Generic smoothness of the moduli of rank two stable bundles over an algebraic surface

by

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§0. Introduction

Let X be a complex algebraic surface with canonical line bundle K, and V be a rank 2 vector bundle over X. We fix an ample line bundle H on X. Recall that V is H-stable, if for all sub-line bundles $L \to V$ we have

$$c_1(L)c_1(H) < c_1(V)c_1(H)/2 \quad .$$

We denote by $M(k)$ the moduli space of all isomorphic classes of H-stable rank 2 vector bundles with fixed determinant bundle D and second Chern class k. It is well known that $M(k)$ is a quasi projective variety, and is not empty if k is sufficently large (see [G] and [M]).

The local structure of $M(k)$ is precisely described by the Kuranishi deformation theorem in the following way. Suppose $V \in M(k)$ and let $\psi : H^1(End_0(V)) \to H^2(End_0(V))$ be Kuranishi map, then the stalk of the structure sheaf $\mathcal{O}_{M(k)}$ at V is naturally isomorphic to the germs of holomorphic functions at $0 \in H^1(End_0(V))$ divided by the ideal generated by the components of ψ. In particular, if $H^2(End_0(V)) = 0$, then $M(k)$ is smooth at V, and the tangent space $T(M(k))_V$ is identified with $H^1(End_0(V))$. The Hirzebruch-Riemann-Roch theorem gives the dimension formula

$$dim\, M(k)_V = -\chi(End_0(V)) = 4k - D^2 - 3\chi(\mathcal{O}_X) \quad .$$

The next step is naturally to study the locus of all $V \in M(k)$ with $H^2(End_0(V) \simeq H^0(End_0(V) \otimes K)^\vee \neq 0$. It is a closed subvariety of $M(k)$. This can be seen by applying the upper semicontinuous theorem to the local universal bundle of $M(k)$. More precisely, Donaldson proved recently the following theorem for the case $D = 0$.

Theorem 1. (Donaldson)

Suppose X is an algebraic surface with canonical line bundle K. Let $M(k)$ be the moduli space of H-stable rank 2 bundles with the trivial determinant bundle and second Chern class k. Then subvariety $\Sigma(k) := \{ V \in M(k) \,|\, H^0(End_0(V) \otimes K) \neq 0\}$ has dimension

$$dim\, \Sigma(k) \leq 3k + A\sqrt{k} + A \quad ,$$

here A is a positive number which depends on the linear system $|2K|$, the Chern classes of X, H only.

Corollary 1.

Every irreducible component of $M(k)$ is reduced, and has dimension $-\chi(End_0(V))$, if k is sufficently large.

Corollary 1 has the following important application in the study of the differentiable structure of 4-manifolds (see [D])

Corollary 2.

The k-th $SU(2)$-invariant on an algebraic surface does not vanish, if k is sufficently large.

In this paper we use the original idea of Donaldson [D] and the important technique due to Friedman [F] and generalize theorem 1 for any case.

Theorem 2.

Let X and K be same as in theorem 1. Let $M(k)$ be the moduli space of H-stable rank 2 bundles with the fixed determinant bundle D and second Chern class k. Then the subvariety $\Sigma(k) := \{ V \in M(k) \,|\, H^0(End_0(V) \otimes K) \neq 0 \}$ has dimension at most $3k + A\sqrt{k} + A$, here A is a positive number which depends on the linear system $|2K|$ the Chern classes of X, H and D only.

Similar as corollary 2 theorem 2 implies immediately the non-vanishing property for the $SO(3)$-invariants on algebraic surfaces. (see [D] and [OV])

The outline of proof for theorem 2 as follows. First, we divide $\Sigma(k)$ into two subsets:
$\Sigma_1(k) := \{ V \in \Sigma(k) \,|\, \exists\, s_{\neq 0} \in H^0(End_0(V) \otimes K), \quad \exists\, t \in H^0(K) \quad \text{s.t.} \quad det(s) + t^2 = 0 \}$, and
$\Sigma_2(k) := \Sigma(k) \setminus \Sigma_1(k)$.
It is easy to see, $\Sigma_1(k)$ is a closed subvariety of $\Sigma(k)$ by looking at the local universal bundle of $M(k)$.
In section 1 we find some special sub-line bundles L of $V \in \Sigma_1(k)$ so that the absolut values $|LH|$ are bounded by a constant depending on KH and DH only. By standard arguments we estimate dimension of the moduli of all extensions

$$0 \longrightarrow \mathcal{O}_X(L) \longrightarrow V \longrightarrow \mathcal{O}_X(-L+D) \otimes I_z \longrightarrow 0 \quad ,$$

hence we get the upper bound of dimension for $\Sigma_1(k)$.
The second section is more interesting. Inspired by the idea of R. Friedman ([F]), and using the spectral surface technique ([BNR], [D] and [Hi]), we show that for any pair (V, s) of $V \in \Sigma_2(k)$ and $s_{\neq 0} \in H^0(End_0(V) \otimes K)$, there exists the following exact sequence on the blowing up $\sigma : \widehat{X} \to X$ at the singularities of the zero locus $(det(s))_0$

