# The Shafarevich conjecture for hyper–Kählerian manifolds

Yves André

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

Institut Henri Poincaré
11, rue Pierre et Marie Curie
F–75231 Paris Cedex 05

France

MPI/91-8

# The Shafarevich conjecture for hyper–Kählerian manifolds

Yves André

1. Introduction. Simply connected projective complex manifolds with trivial canonical class may be considered as the natural higher dimensional generalizations of $K_3$ surfaces. Any such manifold is a finite product of (simply connected) Calabi–Yau manifolds and hyperkählerian manifolds (Bogomolov–Calabi), the latter class being characterize) by the existence of a unique (up to constant) holomorphic two–form, which is non–degenerate at every point.

In a recent paper [6], A. Todorov studies the arithmetic structure of moduli spaces for each of these classes of manifolds, and proposes a number of conjectures, among them the analog of the Shafarevich conjecture. The present note settles the hyperkählerian case.

Theorem 1. Let R be an integral finitely generated $\mathbb{Z}$–algebra, with fraction field k. For any positive integers N, d, there exist only finitely many isomorphy classes of hyperkählerian varieties Y of dimension N defined over K, endowed with the numerical equivalence class of a very ample divisor of degree d, such that Y has good reduction at all primes of R of height one.

We shall prove this by reduction to the Shafarevich conjecture for abelian varieties (solved by G. Faltings [3]). The deduction uses P. Deligne's technique of big monodromy groups [1] [2], applied to a suitable version of the Kuga–Satake construction (see § 4 below). Here the main technical point is:

Theorem 2. Let $(Y,\eta)$ be a polarized hyperkählerian variety defined over some subfield k of $\mathbb{C}$, and assume that for some integer $n > 2$, the Galois module $H^2_{3t}(Y_{\bar{k}}, \mathbb{Z}/n\mathbb{Z})(1)$ is trivial. Then the Kuga–Satake variety attached to $(Y,\eta)$ is defined over k.

We conclude the proof using Todorov's deep results about the Torelli mapping [5], which generalize the work of I. Piatetski–Shapiro and I. Shafarevich on $K_3$ surfaces. In order to make the exposition clearer, we shall have to recall a substantial amount of known results; new material first occurs in § 8.

The paper was written down during a stay at the Max–Planck–Institut für Mathematik, Bonn, under support of the A.–v.–Humboldt–Stiftung. The author thank both institutions for excellent working conditions, and is grateful to A. Todorov for motivating discussions and introduction to the subject of Calabi–Yau and hyperkählerian manifolds.

2. Polarization. Let $(Y,\eta)$ be a polarized complex hyperkählerian manifold of (necessarily even) dimension $N \geq 2$ and degree d; here $\eta \in NS(Y) \subset H^2(Y,\mathbb{Z})(1)$ is the class of some ample line bundle on Y. The lattice $H^2(Y,\mathbb{Z})(1) \simeq \mathbb{Z}^{b_2}$ carries a Hodge structure of type $(-1,1) + (0,0) + (1,-1)$ with $h^{1,-1} = 1$. Let us consider the scalar product $\langle x,y\rangle = -x \wedge y \wedge \underbrace{\eta \wedge \ldots \wedge \eta}_{N-2 \text{ factors}} \in H^{2N}(Y,\mathbb{Z})(N) \simeq \mathbb{Z}$. One has $\langle \eta,y\rangle = d$, and $\langle\ \rangle$ induces polarization on the orthogonal complement $P^2(Y,\eta,\mathbb{Z})(1)$ of $\eta$ inside $H^2(Y,\mathbb{Z})(1)$, i.e. a non–degenerate bilinear form on

$P^2(Y,\eta,\mathbb{Z})(1) \otimes_{\mathbb{Z}} \mathbb{R}$ , positive on the $(0,0)$–component and negative on the $(-1,1)+(1,-1)$–component (which is a plane).

3. Torelli mapping. Let $V_{\mathbb{Z}} = (\mathbb{Z}^{b_2-1}, < >)$ be a non–degenerate quadratic module of signature $((b_2-3)+, 2-)$ , and let us write $V$ for $V \otimes_{\mathbb{Z}} \mathbb{Q}$ . The Hodge structures of type $(-1,1) + (0,0) + (1,-1)$ on $\mathbb{Z}^{b_2-1}$ polarized by $< >$ , with $h^{1,-1} = 1$ , are parametrized by $\Omega^{\pm} := SO(2, b_2-3)/_{SO(2)\times SO(b_2-3)}$ , which is a sum of two copies of a hermitian symmetric domain. Given $(Y,\eta)$ as before, and an isomorphism $\gamma : (P^2(Y,\eta,\mathbb{Z})(1), < >) \xrightarrow{\sim} V_{\mathbb{Z}}$ , one thus attaches a point in $\Omega^{\pm}$ . A weak version of the main result in [5] II, which will suffice here, states that this mapping $((Y,\eta),\gamma) \longrightarrow$ point in $\Omega^{\pm}$ (the so–called Torelli mapping) has finite fibers.

