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MINIMAL INVERSION COMPLETE SETS AND MAXIMAL ABELIAN IDEALS

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ABSTRACT. Let $\mathfrak g$ be a simple Lie algebra, $\mathfrak b$ a fixed Borel subalgebra, and W the Weyl group of $\mathfrak g$. In this note, we study a relationship between the maximal abelian ideals of $\mathfrak b$ and the minimal inversion complete sets of W. The latter have been recently defined by Malvenuto et al. (*J. Algebra*, **424** (2015), 330–356.)

INTRODUCTION

Recently, Malvenuto et al. [9] introduced the *minimal inversion complete sets* (MICS for short) in finite Coxeter groups and determined the maximum cardinality of a MICS for the classical series of Weyl groups W and $\mathbf{G}_2$. They also gave a lower bound on the maximum cardinality of MICS for the other exceptional Weyl groups. It is noticed in [9] that in the simply-laced case this maximum cardinality is related to the maximum dimension of abelian ideals of a Borel subalgebra $\mathfrak b$ of a simple Lie algebra $\mathfrak g$ with Weyl group W. In this article, we elaborate on the relationship between maximal abelian ideals of $\mathfrak b$ and MICS in W. In the simply-laced case, we give a uniform construction of a MICS for **any** maximal abelian ideal of $\mathfrak b$. We also determine the essential set of roots for some MICS obtained. (See definitions below.)

Let Δ be an irreducible crystallographic root system in a real Euclidean vector space V and $W \subset GL(V)$ the corresponding finite reflection group. Let Δ^+ be a set of positive roots, Π the set of simple roots in Δ^+, and θ the highest root. We regard Δ^+ as a poset with respect to the usual order "$\succcurlyeq$". If $w \in W$, then $\mathcal N(w) = \{\gamma \in \Delta^+ \mid w(\gamma) \in -\Delta^+\}$ is the *inversion set* of w.

Definition 1 ([9]). A subset $\mathcal F$ of W is a *minimal inversion complete set* (= MICS), if $\bigcup_{w\in\mathcal F} \mathcal N(w) = \Delta^+$ and the equality fails for any proper subset of $\mathcal F$.

Definition 2 ([9]). A root $\gamma \in \Delta^+$ is said to be *essential* for a given MICS $\mathcal F$, if there is a unique $w \in \mathcal F$ such that $\gamma \in \mathcal N(w)$.

Write $\mathsf{Ess}(\mathcal F)$ for the set of all essential roots. By Definition 1, each $\mathcal N(w)$, $w \in \mathcal F$, contains at least one essential root. Hence $\#\mathsf{Ess}(\mathcal F) \geqslant \#\mathcal F$. Picking just one essential root in every $\mathcal N(w)$ yields a subset of $\mathsf{Ess}(\mathcal F)$ that plays an important role in [9]. However, the whole set

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$\mathsf{Ess}(\mathcal{F})$ is of interest, too. We introduce the *defect* of $\mathcal{F}$ as $\mathrm{defect}(\mathcal{F}) := \#\mathsf{Ess}(\mathcal{F}) - \#\mathcal{F}$ and consider it as another measure of "goodness" of $\mathcal{F}$. That is, for us, the best MICS are those with large cardinality or small defect. If $w_0 \in W$ is the longest element, then $\mathcal{F} = \{w_0\}$ is a MICS and $\mathsf{Ess}(\{w_0\}) = \Delta^+$. Hence this MICS is very bad, since the cardinality is small and defect is large.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}^+$. Here $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}^+$ is a fixed Borel subalgebra, Δ is the set of $\mathfrak{t}$-roots in $\mathfrak{g}$, and Δ^+ is the subset corresponding to $\mathfrak{u}^+$. It is well known and easily seen that if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{b}$, then $\mathfrak{a} \subset \mathfrak{u}^+$ and hence $\mathfrak{a} = \bigoplus_{\gamma \in I(\mathfrak{a})} \mathfrak{g}_\gamma$ for certain subset $I(\mathfrak{a}) \subset \Delta^+$. The subsets of the form $I(\mathfrak{a})$ are characterised by the following two properties:

- If $\gamma \in I(\mathfrak{a})$, $\nu \in \Delta^+$, and $\gamma + \nu \in \Delta$, then $\gamma + \nu \in I(\mathfrak{a})$; [$\mathfrak{a}$ is $\mathfrak{b}$-stable]
- If $\gamma, \gamma' \in I(\mathfrak{a})$, then $\gamma + \gamma' \notin \Delta$. [$\mathfrak{a}$ is abelian]

Theorem 0.1. *If all the roots of Δ have the same length (the* simply-laced *or* ADE *case), then there is a natural MICS associated with* **any** *maximal abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$. For any $\gamma \in I(\mathfrak{a})$, we define the canonical element $\tilde{w}_\gamma \in W$ such that $\mathcal{F}_\mathfrak{a} = \{\tilde{w}_\gamma \mid \gamma \in I(\mathfrak{a})\}$ is a MICS. In particular, $\#(\mathcal{F}_\mathfrak{a}) = \dim \mathfrak{a} = \#I(\mathfrak{a})$. Furtermore, $\mathsf{Ess}(\mathcal{F}_\mathfrak{a}) \supset I(\mathfrak{a})$.*

We provide a full description of $\mathsf{Ess}(\mathcal{F}_\mathfrak{a})$ for two classes of maximal abelian ideals $\mathfrak{a}$. Let $\mathfrak{p} \supset \mathfrak{b}$ be a parabolic subalgebra with abelian nilradical. Then $\mathfrak{p} = \mathfrak{p}_\alpha$ is a maximal parabolic subalgebra that is determined by one simple root α, and its nilradical $\mathfrak{n}_\alpha$ is a maximal abelian ideal of $\mathfrak{b}$.

Theorem 0.2. *Suppose that Δ is simply-laced and $\mathfrak{n}_\alpha$ is an abelian nilradical such that $(\alpha, \theta) = 0$. Then $\mathsf{Ess}(\mathcal{F}_{\mathfrak{n}_\alpha}) = I(\mathfrak{n}_\alpha)$.*

If Δ is of type $\mathbf{A}_n$, then $(\alpha, \theta) \neq 0$ for some abelian nilradicals and $\mathsf{Ess}(\mathcal{F}_{\mathfrak{n}_\alpha}) = \Delta^+$ in those cases, see Example 2.5.

In general, the maximal abelian ideals are naturally parameterised by the long simple roots [10, Corollary 3.8]. (We say more on this correspondence in Section 1.) In particular, in the ADE case, there is a bijection between Π and the maximal abelian ideals. Suppose that θ is a fundamental weight (in the simply-laced case, this means that Δ is of type $\mathbf{D}_n$ or $\mathbf{E}_n$). Write $\hat{\alpha}$ for the unique simple root such that $(\theta, \hat{\alpha}) \neq 0$. The corresponding maximal abelian ideal $\hat{\mathfrak{a}} := \mathfrak{a}_{\hat{\alpha}}$ has the property that $I(\hat{\mathfrak{a}}) \subset \mathcal{H} := \{\gamma \in \Delta^+ \mid (\gamma, \theta) > 0\}$ [10].

Theorem 0.3. *For the above maximal abelian ideal $\hat{\mathfrak{a}}$, we have $\mathsf{Ess}(\mathcal{F}_{\hat{\mathfrak{a}}}) = \mathcal{H}$.*

Explicit computations for $\mathbf{D}_n$ and $\mathbf{E}_n$ ($n \leqslant 6$) suggest that it might be true that if θ is fundamental, then $\mathsf{Ess}(\mathcal{F}_\mathfrak{a}) \subset I(\mathfrak{a}) \cup \mathcal{H}$.

The general theory of abelian ideals (of $\mathfrak{b}$) is based on a relationship with the so-called *minuscule elements* of the affine Weyl group $\widehat{W}$ (the Kostant-Peterson theory, see [8] and also [3]). Proofs in this article heavily rely on some further results obtained in [10, 11]. For instance, the construction of $\mathcal{F}_{\mathfrak{a}}$ (in the ADE case!) exploits the simple root α corresponding to the maximal abelian ideal $\mathfrak{a} = \mathfrak{a}_\alpha$. Furthermore, if $\gamma \in I(\mathfrak{a}_\alpha)$, then $\gamma \succcurlyeq \alpha$ [11, Theorem 3.5], and since $\|\gamma\| = \|\alpha\|$, we are able to introduce the element of minimal length in W taking γ to α, which is denoted by $w_{\gamma,\alpha}$. This is the crucial step in constructing the elements $\tilde{w}_\gamma$ occurring in Theorem 0.1. Actually, the existence of the elements of minimal length $w_{\gamma,\mu}$ is proved for any pair of positive roots such that $\|\gamma\| = \|\mu\|$ and $\gamma \succcurlyeq \mu$, see Prop. 1.2 and Remark 1.3(1).

The article is organised as follows. In Section 1, we provide preliminaries on abelian ideals and elements $w_{\gamma,\mu} \in W$ associated with a pair of positive roots such that $\|\gamma\| = \|\mu\|$ and $\gamma \succcurlyeq \mu$. Theorem 0.1 is proved in Section 2. Section 3 contains some preparatory properties of long roots that are needed in Section 4, where we prove Theorems 0.2 and 0.3. In Section 5, we discuss some conjectures on the essential set of $\mathcal{F}_\alpha$ in the cases that are not covered by Theorems 0.2 and 0.3.

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1. Maximal abelian ideals and simple roots

In what follows, we identify the abelian ideals of $\mathfrak{b}$ with the corresponding sets of roots. Consequently, in place of the dimension of $\mathfrak{a} \subset \mathfrak{u}^+$, we deal with the cardinality of $I(\mathfrak{a}) \subset \Delta^+$, etc. It is proved in [10] that there is a one-to-one correspondence between the maximal abelian ideals and the long simple roots in Δ^+. As our subsequent results on MICS heavily rely on that correspondence, we recall the necessary setup.