$$0 \longrightarrow W \longrightarrow \sigma^* V \longrightarrow Q \longrightarrow 0 \quad ,$$

here W is a rank 2 vector bundle coming from the direct image of a line bundle on a double covering $Y' \to \widehat{X}$ ramified along some components of $\sigma^*(det(s))_0$, and Q is a torsion sheaf, its scheme theoretically support is also some components of $\sigma^*(det(s))_0$.
Using the deformation theorem of torsion sheaves due to Friedman ([F]) we bound dimension of the moduli of all the above extensions, therefore we obtain the upper bound for $\mathbf{\Sigma}_2(k)$.

In section 3 we complete proofs of the claims which are used in the previous sections.

In our paper the symbol A always means a constant positive number which depends on the linear system $|2K|$, the Chern classes of X, H and D only.

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§1. To bound $\dim \mathbf{\Sigma}_1(k)$

The goal of this section is to prove the following

Lemma 1.

$$dim\, \mathbf{\Sigma}_1(k) \le 3k + A\sqrt{k} + A \quad .$$

1.1. The eigen-line bundles of (V, s) from $\mathbf{\Sigma}_1(k)$

Suppose $V \in \mathbf{\Sigma}_1(k)$, taking a non-zero section $s \in H^0(End_0(V) \otimes K)$ with $det(s) + t^2 = 0$, $t \in H^0(K)$ we get the non-trivial maps

$$V \xrightarrow{s - I \otimes t} V \otimes K \quad ,$$

$$V \xrightarrow{s + I \otimes t} V \otimes K \quad .$$

Because their determinant maps are zero map, the kernel $\mathcal{O}_X(L_{\pm})$ of the maps are line bundles on X. Their fibres at a point p are just the eigen-vectors of the linear map $s_p : V_p \to V_p \otimes K_p$, so they will be reasonable called as the eigen-line bundles of (V, s). we have the following exact sequence

$$(1.1) \qquad 0 \longrightarrow \mathcal{O}_X(L_\pm) \longrightarrow V \longrightarrow \mathcal{O}_X(-L_\pm + D) \otimes I_{z_\pm} \longrightarrow 0 \quad ,$$

where $I_{z_\pm}$ are ideal sheaves which define the 0-dimensional subschemes $z_\pm$ of X. A calculation of Chern classes gives

$$(1.2) \qquad -(L_\pm)^2 + (L_\pm)D + |z_\pm| = k \quad .$$

We consider the following commudative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(L_+) & \longrightarrow & V & \longrightarrow & \mathcal{O}_X(-L_+ + D)I_{z_+} & \longrightarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle s - I\otimes t} & & & & \\
0 & \longleftarrow & I_{z_-}\mathcal{O}_X(-L_- + D + K) & \longleftarrow & V \otimes K & \longleftarrow & \mathcal{O}_X(L_- + K) & \longleftarrow & 0 \\
 & & & & \big\downarrow{\scriptstyle s + I\otimes t} & & & & \\
 & & & & V \otimes (2K) \quad . & & & &
\end{array}$$

Noting the composition map $(s + I \otimes t)(s - I \otimes t) = (s^2 - I \otimes t^2) = -I \otimes (det(s) + t^2) = 0$, we get the non-trivial factor map

$$\mathcal{O}_X(-L_+ + D) \to \mathcal{O}_X(L_- + K).$$

This implies that $(-L_+ + D)H \leq (L_- + K)H$. And the stability of V gives $L_\pm H < DH/2$. We put these inequalities together and obtain

$$(1.3) \qquad -KH \leq (L_+ - D/2)H < 0 \quad .$$

Of course, we have also the above inequality for L_-. In the rest of this section we are just interested in one eigen-line bundle of (V, s), so we write $L_\pm$ as L simplely. From (1.2) and (1.3) we have

Claim 1.1

For all $V \in \Sigma_1(k)$, and all $s \in H^0(End_0(V) \otimes K)$, with $det(s) + t^2 = 0$, let L be an eigen-line bundle of (V, s). Then we have

1) $(L - D/2)K \leq A\sqrt{k} + A$

2) $(L - D/2)^2 \leq A$

3) $h^0(-2L+D+K)+h^0(2L-D)\leq A$

4) The subset $\{c_1(L)\}\subset H^2(X,Z)$ is finite.

1.2. The moduli of $\Sigma_1(k)$

From 4) in claim 1.1 and by standard arguments we decompose $\Sigma_1(k)$ into finitely many subvarieties

$$\Sigma_1(k)=\bigcup_i \Sigma_{1,i}(k) \quad ,$$

so that for any $V\in\Sigma_{1,i}(k)$ V comes from the extension (1.1) with same $c_1(L)$.

