RAMIFICATION FILTRATION OF THE GALOIS GROUP OF A LOCAL FIELD. III

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ABSTRACT. Let K be a complete discrete valuation field of characteristic $p > 0$ with a finite residue field and let $\Gamma(p)$ be the Galois group of its maximal p-extension. The main result of the paper describes the image of the ramification filtration of the Galois group of the field K in $\Gamma(p)$ modulo its subgroup of commutators of order $\geq p$ in terms of generators of the group $\Gamma(p)$.

Throughout all this paper K is a complete discrete valuation field of characteristic $p > 0$ with a finite residue field $k \simeq \mathbb{F}_{p^{N_0}}$. We fix a uniformising element t_0 of the field K and use the identification $K = k((t_0))$. Choose a separable closure $K_{\rm sep}$ of K and set $\Gamma = \mathrm{Gal}(K_{\rm sep}/K)$. This group has the decreasing filtration $\{\Gamma^{(v)}\}_{v>0}$ of its normal higher ramification subgroups in upper numbering, cf. [Se, Ch.III]. If $\Gamma(p)$ is the Galois group of the maximal p-extension of the field K, then $\Gamma(p)$ is a free pro-p-group [Sh], and there appears the problem of description of the induced ramification filtration $\{\Gamma(p)^{(v)}\}_{v>0}$ in terms of generators of the group $\Gamma(p)$. In this paper we develop the methods from [Ab1,2] to obtain this description modulo the closure $C_p(\Gamma(p))$ of the subgroup of commutators of order $\geq p$.

For this purpose we construct, cf. n.5.1, the profree Lie algebra $\mathcal{L}^0$ over $\mathbb{Z}_p$ and the identification

$$\bar{\psi} : \Gamma(p)/C_p(\Gamma(p)) \simeq G(\widetilde{\mathcal{L}}^0),$$

where $\widetilde{\mathcal{L}}^0 = \mathcal{L}^0/C_p(\mathcal{L}^0)$ is the maximal quotient of nilpotency class $< p$, and $G(\widetilde{\mathcal{L}}^0)$ is the pro-p-group obtained from elements of $\widetilde{\mathcal{L}}^0$ by the Campbell-Hausdorff composition law. The construction of this identification is based on the nilpotent version of Artin-Schreier theory from [Ab2] and depends on the choice of a uniformising element t_0 and $\alpha \in W(k)$ such that $\mathrm{Tr}\,\alpha = 1$.

The profinite Lie $W(k)$-algebra $\mathcal{L} = \mathcal{L}^0 \otimes W(k)$ has a natural system of generators D_{an}, where $a \in \mathbb{Z}^0(p) = \{n \in \mathbb{N} \mid (n,p) = 1\} \cup \{0\}$, $n \in \mathbb{Z}$, $D_{a,n+N_0} = D_{an}$ and $\sigma D_{an} = D_{a,n+1}$ (σ is the Frobenius automorphism of $W(k)$). If $A \subset \mathbb{Z}^0(p)$ is a finite subset, consider a free Lie $W(k)$-subalgebra $\mathcal{L}(A)$ of $\mathcal{L}$, which is generated by all D_{an} with $a \in A$. Then $\mathcal{L} = \varprojlim\limits_{A} \mathcal{L}(A)$ and $\mathcal{L}^0 = \varprojlim\limits_{A} \mathcal{L}^0(A)$, where $\mathcal{L}^0(A) = \mathcal{L}^0 \cap \mathcal{L}(A)$ is

