it follows that for any connected curve I on X we have const $|f^{-1}(I)| \leq |I| \leq$ const $|f^{-1}(I)|$, where as before const denotes a global positive constant (which may be different in different experessions).

Fix an arbitrary $i' = 1, \ldots, k_0 - 1$ and denote $Z = X_{i'}$, $Y = Y_{i'}$. We will prove that

$$\text{const } \leq |Z|/|Y| \leq \text{const } . \tag{4.3}$$

In the same way one shows that const $\leq |X_{i'}|/|Y_{i'-1}| \leq$ const . Applying these two inequalities finitely many ($\leq k_0$) times, one gets (4.2).

Denote the end points of the segments Z and Y by $q(0)$, $q(1)$ and $q(1)$, $q(2)$, respectively (then $q(0)$, $q(1)$, $q(2)$ are ordered according to the orientation of X). According to the definition of Z, for every point $q \in Z$, the forward billiard trajectory generated by $(q, \nu(q))$ makes at least $n+1$ reflections and the jth reflection point belongs to ∂K_{η_j} for $j \leq n$, while the $(n+1)$st reflection point belongs to $\partial K_{r_{i'}}$. When q runs over Z the $(n+1)$st reflection point of the corresponding trajectory describes a connected part Z' of $\partial K_{r_{i'}}$ with end points $p(0)$ and $p(1)$, where the trajectories of $(q(0), \nu(q(0)))$ and $(q(1), \nu(q(1)))$, respectively, have tangencies. Let $p(s)$, $0 \leq s \leq 1$, be the smooth parametrization of Z' such that $\|\dot{p}(s)\| = \ell_1 = |Z'|$ for all $s \in [0,1]$. This induces a parametrization $q(s)$, $0 \leq s \leq 1$, of Z such that the $(n+1)$st reflection point of the trajectory determined by $(q(s), \nu(q(s)))$ is $p(s)$. Clearly $q(s)$ is continuous for all s and smooth for $s \in (0,1)$.

When q is in the interior of Y, the forward billiard trajectory generated by $(q, \nu(q))$ has only m reflection points and again the jth reflection point belongs to ∂K_{η_j}. If $i' < k_0 - 1$, denote by $p(2)$ the $(n+1)$st reflection point of the trajectory generated by $(q(2), \nu(q(2)))$; this is in fact a point of tangency to $\partial K_{r_{i'+1}}$. In the case when the trajectory generated by $(q(2), \nu(q(2)))$ has only n common points with ∂K, denote by $p(2)$ the intersection point of this trajectory with the boundary of the convex hull of K. Let $p(s)$, $1 \leq s \leq 2$, be the smooth parametrization of the straightline segment $[p(1), p(2)]$ in $\mathbb{R}^2$ with $\|\dot{p}(s)\| = \|p(2) - p(1)\| = \ell_2 \leq 2\mathrm{diam}(K)$. Then we get a corresponding parametrization $q(s)$, $1 \leq s \leq 2$, of Y which is continuous for $1 \leq s \leq 2$ and smooth for $1 < s < 2$.

Given $s \in [0,2]$, denote by $q_j(s)$ $(1 \leq j \leq n)$ the jth *reflection point* of the *forward billiard trajectory* $\gamma(s)$ generated by $(q(s), \nu(q(s)))$ and set $q_0(s) = q(s)$ and $q_{n+1}(s) = p(s)$. Let $\kappa_j(s) > 0$ be the *curvature* of ∂K at $q_j(s)$ and let $\varphi_j(s)$ be the *angle* between the outward normal to ∂K at $q_j(s)$ and the reflected direction of $\gamma(s)$ at $q_j(s)$. Set

$$d_j(s) = \|q_{j+1}(s) - q_j(s)\| \quad , \quad 0 \leq j \leq n.$$

Denote by $k_0(s) > 0$ the curvature of the curve X at $q(s)$, define by induction

$$k_{j+1}(s) = \frac{k_j(s)}{1 + d_j(s)k_j(s)} + 2\frac{\kappa_j(s)}{\cos\varphi_j(s)} \quad , \quad 0 \leq j \leq n, \tag{4.4}$$

and set

$$\delta_j(s) = \frac{1}{1 + d_j(s)k_j(s)} \quad , \quad 0 \leq j \leq n. \tag{4.5}$$

LEMMA 4.2.([St2]) *For all $s \in (0,1) \cup (1,2)$ and all $j = 0, 1, \ldots, n-1$, we have*[7]

$$\text{const } \|\dot{p}(s)\| \delta_{j+1}(s)\delta_{j+2}(s)\ldots\delta_n(s) \leq \|\dot{q}_j(s)\| \leq \text{const } \|\dot{p}(s)\| \delta_{j+1}(s)\delta_{j+2}(s)\ldots\delta_n(s) \; .$$

As one would expect, these inequalities follow from the chain rule for the iterations of the billiard ball map and estimates about the derivative of a single iteration ([Si2], [BSC]). For completeness we sketch a proof of Lemma 4.2 in the Appendix.

Now we continue with the proof of Lemma 4.1. Since ℓ_1 and ℓ_2 are bounded from below and above by positive constants[8], it follows that for $s \in (0,1) \cup (1,2)$, $\|\dot{p}(s)\|$ is bounded from below and above by constants, so Lemma 4.2 implies

$$C'\delta_1(s)\ldots\delta_n(s) \leq \|\dot{q}(s)\| \leq C''\delta_1(s)\ldots\delta_n(s) \quad , \quad s \in (0,1) \cup (1,2), \tag{4.6}$$

for some constants $C'' > C' > 0$. Since the curvature of ∂K is bounded from below and above (by positive constants) and the distances $d_j(s)$ are also uniformly bounded from below and above, it follows easily that there exist constants μ' and μ with $0 < \mu' < \mu < 1$ such that

$$\mu' \leq \delta_j(s) \leq \mu \quad , \quad 0 \leq s \leq 2 \; , \; 1 \leq j \leq n \; .$$

Remark. Using these estimates for $\delta_j(s)$ in Lemma 4.2, one gets const $\mu'^{n-j} \leq \|\dot{q}_j(s)\| \leq$ const μ^{n-j}, which are the usual distortion bounds well-known in hyperbolic dynamics. With $j = 0$ they give const $\mu'^n \leq |Z| \leq$ const μ^n and the same estimate for $|Y|$. Combining these two, one only gets the rough estimates const $(\mu'/\mu)^n \leq |Z|/|Y| \leq$ const $(\mu/\mu')^n$, which are clearly very far from what we want in (4.3), where "const " does not depend on n. To prove (4.3) one has to use Lemma 4.2 (or some similar tool) in a more careful way.

[7] For $j = 0$ one actually gets an exact formula: $\|\dot{q}_0(s)\| = \|\dot{p}(s)\|\delta_1(s)\delta_2(s)\ldots\delta_n(s)$.

[8] For ℓ_2 this follows immediately from its definition. For ℓ_1 one can use for example a simple geometric argument involving $\min_{i \neq j} \text{dist}(\partial K_i, \partial K_j)$, $\text{diam}(K)$ and the radius of the largest circle inscribed in $K_{r_{i'}}$.

Fix for a moment $s \in (0,1)$ and $t \in (1,2)$. We need to estimate the ratio $\delta_j(s)/\delta_j(t)$ from below and above. First, using roughly the estimates in Lemma 4.2, one gets $\|q_j(s)-q_j(t)\| \leq$ const μ^{n-j} for all $j = 1, \ldots, n$. (Here we use the fact that $Z, Y \subset \mathcal{C}_{i_0,j_0}[\eta]$, so for any $s' \in [0,2]$ the trajectory $\gamma(s)$ has n consecutive reflection at the same boundary components $\partial K_{\eta_1}, \ldots, \partial K_{\eta_n}$.) This implies

$$|d_j(s)-d_j(t)| \leq \text{const } \mu^{n-j}, \, |\kappa_j(s)-\kappa_j(t)| \leq \text{const } \mu^{n-j}, \, |\cos\varphi_j(s)-\cos\varphi_j(t)| \leq \text{const } \mu^{n-j}$$

for all $j = 1, \ldots, n$. By induction, $|k_j(s) - k_j(t)| \leq$ const μ^{n-j} which implies $|\delta_j(s)/\delta_j(t) - 1| \leq$ const μ^{n-j} for all $j = 1, \ldots, n$. Changing the roles of s and t, one gets similiar etsimates, so

$$\frac{\delta_j(s)}{\delta_j(t)} \leq 1 + c\mu^{n-j} \quad , \quad \frac{\delta_j(t)}{\delta_j(s)} \leq 1 + c\mu^{n-j} \quad , \quad j = 0, 1, \ldots, n \tag{4.7}$$

for some constant $c > 0$. It follows from the Mean Value Theorem that there exists $s \in (0,1)$ with $|Z| = \int_0^1 \|\dot{q}(u)\| \, du = \|\dot{q}(s)\|$. In the same way, there exists $t \in (1,2)$ with $|Y| = \|\dot{q}(t)\|$. Then combining (4.6) and (4.7), one derives

$$\frac{|Z|}{|Y|} = \frac{\|\dot{q}(s)\|}{\|\dot{q}(t)\|} \leq \frac{C''\delta_1(s)\ldots\delta_n(s)}{C'\delta_1(t)\ldots\delta_n(t)} \leq \frac{C''}{C'} \prod_{j=1}^{n} (1 + c\mu^{n-j}).$$

Hence $\log(|Z|/|Y|) \leq \log(C''/C') + \sum_{j=1}^{n} c\mu^{n-j} < \log(C''/C') + \frac{c}{1-\mu}$, and therefore $\frac{|Z|}{|Y|} \leq \frac{C''}{C'} e^{c/(1-\mu)}$. In the same way, changing the roles of s and t again, we get $\frac{|Y|}{|Z|} \leq \frac{C''}{C'} e^{c/(1-\mu)}$. This proves (4.3) and thus completes the proof of the lemma.■

Next, we deal with the Markov family $R = \cup_{i=1}^{k} R_i$ and the set $U = \cup_{i=1}^{k} U_i$ defined in Sect. 2. Recall that each rectangle R_i is contained in a cross-section D_i to the flow ϕ_t. We will also use the map $\omega : \cup_{i=1}^{k} R_i \longrightarrow M$ defined in Sect. 2. The restriction of ω to any U_i is a diffeomorphism between U_i and $\omega(U_i)$. Consequently, there exist constants $c_2 > c_1 > 0$ such that for any connected curve I contained in some U_i we have

$$c_1|\omega(I)| \leq |I| \leq c_2|\omega(I)|. \tag{4.8}$$

We will assume that $c_1 \leq 1$. Fix an arbitrary $\lambda > 0$ with the property described in Lemma 4.1. Clearly $\lambda \leq 1/(2k_0 - 1)$. Setting

$$\lambda' = \frac{c_1}{c_2}\lambda \quad , \quad \lambda_0 = c_1\lambda^2 \, ,$$

we have $\lambda_0 \leq \lambda^2 \leq 1/(2k_0-1)^2$. If Θ is a connected subcurve of U_i such that $\omega(\Theta) = \mathcal{C}_{i_0,j_0}[\eta]$ for some $i_0 \neq j_0$ and $\eta = (\eta_1, \ldots, \eta_n)$, we will say that Θ is a *cylinder of length* n in $U_i \subset U$.

The first part of the following Corollary will be used in Sect. 5, while the second one is needed for Sect. 6.

COROLLARY 4.3. (a) *If Θ is a cylinder of length $s+1$ contained in some connected component of U and Θ' is a cylinder of length $n+1$ with $\Theta' \subset \Theta$ (so $n \geq s$), then*

$$|\Theta'| \leq (1 - 2k_0\lambda')^{n-s}|\Theta|.$$

(b) *For any* $x \in U \cap \Lambda$ *and any* $r > 0$ *with* $r \leq \mathrm{dist}_U(x, \partial U)$ *there exists an ordered family* $\{\Theta_j\}_{j=1}^{2k_0-1}$ *of connected closed subcurves of* $I(x,r)$ *having disjoint interiors such that:*
(i) $\cup_{j=1}^{2k_0-1}\Theta_j$ *is connected and* $|\Theta_j| \geq \lambda_0 r$ *for all* $j = 1, \ldots, 2k_0 - 1$;
(ii)*For any* $j = 1, \ldots, 2k_0 - 1$, *the interior of* Θ_j *has common points with* Λ *iff* j *is even.*

Proof. (a) Let $\omega(\Theta) = \mathcal{C}_{i_0,j_0}[\eta']$, $\omega(\Theta') = \mathcal{C}_{i_0,j_0}[\eta]$, where $\eta = (\eta_1, \ldots, \eta_n)$ and $\eta' = (\eta_1, \ldots, \eta_s)$ is a part of η. Let $\{\Theta_j\}_{j=1}^{2k_0-1}$ be the sequence of cylinders of length $s+1$ and gaps contained in $\mathcal{C}_{i_0,j_0}[\eta']$. It follows from Lemma 4.1 that $|\Theta_j| \geq \lambda|\mathcal{C}_{i_0,j_0}[\eta']|$. Hence the length of each curve of the form $\Theta'_j = (\omega_{|U_i})^{-1}(\Theta_j)$ is $\geq c_1\lambda\ |\mathcal{C}_{i_0,j_0}[\eta']| \geq \lambda'|\Theta|$. There exists exactly one j' with $\Theta' \subset \Theta'' = \Theta'_{j'} \subset \Theta$. Hence

$$|\Theta''| = |\Theta| - \sum_{j \neq j'} |\Theta'_j| \leq |\Theta| - 2k_0\lambda'|\Theta| = (1 - 2k_0\lambda')|\Theta|.$$

In the same way, one shows that if Θ''' is the only cylinder of length $s+2$ contained in Θ and containing Θ', then $|\Theta'''| \leq (1 - 2k_0\lambda')^2|\Theta|$. Continuing in this way by induction, one gets the desired inequality.