The variety $\Sigma_{1,i}(k)$ has a stratification

$$\Sigma_{1,i}(k)=\bigsqcup_j \Sigma_{1,i,j}(k) \quad ,$$

$\Sigma_{1,i,j}$ is a subvariety of $\Sigma_1(k)$, and for any $V\in\Sigma_{1,i,j}(k)$ the extension group $Ext^1_{\mathcal{O}_X}(\mathcal{O}_X(-L+D)\otimes I_z,\mathcal{O}_X(L))$ has constant dimension j.

Locally see, the moduli $\Sigma_{1,i,j}(k)$ at V is parametrized by two varieties. One is the extension group $Ext^1_{\mathcal{O}_X}(\mathcal{O}_X(-L+D)\otimes I_z,\mathcal{O}_X(L))$. The another is the subvariety of all pairs $(z,L)\in Hilb^{|z|}(X)\times Pic(X)$ satisfying $dim\, Ext^1_{\mathcal{O}_X}(\mathcal{O}_X(-L+D)\otimes I_z,\mathcal{O}_X(L))=j$.

Hence we have roughly the following estimate

$$\begin{aligned}
&dim\,\Sigma_{1,i,j}(k)\\
&\leq dim\,Ext^1(\mathcal{O}_X(-L+D)\otimes I_z,\mathcal{O}_X(L))+2|z|+q(X)\\
(1.4)\quad &=h^1(\mathcal{O}_X(-2L+D+K)\otimes I_z)+2|z|+q(X) && \text{by Serre-Duality}\\
&\leq h^1(-2L+D+K)+3|z|+q(X) && \text{by standard exact sequence}
\end{aligned}$$

Applying Riemann-Roch-theorem to the line bundle $\mathcal{O}_X(-2L+D+K)$, we have

$$\begin{aligned}
&h^1(-2L+D+K)\\
&=-2(L-D/2)^2+(L-D/2)K\\
&\quad +h^0(2L-D)+h^0(-2L+D+K)-\chi(\mathcal{O}_X)\\
(1.5)\quad &\leq -2(L-D/2)^2+A\sqrt{k}+A && \text{by 1), 3) in claim 1.1}\\
&=(L-D/2)^2-3D^2/4+A\sqrt{k}+A+3k-3|z| && \text{by (1.2)}\\
&\leq 3k+A\sqrt{k}+A-3|z| \quad . && \text{by 2) in claim 1.1}
\end{aligned}$$

Finally we put (1.4) and (1.5) together and complete lemma 1.

Remark 1.

In fact, we can prove $dim\, \Sigma_1(k) \leq 3k + A$. But it needs more complicated technical lemmas. For example, the lemma about dimension of varieties of 0-dimensional subschemes in the special position respect to a linear system ([Z], lemma 1).

§2. To bound $dim\, \Sigma_2(k)$

We will prove the following

Lemma 2.

$$dim\, \Sigma_2(k) \leq 3k + A \quad .$$

The proof will be divided into two parts.

2.1. The spectral surface of (V, s) from $\Sigma_2(k)$ (see [BNR], [D] and [Hi])

We take a section $s \in H^0(End_0(V) \otimes K)$ with the non zero determinant $det(s) \in H^0(2K)$. Its zero locus is a curve C in the linear system $|2K|$. We blow up successivelly the singularities of C

$$\widehat{X} \xrightarrow{\ \sigma\ } X \quad , \tag{2.1}$$

which satisfies the following

Condition 2.1

1) All reduced irreducible components of the pull back $\sigma^* C = \sum_i (2p_i + 1) C_i + \sum_j 2q_j C_j$ are smooth curves, and they transversally intersect each other.

2) The irreducible components with the odd multiplicities are disjoint.

Of course, such a blowing up does exist. Its numerical invariants depend on the numerical invariants of the singularities of C only.

We denote by, $\tilde{V} := \sigma^* V$, $\tilde{K} := \sigma^* K$, $\tilde{H} := \sigma^* H$ and $\tilde{s} := \sigma^* s$. It is easy to see that $det(\tilde{s})$ has the zero locus $\sigma^* C$.

By taking the square root $\sqrt{-det(\tilde{s})}$ we get a double covering

$$Y \xrightarrow{\ \pi\ } \widehat{X}$$

with the direct image of the structure sheaf

$$\pi_* \mathcal{O}_Y \simeq \mathcal{O}_{\widehat{X}} \oplus \mathcal{O}_{\widehat{X}}(\tilde{K}^{-1}) \quad .$$

The surface Y is reduced, and in our case is also irreducible, otherwise the section s would have the property $det(s) + t^2 = 0$. In general, Y is not normal, it has exactly the singularities along the curve $\pi^{-1}(\sum_i p_i C_i + \sum_j q_j C_j)$.
By taking the normalization of Y we obtain

$$Y' \xrightarrow{\ \nu\ } Y \xrightarrow{\ \pi\ } \widehat{X}, \qquad \rho = \pi \circ \nu : Y' \to \widehat{X}$$

with

$$\rho_* \mathcal{O}_{Y'} \simeq \mathcal{O}_{\widehat{X}} \oplus \mathcal{O}_{\widehat{X}}(\tilde{K}^{-1} + \sum_i p_i C_i + \sum_j q_j C_j)$$
$$=: \mathcal{O}_{\widehat{X}} \oplus \mathcal{O}_{\widehat{X}}(K'^{-1}) \quad .$$

