Complexity and nilpotent orbits

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Abstract

New formulas for the complexity and the rank of an arbitrary homogeneous space of a reductive group are given. These ones completely reduce the problem to finding of stabilizers of general position of linear representations of reductive groups. As an application the description of spherical (i.e. of complexity zero) nilpotent orbits is obtained.

(0.1) Let G be a reductive algebraic group, defined over algebraically closed field k of the characteristic zero. A famous problem in the theory of algebraic transformation groups is to describe normal G-equivariant embeddings of homogeneous spaces G/H, where H is an arbitrary algebraic subgroup of G. The important role of two numerical parameters of homogeneous spaces hase been recognized during the last decade. These ones are the *rank* $r(G/H)$ and the *complexity* $c(G/H)$. (See [LV], [K], [P1]). These are non-negative integers and the problem of classification of embeddings becomes complicated, if they increase. For instance, if $r(G/H) = 0$, then H is a parabolic subgroup of G and hence $c(G/H) = 0$ [P1]. In this case G/H is a projective variety, which does not admits non-trivial G-equivariant embeddings.

If $c(G/H) = 0$, then G/H is called *spherical.* This condition means that a Borel subgroup of G has an open orbit on G/H. The theory of spherical embeddings is already very extensive and the remarkable theory of toric varieties may be regarded as a particular case of it.

(0.2) In [P1] a method of computation of the complexity and the rank of quasi-affine homogeneous spaces in terms of co-isotropy representation of H

has been proposed. However, if H is not reductive (i.e. G/H is not affine), then this method is not very effective in practical computations. In this case the idea, which is due to M.Brion, appears to be fruitful. It is well-known that for any subgroup $H \subset G$ there exists a parabolic subgroup P such that $H \subset P$ and $H^u \subset P^u$. An inclusion, which satisfies the last condition, is said to be *right*. In [B] a criterion of sphericality of G/H in terms of a right inclusion of H has been obtained. In ch.1 we shall show that combining ideas of [P1] and [B] it is possible to produce general formulas for the rank and the complexity of an arbitrary homogeneous space in terms of a right inclusion of H. Even in the spherical case our formula seems to be better, than the one in [B]. The advantage of this approach is that the problem under consideration is completely reduced to the problem of finding of a *stabilizer of general position (=s.g.p.)* of linear representations of *reductive* groups. But the latter is quite easy.

(0.3) Conjugacy classes of nilpotent elements in semisimple Lie algebras, or simply *nilpotent orbits*, give us the best possible example of non-affine homogeneous spaces such that a right inclusion of the stabilizer is naturally arises.

It is known (see e.g. [SS]) that nilpotent orbits in semisimple Lie algebras are classified by their characteristics, i.e. the Dynkin diagrams with numerical labels. (We recall the construction of the characteristic in ch.2.) Let $\{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots of $\mathfrak{g}$ and $m_1, \ldots, m_l$ be the numerical labels of the characteristic, i.e. m_i stands near the node, which corresponds to the root α_i. I do not know whether it is possible to find a general formulae, which determines the complexity and the rank of a nilpotent orbit via its characteristic. Nevertheless it turns out that for spherical orbits the answer is rather simple. In ch.3,4 we shall prove the following statement.

Theorem. *Let $\Lambda = \sum_{i=1}^{l} n_i\alpha_i$ be the highest root. Then a nilpotent orbit Ge is spherical iff $\mathrm{ht}(e) := \sum_{i=1}^{l} n_i m_i \le 3$.*

Proof of this theorem is almost completely satisfactory, i.e. case-by-case consideration are reduced to a minimum. Namely, it follows from the definition of the characteristic that $\mathrm{ht}(e) \ge 2$. We shall give an a priori proof of the following:

1. If $\mathrm{ht}(e) = 2$, then Ge is spherical;
2. If $\mathrm{ht}(e) \ge 4$, then Ge is not spherical.

Only in the case $\mathrm{ht}(e) = 3$ we have to use case-by-case consideration. Nevertheless I believe that this trouble may be eliminated and this should be related with some interesting properties of stabilizers of these orbits.

For classical Lie algebras the preceding theorem admits a nice reformulation. Namely,
(1) Suppose e is a nilpotent matrix in $\mathfrak{sl}(V)$ or $\mathfrak{sp}(V)$. Then the orbit of e is spherical *iff* $e^2 = 0$.
(2) If $e \in \mathfrak{so}(V)$, then the orbit of e is spherical *iff* $\mathrm{rank}(e^2) \le 1$.

(0.4) As usual the Lie algebras of algebraic groups are denoted by the corresponding small German letters. Notation $(A : B)$ means that an algebraic group A acts regularly on an algebraic variety B.

