BRAIDING IN GROUPS AND MAPPING CLASS GROUPS

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Abstract. The series of the mapping class groups $\Gamma_{g,1}$ of a surface of the genus g with 1 component of the boundary gives a natural example of a braiding in groups, as defined by Ross Street.

Ross Street introduced recently an interesting and natural braiding in the category of groups [St]. The purpose of this note is to show that this kind of braiding naturally arises in the mapping class groups. This language allows clearly express the structure firstly considered by E. Miller [Mil] in the mapping class groups. We remind briefly the R. Street's construction and results.

Let $\mathcal{G}p_*$ be the category of pointed groups. Their objects are pairs (G, g_0) where G is a group and g_0 is its element. A morphism

$$\alpha : (G, g_0) \to (H, h_0)$$

is a group homomorphism $\alpha : G \to H$, such that $\alpha(g_0) = h_0$. The usual monoidal category of groups $\mathcal{G}p$ is identified with the full subcategory of $\mathcal{G}p_*$ consisting of the objects (G, g_0) with g_0 is equal to the unit of the groups G. The monoidal (tensor) structure on $\mathcal{G}p_*$ is given by the free product on groups and group product on elements:

$$(G, g_0) \otimes (H, h_0) = (G * H, g_0 h_0).$$

Theorem (R. Street [St]). *The monoidal category $\mathcal{G}p_*$ admits a braiding*

$$c_{G,H} : (G * H, g_0 h_0) \to (H * G, h_0 g_0)$$

which takes $g \in G$ to $h_0 g h_0^{-1}$ and $h \in H$ to $h \in B * H$.

It is possible to consider monoids pointed by invertible elements instead of pointed groups. Then the larger category is obtained.

Mapping class groups are studied widely in present because of their relations with various fields of mathematics: topology, complex analysis, algebraic geometry, group theory. Alternatively these groups are known as the Teichmüller modular groups [I], and also as the homeotopy group of a surface. The general case is the mapping class group $\Gamma_{g,k}^n$ of a surface S_g of the genus g with k components of the boundary and n fixed (as a set) points. Relations between mapping class groups and

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braid groups were studied also intensively (see the survey [Bi4]). We mention the following of them. At first the group $\Gamma_{0,1}^n$ is the classical braid group on n strings. Then any closed surface of the genus g can be obtained as a branched covering space of the sphere with $2g+2$ branching points and braid homeomorphisms from the base space (corresponding to generators of $\Gamma_{0,1}^{2g+2}$) can be lifted to homeomorphisms of surfaces and generate subgroups of the mapping class group of the surface [BH1], [BH2], [Vi]. Thirdly there are connections between the braid groups of surfaces $Br_m(S_g)$, where the number of strings m is equal either to the number of points n or to the number of connected components of the boundary k. This situation was studied by Joan Birman [Bi1, Bi3] and G. P. Scott [Sc]. Other relations are based on presentations of mapping class groups [Bi2]. The standard generators of the mapping class group which correspond to the Dehn twists along the certain curves on a surface satisfy braid relations (among the others) [W]. This gives possibility to construct homomorphisms from braid groups to mapping class groups. A lot of information about the mapping class groups is contained also in the surveys [Co], [Mis].

The mapping class group $\Gamma_{g,k}^n$ of a surface of the genus g with k components of the boundary and n fixed points is defined geometrically as follows. Let $Diff_+(S_g;k;n)$ be the group of orientation preserving diffeomorphisms of a surface S_g of the genus g which fix pointwise the k components of the boundary and have n points fixed as a set. We denote by $Diff_+^0(S_g;k;n)$ the connected component of the identity diffeomorphism. Then the mapping class group $\Gamma_{g,k}^n$ is defined as the factor group

$$Diff_+(S_g;k;n)/Diff_+^0(S_g;k;n) = \pi_0(Diff_+(S_g;k;n)).$$

There are equivalent definitions of the mapping class groups in terms of homeomorphisms and homotopy equivalences. Let $Homeo(S_g;k;n)$ be the group of homeomorphisms of a surface S_g of the genus g which fix pointwise the k components of the boundary and have n points fixed as a set. A map $f \in Homeo(S_g;k;n)$ is said to be orientation preserving, if the induced homomorphism in two-dimensional homology

$$H(f) : H_2(S_g, \partial S_g; \mathbb{Z}) \to H_2(S_g, \partial S_g; \mathbb{Z})$$

is the identity map of the group $H_2(S_g, \partial S_g; \mathbb{Z}) \cong \mathbb{Z}$. We denote by $Homeo_+(S_g;k;n)$ the subgroup of orientation preserving homeomorphisms from $Homeo(S_g;k;n)$ and by $Homeo_+^0(S_g;k;n)$ the connected component of the identity homeomorphism. Then the mapping class group $\Gamma_{g,k}^n$ is isomorphic to the factor group

$$Homeo_+(S_g;k;n)/Homeo_+^0(S_g;k;n) = \pi_0(Homeo_+(S_g;k;n)).$$

We omit writing zero in the notation $\Gamma_{g,k}^n$, if n or k is equal to zero. The group Γ_g acts on the Teichmüller space T^{3g-3} properly and discontinuously. The quotient of this action is the moduli space of smooth algebraic curves of the genus g.