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Lie $\mathbb{Z}_p$-subalgebra of $\mathcal{L}^0$. For $v > 0$ and $N \in \mathbb{N}$, we define in n.2 the ideals $\mathcal{L}_N^0(A, v)$ of the Lie algebra $\mathcal{L}^0(A)$ as the minimal ideals containing the ideal of commutators of order $\geq p$ and such that $\mathcal{L}_N^0(A, v) \otimes W(k)$ contains elements $\mathcal{F}_{\gamma,-N}$ for all $\gamma \geq v$. These elements $\mathcal{F}_{\gamma,-N}$ are defined by explicit expressions as linear combinations with some p-adic coefficients of commutators $[\dots[D_{a_1 n_1}, D_{a_2 n_2}], \dots, D_{a_s n_s}]$ such that $1 \leq s < p$, $a_1, \dots, a_s \in A$, $n_1 \geq \dots \geq n_s \geq -N$ and $a_1 p^{n_1} + \dots + a_s p^{n_s} = \gamma$. We also show in n.2 that the sequence of ideals $\{\mathcal{L}_N^0(A, v)\}_N$ stabilizes and therefore, determines the ideal $\mathcal{L}^0(A, v)$. This gives for all $v > 0$, the ideals $\mathcal{L}^0(v) = \varprojlim_{A \subset \mathbb{Z}^0(p)} \mathcal{L}^0(A, v)$ of the Lie algebra $\mathcal{L}^0$; and we obtain, cf. Theorem A of n.5.3, the following description of the ramification filtration modulo pth commutators: for any $v > 0$,

$$\bar{\psi}(\Gamma(p)^{(v)} \bmod C_p(\Gamma(p))) = G(\widetilde{\mathcal{L}}^0(v)),$$

where $\widetilde{\mathcal{L}}^0(v) = \mathcal{L}^0(v)/C_p(\mathcal{L}^0)$. The Theorem B of n.5.3 gives also a construction of the above ideals $\widetilde{\mathcal{L}}^0(v)$, which does not use the operation of projective limit.

This result is a consequence of the main theorem of n.4, which gives a description of the image of the ramification filtration $\{\Gamma^{(v)}\}_{v>0}$ under a group epimorphism $\Gamma \longrightarrow G(L)$, where L is a finite Lie algebra over $\mathbb{Z}_p$ of a nilpotency class $< p$ and $G(L)$ is the p-group obtained from L by the Campbell-Hausdorff composition law. This theorem is proved by induction on values of the nilpotency class and the exponent of L and by transfinite decreasing induction on $v > 0$. The main trick is related to the following special property of local fields of characteristic p: if K' is a totally ramified finite extension of K in K_{sep}, then $K \simeq K'$ and therefore, there exists an isomorphism $\mathrm{Gal}(K_{\mathrm{sep}}/K) \simeq \mathrm{Gal}(K_{\mathrm{sep}}/K')$ which is compatible with ramification filtrations. After a suitable choice of the auxillary field K' the induction step (modulo some technical computations with the Campbell-Hausdorff formula from n.3) uses only well-known information about ramification of Artin-Schreier extensions of degree p of the field K'.

The arguments of this paper are based completely on constructions from the papers [Ab1,2]. In n.1.2 we give some commentaries about equivalence of the categories of Lie $\mathbb{Z}_p$-algebras and p-groups with class of nilpotency $< p$. In papers [Ab1,2] there was studied the ramification filtration modulo $I^pC_p(I)$ and modulo $C_3(I)$, respectively, where $I = \bigcup_{v>0} \Gamma^{(v)}$ is the higher ramification subgroup of Γ. The general result about the ramification filtration modulo $C_p(I)$ was only claimed in the paper [Ab2] and was applied for a description of the image of the ramification filtration in the group $\Gamma(p)/C_p(\Gamma(p))$. In this paper we apply our method directly to the group $\Gamma(p)/C_p(\Gamma(p))$ (the modulo $C_p(I)$ description can be now easily recovered). In fact, our approach works also in the case of a ground field K with arbitrary perfect residue field, but in this case the choice of generators of the Lie algebra $\mathcal{L}^0$ is more complicated.

1. Preliminaries.

1.1 *Construction of liftings.*

We use the following construction of liftings from [Ab2], which is a particular case of the general construction from the paper [B-M].