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1 General formulas

(1.1) If an irreducible variety X is acted on by a reductive group G, then the *complexity* of X (relative to G) is the integer $c_G(X) = \min_{x\in X}\mathrm{codim}Bx$, where B is a Borel subgroup of G. In order to define the rank of X (relative to G), one has to consider the characters of all B-semiinvariants in the field of rational functions on X. Obviously these characters form a free abelian group. The rank of this group $r_G(X)$ is called the *rank* of X. Clearly, $r_G(X) \le \mathrm{rk}G$.

Let H be an algebraic subgroup of G. Suppose $H = KH^u$ be a Levi decomposition (i.e. H^u is the unipotent radical of H and K is a maximal reductive (=Levi) subgroup of H). There exists a parabolic subgroup $P \subset G$ with a Levi decomposition $P = LP^u$, such that $H \subset P$ and $H^u \subset P^u$ [Hu]. Then one can also assume that $K \subset L$. An inclusion that satisfies all these conditions is said to be *right*.

Our purpose is to find a method of determination of the rank and the complexity of G/H in terms of a right inclusion of H. Let us recall some necessary results. Let $\mathfrak{k} \subset \mathfrak{l}$ be the inclusion of Lie algebras, which corresponds to $K \subset L$ and $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{l}$ relative to a L-invariant scalar product on $\mathfrak{l}$. The natural representation of K in $\mathfrak{m}$ is said to be the *co-isotropy representation*. Since K is reductive, a s.g.p. S of this representation is also reductive. Being a subgroup of K, S act linearly on $P^u/H^u(\cong \mathfrak{p}^u/\mathfrak{h}^u$ as a K-module). By $B(S)$ denote a Borel subgroup of S.

(1.2) Theorem. (i) $c_G(G/H) = c_L(L/K) + c_S(P^u/H^u)$;

(ii) $r_G(G/H) = r_L(L/K) + r_S(P^u/H^u)$;
(iii) *If* B_* *is a s.g.p. for the action* $(B(S) : P^u/H^u)$, *then* B_* *is also a s.g.p. for the action* $(B : G/H)$.

Proof. (i),(iii). Let us consider G-equivariant fibering $\pi : G/H \to G/P$. By definition of complexity we have $c_G(G/H) = \min_{x\in G/H}\mathrm{codim}Bx$, where $B \subset G$ is a Borel subgroup. The action of B on G/P is locally-transitive and having replaced B on a conjugated subgroup in G one may assume that B-orbit of the point $\{P\} \in G/P$ is open and $B \cap P =: B(L)$ is a Borel subgroup in L. Whence $c_G(G/H) = \min_{y\in\pi^{-1}(\{P\})}\mathrm{codim}B_{\{P\}}y$. Obviously $\pi^{-1}(\{P\}) \cong P/H$ and $B_{\{P\}} = B(L)$. Thus we have already proved that $c_G(G/H) = c_L(P/H)$. There is a natural L-equivariant fibering $\tau : P/H \to L/K$ and $\tau^{-1}(\{K\}) \cong P^u/H^u$. Again we may assume that $\{K\} \in L/K$ is a generic point relative to the $B(L)$-action, that is $\mathrm{codim}_{L/K}B(L)K/K = c_L(L/K)$. Let $\tilde{B} \subset B(L)$ be the stabilizer of $\{K\}$. Then arguing as before, we get $c_L(P/H) = c_L(L/K) + \min_{z\in P^u/H^u}\mathrm{codim}\tilde{B}z$. Our key observation is the following. It has been proved in [P, ch.2] that $\tilde{B} = B(S)$ is a Borel subgroup of S. Therefore the last term of the sum is equal to $c_S(P^u/H^u)$ and the proof of (i) is completed. The preceding chain of arguments also prove (iii).

(ii) It is known [P, ch.1], that $c_G(G/H)+r_G(G/H) = \min_{x\in G/H}\mathrm{codim}Ux$, where U is the unipotent radical of B. Now the arguments, which are parallel to the ones from (i), give us: $c_G(G/H) + r_G(G/H) = c_L(L/K) + r_L(L/K) + \min_{z\in P^u/H^u}\mathrm{codim}\tilde{U}z$, where $\tilde{U}$ is the stabilizer in $U(L)$ of $\{K\}$. By [P, ch.1] we know that $\tilde{U}$ is a maximal unipotent subgroup of S, that is the last term is equal to $c_S(P^u/H^u) + r_S(P^u/H^u)$. Whence comparing with (i) we get the assertion. □

(1.3) *Remark.* The advantage of this theorem is that the problem under consideration is completely reduced to finding of s.g.p. of linear representations of *reductive* groups! It was mentioned before that S is a s.g.p. of coisotropy representation for $K \subset L$ and in order to compute $c_S(P^u/H^u)$ and $r_S(P^u/H^u)$ one have to find a s.g.p. for linear action $(S : P^u/H^u \oplus (P^u/H^u)^*)$ [P1], where the asterisk denote the dual representation.