From this it follows that the induced mapping $(Y,\eta) \longrightarrow$ point in $\Omega^{\pm}/_{SO(V_{\mathbb{Z}})}$ also has finite fibers. We note that the stabilizer $\Gamma \subset SO(V_{\mathbb{Z}})$ of each component of $\Omega^{\pm}$ has index 2 (in fact $SO(V_{\mathbb{Z}})$ is a semi–direct product of $\mathbb{Z}/_{2\mathbb{Z}}$ and $\Gamma$ ), and the quotient $\Omega^{+}/_{\Gamma} \simeq \Omega^{-}/_{\Gamma}$ is a connected algebraic variety.

4. The Kuga–Satake construction applies to any polarized Hodge structure as before on $V_{\mathbb{Z}}$ , see e.g. [1] 4. Let us briefly describe it. The morphism $h : \prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow SO(V_{\mathbb{R}})$ describing the Hodge decomposition on $\mathbb{C}^{b_2-1}$ lifts naturally to a morphism $\tilde{h} : \prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}}$ , where $G$ denotes the Clifford group $C\,Spin\,V$ . Via the action of $G$ on the even Clifford algebra $C^+(V_{\mathbb{R}})$ by left translations, this gives a polarizable Hodge structure $C^+(V)_{sin}$ of type $(1,0) + (0,1)$ on $C^+(V)$ .

Let us denote by $\tilde{\Gamma}$ the preimage of $\Gamma$ in G relative to the exact sequence $0 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow SO(V) \longrightarrow 0$ ; one has a (non–split) exact sequence $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma \longrightarrow 0$ . Let $C^+_{\mathbb{Z}}$ be any lattice in $C^+(V)$ stable under the action of $\tilde{\Gamma}$ . The equality of Hodge structures

$$(*) \qquad C^+_{\mathbb{Z},\sin} = H^1(A,\mathbb{Z})$$

defines an abelian manifold $A = A(Y,\eta,C^+_{\mathbb{Z}})$ up to isomorphism, called "the" Kuga–Satake variety of $(Y,\eta)$ . Moreover the self–action of $C^+(V)$ by right translations respects the Hodge structure, so that A has complex multiplication by $C^+(V)$ . One then has a canonical isomorphism of rational Hodge structures

$$(**) \qquad \overset{\text{even}}{\Lambda} P(Y,\eta,\mathbb{Q})(1) \simeq \operatorname{End}_{C^+} H^1(A,\mathbb{Q}) , \quad \text{see [1] 3.3.}$$

In particular $P^2(Y,\eta,\mathbb{Q})(1)$ occurs as a factor of the Hodge structure $\operatorname{Hom}(\operatorname{End}_{C^+} H^1(A,\mathbb{Q}), \mathbb{Q}(0))$ , since $b_2$ is even.

5. Families. The Kuga–Satake construction also applies in a relative context: let S be a connected algebraic complex manifold, $f : \underline{Y} \longrightarrow S$ a flat morphism whose fibers are hyperkählerian manifolds, and $\underline{\eta}$ a polarization of f , i.e. a section of $\underline{NS}\ Y/S \subset R^2 f_*^{(an)}\mathbb{Z}(1)$ which is a polarization of $Y_s = f^{-1}(s)$ for every $s \in S$ . We denote by $P^2 f_*\mathbb{Z}(1)$ the orthogonal complement of $\underline{\eta}$ inside $R^2 f_*\mathbb{Z}(1)$ , and by $V_{\mathbb{Z}}$ the constant quadratic module obtained from $(P^2 f_*\mathbb{Z}(1), < >)$ by pull–back to the universal covering $\tilde{S}$ of S . The equality of variations of Hodge structure

$$\underline{C}^{+}_{\mathbb{Z}\ \sin} = R^1\tilde{g}_* \mathbb{Z}$$

first defines an analytic family $\tilde{g}$ of abelian manifolds on $\tilde{S}$, whose fibers are the Kuga–Satage varieties attached to the corresponding fibers of $\tilde{f} = f \times_S \tilde{S}$.