Some Notation. We refer to [1],[7] for basic results on root systems and Weyl groups. Write $(\ ,\ )$ for the W-invariant scalar product in V and Δ^+_l (resp. Δ^+_s) for the set of long (resp. short) positive roots. In the ADE case, all roots are assumed to be both long and short.

– $\gamma^\vee = 2\gamma/(\gamma,\gamma)$ and $\sigma_\gamma \in W$ is the reflection with respect to $\gamma \in \Delta$. If $\Pi = \{\alpha_1, \dots, \alpha_n\}$, then we also write σ_i in place of σ_{α_i}.
– $\rho = \frac{1}{2}\sum_{\gamma\in\Delta^+}\gamma$, $\rho^\vee = \frac{1}{2}\sum_{\gamma\in\Delta^+}\gamma^\vee$, and θ is the highest root in Δ^+;
– $\ell(\cdot)$ is the length function on W with respect to $\{\sigma_\alpha \mid \alpha \in \Pi\}$;
– $\mathcal{H} = \{\gamma \in \Delta^+ \mid (\gamma,\theta) \ne 0\} = \{\theta\} \cup \{\gamma \in \Delta \mid (\gamma,\theta^\vee) = 1\}$;
– If $\gamma = \sum_{\alpha\in\Pi} c_\alpha\alpha$, then $\mathsf{ht}(\gamma) = \sum_{\alpha\in\Pi} c_\alpha$.
– The Coxeter number of Δ is $h = h(\Delta) := \mathsf{ht}(\theta) + 1$.

1.1. Let $\mathfrak{Ab}$ (resp. $\overset{o}{\mathfrak{Ab}}$) denote the set of all (resp. nonzero) abelian ideals of $\mathfrak{b}$. We regard $\mathfrak{Ab}$ as a poset with respect to inclusion.

1^o. There is a natural surjective map $\tau : \overset{o}{\mathfrak{Ab}} \to \Delta_l^+$, and each fibre $\tau^{-1}(\mu) =: \mathfrak{Ab}_\mu$ ($\mu \in \Delta_l^+$) contains a unique maximal and a unique minimal ideal [10, Theorem 3.1]. Write $I(\mu)_{max}$ and $I(\mu)_{min}$ for these extreme elements of $\mathfrak{Ab}_\mu$. Say that $I(\mu)_{min}$ (resp. $I(\mu)_{max}$) is the *μ-minimal* (resp. *μ-maximal*) *ideal*.

2^o. The μ-minimal ideals admit the following characterisation:

For $I \in \overset{o}{\mathfrak{Ab}}$, we have $I = I(\mu)_{min}$ for some $\mu \in \Delta_l^+$ if and only if $I \subset \mathcal{H}$ [10, Theorem 4.3]. Furthermore, all other ideals $I \in \mathfrak{Ab}_\mu$ have the property that $I \cap \mathcal{H} = I(\mu)_{min}$ [11, Prop. 3.2]. In particular, $I(\mu)_{max}$ is maximal among all abelian ideals having the prescribed intersection with $\mathcal{H}$. It is also known that $I(\mu)_{min} = I(\mu)_{max}$ if and only if $\mu \in \mathcal{H}$ (i.e., $(\mu, \theta) \neq 0$), see [10, Theorem 5.1].

3^o. For any $\mu \in \Delta_l^+$, W contains a unique element of minimal length taking θ to μ [10, Theorem 4.1]. This element is denoted by w_μ in [10] and here we write $w_{\theta,\mu}$ for it. Its inverse has the following description:

$$\mathcal{N}(w_{\theta,\mu}^{-1}) = \{\nu \in \Delta^+ \mid (\nu, \mu^\vee) = -1\}. \tag{1·1}$$

The μ-minimal ideal can be constructed using $w_{\theta,\mu}$, see [10, Theorem 4.2]. In particular, $\# I(\mu)_{min} = \ell(w_{\theta,\mu}) + 1 = (\rho, \theta^\vee - \mu^\vee) + 1$. See also Lemma 1.1 below.

4^o. If $\mu = \alpha \in \Delta_l^+ \cap \Pi =: \Pi_l$, then $I(\alpha)_{max}$ is not only the maximal element of $\mathfrak{Ab}_\alpha$, but is also a (globally) maximal abelian ideal, and this yields the above-mentioned bijection.

The following is a useful complement to the theory of μ-minimal ideals.

Lemma 1.1. *For any $\mu \in \Delta_l^+$, we have (1) $\mathcal{N}(w_{\theta,\mu}) \subset \mathcal{H} \setminus \{\theta\}$ and*

$$(2) \quad I(\mu)_{min} = \{\theta\} \cup \{\theta - \gamma \mid \gamma \in \mathcal{N}(w_{\theta,\mu})\}.$$

Proof. (1) If $w_{\theta,\mu}(\gamma) = -\nu \in -\Delta^+$, then $\nu \in \mathcal{N}(w_{\theta,\mu}^{-1})$. Therefore $(\nu, \mu^\vee) = -1$ and hence $(-\gamma, \theta^\vee) = -1$. Thus, $\gamma \in \mathcal{H} \setminus \{\theta\}$.

(2) The argument requires an explicit use of the affine root system and affine Weyl group. We give a brief sketch of notation referring to [10, 1.1] for a full account.

Let $\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ be the affine root system and $\widehat{\Pi} = \Pi \cup \{\delta - \theta\}$ the set of affine simple roots. Here $\widehat{\Delta}$ is a subset of the vector space $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, where $(V, \lambda) = (V, \delta) = (\lambda, \delta) = 0$ and $(\delta, \lambda) = 1$. Let σ_0 denote the reflection (in $\widehat{V}$) with respect to $\alpha_0 = \delta - \theta$. The affine Weyl group $\widehat{W}$ is a subgroup of $GL(\widehat{V})$ generated by σ_α, $(\alpha \in \Pi)$ and σ_0. Then $w_{\theta,\mu}\sigma_0 \in \widehat{W}$ is minuscule and its inversion set determines the abelian ideal $I(\mu)_{min}$ [10, Theorem 4.2]. Namely,

$$\gamma \in I(\mu)_{min} \ \text{ if and only if } \ \delta - \gamma \in \mathcal{N}(w_{\theta,\mu}\sigma_0) \tag{1·2}$$

We have $\mathcal{N}(w_{\theta,\mu}\sigma_0) = \{\alpha_0\} \cup \sigma_0(\mathcal{N}(w_{\theta,\mu}))$ and $\mathcal{N}(w_{\theta,\mu}) \subset \mathcal{H} \setminus \{\theta\}$. Therefore $\sigma_0(\nu) = \nu - (\nu, \alpha_0^\vee) = \nu + \alpha_0 = \delta - (\theta - \nu)$ for $\nu \in \mathcal{N}(w_{\theta,\mu})$. Thus,

$$\mathcal{N}(w_{\theta,\mu}\sigma_0) = \{\delta - \theta\} \cup \{\delta - (\theta - \nu) \mid \nu \in \mathcal{N}(w_{\theta,\mu})\}$$

and comparing with Eq. (1·2), we conclude the proof. □

1.2. We regard Δ^+ as a poset with respect to the usual order "$\succcurlyeq$". This means that $\gamma \succcurlyeq \mu$ if and only if $\gamma - \mu$ is a linear combination of simple roots with *nonnegative* coefficients. If $\Xi \subset \Delta^+$, then $\min(\Xi)$ (resp. $\max(\Xi)$) is the set of minimal (resp. maximal) elements of Ξ with respect to "$\succcurlyeq$".

Let Π_l (resp. Π_s) denote the set of long (resp. short) simple roots. We also need the ratio $r = \|\text{long}\|^2/\|\text{short}\|^2$. In the simply-laced case $\Pi = \Pi_l = \Pi_s$ and $r = 1$. If $\gamma = \sum_{\alpha\in\Pi} c_\alpha\alpha \in \Delta_l^+$, then r divides c_α whenever $\alpha \in \Pi_s$ and

$$(\rho, \gamma^\vee) = \sum_{\alpha\in\Pi_l} c_\alpha + \frac{1}{r}\sum_{\alpha\in\Pi_s} c_\alpha. \tag{1·3}$$

Proposition 1.2. *Suppose that $\gamma, \mu \in \Delta_l^+$ and $\gamma \succcurlyeq \mu$. Then there is a unique $w = w_{\gamma,\mu} \in W$ of minimal length such that $w_{\gamma,\mu}(\gamma) = \mu$. In this case, $\ell(w_{\gamma,\mu}) = (\rho, \gamma^\vee) - (\rho, \mu^\vee)$.*

Proof. (1) If $\gamma = \theta$ is the highest root, which is always long, then the required properties of $w_{\theta,\mu}$ are proved in [10, Section 4]. Therefore, we can use below $w_{\theta,\gamma}$ and $w_{\theta,\mu}$.

(2) Since $\|\gamma\| = \|\mu\|$, there is $w \in W$ such that $w(\gamma) = \mu$. For $\alpha \in \Pi$, one computes that $(\rho, (\sigma_\alpha(\gamma))^\vee) - (\rho, \gamma^\vee) = (\alpha, \gamma^\vee)$. Since $(\alpha, \gamma^\vee) \leqslant 1$ if $\gamma \neq \alpha$, one needs at least $(\rho, \gamma^\vee - \mu^\vee)$ steps (simple reflections) in order to reach μ from γ. That is, $\ell(w) \geqslant (\rho, \gamma^\vee - \mu^\vee)$.

Arguing by induction on $m = (\rho, \gamma^\vee - \mu^\vee)$, we shall prove that such an element of length $(\rho, \gamma^\vee - \mu^\vee)$ does exist.

- if $m = 1$, then it follows from Eq. (1·3) that γ and μ differ only in a unique coordinate, say $c_{\tilde\alpha}$, and then $\mu = \sigma_{\tilde\alpha}(\gamma)$.
- Assume that $m > 1$ and $\gamma - \mu = \sum_{\alpha\in\Pi} d_\alpha\alpha$ with $d_\alpha \geqslant 0$. Then $(\gamma - \mu, \alpha) > 0$ for some $\alpha \in \Pi$ and hence $d_\alpha > 0$. Note that r divides d_α whenever α is short.