(1.4) Corollary. *G/H is spherical iff L/K is spherical and a Borel subgroup of S has an open orbit on* $\mathfrak{p}^u/\mathfrak{h}^u$. □

2 Generalities on nilpotent orbits

(2.1) Let $\mathfrak{g}$ be a semisimple Lie algebra, G be its adjoint group, and $\mathfrak{N} \subset \mathfrak{g}$ be the cone of nilpotent elements. Take an arbitrary $e \in \mathfrak{N} \setminus \{0\}$. By the Morozov's theorem there exists $\mathfrak{sl}_2$-triple $\{e, h, f\}$, containing e (i.e. $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$). Semisimple element h determines a natural gradation on $\mathfrak{g}$. Put

$$\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}.$$

Then $\mathfrak{g}(0)$ is a reductive subalgebra of $\mathfrak{g}$ of the same rank and $\mathfrak{g} = \oplus_{i \in \mathbf{Z}} \mathfrak{g}(i)$ is a $\mathbf{Z}$-grading of $\mathfrak{g}$. Put $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}(i)$. This is a parabolic subalgebra of $\mathfrak{g}$, $\oplus_{i \geq 1} \mathfrak{g}(i)$ is its nilpotent radical, and $\mathfrak{l} := \mathfrak{g}(0)$ is a Levi subalgebra of $\mathfrak{p}$. Denote $G_e \subset G$ to be the stabilizer and $Ge \subset \mathfrak{g}$ to be the G-orbit (conjugacy class) of e. Given e, all $\mathfrak{sl}_2$-triples, containing e, form a single G_e-orbit. Therefore properties of $\mathbf{Z}$-grading under consideration reflect really properties of the orbit Ge only.

Definition. The integer $\max\{i \mid \mathfrak{g}(i) \neq 0\}$ is said to be the *height* of e (or the orbit Ge) and would be denoted by $\mathrm{ht}(e)$.

Since $e \in \mathfrak{g}(2)$, we have $\mathrm{ht}(e) \geq 2$ for any $e \in \mathfrak{N} \setminus \{0\}$. There also is the other way to define the height of e. Choose a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ in such a way that $\mathfrak{h} \subset \mathfrak{l}$ and $\mathfrak{b} \subset \mathfrak{p}$. Let Σ_+ be the set of positive roots with respect of $(\mathfrak{b}, \mathfrak{h})$ and $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots. Then $\alpha(h) \geq 0$ for any $\alpha \in \Sigma_+$. The integers $m_i = \alpha_i(h)$, $\alpha_i \in \Pi$ is said to be *numerical labels* of h. The Dynkin diagram of $\mathfrak{g}$ with numerical labels, attached to the corresponding nodes, is said to be the *characteristic* of e. It is known (see e.g. [SS]) that

(a) $m_i \in \{0, 1, 2\}$;

(b) $e, e' \in \mathfrak{N}$ lie in the same G-orbit *iff* their characteristics coincide.

Let $\Lambda \in \Sigma_+$ be the highest root, $\Lambda = \sum_{i=1}^{l} n_i \alpha_i$. The following assertion is evident now:

$$\mathrm{ht}(e) = \Lambda(h) = \sum_{i=1}^{l} m_i n_i \tag{1}$$

Remark. It is worth to mention that not every diagram with numerical labels, satisfying to condition (a), is really corresponds to a nilpotent orbit.

The following proposition contains all necessary results on structure of the stabilizer G_e and the centralizer $\mathfrak{g}_e$ (see e.g.[SS]). As a matter of fact, all the assertions concerning $\mathfrak{g}_e$ are immediate consequences of the theory of $\mathfrak{sl}_2$-representations.

(2.2) Proposition. (i) *Lie algebra $\mathfrak{g}_e$ (resp. $\mathfrak{g}_f$) is positively (resp. negatively) graded; $\mathfrak{g}_e = \oplus_{i\geq 0}(\mathfrak{g}_e)_i$, where $(\mathfrak{g}_e)_i \subset \mathfrak{g}(i)$ and similarily for $\mathfrak{g}_f$;*
(ii) $\mathfrak{k} := (\mathfrak{g}_e)_0 = (\mathfrak{g}_f)_0$ *is a Levi subalgebra of both $\mathfrak{g}_e$ and $\mathfrak{g}_f$;*
(iii) *For every i there are the K-invariant direct sum decompositions:*

$$\mathfrak{g}(i) = (\mathfrak{g}_e)_i + [f, \mathfrak{g}(i+2)], \ \mathfrak{g}(i) = (\mathfrak{g}_f)_i + [e, \mathfrak{g}(i-2)]$$