But the fact that there is no splitting of the exact sequence $0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \tilde{\Gamma} \longrightarrow \Gamma \longrightarrow 0$ prevents us from descending $\tilde{g}$ to $S$ in general. (Note that the problem disappears if one replaces $\tilde{g}$ by the "Kummer family" $\tilde{g}/\{\pm \mathrm{id}\}$ ). However <u>if</u> for some $n > 2$, $R^2f_* \mathbb{Z}/n\mathbb{Z}$ is a constant local system on $S$, then $\tilde{g}$ descends to an abelian scheme $g : A \longrightarrow S$ with complex multiplication by $C^+(V)$, the <u>Kuga–Satake family</u> attached to $f$. Indeed let $\Gamma_n$ (resp. $\tilde{\Gamma}_n$ ) be the principal congruence subgroup of $\Gamma$ (resp. $\tilde{\Gamma}$ ) of level $n$. It is easily checked that the map $\tilde{\Gamma}_n \longrightarrow \Gamma_n$ induced by $G \longrightarrow SO(V)$ is an isomorphism, moreover, our assumption about $R^2f_* \mathbb{Z}/n\mathbb{Z}$ implies that the Torelli mapping $S \longrightarrow \Omega^{\pm}/\Gamma$ factorizes through the smooth quasi–projective variety $\Omega^{\pm}/\Gamma_n = \Omega^{\pm}/\tilde{\Gamma}_n$, and one can argue as in [1] 5.7.

6. <u>Hilbert schemes</u>. By the theory of Chow coordinates or bounded sheaves, one knows that the scheme which parametrizes hyperkählerian varieties $Y$ of dimension $N$ endowed with a <u>very ample</u> divisor of degree $d$ is an open subscheme of a finite disjoint union of suitable Hilbert schemes, hence is quasiprojective. Moreover it follows from the smoothness of the Kuranishi families that the geometric connected components $S_{(j)}$ are <u>smooth</u> ( [5] 2.5.2).

We now fix a point $s \in S_{(j)}$, and drop the subscript $j$. Because the Torelli mapping $S \longrightarrow \Omega^{\pm}/\Gamma$ is <u>dominant</u> ( [5] 2.5.5), the monodromy group of the universal flat family of hyperkählerian varieties $f : \underline{Y} \longrightarrow S$ at $s$ has finite index in $\Gamma$. Define the Galois cover $S_n \longrightarrow S$ via the kernel of the map $\pi_1(S,s) \longrightarrow \mathrm{Aut}\, H^2(\underline{Y}_s, \mathbb{Z}/n\mathbb{Z})(1)$, so that the local system $R^2 f_* \mathbb{Z}/n\mathbb{Z}$ becomes constant on $S_n$; it follows that the monodromy group of the associated Kuga–Satake family $g_n$ is Zariski–dense in Spin V (with the notations of § 5).

7. The Shimura variety attached to the data $(G,\Omega^{\pm})$ is the complex pro–algebraic variety with complex points

$$Sh(G,\Omega^{\pm})(\mathbb{C}) = G(\mathbb{Q})\backslash\Omega^{\pm} \times G(A^f) \ ,$$

where $A^f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. Let $(t_\alpha)$ be a family of tensors for $C^+V$ such that $G$ is the subgroup of $GL(C^+V) \times \mathbb{G}_m$ fixing the $t_\alpha$ (the second projection $G \longrightarrow \mathbb{G}_m$ being the inverse of the Spin norm). We assume for convenience that this collection of tensors includes a basis of $C^+V$ as endomorphism of $C^+V$ by translation on the right. It turns out that $Sh(G,\Omega^{\pm})$ is a fine moduli scheme for triples $(A,(s_\alpha),\gamma)$ up to "isogeny", where $A$ is a complex abelian variety, $s_\alpha$ are Hodge cycles on $A$, and $\gamma$ is an isomorphism $H^1_{et}(A,A^f) \xrightarrow{\sim} C^+(V) \otimes_{\mathbb{Q}} A^f$ mapping each $s_\alpha$ to $t_\alpha$, satisfying the following condition:
(∗∗∗) there exists an isomorphism $i : H^1_B(A,\mathbb{Q}) \longrightarrow C^+V$ mapping each $s_\alpha$ to $t_\alpha$, such that $i^{-1} \circ h \circ i \in \Omega^{\pm}$ (notation $h$ from § 3), see [4] II 3.11 for more details.