(b) Let $x \in U \cap \Lambda$ and $r > 0$ be such that $r \leq \mathrm{dist}_U(x, \partial U)$. Let n be the minimal non-negative integer such that $\omega(I(x,r))$ contains a cylinder $\mathcal{C}_{i_0,j_0}[\eta]$ of length n. (The special form of the curve $I(x,r)$ and $x \in \Lambda$ show that $I(x,r) \cap \Lambda$ contains a non-trivial open subset of $U \cap \Lambda$.) Fix $\eta = (\eta_1, \ldots, \eta_n)$ with this property; then we have $|\mathcal{C}_{i_0,j_0}[\eta]| \leq 2r$. Here we consider the case $n \geq 1$ in detail; the case $n = 0$ follows in the same way (in fact it is an immediate consequence of the above lemma). The minimality of n shows that the cylinder $\mathcal{C}' = \mathcal{C}_{i_0,j_0}[\eta_1, \ldots, \eta_{n-1}]$ of length $n-1$ containing $\mathcal{C}_{i_0,j_0}[\eta]$ (and therefore x) is not contained in $\Theta = I(x,r)$. (When $n = 1$ we have the trivial case $\mathcal{C}' = V_{i_0,j_0}$.) This is only possible when $|\mathcal{C}'| > r$. Combining the latter with Lemma 4.1 and (4.8), it follows that

$$\lambda\, r < \lambda|\mathcal{C}'| \leq |\mathcal{C}_{i_0,j_0}[\eta]|. \tag{4.9}$$

Let $\Theta'_1, \Theta'_2, \ldots, \Theta'_{2k_0-1}$ be the sequence of "gaps" and cylinders of length $n+1$ contained in $\mathcal{C}_{i_0,j_0}[\eta]$. By the definition of these curves, we have that their union is $\mathcal{C}_{i_0,j_0}[\eta]$, so it is connected, and the interior of Θ'_j has common points with M_0 iff j is even. Moreover Lemma 4.1 and (4.8) imply that for $\Theta_j = (\omega_{|U_i})^{-1}(\Theta'_j)$ we have $|\Theta_j| \geq c_1\lambda|\mathcal{C}_{i_0,j_0}[\eta]|$. This and (4.9) give $|\Theta_j| \geq c_1\lambda^2 r = \lambda_0\, r$ for all j. ■

5 The Gibbs measure ν on U

Our aim in this section is to prove that the Gibbs measure ν on U is a Federer measure, that is, it has the following property.

PROPOSITION 5.1. *For every* $m > 0$ *there exists a constant* $D_m > 0$ *such that for all* $x \in U \cap \Lambda$ *and all* $r > 0$ *we have* $\nu(I(x, mr)) \leq D_m\, \nu(I(x,r))$.

We will use the notation from Sect. 2 and 4.

Now we have two symbolic representations at our disposal – the first one is $\Phi : R \longrightarrow \Sigma_A$ given by the Markov family $\mathcal{R}$ (cf. Sect.2), while the second one is $Q : M_0 \longrightarrow \Sigma_{\mathcal{A}}$ determined by the standard "Markov partition" $\Pi = \cup \Pi_{i,j}$ (cf. Sect. 4). Correspondingly, we get two types of cylinders on $U = \cup_{i=1}^k U_i$. The second type are cylinders (we will call them *big cylinders*) in U that correspond, via the projection $\omega : U \longrightarrow M$, to cylinders of the form $\mathcal{C}_{i_0,j_0}[\eta]$ in Π. To describe the first type, consider a finite string $\imath = (i_0, i_1, \ldots, i_p)$ of integers $i_j \in \{1, \ldots, k\}$. We will say that $\imath$ is *A-permissible* if $A_{i_j i_{j+1}} = 1$ for any $j = 0, 1, \ldots, p-1$. Then $\imath$ defines a cylinder

$$\tilde{C}[\imath] = \{(i'_j)_{j=0}^{\infty} : i'_j = i_j, j = 0, 1, \ldots, p\} \subset \Sigma_A^+ .$$

The set $C[\imath]$ of those $x \in U$ such that $\sigma^j(x)$ exists for all $j = 0, 1, \ldots, p$ and $\sigma^j(x) \in U_{i_j}$, $j = 0, 1, \ldots, p$, will be called a *small cylinder* of length p in U. Clearly, $C[\imath]$ is a connected subcurve of U_{i_0} and $(\Phi^+)(C[\imath]) = \tilde{C}[\imath]$. Recall that the projection $\omega(D_q)$ of each cross-section D_q is contained in some $\Pi_{s(q),t(q)}$ $(s(q), t(q) \in \{1, \ldots, k_0\})$. So, $x \in C[\imath]$ means in particular that the first backward reflection of the billiard trajectory determined by x occurs at $\partial K_{s(i_0)}$, while the first forward one occurs at $\partial K_{t(i_0)}$. Then $\sigma(x) \in D_{i_1}$ and the fact that between x and $\mathcal{P}(x)$ there is at most one reflection at ∂K, show that either $s(i_1) = s(i_0)$ and $t(i_0) = t(i_1)$ or $s(i_1) = t(i_0)$. Continuing in this way, we see that for each $j < p$, either the pairs $(s(i_j), t(i_j))$ and $(s(i_{j+1}), t(i_{j+1}))$ are the same or $t(i_j) = s(i_{j+1})$. Transform the sequence $(s(i_0), t(i_0)), (s(i_1), t(i_1)), \ldots, (s(i_p), t(i_p))$ as follows: first remove the repeating successive pairs and then in the new sequence remove the repeating successive numbers. As a result we get a sequence of the form $\mathcal{I}(\imath) = (i'_0, j'_0; \eta_1, \ldots, \eta_n)$, where $i'_0 = s(i_0)$, $j'_0 = t(i_0)$, $n \geq p/k$ and $\eta = (\eta_1, \ldots, \eta_m)$ are such that $\omega(C[\imath]) \subset \mathcal{C}_{i'_0,j'_0}[\eta]$. This gives a map $\mathcal{I}$ from the set of finite A-permissible sequences onto the set of finite $\mathcal{A}$-permissible sequences. Since $|\mathcal{C}_{i'_0,j'_0}[\eta]| \leq \text{const}\ \mu^n$ for some constant $\mu \in (0,1)$ (cf. Sect. 4), it follows that $|C[\imath]| \leq \text{const}\ \mu_1^p$, where $\mu_1 = \mu^{1/k} \in (0,1)$. In particular, $\sigma^p : U \longrightarrow U$ **is expanding** for all sufficiently large p.

The above argument also implies that given an $\mathcal{A}$-permissible sequence $\zeta = (i'_0, j'_0; \eta_1, \ldots, \eta_n)$, we have $\mathcal{C}_{i'_0,j'_0}[\eta] = \cup_{\mathcal{I}(\imath)=\zeta}\ \omega(C[\imath])$, where the union is taken over all A-permissible finite sequences $\imath$ with $\mathcal{I}(\imath) = \zeta$. Choosing minimal sequences $\imath$ with this property, we get a representation of the big cylinder $\mathcal{C}_{i'_0,j'_0}[\eta]$ as a finite disjoint union of small cylinders:

LEMMA 5.2. *For every $\mathcal{A}$-permissible sequence $\zeta = (i'_0, j'_0; \eta_1, \ldots, \eta_n)$ there exists a finite set $\imath^{(1)}, \ldots, \imath^{(\ell)}$ of finite A-permissible sequences such that $\mathcal{I}(\imath^{(j)}) = \zeta$ for all $j = 1, \ldots, \ell$,*

$$\mathcal{C}_{i'_0,j'_0}[\eta] = \cup_{j=1}^{\ell} \omega(C[\imath^{(j)}]),$$

and $C[\imath^{(j)}] \cap C[\imath^{(j')}] = \emptyset$ for $j \neq j'$.■

Recall that ν is the Gibbs measure on U determined by the C^1 function $g = f - P\tau$. Denote

$$g_0 = \max_{z \in R} |g(z)|.$$

According to the definition of Gibbs measures (cf. [Si3] or [PP2]), there exist constants $\alpha > 0$, $\beta > 0$ and $c \in \mathbb{R}$ such that

$$\alpha \leq \frac{\nu(C[\imath])}{e^{pc} e^{g_p(x)}} \leq \beta \tag{5.1}$$

for any small cylinder $C[\imath]$ of length p in U and any $x \in C[\imath]$.

The main tool needed to establish the Federer property of ν is the following.

LEMMA 5.3. *There exists a positive integer ℓ_0 such that for any positive integer n, any big cylinder $\mathcal{C} = \mathcal{C}_{i_0,j_0}[\eta_1,\dots,\eta_n]$ and any two maximal small cylinders $C[\jmath] = C[j_0, j_1,\dots,j_p]$ and $C[\jmath'] = C[j'_0, j'_1,\dots,j'_{p'}]$ whose ω-images are contained in $\mathcal{C}$ and $C[\jmath] \cap \Lambda \neq \emptyset$, $C[\jmath'] \cap \Lambda \neq \emptyset$, we have $|p-p'| \leq k\ell_0$ and $j_i = j'_i$ for $i \leq \min\{p,p'\} - k\ell_0$. Consequently, the number of different maximal small cylinders whose ω-images are contained in $\mathcal{C}$ and have common points with Λ is $\leq k^{2k\ell_0}$ (which is independent of n). Moreover there exists a constant $A_1 > 0$ such that for any two cylinders $C[\jmath]$ and $C[\jmath']$ as above and any $y \in C[\jmath] \cap \Lambda$ and $y' \in C[\jmath'] \cap \Lambda$ we have $|g_p(y) - g_{p'}(y')| \leq A_1$. Thus, $\nu(C[\jmath']) \leq A_2\, \nu(C[\jmath])$ for some constant $A_2 > 0$ independent of n, p and p'.*

Proof. As mentioned in Sect. 4 (this actually follows from Lemma 4.2), there exist constants $\mu \in (0,1)$ and $E > 0$ such that for any big cylinder $\mathcal{C}$ of length ℓ we have $|\mathcal{C}| < E\mu^\ell$. The choice of the Markov family $\mathcal{R} = \{R_i\}$ shows that for each $i = 1,\dots,k$, R_i is contained in the interior of the generalized rectangle $\tilde{R}_i = [U_i, S_i]$ (which is in D_i; cf. Sect. 2), so $\mathrm{dist}(\partial\tilde{R}_i, R_i) > 0$. Since the restriction of the map $\omega : \cup_{i=1}^k D_i \longrightarrow S^*_{\partial K}(\Omega)$ to each D_i is a C^1 diffeomorphism, it follows that there exists $\epsilon_0 > 0$ such that whenever $y, y' \in S^*_{\partial K}(\Omega) \cap \Lambda = M_0$ are such that $y \in \omega(R_i)$ for some i and $\mathrm{dist}(y,y') < \epsilon_0$, then $y' \in \omega(\tilde{R}_i)$, and so $y' \in \omega(R_i)$, too. Fix a positive integer ℓ_0 such that $E\mu^{\ell_0} < \epsilon_0$.

Let $\jmath = (j_0, j_1,\dots,j_p)$ and $\jmath' = (j'_0, j'_1,\dots,j'_{p'})$ be A-premissible sequences of minimal length such that $\omega(C[\jmath]) \subset \mathcal{C}$ and $\omega(C[\jmath']) \subset \mathcal{C}$. We will prove that $|p'-p| \leq k\ell_0$ and $j_i = j'_i$ for all $i = 0,1,\dots,p-k\ell_0$. We assume $p \geq k\ell_0$ or $p' \geq k\ell_0$, otherwise there is nothing to prove. Assume for example that $p \geq k\ell_0$. Fix arbitrary $x \in \omega(C[\jmath] \cap \Lambda)$ and $x' \in \omega(C[\jmath'] \cap \Lambda)$. Then $x \in \omega(U_{j_0}) \cap \Lambda$ and $x' \in \omega(U_{j'_0}) \cap \Lambda$. The assumptions about $C[\jmath]$ and $C[\jmath']$ imply that $s(j_0) = s(j'_0) = i_0$, $t(j_0) = t(j'_0) = j_0$. Set $x_i = B^i(x)$, $x'_i = B^i(x')$, $i \geq 1$, where as before B denotes the billiard ball map on M (cf. Sect. 4). Since $x, x' \in \mathcal{C}$ and $\mathcal{C}$ is a big cylinder of length n, we have $\mathrm{dist}(x,x') < E\mu^n \leq \epsilon_0$ if $n \geq l_0$. So, the choice of ϵ_0 gives that for $n \geq \ell_0$ we have $x \in \omega(R_i)$ for some i iff $x' \in \omega(R_i)$. Thus, the billiard trajectory of x between x and x_1 and that of x' between x' and x'_1 intersect the same set of rectangles R_i. Hence for all i with $i_0 = s(j_0) = s(j_1) = \ldots = s(j_i)$ we have $j_i = j'_i$. Proceeding by induction, assume that for some $i < p - k\ell_0$ we have $j_{i'} = j'_{i'}$ for $i' = 0,1,\dots,i$. Let $s(j_{i+1}) = \eta_q$ for some $q = 1,\dots,n$. If $s(j_i) = s(j_{i+1})$, then $j_{i+1} = j'_{i+1}$ follows as above from the fact that the billiard trajectories of x and x', between x_q and x_{q+1} and x'_q and x'_{q+1}, respectively, will intersect the same set of rectangles. Assume $s(j_i) = s(j'_i) = \eta_{q-1}$. Since between two reflection points x_r and x_{r+1} the billiard trajectory of x meets at most k rectangles R_j (k is the total number of rectangles), it follows that $p - i \leq k(n-q)$. This and $i < p - k\ell_0$ imply $n - q > \ell_0$. Since x_q and x'_q belong to the same cylinder of length $n-q$, it follows that $\mathrm{dist}(x_q, x'_q) < E\mu^{n-q} < E\mu^{\ell_0} \leq \epsilon_0$. Hence the billiard trajectory of x between x_q and x_{q+1} and that of x' between x'_q and x'_{q+1} have common points with the same set of rectangles R_j. In particular, $j_{i+1} = j'_{i+1}$. This proves that $j_i = j'_i$ for all $i \leq p - k\ell_0$. By symmetry, this must be also true for all $i \leq p' - k\ell_0$. Hence $|p - p'| \leq k\ell_0$.