In particular, $\mathrm{ad} f : \mathfrak{g}(i) \to \mathfrak{g}(i-2)$ *is injective, if $i \geq 1$ and surjective, if $i \leq 1$;*
(iv) $[\mathfrak{l}, e] = \mathfrak{g}(2)$ *and $K := L_e$ is a Levi subgroup of G_e;*
(v) $\dim G_e = \dim\mathfrak{g}(0) + \dim\mathfrak{g}(1)$, $\dim(G_e)^u = \dim\mathfrak{g}(1) + \dim\mathfrak{g}(2)$.
□

It follows from the proposition that parabolic subgroup P with $\mathrm{Lie}P = \mathfrak{p}$ yields a right inclusion $G_e \subset P$.

(2.3) Now let us reformulate theorem 1.2 for nilpotent orbits. Keep the previous notation.

Theorem. *Suppose $e \in \mathfrak{N} \setminus \{0\}$, $L = G(0)$, $K = (G_e)_0$, and S is a s.g.p. for the linear action $(K : \mathfrak{g}(2))$. Then*

$$c_G(Ge) = c_L(L/K) + c_S(\oplus_{i\geq 3}\mathfrak{g}(i)),$$

$$r_G(Ge) = r_L(L/K) + r_S(\oplus_{i\geq 3}\mathfrak{g}(i)).$$

Proof. The groups K and L here have the same sense as in 1.2. Since K is the stabilizer of a point from the open L-orbit in $\mathfrak{g}(2)$, the representation $(K : \mathfrak{g}(2))$ is nothing else but the co-isotropy representation of the pair $K \subset L$. Whence S also has the same sense as in 1.2. Finally $S \subset G_e \cap G_f = K$ and it follows from 2.2(iii) that the application of $\mathrm{ad} f$ yields the isomorphism of K- and S-modules $\mathfrak{p}^u/\mathfrak{g}_e^u$ and $\oplus_{i\geq 3}\mathfrak{g}(i)$. □
Recall that a nilpotent element e is called *distinguished*, if $\mathfrak{g}_e$ does not contain semisimple elements, i.e. $\mathfrak{k} = \{0\}$.

(2.4) Corollary. *Suppose e is a distinguished nilpotent element. Then* $c_G(Ge) = \dim Ge - \dim B$ *and* $r_G(Ge) = \mathrm{rk}\mathfrak{g}$.

Proof. It immediately follows that $\mathfrak{s} = \{0\}$. Therefore $r_G(Ge) = r_L(L) = \mathrm{rk}L = \mathrm{rk}G$. Also $c_G(Ge) = c_L(L) + \dim(\oplus_{i\geq 3}\mathfrak{g}(i))$. Then applying 2.2(v) one get the assertion for the complexity. □

3 Spherical nilpotent orbits

(3.1) Let us formulate the main result of the paper.

Theorem. *Let e be a non-zero nilpotent element in a semisimple Lie algebra $\mathfrak{g}$. Then $c_G(Ge) = 0$ iff $\mathrm{ht}(e) = 2$ or 3.*

The proof of the theorem consists of a series of statements for different values of $\mathrm{ht}(e)$. We shall say e is spherical *iff* the orbit Ge is spherical.

(3.2) We start with an auxiliary assertion, which is an application of the general formulas of ch.1. Suppose $\mathfrak{g}$ is reductive and $\mathfrak{g} = \oplus_{i\in\mathbf{Z}}\mathfrak{g}(i)$ is an arbitrary $\mathbf{Z}$-grading, not necessarily arising from a nilpotent element of $\mathfrak{g}$. It has been shown in [Vi] that $G(0)$ has finitely many orbits in $\mathfrak{g}(i)$ for any $i \in\mathbf{Z}$. Whence every $\mathfrak{g}(i)$ contains an open $G(0)$-orbit. Put $d = \max\{i \mid \mathfrak{g}(i) \neq 0\}$.