(a) If $(\gamma, \alpha) > 0$, then $(\rho, (\sigma_\alpha(\gamma)^\vee)) = (\rho, \gamma^\vee) - 1$ and $\gamma \succ \sigma_\alpha(\gamma) \succcurlyeq \mu$.

(b) If $(\mu, \alpha) < 0$, then $(\rho, (\sigma_\alpha(\mu)^\vee)) = (\rho, \mu^\vee) + 1$ and $\gamma \succcurlyeq \sigma_\alpha(\mu) \succ \mu$.
This yields the induction step and existence of w with $\ell(w) = (\rho, \gamma^\vee - \mu^\vee)$.

(3) If $w(\gamma) = \mu$ and $\ell(w) = (\rho, \gamma^\vee - \mu^\vee)$, then $ww_{\theta,\gamma}$ takes θ to μ and $\ell(ww_{\theta,\gamma}) \leqslant \ell(w) + \ell(w_{\theta,\gamma}) = (\rho, \theta^\vee - \mu^\vee)$. Therefore, $ww_{\theta,\gamma} = w_{\theta,\mu}$ and $w = w_{\theta,\mu}w_{\theta,\gamma}^{-1}$ is the unique element of minimal length taking γ to μ. Note also that $\ell(w_{\theta,\mu}) = \ell(w_{\gamma,\mu}) + \ell(w_{\theta,\gamma})$. □

Remark 1.3. 1) Using the passage to the dual root system, one can get a version of Proposition 1.2 for $\gamma,\mu\in\Delta^+_s$. Here $\ell(w_{\gamma,\mu})=(\rho^\vee,\theta-\mu)=\mathsf{ht}(\gamma)-\mathsf{ht}(\mu)$.

2) For our construction of MICS, we only need this Proposition with $\mu\in\Pi_l$.

3) The inversion set $\mathcal N(w_{\theta,\mu}^{-1})$ has an explicit description, see (1·1). In general, we have $w_{\theta,\mu}^{-1}=w_{\theta,\gamma}^{-1}w_{\gamma,\mu}^{-1}$ and $\ell(w_{\theta,\mu}^{-1})=\ell(w_{\theta,\gamma}^{-1})+\ell(w_{\gamma,\mu}^{-1})$. Therefore, $\mathcal N(w_{\gamma,\mu}^{-1})\subset\mathcal N(w_{\theta,\mu}^{-1})$. However, there is no such a description for $\mathcal N(w_{\theta,\mu})$.

Let $w_{\gamma,\mu}=\sigma_{i_m}\cdots\sigma_{i_1}$ be a reduced expression. Here $(\alpha_{i_k},\sigma_{i_{k-1}}\cdots\sigma_{i_1}(\gamma))>0$ for $k=1,2,\dots,m$ and $m=(\rho,\gamma^\vee-\mu^\vee)$. The root sequence

$$\gamma_0=\gamma,\ \gamma_1=\sigma_{i_1}(\gamma_0),\ \gamma_2=\sigma_{i_2}(\gamma_1),\ \dots,\ \gamma_m=\sigma_{i_m}(\gamma_{m-1})=\mu$$

is a *path of minimal length* connecting γ and μ. It follows from Proposition 1.2 that all paths of minimal length yield one and the same element of W, i.e., provide different reduced expressions for $w_{\gamma,\mu}$. If $\alpha_{i_k}\in\Pi_l$, then $\gamma_k=\gamma_{k-1}-\alpha_{i_k}$; and if $\alpha_{i_k}\in\Pi_s$, then $\gamma_k=\gamma_{k-1}-r\alpha_{i_k}$. (Recall that γ is supposed to be long.) Therefore, the multiplicities of simple reflections occurring in a reduced expression for $w_{\gamma,\mu}$ are fully determined by $\gamma-\mu$. That is, if $\gamma-\mu=\sum_{\alpha\in\Pi_l}a_\alpha\alpha+\sum_{\beta\in\Pi_s}b_\beta\beta$, then each σ_α ($\alpha\in\Pi_l$) occurs a_α times and σ_β ($\beta\in\Pi_s$) occurs b_β/r times in a reduced expression for $w_{\gamma,\mu}$.

In the ADE case, we have $r=1$ and $\gamma_k-\gamma_{k-1}$ is always a simple root. Therefore, γ_k covers γ_{k-1} and $(\gamma_0,\ \gamma_1,\dots,\gamma_m)$ is a true path in the Hasse diagram of $(\Delta^+,\succcurlyeq)$. Since $r=1$, we also have $(\rho,\gamma^\vee)=\mathsf{ht}(\gamma)$ and $\ell(w_{\gamma,\mu})=\mathsf{ht}(\gamma)-\mathsf{ht}(\mu)$.

Following C.K. Fan [5], we give the following

Definition 3. An element $w\in W$ is said to be *commutative*, if no reduced expression for w contains a substring $\sigma_\alpha\sigma_\beta\sigma_\alpha$, where α and β are adjacent simple roots.

Every reduced word for $w\in W$ can be obtained from any other by applying a sequence of braid relations [1, Ch. 4, §1.5]. Since we have noticed above that all reduced expressions for $w_{\gamma,\mu}$ have the same distribution of multiplicities of simple reflections, the elements of the form $w_{\gamma,\mu}$ are commutative in the simply-laced case. Below, we provide a more elementary proof of this assertion.

Lemma 1.4. *Suppose that Δ is simply-laced. Then for all $\gamma,\mu\in\Delta^+$ such that $\gamma\succcurlyeq\mu$, the element $w_{\gamma,\mu}$ is commutative.*

Proof. Assume not, and let $\sigma_\alpha\sigma_\beta\sigma_\alpha$ be a substring of a reduced expression for $w_{\gamma,\mu}$, with adjacent $\alpha,\beta\in\Pi$. Then a path of minimal length connecting γ and μ contains a sub-path

$$\dots,\ \nu,\ \sigma_\alpha(\nu)=\nu-\alpha,\ \sigma_\beta(\nu-\alpha)=\nu-\alpha-\beta,\ \nu-2\alpha-\beta,\ \dots$$

Here $(\alpha,\beta)=-1$, $(\nu,\alpha)=1$, and $(\nu-\alpha,\beta)=1$. Then $(\nu-\alpha-\beta,\alpha)=0$, which implies that $\nu-2\alpha-\beta\notin\Delta$. A contradiction! □

Remark 1.5. Elements $w_{\gamma,\mu}$ are not always commutative outside the ADE-realm. For instance, if Δ is of type $\mathbf{F}_4$ and $\Pi = \{\alpha_1, \dots, \alpha_4\}$, with numbering as in [13], then $w = \sigma_3\sigma_2\sigma_3\sigma_4$ is the shortest element taking $\theta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$ to $\mu = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

In general, we can prove that the elements of the form $w_{\gamma,\mu}$, where $\gamma \succcurlyeq \mu$ and $\|\gamma\| = \|\mu\|$, are *fully commutative* in the sense of Stembridge [12]. But this is not needed here.

In the simply-laced case, the following characterisation of commutative elements is given in [6, Theorem 2.4]:

(1·4) *$w \in W$ is commutative if and only if for all $\gamma, \gamma' \in \mathcal{N}(w)$, one has $(\gamma, \gamma') \geqslant 0$.*

Given $w \in W$, we say that w' is a *left factor* of w, if $\ell(w'^{-1}w) = \ell(w) - \ell(w')$. In other words, $w = w'w''$ and $\ell(w) = \ell(w') + \ell(w'')$. Similarly, w'' is a *right factor* of w, if $\ell(ww''^{-1}) = \ell(w) - \ell(w'')$. If $w = w'w''$, then w' is a left factor of w if and only if w'' is a right factor of w. In that case, $\mathcal{N}(w) = \mathcal{N}(w'') \sqcup (w'')^{-1}\mathcal{N}(w')$. In particular, $\mathcal{N}(w'') \subset \mathcal{N}(w)$.

Proposition 1.6. *Suppose that w is commutative and w' is a left factor of w. Then w' is commutative, too, and $\mathcal{N}(w')$ is "located below" $\mathcal{N}(w)$ with respect to "$\succcurlyeq$". The latter means that, for any $\gamma' \in \mathcal{N}(w')$, there exists $\gamma \in \mathcal{N}(w)$ such that $\gamma' \preccurlyeq \gamma$.*

Proof. The commutativity of w' is obvious.

- If $\ell(w'') = 1$, then $w = w'\sigma_\alpha$ for some $\alpha \in \Pi$. Here $\mathcal{N}(w) = \sigma_\alpha(\mathcal{N}(w')) \cup \{\alpha\}$. Take any $\mu' \in \mathcal{N}(w')$. Then $\mu = \sigma_\alpha(\mu') \in \mathcal{N}(w)$ and $(\alpha, \mu) \geqslant 0$ in view of Eq. (1·4). Therefore, $\mu' = \sigma_\alpha(\mu) \preccurlyeq \mu$.
- In general, we argue by downward induction on $\ell(w'')$. If $\ell(w'') > 1$, then $w'' = u\sigma_\alpha$, where $\ell(u) = \ell(w'') - 1$. Now, $w = (w'u)\sigma_\alpha$ and the previous argument shows that $\mathcal{N}(w'u)$ is located below $\mathcal{N}(w)$. By the induction assumption, $\mathcal{N}(w')$ is located below $\mathcal{N}(w'u)$, and we are done. □

Example 1.7. We are going to apply the above proposition in the following situation. Suppose that $\gamma \succ \mu \succ \nu$ are long roots. Then $w_{\gamma,\nu} = w_{\mu,\nu}w_{\gamma,\mu}$ and $w_{\mu,\nu}$ is a left factor of $w_{\gamma,\nu}$. In the simply-laced case, all these elements are commutative and we conclude that $\mathcal{N}(w_{\mu,\nu})$ is located below $\mathcal{N}(w_{\gamma,\nu})$.