Having fixed $C[\jmath]$ and therefore p, the latter implies that for p' we have at most $2k\ell_0$ different possible values. For each $i > \max\{p - k\ell_0, p' - k\ell_0\}$, there are at most k different possible values for j'_i. Thus, there are no more than $k^{2k\ell_0}$ different possible sequences $\jmath'$.

Let $C[\jmath]$ and $C[\jmath']$ be as above and let $y \in C[\jmath] \cap \Lambda$ and $y' \in C[\jmath] \cap \Lambda$. It follows from the C^1 assumption about f that g is Lipschitz. Moreover, the choice of μ and the fact that $y, y' \in C[j_0, j_1, \dots, j_{p-k\ell_0}]$, imply that $\mathrm{dist}_U(\sigma^i(y), \sigma^i(y')) < E\mu_1^{p-k\ell_0-i}$ for all $i = 0, 1, \dots, p - k\ell_0$, where $\mu_1 = \mu^{1/k}$ and $E > 0$ is a constant (which may be different from the one chosen above). Thus,

$$\begin{aligned} |g_p(y) - g_{p'}(y')| &\leq \sum_{i=0}^{p-k\ell_0-1} |g(\sigma^i(y)) - g(\sigma^i(y'))| + \sum_{i=p-k\ell_0}^{p-1} |g(\sigma^i(y))| + \sum_{i=p-k\ell_0}^{p'-1} |g(\sigma^i(y'))| \\ &\leq E\,\|g\|_1 \sum_{i=0}^{p-k\ell_0-1} \mu_1^{p-k\ell_0-i} + k\ell_0 g_0 + 2k\ell_0 g_0 \leq A_1 = \frac{E\,\|g\|_1}{1-\mu_1} + 3k\ell_0 g_0\,, \end{aligned}$$

where $\|g\|_1$ is the Lipschitz constant of g.

Finally, using (5.1) we get

$$\frac{\nu(C[\jmath])}{\nu(C[\jmath'])} \leq \frac{\beta e^{pc} e^{g_p(y)}}{\alpha e^{p'c} e^{g_{p'}(y')}} = \frac{\beta e^{(p-p')c} e^{g_p(y)-g_{p'}(y')}}{\alpha} \leq \frac{\beta e^{k\ell_0|c|} e^{A_1}}{\alpha}.$$

So, we can take $A_2 = \frac{\beta e^{k\ell_0|c|+A_1}}{\alpha}$. ■

Proof of Proposition 5.1. Let $m \geq 1$, and let $x \in U \cap \Lambda$ and $r > 0$. Denote by U' the connected component of U that contains x.

First, assume that $mr < \mathrm{dist}_{U'}(x, \partial U')$. Let $\omega(x) \in V_{i_0,j_0}$. Denote by $n' + 1$ the minimal length of a big cylinder in V_{i_0,j_0} containing $\omega(x)$ and contained in $\omega(I(x, mr))$, and let $\mathcal{C}_{i_0,j_0}[\eta_1, \dots, \eta_{n'}, \eta_{n'+1}]$ be a cylinder with this property. Set $\mathcal{C}' = \mathcal{C}_{i_0,j_0}[\eta]$, where $\eta = (\eta_1, \dots, \eta_{n'})$. The minimality of $n' + 1$ and the fact that $\omega(x) \in \mathcal{C}'$ imply that $\mathcal{C}'$ is not contained in $\omega(I(x, mr))$ which is only possible when $|\mathcal{C}'| > mr/c_2$ (cf. (4.8)). In general however $\mathcal{C}'$ may not contain $\omega(I(x, mr))$, so we still do not have an estimate from above for $\nu(I(x, mr))$. To get such an estimate, notice that $\omega(I(x, mr))$ can have common points with at most one more big cylinder of length n' (if it has common points with three big cylinders of length n', it must contain one of them). Assume that there exists a big cylinder $\hat{\mathcal{C}} \neq \mathcal{C}'$ of length n' which has common points with $\omega(I(x, mr))$. Let n be the maximal length of a big cylinder $\mathcal{C}$ containing both $\hat{\mathcal{C}}$ and $\mathcal{C}'$. Fix a cylinder $\mathcal{C}$ with this property. Then $n < n'$ and $\hat{\mathcal{C}}$ and $\mathcal{C}'$ are parts of big subcylinders of $\mathcal{C}$ of length $n + 1$ separated by a single gap $\mathcal{D}$ in $\mathcal{C}$. From Lemma 4.1 we get $|\mathcal{D}| \geq \lambda\,|\mathcal{C}|$. On the other hand, $\mathcal{D} \subset \omega(I(x, mr))$ implies $c_1|\mathcal{D}| \leq 2mr$, where $c_1 > 0$ is one of the constants in (4.8). Hence

$$\lambda\,|\mathcal{C}| \leq 2mr/c_1\,. \tag{5.2}$$

¿From the choice of $\mathcal{C}$, it is clear that

$$\omega(I(x, mr) \cap \Lambda) \subset \mathcal{C}. \tag{5.3}$$

Finally, applying Corollary 4.3(a) to $\mathcal{C}'$ and $\mathcal{C}$, we get

$$|\mathcal{C}'| \leq A\,\theta^{n'-n} |\mathcal{C}|, \tag{5.4}$$

where $\theta = 1 - 2k_0\lambda' \in (0,1)$. This was derived from the assumption that $\omega(I(x,mr))$ had a common point with another big cylinder of length n'. If this is not the case, then $\omega(I(x,mr)\cap\Lambda) \subset \mathcal{C}'$, so taking $\mathcal{C} = \mathcal{C}'$ and therefore $n = n'$, we get (5.3) and (5.4) trivially satisfied. Moreover, it follows from $\mathcal{C}_{i_0,j_0}[\eta_1,\dots,\eta_{n'},\eta_{n'+1}] \subset \omega(I(x,mr))$, Lemma 4.1 and (4.8) that (5.2) also holds. Hence in the two possible cases there exists a big cylinder $\mathcal{C}$ of length n such that (5.2), (5.3) and (5.4) hold.

Next, we deal with $\omega(I(x,r))$. Let $\mathcal{C}'' = \mathcal{C}_{i_0,j_0}[\zeta]$, $\zeta = (\zeta_1,\dots,\zeta_{n''})$, be a big cylinder of minimal length containing $\omega(x)$ and contained in $\omega(I(x,r))$. Then $|\mathcal{C}'''| \geq r/c_2$, where $\mathcal{C}''' = \mathcal{C}_{i_0,j_0}[\zeta_1,\dots,\zeta_{n''-1}]$. It follows from $\omega(x) \in \mathcal{C}'' \subset \omega(I(x,r)) \subset \omega(I(x,mr))$ and $\omega(x) \in \mathcal{C}'$ that $n'' \geq n'$ and $\zeta_j = \eta_j$ for $j = 1,\dots,n'$. That is, $\mathcal{C}'' \subset \mathcal{C}' \subset \mathcal{C}$. Now Corollary 4.3(a) implies $|\mathcal{C}''| \leq A\theta^{n''-n}|\mathcal{C}|$. Combining the latter with $|\mathcal{C}''| \geq \lambda|\mathcal{C}'''| \geq \lambda r/c_2$ and (5.2), gives $\lambda r/c_2 \leq A\theta^{n''-n}\, 2mr/(\lambda c_1)$, which implies

$$n'' - n \leq \frac{\log(2Amc_2/(\lambda^2 c_1))}{\log(1/\theta)}. \tag{5.5}$$

There exist a small cylinder $C[\imath]$ of minimal length with $\omega(x) \in \omega(C[\imath]) \subset \mathcal{C}''$ and a small cylinder $C[\jmath]$ of minimal length such that $\omega(C[\imath]) \subset \omega(C[\jmath]) \subset \mathcal{C}$. Denote by p'' and p the lengths of $\imath$ and $\jmath$, respectively, and notice that $p \leq p''$ and $p'' - p \leq (n''-n)/k$. Notice that

$$g_{p''}(x) = g(x) + g(\sigma(x)) + \ldots + g(\sigma^{p''-1}(x)) = g_p(x) + g_{p''-p}(y),$$

where $y = \sigma^p(x)$. Clearly, $g_{p''-p}(y) \geq -(p''-p)g_0$. Then (5.1) and (5.5) yield:

$$\frac{\nu(C[\jmath])}{\nu(C[\imath])} \leq \frac{\beta e^{pc} e^{g_p(x)}}{\alpha e^{p''c} e^{g_{p''}(x)}} \leq \frac{\beta}{\alpha}\cdot e^{-(p''-p)c} e^{(p''-p)g_0} \leq \frac{\beta}{\alpha}\cdot e^{(p''-p)|c-g_0|} \leq \frac{\beta}{\alpha}\cdot e^{(n''-n)|c-g_0|/k} \leq D'_m\,,$$

where $D'_m = \frac{\beta}{\alpha}\cdot e^{\frac{|c-g_0|\log(2Amc_2/(\lambda^2 c_1))}{k\log(1/\theta)}}$. Thus,

$$\nu(C[\jmath]) \leq D'_m \nu(C[\imath]) \leq D'_m \nu(I(x,r)). \tag{5.6}$$

It remains to prove that $\nu(\mathcal{C}) \leq \text{const}\ \nu(C[\jmath])$. To do this, we first use Lemma 5.2 and represent $\mathcal{C}\cap\Lambda$ as a finite disjoint union of images (under ω) of sets of the form $C[\jmath']\cap\Lambda$, where $C[\jmath']$ is a maximal small cylinder with $C[\jmath']\cap\Lambda \neq \emptyset$ and $C[\jmath'] \subset \mathcal{C}$. According to Lemma 5.3, the number of such cylinders is $\leq k^{2k\ell_0}$ and for each of them we have $\nu(C[\jmath']) \leq A_2\nu(C[\jmath])$. This, (5.3) and (5.6) imply

$$\nu(I(x,mr)) \leq \nu(\omega^{-1}(\mathcal{C})\cap U) \leq \sum_{\jmath'} \nu(C[\jmath']) \leq A_2 k^{2k\ell_0}\nu(C[\jmath]) \leq D''_m \nu(I(x,r)),$$

where $D''_m = D'_m A_2 k^{2k\ell_0} > 0$.

To deal with the case $mr \geq \text{dist}_{U'}(\partial U', U'\cap\Lambda)$, introduce the constant

$$r' = \min_{1\leq i\leq k} \text{dist}_{U_i}(\partial U_i, U_i\cap\Lambda) > 0.$$

We claim that there exists a constant $D'''_m > 0$ such that for $r'' = r'/m$ we have

$$\nu(I(x,r'')) \geq D'''_m \quad , \quad x \in U\cap\Lambda. \tag{5.7}$$

To prove this we will essentially use some of the above arguments. Consider an arbitrary $x \in U \cap \Lambda$, and let $\mathcal{C}'' = \mathcal{C}_{i_0,j_0}[\zeta]$ be a maximal big cylinder containing $\omega(x)$ and contained in $\omega(I(x,r''))$. If n'' is the length of $\mathcal{C}''$, then as above for a big cylinder $\mathcal{C}'''$ of length $n''-1$ with $\mathcal{C}'' \subset \mathcal{C}'''$ we get $|\mathcal{C}'''| > r''/c_2$. This and Lemma 4.1 imply $|\mathcal{C}''| \geq \lambda |\mathcal{C}'''| \geq \lambda r''/c_2$. On the other hand, we have $|\mathcal{C}''| < E\mu^{n''}$ (cf. the proof of Lemma 5.3). Hence $E\mu^{n''} > \lambda r''/c_2$ and therefore $n'' < \dfrac{\log(Ec_2/(\lambda r''))}{\log(1/\mu)}$. There exist a maximal small cylinder $C[\imath]$ with $\omega(x) \in \omega(C[\imath]) \subset \mathcal{C}''$. If p'' is its length, then we must have $p'' \leq kn''$, so $p'' < \dfrac{k \log \frac{Ec_2}{\lambda r''}}{\log(1/\mu)}$. Now $C[\imath] \subset I(x,r'')$ and (5.1) yield

$$\nu(I(x,r'')) \geq \nu(C[\imath]) \geq \alpha e^{p''c} e^{g_{p''}(x)} \geq \alpha e^{-p''|c-g_0|} \geq D'''_m = \alpha e^{-\frac{k|c-g_0| \log \frac{Ec_2 m}{\lambda r''}}{\log(1/\mu)}} .$$

This proves (5.7).

Setting $D_m = \max\{D''_m, 1/D'''_m\}$, it follows from the above that $\nu(I(x,mr)) \leq D_m \nu(I(x,r))$ for any $r > 0$ and $x \in U \cap \Lambda$. ■

6 The temporal distance function Δ

Recal the definition of the function $\Delta(x,y)$ from Sect. 2. This function has been naturally involved in various constructions in ergodic theory (cf. e.g. [Ra], [KB], [Ch1]). The following proposition is derived by adapting an argument of Katok and Burns [KB].