Proposition. *If $i > d/2$, then the open $G(0)$-orbit in $\mathfrak{g}(i)$ is spherical.*

Proof. Passing to the reductive $\mathbf{Z}$-graded subalgebra $\mathfrak{g}(-i)\oplus\mathfrak{g}(0)\oplus\mathfrak{g}(i) \in \mathfrak{g}$, one immediately reduces to the case $i = d = 1$. Therefore without loss of generality assume that $\mathfrak{g} = \mathfrak{g}(-1)\oplus\mathfrak{g}(0)\oplus\mathfrak{g}(1)$. Define a linear transformation θ of $\mathfrak{g}$ as follows: $\theta\mid_{\mathfrak{g}(-1)\oplus\mathfrak{g}(1)}= -id$, $\theta\mid_{\mathfrak{g}(0)}= id$. Clearly this is a Lie algebra automorphism. Therefore $\mathfrak{g}(0)$ is a *symmetric* subalgebra of $\mathfrak{g}$ and hence $G/G(0)$ is spherical. Making apply theorem 1.2 to the right inclusion $G(0) \subset P$, where $\mathfrak{p} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$, one get: $0 = c_G(G/G(0)) = c_{G(0)}(P/G(0)) = c_{G(0)}(\mathfrak{g}(1))$. But the last term is equal to the complexity of the open $G(0)$-orbit in $\mathfrak{g}(1)$. □

(3.3) Keep the notation of ch.2.

Proposition. *Suppose $\mathrm{ht}(e) \leq 3$. Then $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{l}$, in particular, $c_L(L/K) = 0$.*

Proof. Since $e \in \mathfrak{g}(2)$ and Le is dense in $\mathfrak{g}(2)$, the second assertion already follows from 3.2. In order to prove the first one we use the $\mathfrak{sl}_2$-triple, containing e. Since both K and L are reductive, we have the direct sum decomposition $\mathfrak{l} = \mathfrak{k}\oplus\mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{l}$. It suffices to prove that this one is a $\mathbf{Z}_2$-grading of $\mathfrak{l}$, i.e. $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$. By 2.2(iii) we know that $\mathfrak{m} = (\mathrm{ad}f)\mathfrak{g}(2)$ and $\mathfrak{k} = (\mathfrak{g}_e)_0$. Hence the following relation should be derived. For any $x_1, x_2 \in \mathfrak{g}(2)$

$$[[[f, x_1], [f, x_2]], e] = 0.$$

This is a trivial exercise on the Jacobi identity. The only point, where the condition $\mathrm{ht}(e) \leq 3$ is used, is that $[x, y] = 0$ for any $x, y \in \mathfrak{g}(2)$. (Hence it suffices to assume that $\mathfrak{g}(4) = 0$.) □

Corollary. *If* $\mathrm{ht}(e) = 2$, *then* e *is spherical and* $r_G(Ge) = r_L(L/K)$. □

(3.4) Proposition. *If* $\mathrm{ht}(e) \geq 4$, *then* e *is not spherical.*

Proof. By 2.3 it suffices to prove that $c_S(\oplus_{i\geq 3}\mathfrak{g}(i)) > 0$.
(a) First suppose $d = \mathrm{ht}(e) \geq 5$. Then $\mathfrak{g}(d)$ and $[f, \mathfrak{g}(d)] \subset \mathfrak{g}(d-2)$ are two isomorphic S-modules, which are contained in $\oplus_{i\geq 3}\mathfrak{g}(i)$. Therefore a Borel subgroup of S has a non-constant invariant rational function on $\oplus_{i\geq 3}\mathfrak{g}(i)$, i.e. $c_S(\oplus_{i\geq 3}\mathfrak{g}(i)) \geq 1$.
(b) Assume $d = 4$. Again the application of $\mathrm{ad} f$ give us a K-submodule in $\mathfrak{g}(2)$, which is isomorphic to $\mathfrak{g}(4)$. Hence $\mathfrak{g}(2) \cong \mathfrak{g}(2) \oplus W$, where W is a complementary K-submodule. By definition S is a s.g.p. for the linear action $(K : \mathfrak{g}(2))$ (cf 2.3), i.e. $K(\mathfrak{g}(2)^S)$ is dense in $\mathfrak{g}(2)$. Whence $\mathfrak{g}(4)^S \neq 0$. But the latter clearly implies $c_S(\mathfrak{g}(4)) \geq 1$. □

Corollary *(of the proof.). If* $\mathrm{ht}(e) \geq 5$ *and* $\mathfrak{g}(4) \neq 0$, *then* $c_G(Ge) \geq 2$. □

(3.5) It remains to elaborate the case $\mathrm{ht}(e) = 3$. After 3.3 we know that $c(L/K) = 0$, i.e. in order to finish our consideration we have to prove that $c_S(\mathfrak{g}(3)) = 0$. Unfortunately at this point our arguments use case-by-case considerations. We check directly this condition for all nilpotent orbits of the height 3 in simple Lie algebras. First we shall consider the exceptional simple Lie algebras, while the case of the classical ones would be postponed until ch.4.