The surface Y' is already smooth by the condition 2.1. In fact, it can be also constructed by taking the square root of a section from $H^0(2K')$ with the simple zero locus $\sum_i C_i$.
Looking at the direct image

$$\pi_* \pi^* \tilde{K} \simeq \tilde{K} \otimes \pi_* \mathcal{O}_Y \simeq \tilde{K} \oplus \tilde{K} \otimes \tilde{K}^{-1}$$

we get naturally a section $x \in H^0(\pi^* \tilde{K})$. The Galois-group of the covering operats on x just as multiplies it by -1. It holds $det(\pi^* \tilde{s}) + x^2 = 0$ (see [BNR]).

The twisted endomorphism $\tilde{s} : \tilde{V} \longrightarrow \tilde{V} \otimes \tilde{K}$ gives $\tilde{V}$ an $\mathcal{O}_{\widehat{X}} \oplus \mathcal{O}_{\widehat{X}}(\tilde{K}^{-1})$ ($\simeq \pi_* \mathcal{O}_Y$) module structure. By the general theorem (see [Ha], Chapter 2, prop. 5.2 and [BNR], prop. 3.6) there is a bijective correspondence between isomorphic classes of torsion free sheaves M of rank 1 on Y and isomorphic classes of pairs $(\tilde{V}, \tilde{s})$ where $\tilde{V}$ is a rank 2 bundle over $\widehat{X}$, and $\tilde{s} : \tilde{V} \longrightarrow \tilde{V} \otimes \tilde{K}$ with $Tr(\tilde{s}) = 0$ and $(det(\tilde{s}))_0 = \sigma^* C$. The correspondence is given by associating to any M to the sheaf $\pi_* M$ on $\widehat{X}$ and the natural map $\pi_* M \longrightarrow \pi_*(M \otimes \pi^* \tilde{K}) \simeq \pi_* M \otimes \tilde{K}$ induced by the direct image of the map $I \otimes x : M \longrightarrow M \otimes \pi^* \tilde{K}$.

Fixing $\widehat{X}$, we see that the moduli of $(\tilde{V}, \tilde{s})$ is parametrized by the family of the coverings $Y \longrightarrow \widehat{X}$ plus the family of the torsion free sheaves M on Y. But unfortunately, the second family is not easy to describe. To overcome this difficulty, we replace M by a suitable invertibar sheaf $\mathcal{O}_{Y'}(L)$ on Y', and use its ρ_* direct image to approach $\tilde{V}$ in the following sense

Claim 2.1 (compare [F], Chapter 5)

There exists an invertibar subsheaf $\mathcal{O}_{Y'}(L) \hookrightarrow \rho^* \tilde{V}$ *with the following properties*

1) Let $W := \rho^*(\mathcal{O}_{Y'}(L)) \otimes K'$, *then on* $\widehat{X}$ *there is an exact sequence*

$$0 \longrightarrow W \longrightarrow \tilde{V} \longrightarrow Q \longrightarrow 0 \quad ,$$

Q *is a torsion sheaf, and its scheme theoretically support* E *is some components of the zero locus of* $det(\tilde{s})$.

2) On Y' *there exists a commudative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{Y'}(L) & = & \mathcal{O}_{Y'}(L) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \rho^* W & \longrightarrow & \rho^* \tilde{V} & \longrightarrow & \rho^* Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{Y'}(-L + \rho^* \tilde{D} - \rho^* E) & \longrightarrow & \mathcal{O}_{Y'}(-L + \rho^* \tilde{D}) \otimes I_z & \longrightarrow & \rho^* Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 \; . & & & &
\end{array}$$

2.2. The moduli of pairs $(\tilde{V}, \widehat{X})$

In 2.1. we have constructed the blowing up $\widehat{X}$ and the spectral surface $Y' \to Y \to \widehat{X}$ for any pair (V, s), of $V \in \Sigma_2(k)$, and $s_{\neq 0} \in H^0(End_o(V) \otimes K)$.

Let $M(\widehat{X}, \tilde{V})$ be the moduli of all such pairs $(\widehat{X}, \tilde{V})$. We want to show

Lemma 3

$$dim\, M(\widehat{X}, \tilde{V}) \leq 3k + A \quad .$$

Lemma 2 is a direct consequence from lemma 3 by the surjective map $M(\widehat{X}, \tilde{V}) \to \Sigma_2(k)$.

First by standard arguments we have obviously the following

Lemma 2.2.