PROPOSITION 6.1. *For every $\sigma \in \Lambda \cap S^*_\epsilon(\partial K)$ there exists $\delta > 0$ such that for $x \in W^u_\epsilon(\sigma)$ and $y \in W^s_\epsilon(\sigma) \cap \Lambda$ we have $d_x \Delta(x,y)_{|\, x=\sigma} \neq 0$ whenever $0 < \mathrm{dist}(y,\sigma) < \delta$.*

Proof. Consider the 1-form $\alpha = \sum_{j=1}^2 \xi_j dx_j$ on $T^*(\Omega)$, where $x = (x_1,x_2) \in \Omega$, $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$. Notice that for $\sigma = (x,\xi) \in T^*(\Omega)$, x being in the interior of Ω, and for any $(y,\eta) \in T_\sigma(S^*(\Omega)) = T_x(\Omega) \times T_\xi(S^1)$ we have $\alpha_\sigma(y,\eta) = \langle \xi, y \rangle$. Thus, $\alpha(X(\sigma)) = 1$, where $X(\sigma) = (\xi;0)$ is the Hamilton vector field determined by the Hamilton function $H(z,\zeta) = \frac{1}{2}(\|\zeta\|^2 - 1)$ generating the billiard flow on $S^*(\Omega)$. Also notice that for $\sigma \in \Lambda$ we have

$$\mathrm{Ker}(\alpha_\sigma) = \{(y,\eta) \in T_\sigma(S^*(\Omega)) : y \perp \xi\} = E^u(\sigma) \oplus E^s(\sigma) ,$$

where $E^u(\sigma)$ and $E^s(\sigma)$ are the tangent spaces at σ to $W^u_\epsilon(\sigma)$ and $W^s_\epsilon(\sigma)$, respectively. This follows from the geometric properties of the stable and unstable manifolds for dispersing billiards (cf. [Si2]). Since $d\alpha = \sum_{j=1}^n d\xi_j \wedge dx_j$ is the restriction to $S^*(\Omega)$ of the standard symplectic form on $T^*(\Omega)$, it follows that for any $v \in E^u(\sigma) \setminus \{0\}$ there exists $w \in E^s(\sigma)$ with $d\alpha\,(v,w) \neq 0$.

Let $\sigma \in \Lambda$, $v \in E^u(\sigma)$ and $w \in E^s(\sigma)$ be such that $\|v\| = \|w\| = 1$. Let $x(s)$, $|s| \leq \delta$, and $y(t)$, $|t| \leq \delta$ for some small $\delta > 0$ be local C^1 parametrizations of $W^u_\epsilon(\sigma)$ and $W^s_\epsilon(\sigma)$, respectively, with $x(0) = y(0) = \sigma$, $\|\dot{x}(s)\| = \|\dot{y}(t)\| = 1$ for all s and t and $\dot{x}(0) = v$, $\dot{y}(0) = w$. Using the argument from the proof of Lemma 3.3 in [KB], one gets

$$\Delta(x(s),y(t)) = d\alpha\,(sv,tw) + o(st)$$

as $s \to 0$ and $t \to 0$ with $x(s), y(t) \in \Lambda$. Since $\Delta(x(0), y(t)) = 0$, this yields

$$\frac{d}{ds}\Delta(x(s), y(t))_{|s=0} = t d\alpha\ (v, w) + o(t).$$

Hence this derivative is non-zero whenever $d\alpha(v, w) \neq 0$ and $t \neq 0$ is sufficiently small. It follows from $\dim E^u(\sigma) = \dim E^s(\sigma) = 1$ that v and w span these two subspaces. Clearly $d\alpha(v, w) \neq 0$; otherwise $E^u(\sigma) \oplus E^s(\sigma) \oplus (\mathbb{R} \cdot X(\sigma))$ would be a 3-dimensional Lagrangean subspace of $T_\sigma(\Omega)$ for the symplectic form $d\alpha$ which is impossible. Finally, a simple geometric argument shows that 0 is a limit point of both $\{s \neq 0 : x(s) \in \Lambda\}$ and $\{t \neq 0 : y(t) \in \Lambda\}$.
■

We are now going to use the Markov family $\{R_i\}_{i=1}^k$ discussed in Sect. 2.

The following lemma is similar to Lemma 8 in [D]. The proof is also quite similar to that in [D]. We sketch it for completeness and because the objects constructed in its proof will be used later on.

LEMMA 6.2. *There exist constants $\delta_0 > 0$ and $N_0 \in \mathbb{N}$, such that for any integer $N \geq N_0$ there exists an open neighbourhood $\mathcal{U} = \mathcal{U}(N)$ of $\Lambda \cap U$ in U and C^1 maps $v_1, v_2 : \mathcal{U} \longrightarrow U$ such that $\sigma^N(v_i(u)) = u$ $(i = 1, 2)$ for $u \in \mathcal{U}$, and*

$$\delta_0 \leq \|d_u\,[\tau_N(v_1(u)) - \tau_N(v_2(u))]\| \leq \text{Const} \quad , \ u \in \mathcal{U}.$$

Proof. Set $U^{(n)} = \{\ \mathcal{P}^n(x)\ :\ x \in U\ ,\ \mathcal{P}^j(x) \text{ exists for all } \ j = 1, 2, \ldots, n\ \}$. Notice that $U \cap \Lambda \subset U^{(n)}$ for all $n > 1$ (follows from the conjugacy between σ and the Bernoulli shift on Σ_A^+). For $z, y \in \tilde{R}_i$ we will denote by $d_x\Delta(z, y)$ the differential of the function $\mathcal{F}^u(z) \ni x \mapsto \Delta(x, y) \in \mathbb{R}$ at $x = z$. For $x \in R_1$ denote $\mathcal{F}^u_{\tilde{R}_1}(x) = \tilde{R}_1 \cap \text{pr}_{D_1}(\mathcal{F}^u(x))$, where $\text{pr}_{D_1} : \phi_{[-\epsilon,\epsilon]}(D_1) \longrightarrow D_1$ is the natural projection along the orbits of ϕ_t. It follows from Proposition 6.1 that if $z \in R_1$ and $y \in W^s_\epsilon(z) \cap \Lambda \setminus \{z\} \subset R_1$ is sufficiently close to it, then $d_x\Delta(x, y)_{|x=z} \neq 0$. Since $U^{(n)}$ fills in R_1 densily as $n \to \infty$, taking n_0 large enough, we can find $y_0 \in R_1 \cap U^{(n_0)}$ such that $y_0 \notin U_1$ but y_0 is close to U_1, so that the above remark involving Proposition 6.1 applies with $y = y_0$ and $z = u_0 = p(y_0)$, where $p : \tilde{R}_1 \longrightarrow U_1$ is the projection along the leaves of $\mathcal{F}^s$. That is, $d_{u_1}\Delta(u_1, y_0)_{|u_1=u_0} \neq 0$, where $u_1 \in U_1$. Let $\pi : U_1 \longrightarrow \mathcal{F}^u_{\tilde{R}_1}(y_0)$ be the *projection along the leaves* of $\mathcal{F}^s$. Clearly $u_0 \in \Lambda$, so $u_0 \in R_1$. **Fix $n_0 > 1$, y_0 and u_0 with this property**.

In what folows u_1 and u_2 will be variables in U_1. Since

$$d_{u_1}\Delta(u_1, \pi(u_0))_{u_1=u_0} = d_{u_1}\Delta(u_1, y_0)_{u_1=u_0} \neq 0\ ,$$

there exists an open neighbourhood U_0 of u_0 in U_1 and $\delta' \in (0, 1/3)$ such that

$$\|d_{u_1}\Delta(u_1, \pi(u_2))\| \geq \delta' \qquad u_1, u_2 \in U_0\ . \tag{6.1}$$

Recall that $y_0 \in R_1 \cap U^{(n_0)}$, so $\mathcal{F}^u_{\tilde{R}_1}(y_0) \cap U^{(n_0)} \neq \emptyset$. Let $\mathcal{O}_1$ be a small open connected neighbourhood of y_0 in $\mathcal{F}^u_{\tilde{R}_1}(y_0) \cap U^{(n_0)}$ and let $f_1 : \mathcal{O}_1 \longrightarrow U$ be an arbitrary C^1 map such that $\mathcal{P}^{n_0}(f_1(x)) = x$ for all $x \in \mathcal{O}_1$. Shrinking U_0 if necessary, we may assume that $\pi(U_0) \subset \mathcal{O}_1$. Setting $w_1(z) = f_1(\pi(z))$, $z \in U_0$, we get a C^1 map $w_1 : U_0 \longrightarrow U$ with

$\mathcal{P}^{n_0}(w_1(z)) = \pi(z)$ and therefore $\sigma^{n_0}(w_1(z)) = z$ for all $z \in U_0$. In the same way, using the fact that $u_0 \in U_1 \cap \Lambda \subset U^{(n_0)}$, one constructs a C^1 map $w_2 : U_0 \longrightarrow U$ (shrinking U_0 again if necessary) with $\mathcal{P}^{n_0}(w_2(z)) = z$ for all $z \in U_0$. Then $\sigma^{n_0}(w_2(z)) = z$ for all $z \in U_0$. Thus we got C^1 maps

$$w_i : U_0 \longrightarrow U \quad , \quad \sigma^{n_0}(w_i(z)) = z \ , \ z \in U_0 \quad (i = 1, 2). \tag{6.2}$$

Set $V_i = w_i(U_0)$, $i = 1, 2$. Notice that (6.2) and (2.1) imply $c_0/\gamma_1^{n_0} \leq \|dw_i(z)\| \leq 1/(c_0\gamma^{n_0})$ for all $z \in U_0$ and $i = 1, 2$.

A property of $\Delta(x, y)$ which will be used next is the following ([D]): if $x, y \in U \cap \Lambda$ and the integer $n > 0$ are such that $\mathcal{P}^n(x)$ and $\mathcal{P}^n(y)$ are sufficiently close and therefore belong to the same rectangle R_i, then

$$\tau_n(y) - \tau_n(x) = \Delta(\mathcal{P}^n(x), \mathcal{P}^n(y)) = \Delta(p(\mathcal{P}^n(x)), \mathcal{P}^n(y)) \ ,$$

where as before $p : \tilde{R}_i \longrightarrow U_i$ is the projection along the leaves of $\mathcal{F}^s$. This follows from a simple geometric observation using the definition of Δ. Using this property and the choice of the maps w_i, for $z \in U_0$ and $z' \in U_0$ close to z we get

$$\begin{aligned} & [\tau_{n_0}(w_1(z')) - \tau_{n_0}(w_2(z'))] - [\tau_{n_0}(w_1(z)) - \tau_{n_0}(w_2(z))] \\ = \ & -\Delta(\mathcal{P}^{n_0}(w_1(z')), \mathcal{P}^{n_0}(w_1(z))) + \Delta(\mathcal{P}^{n_0}(w_2(z')), \mathcal{P}^{n_0}(w_2(z))) \\ = \ & -\Delta(\pi(z'), \pi(z)) + \Delta(z', z) = -\Delta(z', \pi(z)) + \Delta(z, \pi(z)). \end{aligned}$$

(Here we used the fact that $\Delta(z', z) = \Delta(z, \pi(z)) = 0$.) Then (6.1) implies

$$\|d_z[\tau_{n_0}(w_1(z)) - \tau_{n_0}(w_2(z))]\| \geq \delta' \quad (z \in U_0). \tag{6.3}$$

Since U_0 is open in U and $U_0 \cap \Lambda \neq \emptyset$, it follows that if $n_1 > 0$ is large enough, then $U = \sigma^{n_1}(U_0)$ (cf. the Convention in Sect. 2). The existence of such n_1 can be easily derived by the symbolic representation obtained for ϕ_t on Λ by means of the Markov family. Shrinking U_0 if necessary (now the new U_0 can no longer be connected; necessarily it will be the union of finitely many connected open sets the distance between any two of them being positive), we may assume that $\sigma^{n_1}(x)$ is well-defined for all $x \in U_0$ and

$$\Lambda \cap U \subset \sigma^{n_1}(U_0). \tag{6.4}$$

We will assume that n_1 is so large that $\sigma^{n_1} : U_0 \longrightarrow U$ is expanding, i.e. $\gamma_1^{n_1} > 1/c_0$ (cf. (2.1)). **Fix U_0 and $n_1 > 0$ with the properties listed above.** Set

$$\mathcal{U} = \sigma^{n_1}(U_0).$$

Since $\sigma^{n_1} : U_0 \longrightarrow \mathcal{U}$ is a C^1 diffeomorphism, its inverse $z(u) = (\sigma^{n_1})^{-1}(u)$ is a C^1 diffeomorphism

$$z : \mathcal{U} \longrightarrow U_0 \ \text{ such that } \ \sigma^{n_1}(z(u)) = u \ , \ u \in \mathcal{U} \ . \tag{6.5}$$

This and (2.1) imply $c_0/\gamma_1^{n_1} \leq \|dz(u)\| \leq 1/(c_0\gamma^{n_1})$ for all $u \in \mathcal{U}$.