The choice of a lattice inside $C^+V \otimes A^f$, for instance in the form $C^+_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$, fixes the universal abelian scheme $\underline{A} \longrightarrow Sh(g,\Omega^{\pm})$ inside the isogeny class. Let $\tilde{\tilde{S}}$ be the pro–

jective limit of commutative diagrams $\begin{array}{ccc} s & \longrightarrow & S' \\ & \searrow & \downarrow \\ & & S \end{array}$ with $S'$ étale finite over $S$ ; the profinite group $\pi_1^{et}(S,s)$ acts on $\tilde{\tilde{S}}$ , with quotient $S$ . The same construction applied to $S_n$ provides the same proalgebraic variety: $\tilde{\tilde{S}}_n \simeq \tilde{\tilde{S}}$ .

By the Kuga–Satake construction and the modular property of $Sh(g,\Omega^{\pm})$ , we get a morphism

$$\chi : \tilde{\tilde{S}} \longrightarrow Sh(G,\Omega^{\pm})$$

and the pull–back of the universal abelian scheme is a Kuga–Satake gamily $\tilde{\tilde{g}}$ . Furthermore $\chi$ passes to the quotient $S_n$ to give a morphism

$$\chi_n : S_n \longrightarrow Sh_{K_n}(G,\Omega^{\pm}) := Sh(G,\Omega^{\pm})/_{K_n} , \quad n > 2 ,$$

where $K_n$ denotes the preimage of the principal congruence subgroup of $SO(V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}})$ of level $n$ inside $G(\mathbf{A}^f)$ (which is isomorphic to its image into $SO(V \otimes_{\mathbb{Q}} \mathbf{A}^f)$ and is a torsion–free congruence subgroup of $G(\mathbf{A}^f)$ ).

8. Descent. It is known that $Sh(G,\Omega^{\pm})$ admits a canonical model over the reflex field $E(G,\Omega^{\pm})$ , see e.g. [4] .

Lemma 1: $E(G,\Omega^{\pm}) = \mathbb{Q}$ .

Proof: One has $E(G,\Omega^{\pm}) \subset E(\tilde{T},x)$ for any special point $x$ with associated rational torus $\tilde{T}$. We construct a special point in the following way: let us choose an orthogonal decomposition $V_{\mathbb{Q}} = V^{+} \perp V^{-}$ where $V^{+}$ (resp. $V^{-}$) is a positive (resp. negative) quadratic subspace. By the inertia theorem, $V^{-}$ has dimension 2, and thus may be identified with the quadratic space defined by the opposite of the norm $N$ on some imaginary quadratic extension $E$ of $\mathbb{Q}$. Let the rational torus $T = \mathrm{Ker}\, N$ act on $V$ trivially upon $V^{+}$ and by homotheties upon $V^{-} \simeq (E,-N)$, so that $T \subset SO(V)$; let $\tilde{T}$ denote the preimage of $T$ inside $G$. The natural lifting $\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow \tilde{T}_{\mathbb{R}}$ of the obvious projection $\prod_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow T_{\mathbb{R}} \simeq U(1,\mathbb{R})$ defines a special point $x$ for which $E(\tilde{T},x) = E$. Furthermore it is plain to change $E$ by moving the subspace $V^{-}$, so that $E(G,\Omega^{\pm}) \neq E$. The lemma follows.

The universal triple $(\underline{A},(s_{\alpha}),\gamma)$ descends to the canonical model ${}_{k}Sh(G,\Omega^{\pm})$ over any subfield $k$ of $\mathbb{C}$ ($s_{\alpha}$ descends to an absolute Hodge cycle). In particular, one obtains an abelian scheme ${}_{k}g : {}_{k}\underline{A} \longrightarrow {}_{k}Sh(G,\Omega^{\pm})$.

Similarly the representation $C^{+}V$ (by left translations) defines a $A^{f}$–sheaf $C^{+}V(A^{f})$ on ${}_{k}Sh(G,\Omega^{\pm})$, see e.g. [4] III 6. In fact, using the lattice $C^{+}_{\hat{\mathbb{Z}}}$, one obtains a $\hat{\mathbb{Z}}$–sheaf $\underline{C}^{+}_{\hat{\mathbb{Z}}} \subset C^{+}V(A^{f})$, together with an isomorphism ${}_{k}\gamma : R^{1}{}_{k}g_{*} \xrightarrow{\sim} \underline{C}^{+}_{\hat{\mathbb{Z}}}$.