Let $M(\widehat{X}, Y', E)$ *be the moduli of all triples* $(\widehat{X}, Y', E)$, *where* $\widehat{X}$ *is a blowing up of* X *at the singularities of a curve* C *from* $|2K|$ *which satisfies the condition 2.1,* Y' *is a smooth double* *covering of $\widehat{X}$ with the branching curve contained in σ^*C, and E is also a curve contained in σ^*C. Then $M(\widehat{X}, Y', E)$ is a quasi projective variety.*

Fixing the blowing up $\widehat{X}$, we see that the moduli of $(\widehat{X}, \tilde{V})$ comes from the following three parts by 1) in claim 2.1

1) The moduli $M(W)$ of the the vector bundles W

2) The moduli $M(Q)$ of the torsion sheaves Q

3) The extension group $Ext^1_{\mathcal{O}_{\widehat{X}}}(Q, W)$

In rest of this section we want to estimate their dimension separately.

1) The moduli $M(W)$

$M(W)$ is just the moduli of pairs (L, Y') by the definition of W. All such Y' form a subvariety $M(Y')$ of the moduli $M(\widehat{X}, Y', E)$ in lemma 2.2, hence it has a bounded dimension

$$dim\, M(Y') \leq A \quad .$$

Fixing Y', there is a stratification for the moduli of all L

$$M(L) = \bigsqcup_i M_i(L)$$

so that all L from one $M_i(L)$ have same Chern class $c_1(L)$. We see easily by lemma 2.2

$$dim\, M_i(L) \leq q(Y') \leq A \quad .$$

Furthermore, we claim

Claim 2.2

$M(L)$ has only finitely many $M_i(L)$

The above inequalities and claim imply the following

Lemma 2.3

$$dim\, M(W) \leq A \quad .$$

2) The moduli $M(Q)$

We start with reviewing the h-th Fitting ideal of a torsion sheaf Q on a surface S and its basic properties (see [F], Chapter 1, (d)).

The 0-th Fitting ideal is just the ideal of the scheme theoretically support E of Q.
The 1-th Fitting ideal $I_{z(Q)}$ is the ideal of a 0-dimensional subscheme $z(Q)$. It can be defined by taking a presentation of Q, but we will not give here (see [F], definition 1.11). We are just interested in the following lemmas.

Lemma 2.4 (see [F], prop. 6.7)
Let $\rho : \tilde{S} \to S$ be a smooth double covering. Then $\rho^* z(Q) = z(\rho^* Q)$, and $|z(Q)| = |z(\rho^* Q)|/2$.

In general, $|z(Q)|$ is not easy to compute. However, we have the following estimate

Lemma 2.5 (see [F], Chapter 1, (d))
Let I_z be an ideal sheaf of a 0-dimensional subscheme z of S, z is a locally complete intersection, $\mathcal{O}_S(-C) \subset I_z$ be an ideal sheaf of a curve $C \subset S$ and $\mathcal{O}_S(D)$ be an invertibar sheaf on S. If $Q \simeq (I_z/\mathcal{O}_S(-C)) \otimes \mathcal{O}_S(D)$, then $I_z \subseteqq I_{z(Q)}$. In particular, $|z(Q)| \leq |z|$.

Using the Fitting ideals of Q we may describe the deformations of Q. This is the following lemma due to Friedman (see [F], Prop. 1.16).

Lemma 2.6
The local deformations of the sheaf Q with the fixed support E has dimension at most

$$h^1(\mathcal{O}_E) + |z(Q)| \quad .$$

Lemma 2.6 shows that the arbitrary local deformations of Q has dimension

$$dim\, Def\,(Q) \leq h^1(\mathcal{O}_E) + |z(Q)| + \text{ dimension of the moduli of } E \quad .$$

Going back to our case. All E have to be contained in some curves from the linear system $|2\tilde{K}|$ by 1) in claim 2.1, hence applying lemma 2.2 to the moduli of E we get

$$h^1(\mathcal{O}_E) + \text{ dimension of the moduli of } E \ \leq A \quad .$$

We want to bound $|z(Q)|$ in terms of k. The exact sequence in the bottom of 2) in claim 2.1 says that $\rho^* Q \simeq (I_z/\mathcal{O}_{Y'}(-\rho^* E)) \otimes \mathcal{O}_{Y'}(-L + \rho^* D)$. And applying Lemma 2.5 and 2.4 we obtain

$$|z(Q)| \leq |z|/2 + A \quad .$$

So we have to bound $|z|$.

Claim 2.3

$$|z| \leq 2k + A \quad .$$

The above four inequalities imply the following

Lemma 2.7

$$dim\, M(Q) \leq k + A \quad .$$

3) The extension group $Ext^1_{\mathcal{O}_{\hat{X}}}(Q, W)$

The following two lemmas are due to Friedman ([F], lemma 6.9, 6.10 and 6.11)

Lemma 2.8

$$dim\, Ext^1_{\mathcal{O}_{\hat{X}}}(Q, W) - dim\, Ext^2_{\mathcal{O}_{\hat{X}}}(Q, W) = \chi(\mathcal{O}_{Y'}) - \chi(\mathcal{O}_{Y'}(-\rho^* E)) + |z| \quad ,$$

here z is the subscheme of Y' in 2) in claim 2.1.