Next, as in the proof of Lemma 9 in [D] one derives that there exists an integer $n_2 > 0$ such that for $n > n_2$ there exist C^1 maps $\tilde{v}_i : V_i \longrightarrow U$ $(i = 1,2)$ with $\sigma^n(\tilde{v}_i(w)) = w$ for all $w \in V_i$ and such that

$$\|d_w \tau_n(\tilde{v}_i(w))\| \leq \frac{c_0^3 \gamma^{n_0+n_1}\, \delta'}{4\gamma_1^{n_1}} \qquad (w \in V_i). \tag{6.6}$$

Fix $n_2 > 0$ and C^1 maps $\tilde{v}_i : V_i \longrightarrow U$ with these properties. Set $N_0 = n_0 + n_1 + n_2$ and let $N > N_0$. Then $n = N - n_0 - n_1 > n_2$, so for each $i = 1,2$ there exists a C^1 diffeomorphism

$$\tilde{v}_i : V_i \longrightarrow U \ \text{ with } \ \sigma^{N-n_0-n_1}(\tilde{v}_i(w)) = w \ , \ w \in V_i \ , \tag{6.7}$$

such that (6.6) holds with $n = N - n_0 - n_1$ and δ'' as above. Now define the C^1 maps

$$v_i : \mathcal{U} \longrightarrow U \quad , \ v_i(u) = \tilde{v}_i(w_i(z(u))). \tag{6.8}$$

Then we have $\sigma^{N-n_1}(v_i(u)) = \sigma^{n_0}(\sigma^{N-n_0-n_1}(v_i(u))) = \sigma^{n_0}(w_i(z(u))) = z(u)$ (which is the same for $i = 1,2$) for all $u \in \mathcal{U}$. Consequently, $\sigma^m(v_1(u)) = \sigma^m(v_2(u))$ for all $m \geq N - n_1$ and $u \in \mathcal{U}$. Thus, $\tau_N(v_1(u)) - \tau_N(v_2(u)) = \tau_{N-n_1}(v_1(u)) - \tau_{N-n_1}(v_2(u))$. Using this, (6.6), (6.3), $\|d_z w_i(z)\| \leq 1/(c_0\gamma^{n_0})$, and the Chain Rule, as in [D] one gets

$$\|d_u\, [\tau_N(v_1(u)) - \tau_N(v_2(u))]\, \| \geq \delta' \frac{c_0}{\gamma_1^{n_1}} - 2\delta''\, \frac{1}{c_0\gamma^{n_0}}\, \frac{1}{c_0\gamma^{n_1}} = \delta' \frac{c_0}{2\gamma_1^{n_1}}$$

for all $u \in \mathcal{U}$. So, we can take $\delta_0 = \delta' \frac{c_0}{2\gamma_1^{n_1}}$. It is also clear that there exists a constant Const > 0 depending on n_0 and n_1 such that $\|d_u\, [\tau_N(v_1(u)) - \tau_N(v_2(u))]\, \| \leq$ Const for all $u \in \mathcal{U}$. This proves the assertion. ■

7 Proof of Lemma 3.4

¿From now until the end of the section we will use the open subset U_0 of U, the integers n_0, n_1, n_2 and $N_0 = n_0 + n_1 + n_2$, the set $\mathcal{U} = \sigma^{n_1}(U_0)$, the number δ_0 and the map $z(u)$ with (6.5) constructed in the proof of Lemma 6.2. In this section we will first use the results from Sect. 4 to prove the existence of special partitions of U_0 (and in fact any connected subcurve of U having common points with Λ).

Fix an arbitrary orientation on each connected component U_i of the U_0. A family $\{\gamma_j\}_{j=1}^q$ of closed conected curves (intervals) in U_0 will be called *ordered* if for every $j = 1,\ldots,q-1$ such that the curves γ_j and γ_{j+1} belong to the same connected component of U_0 we have: γ_{j+1} is after γ_j (with respect to the positive direction along U_0) and $\gamma_j \cap \gamma_{j+1}$ consists of one point which belongs to $\partial\gamma_j \cap \partial\gamma_{j+1}$. We will say that an ordered family $\{\gamma_j\}_{j=1}^q$ of closed connected curves in U_0 has the *triple intersection property* if for every j with $\gamma_j \cap \Lambda \neq \emptyset$ there exists j' with $|j - j'| \leq 1$ such that the curves $\gamma_{j'}$, $\gamma_{j'+1}$, $\gamma_{j'+2}$ are contained in the same connected component of U and all of them have a common point with Λ.

LEMMA 7.1. *There exist constants $0 < c_1 < C_1$ and $\mu_0 \in (0,1/2)$ such that for every real number b with $|b| \geq 1$ and every sufficiently small $\epsilon_1 > 0$ there exist a finite ordered family $\Gamma(b,\epsilon_1) = \{\gamma_j\}_{j=1}^q$ $(q \geq 3k)$ of curves in U_0 diffeomorphic to $[0,1]$ such that:*

(a) $c_1\frac{\epsilon_1}{|b|} \leq |\gamma_j| \leq C_1\frac{\epsilon_1}{|b|}$ *for all* j ;
(b) $U_0 = \cup_{j=1}^q \gamma_j$ *and* $\{\gamma_j\}_{j=1}^q$ *has the triple intersection property.*
(c) $\mathrm{dist}_{\gamma_j}(\partial\gamma_j\ ,\gamma_j\cap\Lambda) \geq 2\mu_0|\gamma_j|$ *for all* j *with* $\gamma_j\cap\Lambda\neq\emptyset$.

Proof. Let $|b|\geq 1$ and let $\epsilon_1 > 0$ be such that

$$\epsilon_1 \leq \min\left\{ \frac{|U'|}{6}\ ,\ \mathrm{dist}_{U'}(\partial U', U'\cap\Lambda)\right\}$$

for any connected component U' of U_0. It is enough to find a representation of each connected component of U_0 as a finite union of subcurves with the properties described in Lemma 7.1.

Let U' be an arbitrary connected component of U_0. Consider an arbitrary representation $U' = \cup_{i=1}^p I'_i$, of U' as a finite union of connected closed (i.e. containing their ends) curves with disjoint interiors such that $\dfrac{\epsilon_1}{|b|} \leq |I'_i| \leq \dfrac{2\epsilon_1}{|b|}$ for all $1\leq i\leq p$, and for each $i<p$, I'_i and I'_{i+1} have a common end point. That is, each I'_i is a subcurve of U' of the form $I'_i = [x_{i-1}, x_i]$, where x_0 and x_p are the initial and the terminal points of U', respectively.

First, we will slightly change the points x_i (whenever necessary) to ensure that $\mathrm{dist}_{U'}(\partial I'_i, I'_i\cap\Lambda)$ is sufficiently large for each i. Fix i for a moment. If the interval $I(x_i,\epsilon_1/6|b|)$ contains no points of Λ, leave x_i unchanged. Assume that $I(x_i,\epsilon_1/6|b|)\cap\Lambda\neq\emptyset$ and choose an arbitrary x'_i in this set. Applying Corollary 4.3(b) to $I(x'_i,\epsilon_1/6|b|)$, we get a sequence of subcurves Θ_j with the properties (i) and (ii). Take an arbitrary Θ_j which is a gap, i.e. its interior does not contain points of Λ, and replace x_i by the middle of Θ_j. In this way we get new intervals I'_i and I'_{i+1}. Repeating this procedure several times by induction starting from I'_1, we end up with a new family of intervals I'_i (we will use the same notation for them) such that

$$\frac{\epsilon_1}{3|b|} \leq |I'_i| \leq \frac{5\epsilon_1}{3|b|} \quad , \quad \mathrm{dist}_{U_0}(\partial I'_i, I'_i\cap\Lambda) \geq \lambda_0\frac{\epsilon_1}{12|b|} \qquad (i=1,\ldots,p).$$

Fix for a moment $i=1,\ldots,p$ with $I'_i\cap\Lambda\neq\emptyset$ and choose an arbitrary $y_i\in I'_i\cap\Lambda$. Then for $r=\lambda_0\frac{\epsilon_1}{12|b|}$ we have $I(y_i,r)\subset I'_i$. Applying Corollary 4.3(b) to $I(y_i,r)$, we get an ordered family $\{\Theta_j\}_{j=1}^{2k_0-1}$ of connected subcurves of $I(y_i,r)$ having the properties listed in Corollary 4.3(b). First, consider the case $k_0>3$. Denote by z_j the middle of the curve Θ_{2j+1} for $1\leq j\leq k_0-2$, and let $z_0=x_{i-1}$ and $z_{k_0-1}=x_i$ (the initial and terminal points of I'_i). For $j=1,\ldots k_0-1$, let $\gamma_j^{(i)}$ be the connected subcurve of I'_i with end points z_{j-1} and z_j. In this way we get a representation of I'_i as a union of k_0-1 closed connected subcurves $\gamma_j^{(i)}$ with disjoint interiors and such that $\gamma_j^{(i)}\cap\Lambda\neq\emptyset$ and $\mathrm{dist}_{U_0}(\partial\gamma_j^{(i)},\gamma_j^{(i)}\cap\Lambda)\geq 2\mu_0\frac{\epsilon_1}{|b|}$ for all $j=1,\ldots,k_0-1$, where $\mu_0=\lambda_0^2/48$ is a positive global constant. Moreover, $|\gamma_j^{(i)}|\leq 2r = \frac{\lambda_0\epsilon_1}{6|b|}$ and $|\gamma_j^{(i)}|\geq \lambda_0 r = \frac{\lambda_0^2\epsilon_1}{12|b|}$. When $k_0>3$ we have $k_0-1\geq 3$, so the family $\{\gamma_j^{(i)}\}_{j=1}^{k_0-1}$ consists of at least three different curves. Thus, the family

$$\{\gamma_j^{(i)} : j=1,\ldots,k_0-1, i=1,\ldots p, I'_i\cap\Lambda\neq\emptyset\}\cup\{I'_i : I'_i\cap\Lambda=\emptyset\}$$

has all the properties listed in Lemma 7.1 for the connected component U' of U_0.

Next, consider the case $k_0 = 3$. Then in the above procedure we only get two subcurves $\gamma_j^{(i)}$ which is not enough. We need to apply Corollary 4.3(b) (or rather its proof) once again, say to each of the subcurves Θ_2 and Θ_4 (in fact it is enough to do this to one of them; the argument in the proof of Corollary 4.3(b) applies easily, since Θ_2 and Θ_4 are cylinders). In this way for $i = 2, 4$ we get a representation $\Theta_i = \cup_{r=1}^5 \Theta_{i,r}$ where $\Theta_{i,r}$ are closed connected subcurves of Θ_i with disjoint interiors such that $|\Theta_{i,r}| \geq \lambda_0 |\Theta_i| \geq \lambda_0^3 \frac{\epsilon_1}{12|b|}$ for all $r = 1, \dots, 5$ and $\Theta_{i,r} \cap \Lambda \neq \emptyset$ iff $r = 2$ or $r = 4$. Now define the family of intervals $\{\gamma_j^{(i)}\}_{j=1}^4$ in such a way that each $\gamma_j^{(i)}$ contains one of the subcurves $\Theta_{i,r}$ with $j, r \in \{2, 4\}$ and the end points of $\gamma_j^{(i)}$ are formed by the end points of the interval I_i' and the middles of the gaps $\Theta_{2,3}$, Θ_3 and $\Theta_{4,3}$. Then $\cup_{i=1}^p \{\gamma_j^{(i)}\}_{j=1}^4$ is a family of subcurves of U' having all the properties listed in Lemma 7.1, where this time $\mu_0 = \lambda_0^3/48$. ■

Let $b \in \mathbb{R}$ be such that $1/|b|^\alpha \leq a_0'$. **Fix an arbitrary** ϵ_1 such that

$$0 < \epsilon_1 \leq \min \left\{ \frac{1}{32 C_0 C_1} \; , \; \mathrm{dist}_{U'}(\partial U', U' \cap \Lambda) \; , \; \frac{|U'|}{6} \right\} \tag{7.1}$$

for any connected component U' of U, and **fix a family** $\Gamma(b, \epsilon_1) = \{\gamma_j\}_{j=1}^q$ of curves in U_0 having the properties listed in Lemma 7.1.

For each $j = 1, \dots, q$ denote by $\tilde{\gamma}_j$ the connected subcurve of γ_j consisting of all points $z \in \gamma_j$ with $\mathrm{dist}_{\gamma_j}(\partial \gamma_j, z) \geq \mu_0 |\gamma_j|$. Then condition (c) in Lemma 7.1 implies

$$\gamma_j \cap \Lambda \subset \tilde{\gamma}_j \quad , \quad 1 \leq j \leq q \; .$$

Given j with $\gamma_j \cap \Lambda \neq \emptyset$ **fix an arbitrary** $z_j \in \gamma_j \cap \Lambda \subset \tilde{\gamma}_j \cap \Lambda$. It follows from (a) and (c) in Lemma 7.1 that $I(z_j, \mu_0 |\gamma_j|) \subset \tilde{\gamma}_j$ and $\mu_0 |\gamma_j| \geq \frac{c_1 \mu_0 \epsilon_1}{|b|}$. Thus,

$$I\left(z_j, \frac{c_1 \mu_0 \epsilon_1}{|b|} \right) \subset \tilde{\gamma}_j \subset \gamma_j \quad , \quad j = 1, \dots, q. \tag{7.2}$$

Consequently, there exists a C^1 function $\eta_j : \gamma_j \longrightarrow [0, 1]$ such that $\eta_j = 0$ on a neighbourhood of $\partial \gamma_j$, $\eta_j = 1$ on $\tilde{\gamma}_j$, and[9]

$$\| d_u \eta_j(u) \| \leq \frac{4|b|}{c_1 \mu_0 \epsilon_1} \quad , \quad u \in \gamma_j \; . \tag{7.3}$$

For each j with $\gamma_j \cap \Lambda \neq \emptyset$ **fix a C^1 function** η_j with the above properties.

Given $j = 1, \dots, q$ and $i = 1, 2$, denote

$$Y_j^i = w_i(\gamma_j) \quad , \quad Z_j = \sigma^{n_1}(\gamma_j) \quad , \quad X_j^i = \{ v_i(u) : u \in Z_j \} \; .$$

It then follows that $\gamma_j = z(Z_j)$, $X_j^i = \tilde{v}_i(Y_j^i)$, and $\mathcal{U} = \cup_{j=1}^q Z_j$. The following diagram may be of some help in figuring out where these subsets belong.