Furthermore all these objects pass through the quotient

$${}_{k}Sh_{K_n}(G,\Omega^{\pm}) := {}_{k}Sh(G,\Omega^{\pm})/K_n .$$

9. Further descent. Let us assume that the Hilbert point $s$ is defined over $k \subset \mathbb{C}$. The geometric connected component containing $s$, that is to say $S$, is then defined over $k$;

let us write $S = {}_kS \otimes_k \mathbb{C}$ , where ${}_kS$ denotes a k–component of the open Hilbert scheme introduced in § 6. One has an exact sequence

$0 \longrightarrow \pi_1^{et}(S,s) \longrightarrow \pi_1^{et}({}_kS,s) \overset{s}{\longrightarrow} \mathrm{Gal}(\bar{k}/k) \longrightarrow 0$ . Let us now <u>assume</u> that the $\mathrm{Gal}(\bar{k}/k)$–module $H^2_{et}(\underline{Y}_s,\mathbb{Z}/n\mathbb{Z})(1)$ is trivial. Define the Galois cover ${}_kS_n \longrightarrow {}_kS$ via the kernel of the map $\pi_1^{et}({}_kS,s) \longrightarrow \mathrm{Aut}\, H^2_{et}(\underline{Y}_s,\mathbb{Z}/n\mathbb{Z})(1)$ ; with the notation of § 6, one has ${}_kS_n \otimes_k \mathbb{C} = S_n$ .

<u>Lemma</u> 2. <u>The morphism</u> $\chi_n : S_n \longrightarrow Sh_{K_n}(G,\Omega^{\pm})$ <u>descends to a morphism</u> ${}_k\chi_n : {}_kS_n \longrightarrow {}_kSh_{K_n}(G,\Omega^{\pm})$ .

<u>Proof</u>: By a standard argument od descent (see [2] 2.3 for details), it is enough to show that $\chi$ is the <u>unique</u> equivariant morphism $\stackrel{\approx}{S} \longrightarrow Sh(G,\Omega^{\pm})$ which is equivariant with respect to the map $\pi_1^{et}(S,s) \longrightarrow K_n$ induced by the original $\chi_n$ (this property being invarinat under $\mathrm{Aut}(\mathbb{C}/k)$ ). As explained before, such an equivariant morphism is equivalent to a triple consisting in an abelian scheme $g : \underline{A} \longrightarrow \stackrel{\approx}{S}$ , a collection of horizontal Hodge cycles $(s_\alpha)$ on $\underline{A}$ including a basis for $C := C^+(V) \cap \mathrm{End}\, \underline{C}^+_{\hat{\mathbb{Z}}}$ (acting on the right on $\underline{C}^+_{\hat{\mathbb{Z}}}$ ) and satisfying a certain condition (***), together with a $\pi_1^{et}(S,s)$–equivariant isomorphism of $\hat{\mathbb{Z}}$–sheaves $\gamma : R^1g_*\hat{\mathbb{Z}} \xrightarrow{\sim} \underline{C}^+_{\hat{\mathbb{Z}}}$ mapping each $s_\alpha$ on $t_\alpha$ (and in particular commuting with the action of $C$ on the right). Because such a triple has no non–trivial automorphism, the unicity of $\mu$ follows from the following statement

(****) if $g_1$ and $g_2$ are two abelian schemes over $\stackrel{\approx}{S}$ such that there are isomorphisms of $\hat{\mathbb{Z}}[\pi_1^{et}(S,s)]$–C–bimodules $R^1g_{1*}\hat{\mathbb{Z}} \xrightarrow{\hat{\mu}} R^1g_{2*}\hat{\mathbb{Z}} \xrightarrow{\sim} \underline{C}^+_{\hat{\mathbb{Z}}}$ , then

$g_1 \simeq g_2$ . Let us now prove (****).

For any prime $\ell$ , the set of bimodule–isomorphisms $R^1g_{1*}\mathbb{Q}_\ell \longrightarrow R^1g_2\mathbb{Q}_\ell$ (resp. $R^1g_{1*}\mathbb{Q} \longrightarrow R^1g_{2*}\mathbb{Q}$) up to constant is a non–empty open subset $U_\ell$ (resp. $U$ ) of a projective space over $\mathbb{Q}_\ell$ (resp. $\mathbb{Q}$ ); indeed it contains (the multiples of) $\hat{u} \otimes 1_{\mathbb{Q}_\ell}$ (resp. it is dense in $U_\ell$ ). We want to show that $U_\ell$ (hence $U$ ) is reduced to one point; this will follow from the absolute irreducibility of the bimodule $\underline{C}^+_{\mathbb{Q}_\ell}$ . Indeed, the image of $\pi_1^{et}(S_n,s)$ in $\mathrm{Aut}(\underline{C}^+_{\mathbb{Q}_\ell})_s$ is Zariski–dense in the Spin group, and $\overline{\mathbb{Q}}_\ell\,[\mathrm{Spin}\ V] = C^+(V_{\overline{\mathbb{Q}}_\ell})$ because $\dim V$ is odd. Now $C^+(V_{\overline{\mathbb{Q}}_\ell})$ is isomorphic to End W , where W stands for the spin representation over $\overline{\mathbb{Q}}_\ell$ , and the described irreducibility reduces to the irreducibility of End W as an End W–End W–bimodule, which is obvious.