Noting z is also a subschem of $\rho^* E$, this induces the natural map

$$\mathcal{O}_{\rho^* E} \otimes \mathcal{O}_{Y'}(\rho^* E) \otimes K_{Y'} \rightarrow \mathcal{O}_z \otimes \mathcal{O}_{Y'}(\rho^* E) \otimes K_{Y'} \quad .$$

Lemma 2.9

$Ext^2_{\mathcal{O}_{\hat{X}}}(Q, W)$ is dual to the kernel of the natural map

$$H^0(\mathcal{O}_{\rho^* E} \otimes \mathcal{O}_{Y'}(\rho^* E) \otimes K_{Y'}) \rightarrow H^0(\mathcal{O}_z \otimes \mathcal{O}_{Y'}(\rho^* E) \otimes K_{Y'}) \quad .$$

Using lemma 2.2 we see that for all pairs (Y', E) the numbers $\chi(\mathcal{O}_{Y'})$, $\chi(\mathcal{O}_{Y'}(-\rho^* E))$ and $H^0(\mathcal{O}_{\rho^* E} \otimes \mathcal{O}_{Y'}(\rho^* E) \otimes K_{Y'})$ are bounded by a constant A. Hence from the lemma 2.8, 2.9 and claim 2.3 we have

Lemma 2.10

$$dim\, Ext^1_{\mathcal{O}_{\hat{X}}}(Q, W) \leq 2k + A \quad .$$

Lemma 2.3, 2.7 and 2.10 together give lemma 3.

§3. To complete proofs of the claims

Proof of claim 1.1

1) We write

$$c_1(L-D/2) = rc_1(H) + c_1(L-D/2)^{\perp} \quad , \tag{3.1}$$

where $c_1(L-D/2)^{\perp}$ is orthogonal to $c_1(H)$, and $|r| = |(L-D/2)H/H^2|$ is bounded by a constant A by (1.3). Therefore we get

$$|c_1(L-D/2)c_1(K)| \leq |c_1(L-D/2)^{\perp}c_1(K)^{\perp}| + A \quad .$$

Because the intersection form is negative definite on the orthogonal complement of $c_1(H)$ in $H^{1,1}(X)$, we have

$$\begin{aligned}
&|c_1(L-D/2)^{\perp}c_1(K)^{\perp}| \\
\leq&\sqrt{-(c_1(L-D/2)^{\perp})^2}\sqrt{-(c_1(K)^{\perp})^2} \\
\leq&A\sqrt{-(c_1(L-D/2)^{\perp})^2} \\
\leq&A\sqrt{-c_1(L-D/2)^2+A} && \text{by (3.1)} \\
=&A\sqrt{-L^2+LD-D^2/4+A} \\
\leq&A\sqrt{k+A} \quad . && \text{by (1.2)}
\end{aligned}$$

The above two inequalities imply 1).

2) By Hodge-index-theorem and (1.3) we have

$$(L-D/2)^2 \leq ((L-D/2)H)^2/H^2 \leq A \quad .$$

3) First, the stability of V shows $(2L-D)H < 0$, hence $h^0(2L-D) = 0$.

Using (1.3) we see that the absolut values $|(-2L+D+K)H|$ are bounded by a constant A'. Because all cuvers $C \subset X$ with bounded degree $CH \leq A'$ form a projective variety. In particular, all $h^0(C)$ are bounded by a constant A.

4) From the proof of 1) we see that $2H^2c_1(L-D/2)^{\perp}$ are integral classes in $H^{1,1}(X)^{\perp} \cap H^2(X,Z)$ with bounded norm $2H^2(\sqrt{k}+A)$, hence thay are finitely many. This implies all $c_1(L-D/2)^{\perp}$ are also finitely many. Noting $c_1(L-D/2) = rc_1(H) + c_1(L-D/2)^{\perp}$ with $|r| \leq A$, we complete 4).

Proof of claim 2.1

The general correspondence gives a sheaf M on Y with $\pi_* M \simeq \tilde{V}$.
Since π is affine, the natural map $\pi^*\pi_* M \to M$ is surjective, and it induces the exact sequence

$$0 \longrightarrow M' \longrightarrow \pi^*\pi_* M \longrightarrow M \longrightarrow 0 \quad .$$

Taking the direct image π_* for the above exact sequence, noting π is affine we get again an exact sequence on $\widehat{X}$

$$0 \longrightarrow \pi_* M' \longrightarrow \pi_*\pi^*\pi_* M \longrightarrow \pi_* M \longrightarrow 0 \quad .$$

The Galois-group $G(Y'/\widehat{X})$ operats on $\pi_*\pi^*\pi_* M$ and induces the ± 1-eigen spaces decompositiom

$$\pi_*\pi^*\pi_* M \simeq \pi_* M \otimes \pi_* \mathcal{O}_Y \simeq \pi_* M \oplus \pi_* M \otimes \tilde{K}^{-1} \quad .$$

The image of the natural map $\pi_* M \hookrightarrow \pi_*\pi^*\pi_* M$ is just the 1-eigen space $\pi_* M$ in the decomposition. And this map is also a section of the projection $\pi_*\pi^*\pi_* M \to \pi_* M$. Hence these show that the composition map

$$\pi_* M' \longrightarrow \pi_*\pi^*\pi_* M \longrightarrow \pi_* M \otimes \tilde{K}^{-1} \tag{3.2}$$

is an isomorphism.