[9]On the real line, given $\delta > 0$, the function $\eta : \mathbb{R} \longrightarrow [0, 1]$, defined by $\eta(x) = 1$ for $x < 0$, $\eta(x) = (\delta^2 - x^2)^2/\delta^4$ for $0 \leq x \leq \delta$, $\eta(x) = 0$ for $x > \delta$, has the desired properties.

$$
\begin{array}{ccccccc}
\mathcal{U} & \underset{\sigma^{n_1}}{\overset{z(u)}{\rightleftarrows}} & U_0 & \underset{\sigma^{n_0}}{\overset{w_i}{\rightleftarrows}} & V_i & \underset{\sigma^{N-n_0-n_1}}{\overset{\tilde{v}_i}{\rightleftarrows}} & U \\
\cup & & \cup & & \cup & & \cup \\
Z_j = \sigma^{n_1}(\gamma_j) & \xrightarrow{z(u)} & \gamma_j & \xrightarrow{w_i} & Y_j^i = w_i(\gamma_j) & \xrightarrow{\tilde{v}_i} & X_j^i = \tilde{v}_i(Y_j^i) = v_i(Z_j)
\end{array}
$$

Let J be a *subset of the set*

$$\Xi = \{ \ (i,j) \ : \ 1 \leq i \leq 2 \ , \ 1 \leq j \leq q \ , \ \gamma_j \cap \Lambda \neq \emptyset \ \}.$$

Given $\mu \in (0,1)$, following [D], we define the function $m = m_{\mu,b,J} : U \longrightarrow [0,1]$ by

$$m(v) = \begin{cases} 1 - \mu\eta_j(z(\sigma^N(v)) & , \ v \in X_j^i \ \text{for some} \ \ (i,j) \in J \ , \\ \qquad 1 & \qquad \text{otherwise} \ . \end{cases}$$

Notice that $v \in X_j^i$ is equivalent to $v = v_i(u)$ for some $u \in Z_j$; then $\sigma^N(v)$ is well-defined and $\sigma^N(v) = \sigma^N(v_i(u)) = u \in Z_j$. Consequently, $z(u)$ is well-defined and $z(u) \in \gamma_j$. Since the curves γ_j have no common interior points and η_j is zero near $\partial\gamma_j = z(\partial Z_j)$, the definition of m is correct and m is C^1. Notice that if $z(\sigma^N(v)) \in \tilde{\gamma}_j$ for some $(i,j) \in J$, then $\eta_j(z(\sigma^N(v))) = 1$, so $m(v) = 1 - \mu$. It is worth mentioning that for $v = v_i(u)$ with $u \in Z_j$ we have

$$\sigma^{N-n_0-n_1}(v) = \sigma^{N-n_0-n_1}(\tilde{v}_i(w_i(z(u))) = w_i(z(u)),$$

so $\sigma^{N-n_1}(v) = \sigma^{n_0}(w_i(z(u))) = z(u) = z(\sigma^N(v))$. Thus, $m(v) = 1 - \mu\eta_j(\sigma^{N-n_1}(v))$ for $v \in X_j^i$. Finally, notice that the definition of m, (7.3) and (2.1) imply

$$\|d_v m(v)\| \leq \mu \frac{4|b|\gamma_1^N}{c_1\mu_0\epsilon_1 c_0} \quad , \ v \in U \ . \tag{7.4}$$

Next, define *the operator* $\mathcal{N} = \mathcal{N}_{ab}^{(J,\mu)} : C^1(U) \longrightarrow C^1(U)$ by

$$(\mathcal{N}h)(u) = \left(\mathcal{M}_{ab}^N(m \cdot h)\right)(u) \quad , \ \ u \in U \ .$$

In the rest of this section and the next one we will choose a number $\mu \in (0,1)$, and a subset J of Λ such that the operator $\mathcal{N} = \mathcal{N}_{ab}^{(J,\mu)}$ has some special properties. Let us say in advance that the only condition for μ will be of the form $\mu \leq \text{const}\ e^{-\text{const}N}$, where const is a positive constant that does not depend on b and N.

The following lemma contains statements similar to Proposition 6 and Lemma 11 in [D]. The proof is almost the same (Lemma 2.2 is applied several times), so we omit it.

LEMMA 7.2. *If* $E > 0$, $1/|b|^\alpha \leq a_0'$, $\mu > 0$ *and* N *satisfy the inequalities*

$$4A_0 \leq E \quad , \quad \gamma^N \geq 4A_0 \quad , \quad \mu \leq \min\left\{ \frac{1}{2} \ , \ \frac{c_0c_1\mu_0\epsilon_1 E}{8\gamma_1^N} \right\} \ , \tag{7.5}$$

then the following hold:

(a) $\mathcal{N}h \in K_{E|b|}$ *for any* $h \in K_{E|b|}$;

(b) *For any* $h, H \in C^1(U)$ *such that* $H \in K_{E|b|}$, $|h(u)| \leq H(u)$ *and* $\|dh(u)\| \leq E|b|H(u)$ *for any* $u \in U$, *we have* $\|d_u(L_{ab}^N h)(u)\| \leq E|b|(\mathcal{N} H)(u)$ *for all* $u \in U$.

In what follows for each integer $m > 0$, D_m will **denote a fixed constant with the property described** in Proposition 5.1.

Modifying a definition in [D], given $r > 0$ and $m > 0$, a subset W of $\mathcal{U}$ will be called (r, m)-*dense* in $\mathcal{U}$ if for every $u \in \Lambda \cap \mathcal{U}$ there exists $v \in \Lambda \cap \mathcal{U}$ such that $I(v, r) \cap \mathcal{U} \subset W \cap I(u, mr)$.

The following is a consequence of Proposition 5.1.

PROPOSITION 7.3. *Let* $E > 0$, $m > 0$ *and let* $\epsilon = \epsilon(m, E) = \dfrac{1}{5\, e^{2mE} D_m}$. *Then for any* $r > 0$, *any* (r, m)*-dense subset* W *of* $\mathcal{U}$ *and any* $H \in K_{E/r}$, *we have*

$$\int_W H^2 \, d\nu \geq \epsilon \int_U H^2 \, d\nu \; .$$

Proof. Since $\mathrm{supp}(\nu) \subset \Lambda \cap U = \Lambda \cap \mathcal{U}$, in the above integrals it is enough to integrate over $\Lambda \cap W$ and $\Lambda \cap \mathcal{U}$, respectively.

Let $r > 0$, W be an (r, m)-dense subset of $\mathcal{U}$ and $H \in K_{E/r}$. Divide each connected component U_i of U into connected curves θ with disjoint interiors and such that $mr/2 \leq |\theta| < mr$. If $mr \leq |U_i|$ for some i, take $\theta = U_i$. Let θ_j $(j = 1, \ldots, p)$ be the (finite) family of these curves that have non-empty intersection with $\Lambda \cap \mathcal{U}$. Thus, $\Lambda \cap \mathcal{U} \subset \cup_{j=1}^p \theta_j$. Denote $m_j = \inf_{u \in \theta_j \cap \mathcal{U}} H(u)$ and $M_j = \sup_{u \in \theta_j \cap \mathcal{U}} H(u)$. Then (3.7) and $H \in K_{E/r}(\mathcal{U})$ imply

$$H(u) \geq m_j \geq M_j e^{-Em} \quad , \;\; u \in \theta_j \cap \mathcal{U}. \tag{7.6}$$

Next, for each $j = 1, \ldots, p$ choose an arbitrary point $u_j \in \theta_j \cap \Lambda \cap \mathcal{U}$. Then $\theta_j \subset I(u_j, mr)$. Different curves $I(u_i, mr)$ and $I(u_j, mr)$ may have common interior points, however every point of U belongs to the interior of at most 5 of these curves. Since $u_j \in \Lambda \cap \mathcal{U}$, it follows from the (r, m)-density of W that there exists $v_j \in \Lambda \cap \mathcal{U}$ such that $I(v_j, r) \cap \mathcal{U} \subset I(u_j, mr) \cap W$. Combining this with Proposition 5.1, gives

$$\nu(I(u_j, mr) \cap W) \geq \nu(I(v_j, r) \cap \mathcal{U})) = \nu(I(v_j, r)) \geq \frac{1}{D_m} \nu(I(u_j, mr)) \geq \frac{1}{D_m} \nu(\theta_j).$$

This and (7.6) yield $\int_W H^2(u) \, d\nu(u) \geq \epsilon \int_U H^2(u) \, d\nu(u)$, which proves the assertion. ∎

Modifying a definition in [D], we will say that the subset J of Ξ is *dense* if for any $j = 1, \ldots, q$ with $\gamma_j \cap \Lambda \neq \emptyset$ there exists $(i, j') \in J$ such that $|j - j'| \leq 2$. (Notice that $(i, j') \in J \subset \Xi$ implies $\gamma_{j'} \cap \Lambda \neq \emptyset$.)

Recall the positive constants a_0', T and γ_1 defined in Sect. 2. Taking a smaller $a_0' > 0$ if necessary, we may assume that there exists $A_2 > 0$ such that $\|f^{(ab)} - f\| \leq a_0' A_2$ and $\|f^{(ab)} - (f - P\tau)\| \leq A_2$ for $|a| \leq a_0'$ and $1/|b|^\alpha \leq a_0'$. Fix a constant $A_2 > 0$ with this property.

The following lemma is the analogue of Lemma 12 in [D].

LEMMA 7.4. *Given the numbers E and μ, there exist $a_0 = a_0(E,\mu,N) > 0$ and $\epsilon_2 = \epsilon_2(E,\mu,N)$ such that*

$$\int_U (\mathcal{N}H)^2 d\nu \leq (1-\epsilon_2)\int_U H^2 d\nu$$

whenever J is dense, $H \in K_{E|b|}$, $|a| \leq a_0$, and $1/|b|^\alpha \leq a_0$. More precisely, we can take

$$a_0 = \min\left\{ (a_0')^{1/\alpha} \ , \ -\frac{\ln(1-\epsilon'\mu e^{-NT})}{2A_2 N} \right\} \quad , \quad \epsilon_2 = \frac{\epsilon'\mu e^{-NT}}{2} \ ,$$

where $\epsilon' = \dfrac{1}{5D_{m'}e^{2m'Ec_0c_1\epsilon_1\mu_0\gamma^{n_1}}}$ *and* $m' = \dfrac{3C_1\gamma_1^{n_1}}{c_0^2 c_1\mu_0\gamma^{n_1}}$.

Sketch of Proof. The definition of $\mathcal{N}$ and the Cauchy-Schwartz inequality imply

$$(\mathcal{N}H)^2(u) = (\mathcal{M}_{ab}^N(mH))^2(u) \leq (\mathcal{M}_{ab}^N m^2)(u)\cdot(\mathcal{M}_{ab}^N H^2)(u) \tag{7.7}$$

for all $u \in U$, where $m = m_{\mu,b,J}$.

Denote by W the set of those $u \in \mathcal{U}$ such that there exists $(i,j) \in J$ with $z(u) \in \tilde{\gamma}_j$. According to the definition of m and the remark just after it, for such u we have $m(v_i(u)) = 1-\mu$. Using the density of J and property (a) in Lemma 7.1, one derives that W is (r,m')-dense in $\mathcal{U}$, where $r = \dfrac{\mu_0 c_0 c_1\epsilon_1\gamma^{n_1}}{|b|}$.

Setting $E' = E\mu_0c_0c_1\epsilon_1\gamma^{n_1}$, we have $H \in K_{E|b|} = K_{\frac{E'}{r}}$. Notice that $\epsilon' = \frac{1}{5e^{2m'E'}D_{m'}}$, so it coincides with $\epsilon(m,E)$ defined in Proposition 7.3 with m and E replaced by m' and E'. It then follows from Proposition 7.3 that $\int_W H^2\,d\nu \geq \epsilon'\int_U H^2\,d\nu$. Since $\mathcal{M}_{ab}^N(H^2) = h^2$ for some C^1 positive function h on U, it follows that

$$\int_W \mathcal{M}_{ab}^N(H^2)\,d\nu \geq \epsilon'\int_U \mathcal{M}_{ab}^N(H^2)\,d\nu. \tag{7.8}$$

Next, $m^2(v) \leq 1$ for all $v \in U$ implies $\mathcal{M}_{ab}^N(m^2) \leq \mathcal{M}_{ab}^N 1 = 1$. Moreover,

$$(\mathcal{M}_{ab}^N(m^2))(u) \leq 1-\mu e^{-NT} \quad , \quad u \in W \ . \tag{7.9}$$

To prove this, consider an arbitrary $u \in W$. It follows from the definition of W that there exists $i = 1,2$ such that $m(v_i(u)) = 1-\mu$. Hence

$$\begin{aligned}(\mathcal{M}_{ab}^N(m^2))(u) &= \sum_{\sigma^N v = u} e^{f_N^{(ab)}(v)} m^2(v) \leq \sum_{\sigma^N v=u, v\neq v_i(u)} e^{f_N^{(ab)}(v)} + (1-\mu)e^{f_N^{(ab)}(v_i(u))} \\ &\leq \sum_{\sigma^N v = u} e^{f_N^{(ab)}(v)} - \mu e^{f_N^{(ab)}(v_i(u))} \leq (\mathcal{M}_{ab}^N\,1)(u) - \mu e^{-NT} = 1-\mu e^{-NT} \ .\end{aligned}$$

This proves (7.9). Using (7.7), (7.8) and (7.9), as in [D], we get the estimate

$$\int_U (\mathcal{N}H)^2\,d\nu \leq (1-\epsilon'\mu e^{-NT})\int_U \mathcal{M}_{ab}^N(H^2)\,d\nu \ . \tag{7.10}$$

As in the proof of Proposition 3.3, for $|a| \leq a_0$ we have

$$\mathcal{M}_{ab}^N(H^2)(u) = L_{f-P\tau}^N \left(e^{(f^{(ab)}-f+P\tau)_N} H^2 \right)(u) ,$$

and $e^{(f^{(ab)}-f^{(0)})_N} \leq e^{a_0 N T}$. Using this and the choice of the constant A_2 (cf. the text before the statement of Lemma 7.4), we get

$$\int_U \mathcal{M}_{ab}^N(H^2)\, d\nu = \int_U L_{f-P\tau}^N \left(e^{(f^{(ab)}-f+P\tau)_N} H^2 \right)\, d\nu \leq e^{a_0 A_2 N} \int_U L_{f-P\tau}^N(H^2)\, d\nu = e^{a_0 A_2 N} \int_U H^2\, d\nu .$$

This and (7.10) yield $\int_U (\mathcal{N} H)^2\, d\nu \leq (1-\epsilon_2) \int_U H^2\, d\nu$, since $(1-\epsilon' \mu e^{-NT}) e^{a_0 A_2 N} \leq 1-\epsilon_2$. ■

In what follows we assume that $h, H \in C^1(U)$ are such that

(7.11) $$H \in K_{E|b|} \ , \ |h(u)| \leq H(u) \ , \ \|dh(u)\| \leq E|b| H(u) \text{ for all } u \in U .$$

Given $\mu \in (0, 1/2)$, following [D], define the functions $\chi_\mu^{(i)} : \mathcal{U} \longrightarrow \mathbf{C}$ $(i = 1, 2)$ by

$$\chi_\mu^{(1)}(u) = \frac{\left| e^{(f_N^{(ab)} + \mathbf{i} b \tau_N)(v_1(u))} h(v_1(u)) + e^{(f_N^{(ab)} + \mathbf{i} b \tau_N)(v_2(u))} h(v_2(u)) \right|}{(1-2\mu) e^{f_N^{(ab)}(v_1(u))} H(v_1(u)) + e^{f_N^{(ab)}(v_2(u))} H(v_2(u))} ,$$

$$\chi_\mu^{(2)}(u) = \frac{\left| e^{(f_N^{(ab)} + \mathbf{i} b \tau_N)(v_1(u))} h(v_1(u)) + e^{(f_N^{(ab)} + \mathbf{i} b \tau_N)(v_2(u))} h(v_2(u)) \right|}{e^{f_N^{(ab)}(v_1(u))} H(v_1(u)) + (1-2\mu) e^{f_N^{(ab)}(v_2(u))} H(v_2(u))} .$$

The following lemma is analogous to Lemma 13 in [D].