Since $U_\ell$ and $U$ are reduced to one point, one can normalize $\hat{u}$ so that $\hat{u} \otimes_{\hat{\mathbb{Z}}} 1_{A^f} = u \otimes_{\mathbb{Q}} 1_{A^f}$ for some $u \in U$ . Then u induces an isomorphism of local systems $R^1g_{1*}\mathbb{Z} \xrightarrow{\sim} R^1g_{2*}\mathbb{Z}$ , unique up to sign, thus respecting the Hodge structure, hence coming from an isomorphism of abelian schemes. This proves (****), and the lemma.

10. Proof of theorem 2. Let $(Y,\eta)$ be a polarized hyperkählerian variety of dimension N over a field $k \subset \mathbb{C}$ satisfying the assumption in theorem 2. Let $\delta.\eta$ be a very ample multiple of $\eta$ . To the Jata $(Y,\delta_\eta)$ , one attaches its Hilbert point $s_1 \in {}_kS(k)$ , and a suitable geometric point $s \in {}_kS_n(\bar{k})$ lying above $s_1$ .

Lemma 3. The point $s$ comes from a rational point $s_n \in {}_kS_n(k)$ .

Proof: From the exact sequence

$$0 \longrightarrow \pi_1^{et}(S,\bar{s}_1) \longrightarrow \pi_1^{et}({}_kS,\bar{s}_1) \xrightarrow[\hookleftarrow]{\bar{s}_1} \mathrm{Gal}(\bar{k}/k) \longrightarrow 0$$

and the assumption that $\bar{s}_1(\mathrm{Gal}(\bar{k}/k))$ acts trivially on $(R^2_{et}f_*\mathbb{Z}/n\mathbb{Z}(1))_{\bar{s}_1} \simeq H^2_{et}(Y_{\bar{k}},\mathbb{Z}/n\mathbb{Z})(1)$ , one deduces a split exact sequence

$$0 \longrightarrow \pi_1^{et}(S_n,s) \longrightarrow \pi_1^{et}({}_kS_n,s) \xrightarrow{\hookleftarrow - - -} \mathrm{Gal}(\bar{k}/k) \longrightarrow 0$$

, the splitting being given by $s$ ; this means that the decomposition group of $s$ in $\pi_1^{et}({}_kS_n,s)$ projects isomorphically onto the full Galois group $\mathrm{Gal}(\bar{k}/k)$ , hence $s$ is rational over $k$ .

Remark. More generally, any k–family of polarized hyperkählerian varieties $f : \underline{Y} \longrightarrow T$ of the right dimension and degree, such that $R^2_{et}f_* \mathbb{Z}/n\mathbb{Z}\,(1)$ is a constant torsion sheaf, gives rises to a morphism $T \longrightarrow {}_kS_n$ such that $f$ is the pull–back of the standard hyperkählerian family over ${}_kS_n$ .

Indead, let $t$ be a geometric point of $T$ (which we assume to be connected), and let $\tilde{\tilde{T}}$ (resp. ${}_k\tilde{\tilde{S}}$ ) be the projective limit of commutative diagrams

$$\begin{array}{ccc} t & \longrightarrow & T' \\ & \searrow & \downarrow \\ & & T \end{array}$$

(resp.

$$\begin{array}{ccc} t & \longrightarrow & S' \\ & \searrow & \downarrow \\ & & {}_kS \end{array}$$

) with $T'$ , resp. $S'$ étale finite over $T$ , resp. ${}_kS$ . There is a commutative diagram with exact rows

$$\begin{array}{l} 0 \longrightarrow \pi_1^{et}(T,t) \xrightarrow{\sim} \pi_1^{et}(T,t) \searrow \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Aut}(R^e_{et} f_* \mathbb{Z}/n\mathbb{Z})(1)_t \\ 0 \longrightarrow \pi_1^{et}({}_kS_n,t) \longrightarrow \pi_1^{et}({}_kS,t) \nearrow \end{array}$$

which shows that the Hilbert map $T \longrightarrow {}_kS$ induces a map of étale fundamental groups $\pi_1^{et}(T,t) \longrightarrow \pi_1^{et}({}_kS_n,t)$.