Let $F(S_g, n)$ be the space of n-tuples of pairwise different points in S_g:

$$F(S_g, n) = \{p_1, ..., p_n) \in S_g^n : p_i \neq p_j \text{ for } i \neq j\}.$$

If we fix n different points on a surface S_g then the evaluation map

$$Homeo_+(S_g; 0; 0) \to F(S_g, n)$$

is defined. It was proved by Joan Birman [Bi1, Bi3], that this map is a locally trivial fibering with the fiber equal to $Homeo_+(S_g; 0; n)$. The exactness of the following sequence is a corollary of this fact:

$$... \to \pi_1(F(S_g, n)) \to \Gamma_g^n \to \Gamma_g \to e. \tag{1}$$

In the case when $n = 1$, $g \geq 2$, the following short exact sequence

$$e \to \pi_1(S_g) \to \Gamma_g^1 \to \Gamma_g \to e$$

is obtained from (1). We have also the exact sequence

$$e \to \mathbb{Z} \to \Gamma_{g,1} \to \Gamma_g^1 \to e, \tag{2}$$

where the generator of $\mathbb{Z}$ corresponds to the Dehn twist along the boundary of the disk. Using the construction of "pair of pants" one can obtain an inclusion

$$\Gamma_{g,1} \to \Gamma_{g+1,1} \tag{3}$$

and a pairing

$$\mu_{g,q} : \Gamma_{g,1} \times \Gamma_{q,1} \to \Gamma_{g+q,1}. \tag{4}$$

Using the inclusions (3) one defines the *stable* mapping class group

$$\Gamma_{\infty,1} = \cup_g \Gamma_{g,1}.$$

The mapping class group Γ_g^n of a surface of the genus g with n points deleted may be interpreted algebraically as follows. Let $\Pi(g, n)$ be the group with generators $a_1, b_1, ..., a_g, b_g, u_1, ..., u_n$ and one defining relation

$$u_1 u_2 ... u_n a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} = 1.$$

We denote by $A(g, n)$ the group of those automorphisms of the group $\Pi(g, n)$ which map each generator u_i into its conjugate or to the conjugate of another such generator u_j. Let $I(g, n)$ be the group of inner automorphisms of the group $\Pi(g, n)$. Then $I(g, n)$ is the normal subgroup of $A(g, n)$ and there is an isomorphism (see, for example, [MKS, p. 175]):

$$\Gamma_g^n \cong A(g, n)/I(g, n). \tag{5}$$

Suppose now that $n = 1$. Then we interpret the group $A(g, n)$ as the group of automorphisms of the free group F_{2g} with generators $a_1, b_1, ..., a_g, b_g$, such that the elements of $A(g, n)$ map the commutator $a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}$ into its conjugate. Then from the formulas (2) and (5) one can identify the group $\Gamma_{g,1}$ with the group of automorphisms ϕ of the free group F_{2g} for which the commutator is invariant

$$(a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1})\phi = a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1}.$$

The generator of the group $\mathbb{Z}$ by the map from (2) corresponds to an element in $A(g,n)$, which is conjugation by the commutator $a_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1}$.

The canonical imbeddings

$$\mathrm{Aut}\, F_{2g} \times \mathrm{Aut}\, F_{2q} \to \mathrm{Aut}\, F_{2(g+q)}$$

induce the pairing for the system of groups $\Gamma_{g,1}$:

$$\mu_{g,q} : \Gamma_{g,1} \times \Gamma_{q,1} \to \Gamma_{g+q,1},$$

in such a way that the following diagram

$$\begin{array}{ccc}
\Gamma_{g,1} \times \Gamma_{q,1} & \xrightarrow{\mu_{g,q,}} & \Gamma_{g+q,1} \\
\downarrow & & \downarrow \\
\mathrm{Aut}\, F_{2g} \times \mathrm{Aut}\, F_{2q} & \longrightarrow & \mathrm{Aut}\, F_{2(g+q)}
\end{array}$$

is commutative. This pairing is associative in a usual way:

$$\begin{array}{ccc}
\Gamma_{g,1} \times \Gamma_{q,1} \times \Gamma_{l,1} & \xrightarrow{Id\times\mu_{q,l}} & \Gamma_{g,1} \times \Gamma_{q+l,1} \\
\downarrow \mu_{g,q} \times Id & & \downarrow \mu_{g,q+l} \\
\Gamma_{g+q,1} \times \Gamma_{l,1} & \xrightarrow{\mu_{g+q,l}} & \Gamma_{g+q+l}.
\end{array}$$

It coincides with the pairing (4).