Lemma 7.5. *If $\epsilon_1 > 0$ and $\mu > 0$ are sufficiently small, then for any $j = 1, \ldots, q$ with $\gamma_j \cap \Lambda \neq \emptyset$ there exist $j' = 1, \ldots, q$ with $|j' - j| \leq 2$ and $\gamma_{j'} \cap \Lambda \neq \emptyset$ and $i = 1, 2$ such that*

(7.12) $$\chi_\mu^{(i)}(u) \leq 1 \quad , \quad u \in Z_{j'} .$$

We will prove this lemma first and then show how to derive Lemma 3.4 from it. The following lemma coincides with Lemma 14 in [D] and we omit its proof.

Lemma 7.6. *If h and H satisfy* (7.11) *and*

(7.13) $$0 < \epsilon_1 \leq \frac{1}{8 E C_1},$$

then for all $j = 1, \ldots, q$ and $i = 1, 2$ we have:

(a) $\frac{1}{2} \leq \frac{H(v_i(u'))}{H(v_i(u''))} \leq 2$ *for all $u', u'' \in Z_j$;*

(b) *Either for all $u \in Z_j$ we have $|h(v_i(u))| \leq \frac{3}{4} H(v_i(u))$, or $|h(v_i(u))| \geq \frac{1}{4} H(v_i(u))$ for all $u \in Z_j$.*

As in [D] (cf. Lemma 15 there), using Lemma 6.2, one derives the following

LEMMA 7.7. *Let $\psi(u) = b[\tau_N(v_1(u)) - \tau_N(v_2(u))]$ for $u \in \mathcal{U}$, and let $j, j' \in \{1, 2, \dots, q\}$ be such that $|j - j'| = 2$ and γ_j and $\gamma_{j'}$ belong to the same connected component of U_0. Then $c_2\epsilon_1 \leq |\psi(u) - \psi(u')|$ for $u \in Z_j$ and $u \in Z_{j'}$, where $c_2 = \delta_0 c_0 c_1 \gamma^{n_1}$.* ■

Proof of Lemma 7.5. Assume that (7.13) holds and $\gamma_j \cap \Lambda \neq \emptyset$. It follows from the triple intersection property of the family $\{\gamma_i\}$ that three consecutive intervals γ_i including γ_j belong to the same connected component of U_0 and have common points with Λ. Suppose that γ_j, γ_{j+1}, γ_{j+2} is such a triple; the other cases are similar. Set

$$\tilde{Z}_j = Z_j \cup Z_{j+1} \cup Z_{j+2} .$$

If there exist $j' \in \{j, j+1, j+2\}$ and $i = 1, 2$ such that the first alternative in Lemma 7.6 (b) holds for $Z_{j'}$ and i, then $\chi_\mu^{(i)}(u) \leq 1$ for any $u \in Z_{j'}$ follows immediately.

Assume that for every $j' \in \{j, j+1, j+2\}$ and every $i = 1, 2$ the second alternative in Lemma 7.6 (b) holds for $Z_{j'}$ and i. As in [D], consider the function

$$\Psi(u) = \mathrm{Arg}\left(e^{\mathbf{i}b\tau_N(v_1(u))} h(v_1(u))\right) - \mathrm{Arg}\left(e^{\mathbf{i}b\tau_N(v_2(u))} h(v_2(u))\right) .$$

Since $|h(v_i(u))| \geq \frac{1}{4} H(v_i(u)) > 0$, we can choose real continuous functions $\theta_i(u)$, $u \in \tilde{Z}_j$, $i = 1, 2$, such that $h(v_i(u)) = e^{\mathbf{i}\theta_i(u)} |h(v_i(u))|$ for all $u \in \tilde{Z}_j$. Then $\ln h(v_i(u))$ is well-defined with $\mathrm{Im}(\ln(h(v_i(u))) = \theta_i(u)$. Using the second alternative in Lemma 7.6(b), Lemma 2.1, $\sigma^N(v_i(u)) = u$ and (7.11), we get

$$\|d_u \ln h(v_i(u))\| \leq \frac{\|d_v h(v)\|}{|h(v_i(u))|} \, \|dv_i(u))\| \leq \frac{4E|b|}{c_0 \gamma^N}$$

for $u \in \tilde{Z}_j$. In particular, for $u, u' \in \tilde{Z}_j$ we have

$$|\ln h(v_i(u)) - \ln h(v_i(u'))| \leq \frac{4E|b|}{c_0\gamma^N} \mathrm{dist}_U(u, u') \leq \frac{4E|b|}{c_0\gamma^N} \cdot \frac{3C_1\gamma_1^{n_1}\epsilon_1}{c_0|b|} = \frac{C_2 E\epsilon_1}{\gamma^N},$$

where $C_2 = \frac{12 C_1 \gamma_1^{n_1}}{c_0^2} > 0$ is a constant. This yields

$$|\theta_i(u) - \theta_i(u')| \leq |\ln h(v_i(u)) - \ln h(v_i(u'))| \leq \frac{C_2 E \epsilon_1}{\gamma^N} \quad , \quad u, u' \in \tilde{Z}_j . \tag{7.14}$$

Let $u \in Z_j$ and $u' \in Z_{j+2}$. It follows from Lemma 7.7 and (7.14) that

$$\begin{aligned} |\Psi(u) - \Psi(u')| &\geq |\psi(u) - \psi(u')| - |\theta_1(u) - \theta_1(u')| - |\theta_2(u) - \theta_2(u')| \\ &\geq c_2\epsilon_1 - 2C_2 \frac{E\epsilon_1}{\gamma^N} \geq \frac{c_2}{2}\epsilon_1, \end{aligned}$$

provided

$$4\frac{C_2}{c_2} E \leq \gamma^N . \tag{7.15}$$

Thus, assuming (7.15), we have $|\Psi(u) - \Psi(u')| \geq \frac{c_2}{2}\epsilon_1$ for all $u \in Z_j$ and $u' \in Z_{j+2}$. Hence either $|\Psi(u)| \geq \frac{c_2}{4}\epsilon_1$ for all $u \in Z_j$ or $|\Psi(u')| \geq \frac{c_2}{4}\epsilon_1$ for all $u' \in Z_{j+2}$.

Assume for example that $|\Psi(u)| \geq \frac{c_2}{4}\epsilon_1$ for all $u \in Z_j$. We will show that $\chi_\mu^{(i)}(u) \leq 1$ for all $u \in Z_j$, $i = 1, 2$. Given $u \in Z_j$, consider the points

$$\rho_1 = e^{(f_N^{(ab)} + \mathbf{i}b\tau_N)(v_1(u))} h(v_1(u)) \quad , \quad \rho_2 = e^{(f_N^{(ab)} + \mathbf{i}b\tau_N)(v_2(u))} h(v_2(u))$$

in the complex plane $\mathbf{C}$. Then $|\mathrm{Arg}(\rho_1) - \mathrm{Arg}(\rho_2)| = |\Psi(u)| \geq \epsilon = \frac{c_2}{4}\epsilon_1$, while the second alternative in Lemma 7.6 (b) (which holds by assumption) and Lemma 7.6(a) imply

$$\frac{|\rho_1|}{|\rho_2|} = e^{f_N^{(ab)}(v_1(u)) - f_N^{(ab)}(v_2(u))} \frac{|h(v_1(u))|}{|h(v_2(u))|} \leq \omega = 8e^{a_0 A_3 N} \leq 8e^{A_3 N} \ ,$$

for some constant $A_3 > 0$ (as in the proof of Lemma 7.4). By symmetry $\frac{|\rho_2|}{|\rho_1|} \leq \omega$. This yields

$$|\rho_1 + \rho_2| \leq \min(\ |\rho_1| + (1-\delta)|\rho_2| \ , \ (1-\delta)|\rho_1| + |\rho_2| \ ) \ , \tag{7.16}$$

where

$$\delta = \frac{2}{\omega} \sin^2 \frac{\epsilon}{4} = \frac{1}{4e^{A_3 N}} \sin^2 \left(\frac{c_2 \epsilon_1}{16} \right) \ .$$

Indeed, assume that $|\rho_1| \leq |\rho_2|$; the other case is similar. Then $\frac{|\rho_2|}{\omega} \leq |\rho_1| \leq |\rho_2|$, and we have to show that $|\rho_1 + \rho_2| \leq |\rho_1| + (1-\delta)|\rho_2|$. Clearly the angle α between the vectors ρ_1 and ρ_2 is $\alpha = |\mathrm{Arg}(\rho_1) - \mathrm{Arg}(\rho_2)| \geq \epsilon$. Consider the triangle with vertices $A = 0$, $B = \rho_1$, $C = \rho_1 + \rho_2$. Then $\angle ABC = \pi - \alpha \leq \pi - \epsilon$. Moreover, $\frac{|BC|}{\omega} \leq |AB| \leq |BC|$, $\omega \geq 1$. Then it follows from elementary geometry (apply the sine theorem twice) that $|AC| \leq |AB| + \left(1 - \frac{2}{\omega}\sin^2\frac{\epsilon}{4}\right)|BC|$. That is,

$$|\rho_1 + \rho_2| = |AC| \leq |AB| + \left(1 - \frac{2}{\omega}\sin^2\frac{\epsilon}{4}\right)|BC| = |\rho_1| + (1-\delta)|\rho_2|.$$

This proves (7.16). Choosing $\mu \leq \frac{\delta}{2} = \frac{1}{\omega}\sin^2\frac{\epsilon}{4}$, i.e.

$$\mu \leq \frac{1}{8e^{A_3 N}} \sin^2 \left(\frac{c_0 \epsilon_1}{16} \right) \ , \tag{7.17}$$

we get $\chi_\mu^{(2)}(u) \leq 1$ for all $u \in Z_j$. In the same way, using $|\rho_1 + \rho_2| \leq (1-\delta)|\rho_1| + |\rho_2|$, one gets $\chi_\mu^{(1)}(u) \leq 1$ for all $u \in Z_j$. This proves the lemma. ■

Proof of Lemma 3.4. Take $E = 4A_0$, and fix a positive integer $N \geq n_0 + n_1 + n_2$ such that

$$\gamma^N \geq \max \left\{ 4A_0 \ , \ 48C_2A_0 \ , \ 16\frac{C_2A_0}{c_2} \right\} \ . \tag{7.18}$$

Take $\epsilon_1 > 0$ with (7.1) and (7.13). Then fix an arbitrary $\mu = \mu(N)$ with

$$0 < \mu \leq \min \left\{ \frac{1}{2} \ , \ \frac{c_0 c_1 \mu_0 \epsilon_1 A_0}{2\gamma_1^N} \ , \ \frac{1}{8e^{A_3 N}} \sin^2 \left(\frac{c_0 \epsilon_1}{16} \right) \right\} \ .$$

Notice that with this choice of E, N, ϵ_1 and μ, the inequalities (7.5), (7.13), (7.15) and (7.17) are satisfied. Finally, define $a_0 = a_0(\mu, N) > 0$ and $\epsilon_2 = \epsilon_2(\mu, N) > 0$ as in Lemma 7.4. Then choose U_0, $z(u)$, $v_i(u)$, etc. as in the proof of Lemma 6.2 and a family $\{\gamma_j\}$ of subcurves of U_0 as in the beginning of Sect. 7.

Next, denote by J the set of those $(i,j) \in \Xi$ such that $\chi_\mu^{(i)}(u) \leq 1$ for all $u \in Z_j$. It follows from Lemma 7.5 (cf. also its proof) that J is dense. Consider the operator $\mathcal{N} = \mathcal{N}_{ab}^{J,\mu} : C^1(U) \longrightarrow C^1(U)$, where $|a| \leq a_0$, and $1/|b| \leq a_0$. Then Lemma 7.2 (a) implies property (a) in Lemma 3.4, while Lemma 7.4 gives property (b) in Lemma 3.4 with $\rho = 1 - \epsilon_2 = \rho(\mu, N)$.