2. MICS ASSOCIATED WITH A MAXIMAL ABELIAN IDEAL

In this section, we fix $\alpha \in \Pi_l$ and consider the corresponding maximal abelian ideal $I(\alpha)_{max} \in \mathfrak{Ab}_\alpha$. Eventually, we stick to the ADE case (i.e., $\Pi = \Pi_l$), but Lemmas 2.1 and 2.2 are valid in the general setting. Recall that $I(\alpha)_{min} = I(\alpha)_{max} \cap \mathcal{H}$. In the special case $\mu = \alpha$, there is the following complement to Lemma 1.1.

Lemma 2.1. *For $\alpha \in \Pi_l$, one has $\mathcal{N}(w_{\theta,\alpha}) = \mathcal{H} \setminus I(\alpha)_{max} = \mathcal{H} \setminus I(\alpha)_{min}$. In other words, $\mathcal{H} = I(\alpha)_{min} \sqcup \mathcal{N}(w_{\theta,\alpha})$. Furthermore, $\mu \in \mathcal{N}(w_{\theta,\alpha})$ if and only if $\theta - \mu \in I(\alpha)_{min} \setminus \{\theta\}$.*

Proof. By [10, Sect. 4], we have $\#I(\alpha)_{min} = \ell(w_{\theta,\alpha}) + 1 = (\rho, \theta^\vee)$ (see also Lemma 1.1(2)). On the other hand,

$$(2\rho, \theta^\vee) = \sum_{\gamma\in\Delta^+} (\gamma, \theta^\vee) = \#\{\gamma \mid (\gamma, \theta^\vee) = 1\} + (\theta, \theta^\vee) = (\#\mathcal{H} - 1) + 2 = \#\mathcal{H} + 1.$$

Therefore $\#I(\alpha)_{min} + \#\mathcal{N}(w_{\theta,\alpha}) = 2(\rho, \theta^\vee) - 1 = \#\mathcal{H}$. Since both $I(\alpha)_{min}$ and $\mathcal{N}(w_{\theta,\alpha})$ lies in $\mathcal{H}$, comparing with Lemma 1.1 shows that the only possibility is that $\mathcal{H}$ is a disjoint union of $I(\alpha)_{min}$ and $\mathcal{N}(w_{\theta,\alpha})$. Indeed, if $\nu \in I(\alpha)_{min} \cap \mathcal{N}(w_{\theta,\alpha})$, then $\theta - \nu \in I(\alpha)_{min}$, which contradicts the fact that $I(\alpha)_{min}$ is abelian.

The last assertion stems from the fact that $\mathcal{H} \setminus \{\theta\}$ is the union of pairs of the form $\{\mu, \theta - \mu\}$. □

Lemma 2.2. *For any $\mu' \in \Delta^+ \setminus I(\alpha)_{max}$, there exists $\mu \in I(\alpha)_{min}$ such that $\mu + \mu'$ is a root; and thereby $\mu + \mu' \in I(\alpha)_{min}$.*

Proof. If $\mu' \in \max(\Delta^+ \setminus I(\alpha)_{max})$, then $\mu' \in \mathcal{H}$ and $\mu = \theta - \mu' \in I(\alpha)_{min}$ [11, Theorem 4.7]. That is, the required property holds for the maximal elements of $\Delta^+ \setminus I(\alpha)_{max}$.

Assume that the required property is not satisfied for all of $\Delta^+ \setminus I(\alpha)_{max}$. Let $\nu \in \Delta^+ \setminus I(\alpha)_{max}$ be a maximal root among those that do not have the required property. Since $\nu \notin \max(\Delta^+ \setminus I(\alpha)_{max})$, there is $\beta \in \Pi$ such that $\nu + \beta \in \Delta^+ \setminus I(\alpha)_{max}$. Then $\nu + \beta$ has the required property and one can find $\mu \in I(\alpha)_{min}$ such that $(\nu + \beta) + \mu$ is a root. Here one easily proves that $\nu + \mu$ or $\beta + \mu$ is a root. In the second case, one obtains $\beta + \mu \in I(\alpha)_{min}$. Therefore, both possibilities contradict the assumption on ν. Thus, all roots in $\Delta^+ \setminus I(\alpha)_{max}$ must have the required property. □

For any $\gamma \in I(\alpha)_{max}$, one has $\gamma \succcurlyeq \alpha$ [11, Theorem 3.5]. Therefore, using Proposition 1.2, we obtain the shortest element $w_{\gamma,\alpha}$ for each *long* γ.

In the rest of the section, we assume that Δ is simply-laced.

Lemma 2.3. *For any $\gamma \in I(\alpha)_{max}$, we have $\mathcal{N}(w_{\gamma,\alpha}) \subset \Delta^+ \setminus I(\alpha)_{max}$.*

Proof. If $\gamma = \theta$, then $\mathcal{N}(w_{\theta,\alpha}) = \mathcal{H} \setminus I(\alpha)_{max}$ by Lemma 2.1. For arbitrary $\gamma \in I(\alpha)_{max}$, we have $\theta \succcurlyeq \gamma \succcurlyeq \alpha$ and $w_{\theta,\alpha} = w_{\gamma,\alpha} w_{\theta,\gamma}$ (see the proof of Proposition 1.2). Then Proposition 1.6 and Example 1.7 show that $\mathcal{N}(w_{\gamma,\alpha})$ is located below $\mathcal{N}(w_{\theta,\alpha})$. As $I(\alpha)_{max}$ is an ideal and $\mathcal{N}(w_{\theta,\alpha})$ does not intersect $I(\alpha)_{max}$, so does $\mathcal{N}(w_{\gamma,\alpha})$. □

Now, we are ready to provide an accurate construction of MICS associated with $\alpha \in \Pi$. Having the shortest elements $w_{\gamma,\alpha}$ ($\gamma \in I(\alpha)_{max}$) at our disposal, we set $\tilde{w}_\gamma = \sigma_\alpha w_{\gamma,\alpha}$. Then $\mathcal{N}(\tilde{w}_\gamma) = \mathcal{N}(w_{\gamma,\alpha}) \cup \{w_{\gamma,\alpha}^{-1}(\alpha)\} = \mathcal{N}(w_{\gamma,\alpha}) \cup \{\gamma\}$. In particular, $\ell(\tilde{w}_\gamma) = \ell(w_{\gamma,\alpha}) + 1 = \mathsf{ht}(\gamma)$. Note also that $\tilde{w}_\gamma$ is the shortest element of W taking γ to $-\alpha$.

Theorem 2.4. *Suppose that Δ is simply-laced. For $\alpha \in \Pi$ and the corresponding maximal abelian ideal $I(\alpha)_{max}$, the family $\mathcal{F}_\alpha = \{\tilde{w}_\gamma \mid \gamma \in I(\alpha)_{max}\}$ is a MICS in W and $\mathsf{Ess}(\mathcal{F}_\alpha) \supset I(\alpha)_{max}$.*

Proof. We already know that $\mathcal{N}(\tilde{w}_\gamma)=\mathcal{N}(w_{\gamma,\alpha})\cup\{\gamma\}$ and

$$\mathcal{N}(w_{\gamma,\alpha})\subset\Delta^+\setminus I(\alpha)_{max}\ \text{ (Lemma 2.3)}.$$

Consequently,

$$\bigcup_{\gamma\in I(\alpha)_{max}}\mathcal{N}(\tilde{w}_\gamma)=I(\alpha)_{max}\sqcup\Bigl(\bigcup_{\gamma\in I(\alpha)_{max}}\mathcal{N}(w_{\gamma,\alpha})\Bigr).$$

Take an arbitrary $\mu'\in\Delta^+\setminus I(\alpha)_{max}$. By Lemma 2.2, there is $\mu\in I(\alpha)_{min}$ such that $\gamma:=\mu+\mu'$ is a root, and thereby $\gamma\in I(\alpha)_{min}\subset I(\alpha)_{max}$. Then

$$\alpha=w_{\gamma,\alpha}(\gamma)=w_{\gamma,\alpha}(\mu)+w_{\gamma,\alpha}(\mu').$$

By Lemma 2.3, $w_{\gamma,\alpha}(\mu)$ is a positive root. Hence $w_{\gamma,\alpha}(\mu')$ must be negative, i.e., $\mu'\in\mathcal{N}(w_{\gamma,\alpha})$. It follows that $\bigcup\limits_{\gamma\in I(\alpha)_{min}}\mathcal{N}(w_{\gamma,\alpha})=\Delta^+\setminus I(\alpha)_{max}$ and hence $\bigcup\limits_{\gamma\in I(\alpha)_{max}}\mathcal{N}(\tilde{w}_\gamma)=\Delta^+$. Note that $\tilde{w}_\gamma$ is the only element of $\mathcal{F}_\alpha$ whose inversion set contains $\gamma\in I(\alpha)_{max}$. Therefore, $\mathcal{F}_\alpha$ is a MICS and $I(\alpha)_{max}\subset\mathsf{Ess}(\mathcal{F}_\alpha)$. $\square$

Remark. Our use of the canonical elements $w_{\gamma,\alpha}$ and $\tilde{w}_\gamma$ ($\gamma\in I(\alpha)_{max}$) replaces the ad hoc considerations and case-by-case calculations of [9].

The inclusion $\mathsf{Ess}(\mathcal{F}_\alpha)\supset I(\alpha)_{max}$ can be strict.

Example 2.5. Let Δ be of type $\mathbf{A}_n$ and $\Delta^+=\{\varepsilon_i-\varepsilon_j\mid 1\leqslant i<j\leqslant n+1\}$. Here $\theta=\varepsilon_1-\varepsilon_{n+1}$ and $\Pi=\{\alpha_1,\dots,\alpha_n\}$ with $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$.