These data define in a standard way a strict monoidal category $\mathcal{GM}$ whose objects are free groups F_{2g}, $g = 0, 1...$, and morphisms are defined by the formula:

$$hom_{\mathcal{GM}}(F_{2k}, F_{2l}) = \begin{cases} \Gamma_{k,1}, & \text{if} \quad k = l, \\ \emptyset, & \text{if} \quad k \neq l. \end{cases}$$

We suppose that $\Gamma_{0,1} = \{e\}$.

Our considerations can be summarized by the following

Proposition 1. *The monoidal category $\mathcal{GM}$ naturally imbeds into the monoidal category of pointed groups $\mathcal{G}p_*$ by the functor which maps F_{2k} to the pair*

$$(F_{2k}, a_1b_1a_1^{-1}b_1^{-1}...a_kb_ka_k^{-1}b_k^{-1})$$

and imbeds $\Gamma_{k,1}$ into the subgroup of automorphisms ϕ of the free group F_{2k} for which the commutator is invariant

$$(a_1b_1a_1^{-1}b_1^{-1}...a_kb_ka_k^{-1}b_k^{-1})\phi = a_1b_1a_1^{-1}b_1^{-1}...a_kb_ka_k^{-1}b_k^{-1}.$$

The braiding in $\mathcal{G}p_*$ defines a braiding in $\mathcal{GM}$, so that the later becomes a braided monoidal category.

The classical representation

$$\rho_g : \Gamma_{g,1} \to Sp_{2g}(\mathbb{Z})$$

geometrically is defined by the action of a homeomorphism or homotopy equivalence on the first homology group of the underlying surface. In the interpretation that we use the representation ρ_g is obtained from the canonical epimorphism

$$F_{2g} \to A_{2g},$$

where A_{2g} is a free abelian group of the rank $2g$. We denote by $\mathcal{S}p$ the analogous to $\mathcal{GM}$ category defined with the help of the modular groups. Representations ρ_g define a morphism of monoidal categories:

$$P : \mathcal{GM} \to \mathcal{S}p.$$

It follows from [JS] that the braid category $\mathcal{B}$ is the free braided monoidal category generated by the single object $\bar{1}$. In the category $\mathcal{B}$ objects $\{\bar{0}, ..., \bar{k}, ...\}$ correspond to integer numbers from zero to infinity and morphisms are defined by the formula:

$$hom_{\mathcal{B}}(\bar{k}, \bar{l}) = \begin{cases} Br_k, & \text{if} \quad k = l, \\ \emptyset, & \text{if} \quad k \neq l, \end{cases}$$

where Br_k is the braid group on k strings. The monoidal structure is defined by addition of numbers and addition of braids.

Let us consider the free group F_2 on the generators a and b as an object of $\mathcal{G}p_*$ by choosing the commutator $aba^{-1}b^{-1}$ to be the distinguished point. The freeness of $\mathcal{B}$ allows to define a strong monoidal functor $R : \mathcal{B} \to \mathcal{G}p_*$ unique up to isomorphism with $R(\bar{1}) = F_2$. It follows that there is a cannonical isomorphism $R(\bar{k}) \cong F_{2k}$, where the distinguished point in F_{2k} is the product $a_1 b_1 a_1^{-1} b_1^{-1} ... a_k b_k a_k^{-1} b_k^{-1}$.

Proposition 2. *The functor*

$$R : \mathcal{B} \to \mathcal{G}p_*$$

is faithful.

The proof is the same as in [V].

Remark. These consideration are completely analogous to those from the Proposition 2 in [St].

The category $\mathcal{S}p$ can also be considered as a subcategory of $\mathcal{G}p_*$ where the fixed elements of the groups A_{2k} are the units. Then the functor $P : \mathcal{GM} \to \mathcal{S}p$ respects the distinguished elements in groups.

If the fixed element $g_0 \in G$ is equal to the unit, then the symmetry as in the Theorem of R. Street defines a permutative category structure on $\mathcal{G}p$. Let us consider its subcategory $\mathcal{A}$ consisting of the free groups F_n as objects (one for each n) and morphisms defined by the formula:

$$hom_{\mathcal{A}}(F_k, F_l) = \begin{cases} \mathrm{Aut}\, F_k, & \text{if} \quad k = l, \\ \emptyset, & \text{if} \quad k \neq l. \end{cases}$$

Then it will be a symmetric category and analogously to the Proposition 2 there exists a functor from the category Σ defined by symmetric groups to $\mathcal{A}$:

$$\Sigma \to \mathcal{A}$$

which is faithful. This expresses the fact that there is a canonical inclusion of the symmetric group in the group of automorphisms of the free group.

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