To check (c) in Lemma 3.4, assume that h and H satisfy (7.11). Then Lemma 7.2 (b) implies $\|d_u(L_{ab}^N h)(u)\| \leq E|b|(\mathcal{N}H)(u)$ for all $u \in U$. So, it remains to show that

$$\left|(L_{ab}^N h)(u)\right| \leq (\mathcal{N}H)(u) \quad , \quad u \in U. \tag{7.19}$$

Let $u \in U$ and let $m = m_{\mu,b,J}$. If u does not belong to the *interior* $\mathrm{Int}(Z_j)$ of Z_j for any $(i,j) \in J$, then $m(v) = 1$ whenever $\sigma^N v = u$ (since $v \in X_j^i$ implies $u = \sigma^N v \in Z_j$). Hence

$$\left|(L_{ab}^N h)(u)\right| = \left|\sum_{\sigma^N v = u} e^{(f_N^{(ab)} + \mathbf{i}b\tau_N)(v)} h(v)\right| \leq (\mathcal{M}_{ab}^N(mH))(u) = (\mathcal{N}H)(u).$$

Assume that $u \in \mathrm{Int}(Z_j)$ for some $(i,j) \in J$; then $u \in \mathcal{U}$. Without loss of generality we may assume that $(1,j) \in J$ (it may happen that both $(1,j)$ and $(2,j)$ belong to J). Then $\chi_\mu^{(1)}(u) \leq 1$. Notice that if $\sigma^N v = u$ for some v and $m(v) < 1$, then we must have $v = v_{i'}(u') \in \mathrm{Int}(X_{j'}^{i'})$ for some $(i',j') \in J$. Then $u = \sigma^N v = \sigma^N(v_{i'}(u')) = u'$ shows that $j' = j$ (since $\mathrm{Int}(Z_j) \cap \mathrm{Int}(Z_{j'}) = \emptyset$ for $j \neq j'$). Thus, $\sigma^N v = u$ and $m(v) < 1$ imply $v = v_1(u)$ or $v = v_2(u)$.

Using $\chi_\mu^{(1)}(u) \leq 1$, we get

$$\begin{aligned}
\left|(L_{ab}^N h)(u)\right| &\leq \left|\sum_{\sigma^N v = u,\ v \neq v_1(u), v_2(u)} e^{(f_N^{(ab)} + \mathbf{i}b\tau_N)(v)} h(v)\right| \\
&\quad + \left|e^{(f_N^{(ab)} + \mathbf{i}b\tau_N)(v_1(u))} h(v_1(u)) + e^{(f_N^{(ab)} + \mathbf{i}b\tau_N)(v_2(u))} h(v_2(u))\right| \\
&\leq \sum_{\sigma^N v = u,\ v \neq v_1(u), v_2(u)} e^{f_N^{(ab)}(v)} |h(v)| \\
&\quad + \left[(1 - 2\mu) e^{f_N^{(ab)}(v_1(u))} H(v_1(u)) + e^{f_N^{(ab)}(v_2(u))} H(v_2(u))\right]. \qquad (7.20)
\end{aligned}$$

First, consider the case when $(1,j) \in J$ but $(2,j) \notin J$. Then amongst the v's with $\sigma^N v = u$ there is only one for which $m(v) < 1$; this is $v = v_1(u)$ for which we have $m(v_1(u)) \geq 1 - \mu > 1 - 2\mu$. Thus, (7.20) implies

$$\begin{aligned}
\left|(L_{ab}^N h)(u)\right| &\leq \sum_{\sigma^N v = u,\ v \neq v_1(u), v_2(u)} e^{f_N^{(ab)}(v)} m(v) H(v)| \\
&\quad + \left[e^{f_N^{(ab)}(v_1(u))} m(v_1(u)) H(v_1(u)) + e^{f_N^{(ab)}(v_2(u))} m(v_2(u)) H(v_2(u))\right] = (\mathcal{N}H)(u).
\end{aligned}$$

So, (7.19) holds in this case.

Next, assume that both $(1,j)$ and $(2,j)$ belong to J. Then both inequalities $\chi_\mu^{(1)}(u) \leq 1$ and $\chi_\mu^{(2)}(u) \leq 1$ hold. So, apart from (7.20), we also have

$$\left|(L_{ab}^N h)(u)\right| \leq \sum_{\sigma^N v = u,\ v \neq v_1(u), v_2(u)} e^{f_N^{(ab)}(v)} |h(v)| + \left[e^{f_N^{(ab)}(v_1(u))} H(v_1(u)) + (1-2\mu) e^{f_N^{(ab)}(v_2(u))} H(v_2(u))\right].$$

Taking the average of this and (7.20), one gets $\left|(L_{ab}^N h)(u)\right| \leq (\mathcal{N} H)(u)$. Thus, (7.19) holds in all possible cases. This completes the proof of Lemma 3.4.■

8 Exponential decay of correlations

In this section we very briefly describe the procedure from [D] that uses Corollary 3.3 and leads to the proof of Theorem 1.2. It works in exactly the same way, since it is practically carried out on the symbol space Σ_A. Therefore we omit the details.

Let $F \in \mathcal{F}_\alpha(\Lambda)$ and let $f \in \mathcal{F}_\alpha(U \cap \Lambda)$ be defined by $f(x) = \int_0^{\tau(x)} F(\phi_s(x))\, ds$ (cf. Sect. 2). Given continuous functions A and B on Λ, let

$$\rho_{A,B}(t) = \int_\Lambda A(x) B(\phi_t(x))\, d\nu_F(x)$$

be the corresponding *correlation function*, where as in Theorem 1.2, ν_F is the Gibbs measure on Λ determined by F. One also needs the *Laplace transform* $\hat{\rho}_{A,B}(z) = \int_0^\infty \rho_{A,B}(t) e^{-zt}\, dt$ of $\rho_{A,B}$. Here $z = a + \mathbf{i}b$ is a complex variable. As before, we will denote by ν the Gibbs measure on U (with support in $\Lambda \cap U$) determined by $f - P\tau$.

The first step comprises Propositions 3 and 4 in [D] and is based on previous arguments of Pollicott [Po] and Ruelle [R1].

Step 1. *There exist constants $\epsilon_3 > 0$, $\theta \in (0,1)$, $M > 0$ and $C_3 > 0$ such that for every $z = a + \mathbf{i}b$ with $|a| \leq \epsilon_3$ and $1/|b| \leq \epsilon_3$ and every integer $n \geq 0$ there exist linear operators $Q_n(z), R_n(z) : \mathcal{F}_\alpha(\Lambda) \longrightarrow \mathcal{F}_\alpha(\Lambda \cap U)$ such that:*

(a) $\|Q_n(z)A\|_0 \leq C_3 \theta^n |b|\, \|A\|_\alpha$ *and* $\|R_n(z)A\|_0 \leq C_3 \theta^n |b|\, \|A\|_\alpha$ *for all* $A \in \mathcal{F}_\alpha(\Lambda)$;

(b) $\|Q_n(z)A\|_\alpha \leq C_3 M^n |b|^2\, \|A\|_\alpha$ *and* $\|R_n(z)A\|_\alpha \leq C_3 M^n |b|^2\, \|A\|_\alpha$ *for all* $A \in \mathcal{F}_\alpha(\Lambda)$;

(c) *The function*

$$\frac{1}{\|A\|_0 \|B\|_0} \left[\hat{\rho}_{A,B}(z) - \sum_{i,j=0}^{\infty} \int_{\Lambda \cap U} \left(L_{f-(P+z)\tau}^{i+j} (1 - L_{f-(P+z)\tau})^{-1} [Q_i(z)A]\right) [R_j(z)B]\, d\nu\right]$$

is uniformly bounded.

The next step is given by Corollary 3 (b) in [D] (cf. also Lemma 18 there).

Step 2. *For every $\beta \in (0,\alpha)$ there exists a constant $C_4 > 0$ such that for $z = a + \mathbf{i}b$ with $|a| \leq \epsilon_3$ and $1/|b| \leq \epsilon_3$ we have $\|(1 - L_{f-(P+z)\tau})^{-1} h\|_0 \leq C_4 \ln|b|\, \|h\|_\beta$ for all $h \in \mathcal{F}_\beta(\Lambda \cap U)$.*

Then Corollary 4 in [D] gives

Step 3. *For all $\beta \in (0, \alpha)$ and $A, B \in \mathcal{F}_\beta(\Lambda)$ there exists $\epsilon_3 > 0$ such that $\hat{\rho}_{A,B}(z)$ is analytic in the region $|a| \leq \epsilon_3$, $1/|b| \leq \epsilon_3$. Moreover, $|\hat{\rho}_{A,B}(z)| \leq C_5|b|^2 \ln|b| \; \|A\|_\beta \; \|B\|_\beta$ for some constant $C_5 > 0$.*

As in [D], this yields the exponential decay of the function $\rho_{A,B}(t) - (\int_\Lambda A \, d\nu_F)(\int_\Lambda B \, d\nu_F)$.

Appendix

Sketch of Proof of Lemma 4.2. We will use the following simple fact (cf. e.g. Lemma 3.7 in [I1]).

SUBLEMMA. *Let W' and W'' be the boundaries of two disjoint compact strictly convex domains in $\mathbb{R}^2$. Suppose W' and W'' are C^4-smooth and denote by ν' and ν'', respectively, their outer unit normal fields. Given $x \in W'$, let $x(t)$ $(t \geq 0)$ be the shift of x along the billiard trajectory in the exterior of W'' issued from $(x, \nu'(x))$. Suppose $x, y \in W'$ are such that the trajectories $\{x(t) : t \geq 0\}$ and $\{y(t) : t \geq 0\}$ hit transversally W'' at $x(\tau_1)$ and $y(\tau_2)$, respectively. Denote by k the curvature of W' at x. Assuming that the angle $\varphi(x)$ between the reflected trajectory $\{x(t) : t \geq \tau_1\}$ and the unit normal $\nu''(x(\tau_1))$ is bounded from above by a constant $\varphi_0 \in (0, \pi/2)$, we have*

$$(1 + \tau_1 k)\|x - y\| - c\|x - y\|^2 \leq \|x(\tau_i) - y(\tau_i)\| \leq (1 + \tau_1 k)\|x - y\| + C\|x - y\|^2$$

for $i = 1, 2$, where $c > 0$ and $C > 0$ are constants depending only on the minimum and maximum of the curvatures of W' and W'' and on the constant φ_0. Moreover, choosing c and C appropriately, for $i = 1, 2$ we have $c\|x(\tau_i) - y(\tau_i)\| \leq \|x(\tau_1) - y(\tau_2)\| \leq C\|x(\tau_i) - y(\tau_i)\|$.

We now proceed with the proof of Lemma 4.2. For $s \in [0, 2]$ and $t \geq 0$, denote by $u_t(s)$ the shift of $(q(s), \nu(q(s)))$ along the forward billiard trajectory $\gamma(s)$ after time t and by $v_t(s) \in S^1$ the direction of $\gamma(s)$ at $u_t(s)$. Let $t_0(s) = 0$ and let $t_1(s) < \ldots < t_m(s)$ be the times of the consecutive reflections of the trajectory $\gamma(s)$ at ∂K. Then $t_j(s) = d_0(s) + d_1(s) + \ldots + d_{j-1}(s)$ and $q_j(s) = u_{t_j}(s)$ for $j = 1, \ldots, n$. Set $t_{n+1}(s) = t_n(s) + d_n(s)$. For $t \neq t_j(s)$ $(j = 1, 2 \ldots, n)$, locally near $u_t(s)$, $X_t = \{u_t(s) : s \in [0, 2]\}$ is a smooth convex curve with "outward" normal field $\{v_t(s) : s \in [0, 2]\}$. Denote by $k_t(s)$ the curvature of X_t at $u_t(s)$. It follows from old results in the theory of billiards (cf. e.g. [Si2]; see also the Appendix of [St1]) that for $t = t_j(s)$ for some j we have $k_j(s) = \lim_{t \downarrow t_j(s)} k_t(s)$, where $k_j(s)$ is defined by (4.4). Moreover, for $t_j(s) < t < t_{j+1}(s)$, the curvature of X_t at $u_t(s)$ is $k_t(s) = \dfrac{k_j(s)}{1 + (t - t_j(s))k_j(s)}$.

Fix for a moment an arbitrary $s_0 \in (0, 1) \cup (1, 2)$ and an arbitrary $j = 1, \ldots, n$. Given s close to s_0, assume $t_j(s_0) \geq t_j(s)$ and $t_{j+1}(s_0) \leq t_{j+1}(s)$; the other cases are similar. Applying the Sublemma to $x = u_{t_j(s_0)}(s_0) = q_j(s_0)$ and $y = u_{t_j(s_0)}(s)$ with $W' = X_{t_j(s_0)}$ and $W'' = \partial K$ near $u_{t_{j+1}(s_0)}(s_0) = q_{j+1}(s_0)$, and using the notation $\delta_j = \delta_j(s_0)$ and $\epsilon_j = \|u_{t_j(s_0)}(s) - u_{t_j(s_0)}(s_0)\|$, we get $\dfrac{\epsilon_j}{\delta_j} - \text{const } \epsilon_j^2 \leq \epsilon_{j+1} \leq \dfrac{\epsilon_j}{\delta_j} + \text{const } \epsilon_j^2$, which is equivalent to $\delta_j \epsilon_{j+1}/(1 + \text{const } \delta_j \epsilon_j) \leq \epsilon_j \leq \delta_j \epsilon_{j+1}/(1 - \text{const } \delta_j \epsilon_j)$. Applying these inequalities recursively, it follows that

$$\text{const } \|p(s) - p(s_0)\| \prod_{i=j}^{m} \frac{\delta_i}{1 + \text{const } \delta_i \epsilon_i} \leq \epsilon_j \leq \text{const } \|p(s) - p(s_0)\| \prod_{i=j}^{n} \frac{\delta_i}{1 - \text{const } \delta_i \epsilon_i} \,.$$

It follows from the last sentence in the Sublemma that $c\epsilon_j \leq \|q_j(s) - q_j(s_0)\| \leq C\epsilon_j$. Combining this with the above estimates for ϵ_j, we get

$$\text{const } \|p(s)-p(s_0)\| \prod_{i=j}^{n} \frac{\delta_i}{1 + \text{const } \delta_i \epsilon_i} \leq \|q_j(s)-q_j(s_0)\| \leq \text{const } \|p(s)-p(s_0)\| \prod_{i=j}^{n} \frac{\delta_i}{1 - \text{const } \delta_i \epsilon_i} \,.$$

Dividing all sides of the last row by $s - s_0$ and letting $s \to s_0$, proves the lemma. ∎

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