On the other hand, we look at the pull back $\nu^* M' \hookrightarrow \nu^*\pi^*\pi_* M$ on the smooth surface Y'. It induces the diagram

$$\begin{array}{ccc} & \mathcal{O}_{Y'}(L_1) & \\ \nearrow & & \searrow \\ \nu^* M' & \longrightarrow & \nu^*\pi^*\pi_* M \end{array} \quad ,$$

here $\mathcal{O}_{Y'}(L_1)$ is an invertibar subsheaf of $\nu^*\pi^*\pi_* M$ with the torsion free cokernel.
Taking the direct image ν_* for the diagram we obtain the diagram on Y

$$\begin{array}{ccc} & \nu_*\mathcal{O}_{Y'}(L_1) & \\ \nearrow & & \searrow \\ \nu_*\nu^* M' & \longrightarrow & \nu_*\nu^*\pi^*\pi_* M \\ \uparrow & & \uparrow \\ M' & \longrightarrow & \pi^*\pi_* M \end{array} \quad .$$

Furthermore, we take π_* for the above diagram, and get

$$\begin{array}{ccccc} \rho_*\mathcal{O}_{Y'}(L_1) & \longrightarrow & \rho_*\rho^*\pi_*M & \xrightarrow{\sim} & \pi_*M \oplus \pi_*M \otimes K'^{-1} \\ \uparrow & & \uparrow & & \uparrow \\ \pi_*M' & \longrightarrow & \pi_*\pi^*\pi_*M & \xrightarrow{\sim} & \pi_*M \oplus \pi_*M \otimes \tilde{K}^{-1} \end{array} \quad .$$

The last vertical map splits, it maps 1-eigen space π_*M to 1-eigen space π_*M identically, and maps -1-eigen space $\pi_*M \otimes \tilde{K}^{-1}$ to -1-eigen space $\pi_*M \otimes K'^{-1}$ as the identical map multiplied by the natural map $\tilde{K}^{-1} \to K'^{-1}$ of the zero locus $\sum_i p_i C_i + \sum_j q_j C_j$. Therefore, the above diagram and (3.2) induce the following diagram

$$\begin{array}{ccccc} & & \rho_*\mathcal{O}_{Y'}(L_1) & & \\ & \nearrow & & \searrow & \\ \pi_*(M) \otimes \tilde{K}^{-1} & & \hookrightarrow & & \pi_*(M) \otimes K'^{-1} \end{array} \quad .$$

Twisting the diagram by K', and let $W_1 =: \rho_*(\mathcal{O}_{Y'}(L_1)) \otimes K'$ we obtain

(3.3) $$0 \longrightarrow W_1 \xrightarrow{\varphi_1} \tilde{V} \longrightarrow Q_1 \longrightarrow 0 \quad ,$$

Q_1 is a torsion free sheaf, and its scheme theoretically support is some components of the curve $2(\sum_i p_i C_i + \sum_j q_j C_j)$.

We see that L_1 is a line bundle which satisfies property 1) in claim 2.1.

Consider the pull back ρ^* for (3.3), noting the flatness of ρ, it is againg exact. Hence the natural map $\rho_*(W_1) \to \mathcal{O}_{Y'}(L_1 + \rho^*K')$ induces the following diagram

(3.4) $$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_{Y'}(L_1') & \longrightarrow & \rho^*W_1 & \longrightarrow & \mathcal{O}_{Y'}(L_1 + \rho^*K') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \rho^*\varphi_1} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{Y'}(L_2) & \longrightarrow & \rho^*\tilde{V} & \longrightarrow & \rho^*\tilde{V}/\mathcal{O}_{Y'}(L_2) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \rho^*Q_1 & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \quad ,$$

here $\mathcal{O}_{Y'}(L_2)$ is an invertibar subsheaf of $\rho^*\tilde{V}$ with the torsion free cokernel.