(3.6) Inspecting the Elashvili's tables of the characteristics of nilpotent orbits in exceptional Lie algebras [E], one easily finds by using (1) all $e \in \mathfrak{N}\backslash 0$ with $\mathrm{ht}(e) \leq 3$. Let us indicate the numbers of orbits obtained (in the brackets the numbers of e with $\mathrm{ht}(e) = 3$ are given).
$\mathbf{G}_2$ - 2(1), $\mathbf{F}_4$ - 3(1), $\mathbf{E}_6$ - 3(1), $\mathbf{E}_7$ - 5(2), $\mathbf{E}_8$ - 4(2).
The necessary information concerning the orbits of the height 3 is gathered in the table below. We draw the Dynkin diagram in such a way that the *left* node always corresponds to a *short* root. A remarkable a posteriori fact is that the representation $(S : \mathfrak{g}(3))$, modulo uneffectivity kernel, always appears to be the sum of copies of 2-dimensional representations of different groups SL_2. This clearly implies that $B(S)$ has an open orbit in $\mathfrak{g}(3)$, i.e. $c_S(\mathfrak{g}(3)) = 0$. In the column $(S : (\mathfrak{g}(3))$ the highest weights of irreducible S-submodules of $\mathfrak{g}(3)$ are indicated. The Lie algebra of an n-dimensional torus is denoted by $\mathfrak{t}_n$.

Table

Group	Characteristic	$\mathfrak{k}$	$\mathfrak{s}$	$(S:\mathfrak{g}(3))$	$r(Ge)$
$\mathbf{G}_2$	1-0	$\mathbf{A}_1$	$\mathbf{A}_1$	φ_1	2
$\mathbf{F}_4$	0-0-1-0	$\mathbf{A}_1\times\mathbf{A}_1$	$\mathbf{A}_1$	φ_1	4
$\mathbf{E}_6$	$\overset{0}{\text{0-0-1-0-0}}$	$\mathbf{A}_2\times\mathbf{A}_1$	$\mathbf{A}_1\times\mathfrak{t}_2$	φ_1	4
$\mathbf{E}_7$	$\overset{0}{\text{0-0-0-0-1-0}}$	$\mathbf{C}_3\times\mathbf{A}_1$	$(\mathbf{A}_1)^4$	$\varphi_1^{(1)}$	4
	$\overset{1}{\text{1-0-0-0-0-0}}$	$\mathbf{C}_3$	$(\mathbf{A}_1)^3$	$\varphi_1^{(1)}+\varphi_1^{(2)}+\varphi_1^{(3)}$	7
$\mathbf{E}_8$	$\overset{0}{\text{0-1-0-0-0-0-0}}$	$\mathbf{F}_4\times\mathbf{A}_1$	$\mathbf{D}_4\times\mathbf{A}_1$	$1\otimes\varphi_1$	4
	$\overset{1}{\text{0-0-0-0-0-0-0}}$	$\mathbf{C}_4$	$(\mathbf{A}_1)^4$	$\varphi_1^{(1)}+\ldots+\varphi_1^{(4)}$	8

4 Spherical nilpotent orbits in classical Lie algebras

(4.1) In $\mathfrak{sl}(V)$, $\mathfrak{sp}(V)$, $\mathfrak{so}(V)$ it is more natural to describe nilpotent matrices by the sizes of the blocks in the Jordan normal form. We shall identify the nilpotent orbits and the associated partitions $(a_1,\ldots,a_t)$, where $a_1\geq a_2\geq\ldots\geq a_t$ and $\sum_{i=1}^t a_i=\dim V$. The only restriction is that for $\mathfrak{sp}(V)$ the blocks of *odd* size occur *pairwise* and for $\mathfrak{so}(V)$ the blocks of *even* size occur *pairwise*. There is also a delicate point for $\mathfrak{so}(V)$, $\dim V\equiv 0(\text{mod}4)$. Partitions with all blocks of even sizes (occurring pairwise) comes from two different $SO(V)$-orbit. But the inner automorphism of $SO(V)$ permutes these orbits and therefore they have the same rank and complexity.

There exist simple algorithms how to produce the characteristic of a nilpotent matrix from its Jordan normal form [SS, ch.4]. Making use this way we derive a more transparent characterization of spherical nilpotent orbits.

(4.2) $\mathfrak{g}=\mathfrak{sl}_{n+1}=\mathbf{A}_n$ or $\mathfrak{g}=\mathfrak{sp}_{2n}=\mathbf{C}_n$.
Characteristics of nilpotent matrices in $\mathfrak{sl}_{n+1}$ are invariant with respect of the automorphism of the Dynkin diagram and the ones in $\mathfrak{sp}_{2n}$ have even numerical label on the long simple root. Whence a direct computation allow us to derive the following curious fact.

Assertion. There are no nilpotent elements of the height 3 in $\mathbf{A}_n$ and $\mathbf{C}_n$. Let us consider nilpotent orbits of the height 2.