Consider the MICS associated with α_1. In this case,

$$I(\alpha_1)_{max}=I(\alpha_1)_{min}=\{\alpha_1,\alpha_1+\alpha_2,\dots,\alpha_1+\dots+\alpha_n=\theta\}.$$

Then $w_{\theta,\alpha_1}=\sigma_2\cdots\sigma_n$, $\tilde{w}_\theta=\sigma_1\sigma_2\cdots\sigma_n$, and $\tilde{w}_{\alpha_1+\dots+\alpha_i}=\sigma_1\cdots\sigma_i$ for $1\leqslant i\leqslant n$. Thus,

$$\mathcal{N}(\tilde{w}_{\alpha_1+\dots+\alpha_i})=\{\alpha_i,\alpha_i+\alpha_{i-1},\dots,\alpha_i+\dots+\alpha_1\}$$

and $\Delta^+=\bigsqcup_{i=1}^n\mathcal{N}(\tilde{w}_{\alpha_1+\dots+\alpha_i})$ (the disjoint union!). Therefore, $\mathsf{Ess}(\mathcal{F}_{\alpha_1})=\Delta^+$.

The similar situation occurs for α_n. On the other hand, if $2\leqslant i\leqslant n-1$, then $(\theta,\alpha_i)=0$ and therefore $\mathsf{Ess}(\mathcal{F}_{\alpha_i})=I(\alpha_i)_{max}$, see Theorem 4.1(ii) below.

Remark 2.6. If Δ has two root lengths and $\mu\in I(\alpha)_{max}$ is short, then α and μ are not W-conjugate. Therefore, one cannot define an element $w_{\mu,\alpha}$ and our approach, as it is, fails. Of course, for each $\mu\in I(\alpha)_{max}\cap\Delta_s$, one can find $\beta\in\Pi_s$ such that $\mu\succcurlyeq\beta$ and then take the shortest element $w_{\mu,\beta}$, etc. But this is not canonical and, moreover, this merely does not provide a MICS! In general, Malvenuto et al. suggest that the maximum cardinality of a MICS equals the maximum cardinality of a strongly abelian subset of Δ^+ [9, Sect. 9]. Here $\mathcal{A}\subset\Delta^+$ is said to be *strongly abelian*, if $(\mathbb{R}_{>0}\gamma+\mathbb{R}_{>0}\mu)\cap\Delta=\varnothing$ for any $\gamma,\mu\in\mathcal{A}$.

Example 2.7. For Δ of type $\mathbf{F}_4$, $\#\mathcal{A} \leqslant 6$ and there is a MICS of cardinality 6 [9, (7.1)], whereas the maximal abelian ideals are of cardinality 8 and 9. From our point of view, that specific MICS can be obtained as follows. Write $[n_1n_2n_3n_4]$ for $\gamma = \sum_{i=1}^{4} n_i\alpha_i \in \Delta^+(\mathbf{F}_4)$. The numbering of Π is as in [13, Tables], so that $\Pi_s = \{\alpha_1, \alpha_2\}$ and $\Pi_l = \{\alpha_3, \alpha_4\}$. Then

$$\mathcal{A} = \{[2432], [2431], [2421], [2321], [1321], [1221]\}$$

is a maximal strongly abelian subset, where the last three roots are short. For $\gamma \in \mathcal{A}_l$ (resp. $\gamma \in \mathcal{A}_s$), we take the element of minimal length w_{γ,α_3} (resp. w_{γ,α_1}) and set $\tilde{w}_\gamma = \sigma_3 w_{\gamma,\alpha_3}$ ($\gamma \in \mathcal{A}_l$) and $\tilde{w}_\gamma = \sigma_1 w_{\gamma,\alpha_1}$ ($\gamma \in \mathcal{A}_s$). The resulting six elements form a MICS pointed out in [9]. This procedure strongly resembles our construction of MICS in Theorem 2.4, but we do not have a general theoretical explanation for this. Since this construction exploits simple roots α_1 and α_3, we denote the resulting MICS by $\mathcal{F}_{1,3}$. Furthermore, a similar construction can be performed with the other simple roots in suitable combinations. That is, we use either α_3 or α_4 for $\gamma \in \mathcal{A}_l$ and either α_1 or α_2 for $\gamma \in \mathcal{A}_s$. This yields four different MICS of cardinality six: $\mathcal{F}_{1,3}, \mathcal{F}_{1,4}, \mathcal{F}_{2,3}, \mathcal{F}_{2,4}$. But the essential sets for them are essentially different! We have $\#\mathsf{Ess}(\mathcal{F}_{1,3}) = \#\mathsf{Ess}(\mathcal{F}_{2,4}) = 12$, $\#\mathsf{Ess}(\mathcal{F}_{2,3}) = 14$, and $\#\mathsf{Ess}(\mathcal{F}_{1,4}) = 10$. For instance, $\mathsf{Ess}(\mathcal{F}_{1,4}) = \mathcal{A} \cup \{\alpha_1, \alpha_4, [1211], [2221]\}$. Thus, $\mathcal{F}_{1,4}$ has the smallest defect among them (=4).

3. Some properties of root systems

Here we gather preparatory results on long roots that are needed in the following section, where we describe the essential set for some MICS constructed in Section 2.

3.1. Given $\gamma \in \Delta^+$, consider the set $\Gamma_\gamma = \{\nu \in \Delta^+ \mid \gamma - \nu \in \Delta^+\}$. Clearly, if $\nu \in \Gamma_\gamma$, then $\gamma - \nu \in \Gamma_\gamma$ as well and $\gamma - \nu \neq \nu$. Therefore, $\#\Gamma_\gamma$ is even, and if it is $2k$, then there are exactly k different ways to write γ as a sum of two positive roots.

Lemma 3.1. *If $\gamma \in \Delta_l^+$ and $\sigma_\gamma \in W$ is the reflection with respect to γ, then $\mathcal{N}(\sigma_\gamma) = \Gamma_\gamma \cup \{\gamma\}$. In particular, $\#\Gamma_\gamma + 1 = \ell(\sigma_\gamma)$.*

Proof. Since $\sigma_\gamma(\nu) = \nu - (\nu, \gamma^\vee)\gamma$ and $(\nu, \gamma^\vee) = 2$ if and only if $\nu = \gamma$, we have

$$\mathcal{N}(\sigma_\gamma) = \{\gamma\} \cup \{\mu \in \Delta^+ \mid (\mu, \gamma^\vee) = 1 \ \& \ \mu - \gamma \in -\Delta^+\} = \\ \{\gamma\} \cup \{\mu \in \Delta^+ \mid \gamma - \mu \in \Delta^+\} = \{\gamma\} \cup \Gamma_\gamma. \quad \Box$$

Remark. If Δ is of type BCF and γ is short, then there is still a natural bijection between $\mathcal{N}(\sigma_\gamma)$ and $\Gamma_\gamma \cup \{\gamma\}$, hence the equality $\#\Gamma_\gamma + 1 = \ell(\sigma_\gamma)$ survives. But this is not true for the dominant short root in type $\mathbf{G}_2$.

Definition 4 ([2]). The *depth* of $\gamma \in \Delta^+$ is $\mathrm{dp}(\gamma) = \min\{\ell(w) \mid w \in W,\ w(\gamma) \in -\Delta^+\}$.

This notion has been studied in the context of arbitrary (possibly infinite) Coxeter groups [2, 4], but we only use it for Weyl groups. It is easily seen that if the required minimum of $\ell(w)$ is achieved, then $w(\gamma) \in -\Pi$. Furthermore, if $w(\gamma) = -\alpha \in -\Pi$, then $w = \sigma_\alpha w'$, where $w'(\gamma) = \alpha$ and $\ell(w) = \ell(w') + 1$. Combining this with Proposition 1.2 with $\mu = \alpha$, we see that $w' = w_{\gamma,\alpha}$. Hence

Lemma 3.2. *If Δ is simply-laced, then $\mathrm{dp}(\gamma) = (\rho, \gamma^\vee) = \mathsf{ht}(\gamma)$ for all $\gamma \in \Delta^+$.*

Then using the general equality $\ell(\sigma_\gamma) = 2\mathrm{dp}(\gamma) - 1$ [4, Lemma 1.2], we obtain

$$\#\Gamma_\gamma = \ell(\sigma_\gamma) - 1 = 2\mathrm{dp}(\gamma) - 2 = 2\mathsf{ht}(\gamma) - 2.$$

In other words,

Proposition 3.3. *If Δ is simply-laced, then there are $\mathsf{ht}(\gamma) - 1$ ways to write $\gamma \in \Delta^+$ as a sum of two positive roots.*

Remark. More generally, if Δ is arbitrary and $\gamma \in \Delta^+_s$, then $\mathrm{dp}(\gamma) = \mathsf{ht}(\gamma)$. This allows to conclude that if $\gamma \in \Delta^+$ is also arbitrary, then $\mathrm{dp}(\gamma) = \min\{\mathsf{ht}(\gamma), \mathsf{ht}(\gamma^\vee)\}$, where $\mathsf{ht}(\gamma^\vee)$ is the height of $\gamma^\vee$ in the dual root system $\Delta^\vee$. But we do need this here.

3.2. Consider a sort of dual object to Γ_γ. For $\gamma \in \Delta^+$, set $\Phi_\gamma = \{\nu \in \Delta^+ \mid \nu - \gamma \in \Delta^+\}$.

Lemma 3.4. *If $\gamma \in \Delta^+_l$, then $\#\Phi_\gamma = (\rho, \theta^\vee - \gamma^\vee)$. In particular, if Δ is simply-laced, then $\#\Phi_\gamma = \mathsf{ht}(\theta) - \mathsf{ht}(\gamma) = h - 1 - \mathsf{ht}(\gamma)$.*

Proof. Consider the shortest element $w_{\theta,\gamma} \in W$. Then $\ell(w_{\theta,\gamma}) = (\rho, \theta^\vee - \gamma^\vee)$ (Proposition 1.2) and $\mathcal{N}(w_{\theta,\gamma}^{-1}) = \{\mu \in \Delta^+ \mid (\mu, \gamma^\vee) = -1\}$, see Eq. (1·1). That is, the number of positive roots μ such that $\gamma + \mu$ is a root equals $\ell(w_{\theta,\gamma})$. It remains to observe that $(\mu, \gamma^\vee) = -1$ if and only if $\gamma + \mu \in \Phi_\gamma$. □

3.3. For $\alpha \in \Pi$ and $\nu \in \Delta$, let $[\nu : \alpha]$ denote the coefficient of α in the sum $\nu = \sum_{\beta \in \Pi} c_\beta \beta$. We also say that $[\nu : \alpha]$ is the *α-height* of ν. Set $\Delta_\alpha(i) = \{\gamma \in \Delta \mid [\gamma : \alpha] = i\}$ and $\Delta_\alpha(0)^+ = \Delta_\alpha(0) \cap \Delta^+$. The Weyl group of $\Delta_\alpha(0)$ is $W_\alpha(0) = \langle \sigma_\beta \mid \beta \in \Pi \setminus \{\alpha\} \rangle$.