Therefore the universal covering $\overset{\approx}{T} \longrightarrow {}_k\overset{\approx}{S}$ of the Hilbert map passes to the quotient $\overset{\approx}{T}/\pi_1^{et}(T,t) \longrightarrow {}_k\overset{\approx}{S}/\pi_1^{et}({}_kS_n,t)$, i.e. furnishes a lifting $T \longrightarrow {}_kS_n$ of the Hilbert map. (We refer to [4] II 10 for details about k–schemes with a continuous action of a locally profinite group).

Applying the previous two lemmata, one obtains a composed morphism

$$\mathrm{Spec}\, k \xrightarrow{s_n} {}_kS_n \xrightarrow{{}_k\chi_n} {}_k\mathrm{Sh}_{K_n}(G,\Omega^{\pm}) \ .$$

Pulling back the standard abelian scheme on ${}_k\mathrm{Sh}_{K_n}(G,\Omega^{\pm})$ (attached to a suitable lattice $C^+_{\mathbb{Z}}$ ) gives a k–model of the Kuga–Satake variety $A(Y, \delta.\eta, C^+_{\mathbb{Z}})$, with $C^+_{\mathbb{Z}} \otimes \mathbb{Q} = C^+(V,< >_{\delta_\eta})$.

It remains to remark that $< >_{\delta_\eta} = (\delta^{\frac{N}{2}-1})^2 < >_\eta$, so that the publication by $\delta^{\frac{N}{2}-1}$ provides an isomorphism $C^+(V,< >_{\delta_\eta}) \simeq C^+(V,< >_\eta)$. If we still denote by

$C^+_{\mathbb{Z}}$ the image of $C^+_{\mathbb{Z}}$ under this isomorphism, we obtain that the Kuga–Satake variety $A(Y,\eta,C^+_{\mathbb{Z}})$ is defined over $k$, in conformity with theorem 2.

11. Good reduction. In fact, the k–model $A$ of the Kuga–Satake variety just constructed enjoys a nice extra property: the pull–back of ${}_k\gamma$ by ${}_k\chi_n \circ s_n$ is an isomorphism of $\mathrm{Gal}(\bar{k}/k)$–modules $H^1_{et}(A_{\bar{k}},\hat{\mathbb{Z}}) \xrightarrow{\sim} s_n^* \, {}_k\chi_n^* \, \underline{C}^+_{\hat{\mathbb{Z}}}$.

Lemma 4. In the situation where $H^2_{et}(Y_{\bar{k}}, \mathbb{Z}/n\mathbb{Z})(1)$ is a trivial Galois module for some $n > 2$ and $Y$ has good reduction at a prime $p$ of $\bar{k}$ which does not divide $n$, $A$ has good reduction at $p$.

Proof: Replacing $n$ by a factor, one may assume that $n$ is prime. The torsion sheaf $\underline{C}^+_{\hat{\mathbb{Z}}} \otimes \mathbb{Z}/n\mathbb{Z}$ is constant, so that the Galois module $H^1_{et}(A_{\bar{k}},\mathbb{Z}/n\mathbb{Z})$ is trivial. Therefore the n–torsion points of $A_{\bar{k}}$ are rational over $k$, and by the theory of semi–stable reduction, the action of the inertia group $I$ at $p$ is unipotent on $H^1_{et}(A_{\bar{k}},\mathbb{Z}_n)$. On the other side $I$ acts trivially on $H^2_{et}(Y_{\bar{k}},\mathbb{Z}_n)(1)$ because $Y$ has good reduction at $p$, hence $I$ acts trivially on the even Clifford algebra $C^+(P^2_{et}(Y_{\bar{k}},\mathbb{Z}_n)(1), < >_\eta \otimes 1_{\mathbb{Z}_\ell})$, which is isomorphic to the Galois module of all endomorphisms of $H^1_{et}(A_{\bar{k}},\mathbb{Z}_n)$ which commute to the complex multiplication $C$ (see [1] 6.6 for more details); therefore $I$ acts on $H^1_{et}(A_{\bar{k}},\mathbb{Z}_n)$ through the center of $C$. At last, we find that $I$ acts trivially on $H^1_{et}(A_{\bar{k}},\mathbb{Z}_n)$, it follows that $A$ has good reduction at $p$.