(a) $\mathbf{A}_n$. If a nilpotent matrix has the height 2, then its characteristic have to look as follows:

$$h_i = (\underbrace{0\ldots01}_{i}0\ldots0\underbrace{10\ldots0}_{i}),\ 1 \leq i < \frac{n+1}{2} \text{ or}$$

$h_{\frac{n+1}{2}} = (0\ldots020\ldots0)$. These are really characteristics and the associated partition is $e_i = (2^i, 1^{n-2i+1})$. Evidently all nilpotent matrices without blocks of the size > 2 occur in this way. By using 3.2 Corollary it is easy to determine the rank of Ge_i. Here $L = (SL_i)^2 \times SL_{2n-2i+1} \times (k^*)^2$ and $K = SL_i \times SL_{2n-2i+1} \times k^*$, where $K \supset SL_i \cong \Delta \subset (SL_i)^2 \subset L$. Therefore $r_G(Ge_i) = r_L(L/K) = i = \text{rank}(e_i)$ (the obvious rank of a matrix).

(b) $\mathbf{C}_n$. If a nilpotent matrix has the height 2, then its characteristic is of the form

$$h_i = (\underbrace{0\ldots01}_{i}0\ldots0),\ i < n \text{ or } h_n = (0\ldots02).$$

The corresponding nilpotent matrix e_i has the Jordan form $(2^i, 1^{2n-2i})$. Here $L = SL_i \times Sp_{2n-2i} \times k^*$ and $K = O_i \times Sp_{2n-2i}$. Therefore $r_G(Ge_i) = r_L(L/K) = i = \text{rank}(e_i)$. Again these are all nilpotent symplectic matrices without blocks of the size > 2. Summarazing we get

Theorem. *Let e be a nilpotent matrix in $\mathfrak{sl}(V)$ or $\mathfrak{sp}(V)$.*

1. *The following conditions are equivalent:*
   (i) *e is spherical;*
   (ii) *$e^2 = 0$ (the obvious matrix multiplication);*
   (iii) $\text{ht}(e) = 2$.
2. *If $e = e_i$, then $r(Ge_i) = i$.* □

(4.3) $\mathfrak{g} = \mathfrak{so}(V)$ and $\dim V \geq 7$, i.e. $\mathfrak{g} = \mathbf{B}_n$ or $\mathbf{D}_n$.

Straightforward computations give us:

(a) $\text{ht}(e) \leq 3$ *iff* e has at most 1 block of the size 3 and all others have the size ≤ 2 *iff* $\text{rank}(e^2) \leq 1$.

(b) $\text{ht}(e) = 3$ *iff* e has exactly 1 block of the size 3 and at least 2 blocks of the size 2 *iff* $\text{rank}(e^2) = 1$ and $\text{rank}(e) > 2$.

In order to finish the proof of theorem 3.1 we explicitly describe the situation around nilpotent matrices of the height 3. Suppose $e = (3, 2^{2t}, 1^l)$ and $t > 0$. If $l = 2s$, then $e \in \mathbf{B}_{2t+s+1}$ and the numerical labels of the characteristic are:

$$m_i = \begin{cases} 1, & \text{if } i = 1, 2t+1 \\ 0, & \text{otherwise} \end{cases}$$

If $l = 2s+1$, then $e \in \mathbf{D}_{2t+s+2}$ and in the case $s > 0$ the numerical labels are the same as for $\mathbf{B}$. If $s = 0$, then

$$m_i = \begin{cases} 1, & \text{if } i = 1, 2t+1, 2t+2 \\ 0, & \text{otherwise} \end{cases}$$

However by using [SS, ch.4] it is easy to write explicitly the diagonal matrix h and to determine the subalgebra $\mathfrak{l} = \mathfrak{g}(0)$ and $\mathfrak{g}(0)$-modules $\mathfrak{g}(i)$.

If $e = (3, 2^{2t}, 1^l)$, then $\mathfrak{l} = \mathfrak{sl}_{2t} \times \mathfrak{so}_{l+1} \times \mathfrak{t}_2$, $\mathfrak{k} = \mathfrak{sp}_{2t} \times \mathfrak{so}_l$, and $\mathfrak{s} = (\mathfrak{sl}_2)^t \times \mathfrak{so}_{l-1}$. Here $\dim\mathfrak{g}(3) = 2t$ and the representation $(S : \mathfrak{g}(3))$ is the direct sum of t copies of the 2-dimensional representations of different SL_2's (SO_{l-1} does not act on $\mathfrak{g}(3)$). Whence $B(S)$ has an open orbit in $\mathfrak{g}(3)$. Thus the proof of theorem 3.1 is completed.