Although, we are not aware of a general description for $\mathcal{N}(w_{\gamma,\alpha})$, we do have a nice formula if $[\gamma : \alpha] = 1$, i.e., $\gamma \in \Delta_\alpha(1)$.

Proposition 3.5. *Suppose that Δ is simply-laced. If $\gamma \succcurlyeq \alpha$ and $[\gamma : \alpha] = 1$, then $\mathcal{N}(w_{\gamma,\alpha}) = \{\mu \in \Delta_\alpha(0)^+ \mid \gamma - \mu \in \Delta^+\}$.*

Proof. Since γ and α have the same α-height, any reduced expression for $w_{\gamma,\alpha}$ does not contain σ_α. Therefore, $w_{\gamma,\alpha} \in W_\alpha(0)$ and $\mathcal{N}(w_{\gamma,\alpha}) \subset \Delta_\alpha(0)^+$. Suppose that $\gamma = \mu_1 + \mu_2$ is a sum of positive roots. Then $\mu_1 \in \Delta_\alpha(0)^+$ and $\mu_2 \in \Delta_\alpha(1)$. Since

$$\alpha = w_{\gamma,\alpha}(\gamma) = w_{\gamma,\alpha}(\mu_1) + w_{\gamma,\alpha}(\mu_2),$$

exactly one summand in the right hand side is negative. The above discussion implies that $w_{\gamma,\alpha}(\mu_1) \in -\Delta^+$, i.e., $\mu_1 \in \mathcal{N}(w_{\gamma,\alpha})$. This means that every presentation of γ as a sum of two positive roots yields an element of $\mathcal{N}(w_{\gamma,\alpha})$. On the other hand, we know that

(i) $\#\mathcal{N}(w_{\gamma,\alpha}) = \mathsf{ht}(\gamma) - 1$ (Proposition 1.2) and

(ii) the total number of such presentations is $\mathsf{ht}(\gamma) - 1$ (Proposition 3.3).

Thus, each element of $\mathcal{N}(w_{\gamma,\alpha})$ lies in $\Delta_\alpha(0)^+$ and is associated with a presentation of γ as a sum of two roots. □

4. On the essential set of $\mathcal{F}_\alpha$

Any MICS $\mathcal{F} \subset W$ determines a natural statistic on Δ^+:

$$\gamma \mapsto n_{\mathcal{F}}(\gamma) = \#\{w \in \mathcal{F} \mid \gamma \in \mathcal{N}(w)\}.$$

We call it the $\mathcal{F}$-*statistic* on Δ^+ and say that $n_{\mathcal{F}}(\gamma)$ is the $\mathcal{F}$-*multiplicity* of γ. Clearly, $\mathsf{Ess}(\mathcal{F}) = \{\gamma \in \Delta^+ \mid n_{\mathcal{F}}(\gamma) = 1\}$. For $\mathcal{F} = \mathcal{F}_\alpha$ as above, we write $n_\alpha(\gamma)$ in place of $n_{\mathcal{F}_\alpha}(\gamma)$.

In this section, we describe $\mathsf{Ess}(\mathcal{F}_\alpha)$ in the following two cases:

(A) $I(\alpha)_{max}$ is the set of roots of the nilradical of a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. (That is, the nilradical of $\mathfrak{p}$ is abelian.)

(B) θ is a fundamental weight and $\alpha = \hat\alpha$ is the unique simple root such that $(\theta, \hat\alpha) \ne 0$.

We also provide a full description of the $\mathcal{F}_\alpha$-statistic in case (A). Recall the necessary setup. Let $\mathfrak{p}_\alpha$ be the maximal parabolic subalgebra associated with $\alpha \in \Pi$. Then $\Delta_\alpha(0)$ is the root system of the standard Levi subalgebra of $\mathfrak{p}_\alpha$ and $\bigcup_{i\geqslant 1} \Delta_\alpha(i)$ is the set of roots of the nilradical $\mathfrak{n}_\alpha$ of $\mathfrak{p}_\alpha$. The system $\Delta_\alpha(0)$ can be reducible, and we need its partition into irreducible subsystems $\Delta_\alpha(0) = \bigsqcup_{j\in J} \Delta_\alpha(0)_j$.

For (A): Suppose that α has the property that $[\theta : \alpha] = 1$. Then $\Delta^+ = \Delta_\alpha(0)^+ \cup \Delta_\alpha(1)$ and $\mathfrak{n}_\alpha$ is an abelian ideal of $\mathfrak{b}$. Moreover, $\Delta_\alpha(1) = I(\alpha)_{max}$. Note that $\alpha \in \Delta_\alpha(1)$.

As is well-known, the simple roots α such that $[\theta : \alpha] = 1$ are:

- $\alpha_1, \dots, \alpha_n$ for $\mathbf{A}_n$;
- $\alpha_1, \alpha_{n-1}, \alpha_n$ for $\mathbf{D}_n$, $n \geqslant 4$;
- α_1, α_5 for $\mathbf{E}_6$ and α_1 for $\mathbf{E}_7$; there are no such simple roots in $\mathbf{E}_8$. (We use the numbering of simple roots from [13, Tables].)

Furthermore, $\#J = 2$ for $(\mathbf{A}_n, \alpha_i)$ with $2 \leqslant i \leqslant n-1$ and $\#J = 1$ in the other cases.

Theorem 4.1. *Suppose that $[\theta : \alpha] = 1$ and $\Delta^+ = \Delta_\alpha(0)^+ \sqcup \Delta_\alpha(1) = \Delta_\alpha(0)^+ \sqcup I(\alpha)_{max}$, as above. Consider the MICS $\mathcal{F}_\alpha = \{\tilde w_\gamma \mid \gamma \in \Delta_\alpha(1)\}$ constructed in Section 2.*

(i) *For $\mu \in \Delta_\alpha(0)^+$, the $\mathcal{F}_\alpha$-multiplicity of μ depends only on the irreducible subsystem $\Delta_\alpha(0)_j$ to which μ belongs. Namely, if h (resp. h_j) is the Coxeter number of Δ (resp. $\Delta_\alpha(0)_j$), then $n_\alpha(\mu) = h - h_j$.*

(ii) *In particular, if* $(\theta, \alpha) = 0$ *(which covers all* $\mathbf{D}_n$ *and* $\mathbf{E}_n$ *cases), then* $\mathsf{Ess}(\mathcal{F}_\alpha) = I(\alpha)_{max}$ *and thereby* $\mathrm{defect}(\mathcal{F}_\alpha) = 0$.

Proof. (i) By Theorem 2.4, $n_\alpha(\mu) = 1$ for $\mu \in \Delta_\alpha(1)$, and we only have to handle the case in which $\mu \in \Delta_\alpha(0)^+$. The very construction of $\mathcal{F}_\alpha$ shows that $n_\alpha(\mu)$ equals the number of $\gamma \in \Delta_\alpha(1)$ such that $\mu \in \mathcal{N}(w_{\gamma,\alpha})$. According to Proposition 3.5, the latter equals the number of $\gamma \in \Delta_\alpha(1)$ such that $\gamma - \mu \in \Delta^+$.

Recall that $\Phi_\mu = \{\gamma \in \Delta^+ \mid \gamma - \mu \in \Delta^+\}$ and $\#\Phi_\mu = h - 1 - \mathsf{ht}(\mu)$ (Lemma 3.4). Here $\Phi_\mu = \Phi_\mu(0) \cup \Phi_\mu(1)$, according to whether γ lies in $\Delta_\alpha(0)^+$ or $\Delta_\alpha(1)$, and $n_\alpha(\mu) = \#\Phi_\mu(1)$. If $\gamma \in \Delta_\alpha(0)^+$, then γ and μ must belong to one and the same irreducible subsystem of $\Delta_\alpha(0)$. Therefore, $\#\Phi_\mu(0) = h_j - 1 - \mathsf{ht}(\mu)$ for $\mu \in \Delta_\alpha(0)_j$ and hence $\#\Phi_\mu(1) = h - h_j$.

(ii) If $(\alpha, \theta) = 0$ and $[\theta : \alpha] = 1$, then the highest root θ_j in $\Delta_\alpha(0)_j$ has the height at least two less than the height of θ, i.e., $h - h_j \geqslant 2$. (Indeed, at the very first step down from θ to θ_j, one subtracts certain $\beta \in \Pi$ such that $(\beta, \theta) \ne 0$, and one also must subtract α once.) □

Example 4.2. For all $\alpha \in \Pi$ with $[\theta : \alpha] = 1$, we point out the differences $h - h_j$ and thereby $\mathcal{F}_\alpha$-multiplicities of the roots in $\Delta_\alpha(0)^+$. Recall that $h = h(\Delta)$.