We take againg the direct image ρ_* for the diagram

$$
\begin{array}{ccc}
\mathcal{O}_{Y'}(L_1') & \longrightarrow & \rho^* W_1 \\
\big\downarrow & & \big\downarrow{\scriptstyle \rho^*\varphi_1} \\
\mathcal{O}_{Y'}(L_2) & \longrightarrow & \rho^* \tilde{V}
\end{array}
$$

and get

$$
\begin{array}{ccccccc}
\rho_*\mathcal{O}_{Y'}(L_1') & \longrightarrow & W_1 \otimes \rho_*\mathcal{O}_{Y'} & \longrightarrow & W_1 \otimes K'^{-1} & = & \rho_*\mathcal{O}_{Y'}(L_1) \\
\big\downarrow & & {\scriptstyle \rho_*\rho^*\varphi_1 \simeq \varphi_1 \otimes I}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_1 \otimes I} & & \\
\rho_*\mathcal{O}_{Y'}(L_2) & \longrightarrow & \tilde{V} \otimes \rho_*\mathcal{O}_{Y'} & \longrightarrow & \tilde{V} \otimes K'^{-1} & . &
\end{array}
$$

Similar as (3.2), the upper composition map in the above diagram is an isomorphism, therefore we obtain the diagram

(3.5)
$$
\begin{array}{ccccc}
 & & \rho_*\mathcal{O}_{Y'}(L_2) & & \\
 & \nearrow & & \searrow & \\
\rho_*\mathcal{O}_{Y'}(L_1) & & \hookrightarrow & & \tilde{V} \otimes K'^{-1} \quad .
\end{array}
$$

Generally, we repeat the above process $n+1$ times, and get the diagrams (3.4) and (3.5) for the pair of line bundles (L_n, L_{n+1}) on Y'. Hence there is an increased sequence of line bundles with the upper bound

$$
det(\tilde{V} \otimes \tilde{K}^{-1}) \hookrightarrow det(\rho_*\mathcal{O}_{Y'}(L_1)) \hookrightarrow det(\rho_*\mathcal{O}_{Y'}(L_2)) ... \hookrightarrow det(\rho_*\mathcal{O}_{Y'}(L_n)) ... \hookrightarrow det(\tilde{V} \otimes K'^{-1}).
$$

We see that in certain step, it has to be $det(\rho_*\mathcal{O}_{Y'}(L_i)) \simeq det(\rho_*\mathcal{O}_{Y'}(L_{i+1}))$.
This implies $\rho_*\mathcal{O}_{Y'}(L_i) \simeq \rho_*\mathcal{O}_{Y'}(L_{i+1})$ in (3.5), hence $\mathcal{O}_{Y'}(L_i') \simeq \mathcal{O}_{Y'}(L_{i+1})$ in (3.4).
Let $L := L_{i+1}$, then L has the both properties in claim 2.1.

Proof of claim 2.3

We look at the diagram 2) in claim 2.1. The middle vertical sequence gives

(3.6)
$$
\begin{aligned}
|z| &= 2k + L^2 - L\rho^*\tilde{D} \\
&= 2k + (L - \rho^*\tilde{D}/2)^2 - \tilde{D}^2/2 \quad .
\end{aligned}
$$

Noting the determinant formula (see [F], chapter 5)

$$det(\rho^* W) = L + i^* L + \rho^* K' \quad ,$$

here i is the involution on Y', the middle horizontal sequence gives

$$L + i^* L - \rho^* \tilde{D} = -\rho^*(E + K'),$$

hence

(3.7)
$$\begin{aligned}(2L - \rho^* \tilde{D})\rho^* \tilde{H} &= L\rho^* \tilde{H} + i^* L \rho^* \tilde{H} - \rho^* \tilde{D} \rho^* \tilde{H} \\ &= -\rho^*(E + K')\rho^* \tilde{H} \\ &= -2(E + K')\tilde{H} \quad .\end{aligned}$$

Using Hodge-index-theorem and the above equality we get

(3.8)
$$\begin{aligned}(2L - \rho^* \tilde{D})^2 &\leq ((2L - \rho^* \tilde{D})\rho^* \tilde{H})^2 / (\rho^* \tilde{H})^2 \\ &= 2(E + K')\tilde{H} / \tilde{H}^2 \\ &\leq A \quad . \qquad \text{by lemma 2.2}\end{aligned}$$

(3.6) and (3.8) imply claim 2.3.

Proof of claim 2.2

For any such a line bundle L in claim 2.1, similar as in the proof of claim 1.1, we have the following orthogonal decomposition respect to $c_1(\rho^* \tilde{H})$

$$c_1(2L - \rho^* \tilde{D}) = r c_1(\rho^* \tilde{H}) + c_1(2L - \rho^* \tilde{D})^{\perp} \quad ,$$

here $r = (2L - \rho^* \tilde{D})\rho^* \tilde{H} / (\rho^* \tilde{H})^2$. Using (3.7) $|r|$ is bounded by a constant. And using (3.6) the integral class $(\rho^* \tilde{H})^2 c_1(2L - \rho^* \tilde{D})^{\perp}$ has bounded norm $A\sqrt{k} + A$ in $H^{1,1}(Y')^{\perp} \cap H^2(Y', Z)$. These show that there are only finitely many $c_1(L)$.

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