Remark. For all nilpotent orbits of the height 3 representations $(S : \mathfrak{g}(3))$ always have the same structure (cf. also 3.6). I think there exists a reasonable explanation of this phenomenon.

(4.5) There are no difficulties in computing the ranks of nilpotent $SO(V)$-orbits; e.g. all data for the case of the height 3 are given above. Afterwards, our results for $G = SO(V)$ may be summarized as follows.

Theorem. *Let $e \in \mathfrak{so}(V)$ be a nilpotent matrix.*

1. *The following conditions are equivalent:*
   (i) *e is spherical;*
   (ii) $\operatorname{rank}(e^2) \le 1$.
2. *If $e^2 = 0$ and* $\operatorname{rank}(e) = 2t$, *then $r(Ge) = t$.*
3. *If* $\operatorname{rank}(e^2) = 1$ *and $e = (3, 2^{2t}, 1^l)$, then* $\operatorname{rank}(e) = 2t+2$ *and*

$$r(Ge) = \begin{cases} 2t+2, & \text{if } l > 0 \\ 2t+1, & \text{if } l = 0. \end{cases} \quad \square$$

5 Concluding remarks

(5.1) As usual in theory of semisimple Lie algebras, any problem on properties of adjoint orbits may be reduced to semisimple or nilpotent ones. Not surprising that this is the case for the complexity and the rank.

In this section it would be shown that complexity and rank are constant along sheets of $\mathfrak{g}$. Recall that a *sheet* of $\mathfrak{g}$ is an irreducible component of the variety of adjoint orbits of a fixed dimension. Each sheet contains exactly one nilpotent orbit [BK].

Let $\mathcal{O} \subset \mathfrak{g} \setminus \mathfrak{N}$ be an orbit. There are 2 ways to construct the nilpotent orbit of the sheet that contains $\mathcal{O}$. The first one uses the parabolic induction [LS], while the second one, which we shall use, is more geometric (cf. [BK]).

Let us recall the related constructions. Put $X = \overline{k^*\mathcal{O}} \subset \mathfrak{g}$. This is a closed G-invariant cone and $\dim X = \dim \mathcal{O} + 1$. The algebra of G-invariant regular functions on X is generated by a (homogeneous) polynomial π. Let $\pi : X \to \mathrm{A}^1$ be the corresponding morphism. Denote X_τ to be the fiber of π over $\tau \in \mathrm{A}^1$.

(5.2) Proposition. (cf. [BK])
(i) $\overline{\mathcal{O}} = X_\tau$ for some $\tau \neq 0$;
(ii) $X_0 = X \cap \mathfrak{N}$;
(iii) X_0 is irreducible and contains a unique dense G-orbit, say $\mathcal{O}_n$.
□
It immediately follows from the proposition that $\mathcal{O}$ and $\mathcal{O}_n$ belong to the same sheet and $\mathcal{O}_n$ is the nilpotent orbit of this sheet.

(5.3) Proposition. *$c_G(X_\tau) = c_G(X_0)$ and $r_G(X_\tau) = r_G(X_0)$ for any $\tau \in \mathrm{A}^1 \setminus \{0\}$.*

Proof. Since π is homogeneous, one can see that $X_\tau \cong X_{\tau'}$, if $\tau, \tau' \in \mathrm{A}^1 \setminus \{0\}$. Therefore, if B_* is a s.g.p. for the action $(B : X)$, then it also is a s.g.p. for the action $(B : X_\tau)$, $\tau \neq 0$. Whence $\dim B_y \geq \dim B_*$ for any $y \in X_0$ and $c_G(X_\tau) \leq c_G(X_0)$. Having replaced B on U in preceding arguments, one get $c_G(X_\tau) + r_G(X_\tau) \leq c_G(X_0) + r_G(X_0)$. (cf. ch.1)

In order to prove the converse we compare the *multiplicities* of irreducible G-modules in the algebras of regular functions on X_τ and X_0. We let $m_V(X)$ denote the multiplicity of an irreducible G-module V in the algebra $k[X]$. In the notations of [BK] we have $X_0 = \mathcal{K}X_\tau$ (the associated cone), and therefore by lemma 3.2 in [loc. cit.]

$$m_V(X_\tau) \geq m_V(X_0) \tag{2}$$

for any V. This unequality implies $r_G(X_\tau) \geq r_G(X_0)$. It has been shown in [P2] that complexity is fully determined through the growth of multiplicities. Whence (2) implies $c_G(X_\tau) \geq c_G(X_0)$. □

(5.4) It easily follows from the definitions that the complexity and the rank are birational characteristics of actions. Therefore 5.2 and 5.3 imply

Proposition. *The rank and the complexity of orbits are constant along the sheets of the adjoint representation* $(G : \mathfrak{g})$. □

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