12. Proof of theorem 1. We turn back to the notations and assumptions of theorem 1. Let us first remark that by localization, we may assume that $R$ is a regular ring; we

choose a prime number $n > 2$, and we also assume that $n$ is invertible in $R$. Since $Y$ has good reduction at all primes of $R$ of height one, it then follows from the purity of the branch–locus that the representation of $Gal(\bar{K}/K)$ on $H^2_{et}(Y_{\bar{K}},\mathbb{Z}_n)(1)$ factors through $\pi_1(\mathrm{Spec}\, R)$.

On the other side, since we fixed the pair $(N,d)$, there may occur only finitely many quadratic lattices $V_{\mathbb{Z},(j)} = (\mathbb{Z}^{b_2-1},< >)$ (notation of § 2); this follows from the Hilbert scheme argument of § 6. According to Hermite–Minkowski, there exists only finitely many continuous homomorphisms $\pi_1(\mathrm{Spec}\, R) \longrightarrow \prod O(V_{\mathbb{Z},(j)} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z})$. Denoting by $r$ the (unramified) finite extension of $R$ determined by the intersection of the kernels of these homomorphisms, and by $k$ its fraction field, we have proved the following

<u>Lemma 5</u>. <u>There exists a finite extension</u> $r$ <u>of</u> $R$, <u>depending only on the pair</u> $(N,d)$, <u>such that for any</u> $(Y,\eta)$ <u>as in theorem 1, the</u> $Gal(\bar{k}/k)$–<u>module</u> $H^2_{et}(Y_{\bar{k}},\mathbb{Z}/n\mathbb{Z})(1)$ <u>is trivial</u>.

Using lemma 4, we see that the representation of $Gal(\bar{k}/k)$ on $H^1_{et}(A_{\bar{k}},\mathbb{Z}_n)$ factors through $\pi_1(\mathrm{Spec}\, r)$, i.e. $A$ has good reduction at any prime $p$ of the integral closure of $r$ in $\bar{k}$. By Falting's theorem [3], there are only <u>finitely many</u> such abelian varities $A$.

Note that the complex abelian manifolds $A_{\mathbb{C}}$ which occur are described by a Hodge structure $C^+_{\mathbb{Z},\sin}$ (formula (*)) on a previously chosen lattice $C^+_{\mathbb{Z},(j)}$ inside $C^+(V_{(j)})$. Using formula (**), we know that there are only finitely many rational Hodge structures $P^2(Y,\eta,\mathbb{Q})(1)$ on $V_{(j)}$ which are the image under the Torelli mapping, tensored with $\mathbb{Q}$, of polarized hyperkähler varieties satisfying the assumptions of theorem 1; this leaves only finitely many integral polarized Hodge structures

$P^2(Y,\eta,\mathbb{Z})(1)$ on $V_{\mathbb{Z},(j)}$. Because the Torelli mapping has finite fibres, we see that there are only finitely many possibilities for $(Y \otimes_K \mathbb{C}, \eta \otimes_K \mathbb{C})$. By Galois descent, K–forms of $(Y \otimes_K \mathbb{C}, \eta \otimes_K \mathbb{C})$ are described by the set $H^1(\mathrm{Gal}(\bar{k}/k), \underline{\mathrm{Aut}}(Y,\eta))$, which is finite (like $\underline{\mathrm{Aut}}(Y,\eta)$ ). We conclude that there are only finitely many isomorphy classes of hyperkählerian varieties Y of dimension N defined over K, endowed with the numerical class of a very ample divisor of degree d, such taht Y has good reduction at all primes of R of height one.

## References

[1] Deligne P., La conjecture de Weil pour les surfaces $K_3$, Inventiones Math. 15, 206–226 (1972).

[2] Deligne P., Les intersections complètes de niveau de Hodge un, Inventiones Math. 15, 237–250 (1972).

[3] Faltings G., Wüstholz G., Rational points, Aspects of Math. EG, Vieweg Verlag Wiesbaden, 2nd edition 1986.

[4] Milne J., Canonical models of (mixed) Shimura varieties and automorphic vector bundles, preprint (part of a forthcoming book).

[5] Todorov A., Moduli for hyper–Kählerian manifolds, I, II; preprints Max–Planck–Institut für Mathematik, Bonn, 1990.

[6] Todorov A., Arithmetic height on the moduli space of Calabi–Yau manifolds; preprint Max–Planck–Institut für Mathematik, Bonn, 1990.