(1) For $\mathbf{E}_6$ and $\alpha = \alpha_1$ or α_5, $\Delta_\alpha(0)$ is of type $\mathbf{D}_5$ and $h - h(\mathbf{D}_5) = 4$;
(2) For $\mathbf{E}_7$ and $\alpha = \alpha_1$, $\Delta_\alpha(0)$ is of type $\mathbf{E}_6$ and $h - h(\mathbf{E}_6) = 6$;
(3) For $\mathbf{D}_n$ and $\alpha = \alpha_1$, $\Delta_\alpha(0)$ is of type $\mathbf{D}_{n-1}$ and $h - h(\mathbf{D}_{n-1}) = 2$;
(4) For $\mathbf{D}_n$ and $\alpha = \alpha_{n-1}$ or α_n, $\Delta_\alpha(0)$ is of type $\mathbf{A}_{n-1}$ and $h - h(\mathbf{A}_{n-1}) = n - 2$;
(5) For $\mathbf{A}_n$ and $\alpha = \alpha_i$ $(2 \leqslant i \leqslant n-1)$, $\Delta_\alpha(0)$ is of type $\mathbf{A}_{n-i} \times \mathbf{A}_{i-1}$. Therefore, the $\mathcal{F}_{\alpha_i}$-multiplicity of $\mu \in \Delta_\alpha(0)^+$ is either $h - h(\mathbf{A}_{n-i}) = i$ or $h - h(\mathbf{A}_{i-1}) = n - i + 1$.
(6) For $\mathbf{A}_n$ and $\alpha = \alpha_1$ or α_n, $\Delta_\alpha(0)$ is of type $\mathbf{A}_{n-1}$ and $h - h(\mathbf{A}_{n-1}) = 1$. Therefore here $\mathsf{Ess}(\mathcal{F}_{\alpha_1}) = \mathsf{Ess}(\mathcal{F}_{\alpha_n}) = \Delta^+$, as already computed in Example 2.5.

The last item represents the only possibilities in which $(\theta, \alpha) \ne 0$.

For (B): Suppose now that θ is a fundamental weight. Write $\hat\alpha$ for the unique simple root that is not orthogonal to θ. By assumption, $(\theta, \hat\alpha^\vee) = 1$. Therefore $\hat\alpha$ is long, $(\hat\alpha, \theta^\vee) = 1$, and $(\beta, \theta^\vee) = 0$ for $\beta \in \Pi \setminus \{\hat\alpha\}$. Hence $2 = (\theta, \theta^\vee) = [\theta : \hat\alpha]$ and $(\gamma, \theta^\vee) = [\gamma : \hat\alpha]$ for any $\gamma \in \Delta^+$. In this situation, $\mathcal{H} = \Delta_{\hat\alpha}(1) \cup \Delta_{\hat\alpha}(2)$ and $\Delta_{\hat\alpha}(2) = \{\theta\}$. Clearly, $\theta - \hat\alpha$ is the unique root covered by θ and $\theta \succ \theta - \hat\alpha \succ \hat\alpha$. Since $\theta - \hat\alpha \in \Delta_l^+$, the shortest element $\hat w := w_{\theta-\hat\alpha,\hat\alpha}$ is well-defined. By the very definition of $\hat w$, we have $\hat w(\theta - \hat\alpha) = \hat\alpha$. We are going to prove that $\hat w$ is an involution, and the next assertion yields a first step towards that goal.

Lemma 4.3. $\hat w(\hat\alpha) = \theta - \hat\alpha$.

Proof. Since $\theta - \hat\alpha$ and $\hat\alpha$ have the same $\hat\alpha$-height, a reduced expression for $\hat w$ does not contain $\sigma_{\hat\alpha}$. Therefore, $\hat w \in W_{\hat\alpha}(0) = \langle \sigma_\beta \mid \beta \in \Pi \setminus \{\hat\alpha\}\rangle$ and $\hat w(\theta) = \theta$. Applying $\hat w$ to the equality $(\theta - \hat\alpha) + \hat\alpha = \theta$, we obtain $\hat\alpha + \hat w(\hat\alpha) = \theta$, as required. □

Proposition 4.4. *If θ is a fundamental weight, then $\hat{w}^2 = 1$.*

Proof. It suffices to prove that $\mathcal{N}(\hat{w}^{-1}) \subset \mathcal{N}(\hat{w})$. Since $\sigma_{\hat\alpha}(\theta) = \theta - \hat\alpha$, $\hat{w}\sigma_{\hat\alpha} = w_{\theta,\hat\alpha}$ is the shortest element of W taking θ to $\hat\alpha$. Therefore, using Lemma 4.3, we obtain

$$\mathcal{N}(\hat{w}^{-1}) = \mathcal{N}(w_{\theta,\hat\alpha}^{-1}) \setminus \{\hat{w}(\hat\alpha)\} = \mathcal{N}(w_{\theta,\hat\alpha}^{-1}) \setminus \{\theta - \hat\alpha\}.$$

Recall that $\mathcal{N}(w_{\theta,\hat\alpha}^{-1}) = \{\nu \in \Delta^+ \mid \nu + \hat\alpha \in \Delta^+\}$, see Eq. (1·1). It follows that

$$\mathcal{N}(\hat{w}^{-1}) = \{\nu \in \Delta^+ \mid \nu + \hat\alpha \in \Delta^+ \setminus \{\theta\}\}.$$

Because θ is the only root with $\hat\alpha$-height equal to 2, this means that $[\nu : \hat\alpha] = 0$. That is,

$$\mathcal{N}(\hat{w}^{-1}) = \{\nu \in \Delta_{\hat\alpha}(0)^+ \mid \nu + \hat\alpha \in \Delta_{\hat\alpha}(1)\}.$$

On the other hand, Proposition 3.5 with $\alpha = \hat\alpha$ shows that

$$\mathcal{N}(\hat{w}) = \{\mu \in \Delta_{\hat\alpha}(0)^+ \mid (\theta - \hat\alpha) - \mu \in \Delta_{\hat\alpha}(1)\}.$$

Finally, if $\nu \in \mathcal{N}(\hat{w}^{-1})$, then $\nu + \hat\alpha \in \Delta_{\hat\alpha}(1)$, hence $\theta - (\nu + \hat\alpha)$ is a root (in $\Delta_{\hat\alpha}(1)$), i.e., $(\theta - \hat\alpha) - \nu \in \Delta_{\hat\alpha}(1)$. Thus, $\nu \in \mathcal{N}(\hat{w})$ and we are done. □

Theorem 4.5. *Suppose that Δ is simply-laced, θ is fundamental, and $\hat\alpha$ is the unique simple root that is not orthogonal to θ. Then* $\mathsf{Ess}(\mathcal{F}_{\hat\alpha}) = \mathcal{H}$ *and* $\mathrm{defect}(\mathcal{F}_{\hat\alpha}) = h - 2$.

Proof. Recall that $\mathcal{H} = I(\hat\alpha)_{min} \sqcup \mathcal{N}(w_{\theta,\hat\alpha})$ (Lemma 2.1). Since $\hat\alpha \in \mathcal{H}$, we have $I(\hat\alpha)_{max} = I(\hat\alpha)_{min}$ [10, Theorem 5.1] and write below $I(\hat\alpha)$ for this common ideal.

1) We first prove that $\mathcal{H} \subset \mathsf{Ess}(\mathcal{F}_{\hat\alpha})$. As $I(\hat\alpha) \subset \mathsf{Ess}(\mathcal{F}_{\hat\alpha})$ (Theorem 2.4), our task is to prove that $\mathcal{N}(w_{\theta,\hat\alpha}) \subset \mathsf{Ess}(\mathcal{F}_{\hat\alpha})$.

The proof of Theorem 2.4 shows that $\bigcup_{\gamma\in I(\hat\alpha)} \mathcal{N}(w_{\gamma,\hat\alpha}) = \Delta^+ \setminus I(\hat\alpha)$ and the $\mathcal{F}_{\hat\alpha}$-multiplicity of $\mu \in \Delta^+ \setminus I(\hat\alpha)$ equals the number of $\gamma \in I(\hat\alpha)$ such that $\mu \in \mathcal{N}(w_{\gamma,\hat\alpha})$. Therefore, the required inclusion is equivalent to that

$$\mathcal{N}(w_{\theta,\hat\alpha}) \cap \Big(\bigcup_{\gamma\in I(\hat\alpha)\setminus\{\theta\}} \mathcal{N}(w_{\gamma,\hat\alpha})\Big) = \varnothing.$$

We have $\mathcal{N}(w_{\theta,\hat\alpha}) \subset \Delta_{\hat\alpha}(1)$, see Lemma 1.1. Therefore, it suffices to prove that $\mathcal{N}(w_{\gamma,\hat\alpha}) \subset \Delta_{\hat\alpha}(0)^+$ for $\gamma \neq \theta$. First, consider $\gamma = \theta - \hat\alpha$, the only root covered by θ. As above, we set $\hat{w} = w_{\theta-\hat\alpha,\hat\alpha}$. Then $\hat{w}\sigma_{\hat\alpha} = w_{\theta,\hat\alpha}$ and

$$\mathcal{N}(\hat{w}) = \sigma_{\hat\alpha}\big(\mathcal{N}((w_{\theta,\hat\alpha}) \setminus \{\hat\alpha\}\big).$$

Here we have to prove that if $\nu \in \mathcal{N}((w_{\theta,\hat\alpha}) \setminus \{\hat\alpha\}$, then $(\nu, \hat\alpha) > 0$ and thereby $\sigma_{\hat\alpha}(\nu) = \nu - \hat\alpha \in \Delta_{\hat\alpha}(0)^+$. Let us exclude the possibility that $(\nu, \hat\alpha) \leqslant 0$.

(a) Assume that $(\nu, \hat\alpha) < 0$. Then $\hat\alpha + \nu$ has the $\hat\alpha$-height 2, i.e., $\hat\alpha + \nu = \theta$. However, $\theta - \hat\alpha \in I(\hat\alpha)$, since $\hat\alpha \in \mathcal{N}(w_{\theta,\hat\alpha})$, see Lemma 2.1. A contradiction!

(b) Assume that $(\nu, \hat\alpha) = 0$. Then $\hat{w}(\nu) = w_{\theta,\hat\alpha}(\nu)$ is negative. Since $\hat{w}$ is an involution (Prop. 4.4), $\nu \in \mathcal{N}(\hat{w}^{-1}) \subset \mathcal{N}(w_{\theta,\hat\alpha}^{-1})$. Hence one must have $(\nu, \hat\alpha^\vee) = -1$. A contradiction!

This proves that $\mathcal{N}(\hat{w}) \subset \Delta_{\hat{\alpha}}(0)^+$. For any other $\gamma \in I(\hat{\alpha})$, we have $\gamma \prec \theta - \hat{\alpha}$. Then $w_{\gamma,\hat{\alpha}}$ is a left factor of $\hat{w}$ and hence $\mathcal{N}(w_{\gamma,\hat{\alpha}})$ is located below $\mathcal{N}(\hat{w})$, see Proposition 1.6 and Example 1.7. Therefore, $\mathcal{N}(w_{\gamma,\hat{\alpha}})$ lies in $\Delta_{\hat{\alpha}}(0)^+$, too.

2) Let us prove that if $\mu \in \Delta_{\hat{\alpha}}(0)^+$, then $\mu \notin \mathsf{Ess}(\mathcal{F}_{\hat{\alpha}})$. In other words, we have to prove that $n_{\hat{\alpha}}(\mu) \geqslant 2$. We use the same approach as in the proof of Theorem 4.1. Consider

$$\Phi_\mu = \{\gamma \in \Delta^+ \mid \gamma - \mu \in \Delta^+\} = \Phi_\mu(0) \cup \Phi_\mu(1) \cup \Phi_\mu(2),$$

where $\Phi_\mu(i)$ denotes the subset in which $\gamma \in \Delta_{\hat{\alpha}}(i)$. Note that $\Delta_{\hat{\alpha}}(2) = \{\theta\}$ and $\theta - \mu$ is not a root. Hence $\Phi_\mu(2) = \varnothing$. Recall that $\Delta_{\hat{\alpha}}(0)$ is the disjoint union of the irreducible subsystems $\Delta_{\hat{\alpha}}(0)_j,\ j \in J$, and h_j is the Coxeter number of $\Delta_{\hat{\alpha}}(0)_j$. By Lemma 3.4, $\#\Phi_\mu = h - 1 - \mathsf{ht}(\mu)$ and $\#\Phi_\mu(0) = h_j - 1 - \mathsf{ht}(\mu)$ if $\mu \in \Delta_{\hat{\alpha}}(0)_j$. Hence $\#\Phi_\mu(1) = h - h_j$. However, $\Delta_{\hat{\alpha}}(1) = I(\hat{\alpha}) \cup \mathcal{N}(w_{\theta,\hat{\alpha}})$, and we only need to count the subset of $\Phi_\mu(1)$ corresponding to $\gamma \in I(\hat{\alpha})$.

Given $\mu \in \Delta_{\hat{\alpha}}(0)^+$, a sum $\mu + \mu' = \gamma \in \Delta_{\hat{\alpha}}(1)$ is said to be *good* (resp. *bad*) if $\gamma \in I(\hat{\alpha})$ (resp. $\gamma \in \mathcal{N}(w_{\theta,\hat{\alpha}})$). Accordingly, $\Phi_\mu(1) = \Phi_\mu(1)^{good} \cup \Phi_\mu(1)^{bad}$ and $n_{\hat{\alpha}}(\mu) = \#\Phi_\mu(1)^{good}$. If $\mu + \mu'$ is bad, then $\nu = \theta - (\mu + \mu')$ is a root and, moreover, $\nu \in I(\hat{\alpha})$, see Lemma 2.1. Since $\mu + \mu' + \nu = \theta$ and $\mu' \in \Delta_{\hat{\alpha}}(1)$, $\mu + \nu$ is also a root. Furthermore, since $\nu \in I(\hat{\alpha})$, we have $\nu + \mu \in I(\hat{\alpha})$, too. Thus, starting with any bad sum $\mu + \mu'$, we obtain a good sum $\mu + \nu$. This yields an injection $\Phi_\mu(1)^{bad} \hookrightarrow \Phi_\mu(1)^{good}$ and hence

$$n_{\hat{\alpha}}(\mu) = \#\Phi_\mu(1)^{good} \geqslant \frac{1}{2}(h - h_j).$$

It remains to prove that $h - h_j \geqslant 4$ for any irreducible subsystem $\Delta_{\hat{\alpha}}(0)_j$ of $\Delta_{\hat{\alpha}}(0)$. In other words, $\mathsf{ht}(\theta) - \mathsf{ht}(\mu) \geqslant 4$ for any $\mu \in \Delta_{\hat{\alpha}}(0)^+$. Indeed, since $[\theta : \hat{\alpha}] = 2$, a passage from θ to μ must contain two subtractions of $\hat{\alpha}$. Because $\theta - 2\hat{\alpha} \notin \Delta$, such a passage should also include a subtraction of another $\alpha \in \Pi$, necessarily adjacent to $\hat{\alpha}$. Then one might assume that $\theta - 2\hat{\alpha} - \alpha$ was a root in $\Delta_{\hat{\alpha}}(0)^+$. But $\theta - 2\hat{\alpha} - \alpha$ is not a root in the simply-laced case! (Cf. the proof of Lemma 1.1.) Thus, one needs at least 4 subtractions. See also the following Example.

3) Here $\mathrm{defect}(\mathcal{F}_{\hat{\alpha}}) = \#\mathcal{N}(w_{\theta,\hat{\alpha}}) = \mathsf{ht}(\theta) - 1 = h - 2$. □

Example 4.6. We provide the differences $h - h_j$ for the irreducible subsystems of $\Delta_{\hat{\alpha}}(0)$ in all simply-laced cases with fundamental θ.

(1) For $\mathbf{E}_6$, $\Delta_{\hat{\alpha}}(0)$ is of type $\mathbf{A}_5$ and $h - h(\mathbf{A}_5) = 12 - 6 = 6$;
(2) For $\mathbf{E}_7$, $\Delta_{\hat{\alpha}}(0)$ is of type $\mathbf{D}_6$ and $h - h(\mathbf{D}_6) = 18 - 10 = 8$;
(3) For $\mathbf{E}_8$, $\Delta_{\hat{\alpha}}(0)$ is of type $\mathbf{E}_7$ and $h - h(\mathbf{E}_7) = 30 - 18 = 12$;
(4) For $\mathbf{D}_n$ ($n \geqslant 4$), $\Delta_{\hat{\alpha}}(0)$ is of type $\mathbf{A}_1 \times \mathbf{D}_{n-2}$. Then $h - h(\mathbf{D}_{n-2}) = (2n-2) - (2n-6) = 4$ and $h - h(\mathbf{A}_1) = (2n-2) - 2 = 2n - 4 \geqslant 4$.

Remark 4.7. Unlike the case in which $I(\alpha)_{max}$ corresponds to an abelian nilradical, see Theorem 4.1, the $\mathcal{F}_{\hat\alpha}$-multiplicities are not constant on the irreducible subsystems of $\Delta_{\hat\alpha}(0)$. The reason is that different μ can participate in a different number of *good* and *bad* sums.

5. Some speculations

Our previous results on MICS of the form $\mathcal{F}_\alpha$ ($\alpha \in \Pi = \Pi_l$) describe $\mathsf{Ess}(\mathcal{F}_\alpha)$ for

- all simple roots for $\mathbf{A}_n$;
- $\alpha_1, \alpha_2 = \hat\alpha, \alpha_{n-1}, \alpha_n$ for $\mathbf{D}_n$ ($n \geqslant 4$);
- $\alpha_1, \alpha_5, \alpha_6 = \hat\alpha$ for $\mathbf{E}_6$;
- $\alpha_1, \alpha_6 = \hat\alpha$ for $\mathbf{E}_7$;
- $\alpha_1 = \hat\alpha$ for $\mathbf{E}_8$.

The remaining cases concern the series $\mathbf{D}_n$ and $\mathbf{E}_n$, where θ is fundamental. Our partial computations suggest the following

Conjecture 5.1. *If Δ is simply-laced and θ is fundamental, then $\big(\mathsf{Ess}(\mathcal{F}_\alpha) \setminus I(\alpha)_{max}\big) \subset \mathcal{H}$.*

Since $\mathcal{H} = I(\alpha)_{min} \sqcup \mathcal{N}(w_{\theta,\alpha})$ (Lemma 2.1) and $I(\alpha)_{min} = I(\alpha)_{max} \cap \mathcal{H}$, it can also be restated as $\mathsf{Ess}(\mathcal{F}_\alpha) \subset I(\alpha)_{max} \sqcup \mathcal{N}(w_{\theta,\alpha})$. We also know that $\#\{\gamma \in \Delta^+ \mid (\gamma, \theta^\vee) = 1\} = 2\big((\rho,\theta^\vee) - 1\big)$ and

$$\#I(\alpha)_{max} = \#\mathcal{N}(w_{\theta,\alpha}) = (\rho, \theta^\vee) - 1 = h - 2.$$

Therefore, Conjecture 5.1 would imply that $\#\mathsf{Ess}(\mathcal{F}_\alpha) - \#\mathcal{F}_\alpha \leqslant h-2$. Then a companion conjecture is

Conjecture 5.2. *(1) If Δ is simply-laced and θ is fundamental, then $\mathrm{defect}(\mathcal{F}_\alpha) \leqslant h-2$;*

(2) Furthermore, $\mathrm{defect}(\mathcal{F}_\alpha) = h-2$ if and only if $\alpha = \hat\alpha$ (i.e., $(\theta,\alpha) \ne 0$).

(3) $\mathrm{defect}(\mathcal{F}_\alpha) = 0$ if and only if $\alpha \in \Pi$ is an endpoint of the Dynkin diagram that is different from $\hat\alpha$.

Note that if $[\theta : \alpha] = 1$, then $\mathrm{defect}(\mathcal{F}_\alpha) = 0$ (Theorem 2.4) and α is always an endpoint of the Dynkin diagram for $\mathbf{D}_n$ and $\mathbf{E}_n$.

Example 5.3. For Δ of type $\mathbf{E}_6$, straightforward computations show that both Conjectures are true, $\mathrm{defect}(\mathcal{F}_{\alpha_2}) = \mathrm{defect}(\mathcal{F}_{\alpha_4}) = 2$, and $\mathrm{defect}(\mathcal{F}_{\alpha_3}) = 4$.

Another curious observation is that in all cases, where the explicit description is known, $\mathsf{Ess}(\mathcal{F}_\alpha)$ appears to be the set of roots of a $\mathfrak{b}$-*stable subspace* of $\mathfrak{u}^+$. This should be compared with the fact all MICS in Example 2.7 do not have this property, and the reason is that $\mathcal{A}$ there is not an abelian ideal!

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