For a field $\mathcal{E}$ such that $K \subset \mathcal{E} \subset K_{\mathrm{sep}}$ and a natural number N consider the $\mathbb{Z}/p^N\mathbb{Z}$-algebra

$$O_N(\mathcal{E}) = W_N(\sigma^{N-1}\mathcal{E})[t\ ] \subset W_N(\mathcal{E}),$$

where W_N is the functor of Witt vectors of length N, σ is the Frobenius, and $t = [t_0] \in W_N(K) \subset W_N(\mathcal{E})$ is the Teichmüller representative of t_0. The algebra $O_N(\mathcal{E})$ is a lifting of the field $\mathcal{E}$ modulo p^N, i.e. $O_N(\mathcal{E})$ is a flat $\mathbb{Z}/p^N\mathbb{Z}$-algebra such that $O_N(\mathcal{E})/pO_N(\mathcal{E}) = \mathcal{E}$, and its construction essentially depends on the initial choice of the uniformising element of the field K. For any $N \in \mathbb{N}$, we have the algebra epimorphisms of reduction modulo p^N

$$O_{N+1}(\mathcal{E}) \longrightarrow O_N(\mathcal{E}) = O_{N+1}(\mathcal{E}) \otimes \mathbb{Z}/p^N\mathbb{Z}.$$

If $O(\mathcal{E}) = \varprojlim O_N(\mathcal{E})$ with respect to these epimorphisms, then $O(\mathcal{E})$ is the valuation ring of an absolutely unramified field with the residue field $\mathcal{E}$. The Frobenius of Witt vectors induces the system of Frobenius morphisms $\sigma = \sigma_{\mathcal{E}} : O(\mathcal{E}) \longrightarrow O(\mathcal{E})$, which is compatible on fields $\mathcal{E}$. We note, that $O(K) = W(k)((t))$ and $\sigma t = t^p$. Clearly, there is a natural action of Γ on $O(K_{\mathrm{sep}})$. If H is an open subgroup in Γ and $K_{\mathrm{sep}}^H = \mathcal{E}$, then $O(K_{\mathrm{sep}})^\Gamma = O(\mathcal{E})$.

1.2. *Groups and Lie algebras.*

Let $L(X,Y)$ be the completion of the free Lie algebra with the generators X and Y with respect to its lower central series. Denote by $A(X,Y)$ the Magnus algebra in variables X and Y with integral coefficients (this is the completion by powers of the augmentation ideal of the free associative algebra generated by X and Y). Then we have a natural inclusion $L(X,Y) \subset A(X,Y)$ and in $A(X,Y)\hat{\otimes}\mathbb{Q}$ by the Campbell-Hausdorff formula it holds

$$\exp(X)\exp(Y) = \exp(X \circ Y),$$

where $X \circ Y = X + Y + \frac{1}{2}[X,Y] + \cdots \in L(X,Y)\hat{\otimes}\mathbb{Q}$. We note that if $c_i(X,Y)$ is the component of $X \circ Y$ of total degree i, then $c_i(X,Y)$ is p-integral for $1 \le i < p$.

Consider the category $Lie(p)$ of finite Lie algebras over $\mathbb{Z}_p$ of class of nilpotency $< p$ and the category $Gr(p)$ of finite p-groups with the same condition for nilpotency class. If $s \ge 1$ and $L \in Lie(p)$, we use the notation $C_s(L)$ for the ideal of commutators of order $\ge s$ in L. Similarly, if $G \in Gr(p)$, then $C_s(G)$ will denote the normal subgroup of commutators of order $\ge s$ in G. So, with the above notation we have always $C_p(L) = 0$ and $C_p(G) = e$.

If $L \in Lie(p)$, denote by $G(L)$ the p-group given by the composition law $(l_1, l_2) \mapsto l_1 \circ l_2$ on elements l_1, l_2 of the Lie algebra L. The correspondence $L \mapsto G(L)$ gives the functor $G : Lie(p) \longrightarrow Gr(p)$, and this functor is an equivalence of categories, cf. [La]. For our purposes we give below an interpretation of this equivalence in terms related to envelopping algebras of Lie algebras from $Lie(p)$. Our arguments use information about "dimension subgroups modulo n" from the paper [Mo].

Let $H \in Gr(p)$ and let $J_{\mathbb{Z}}[H]$ be the augmentation ideal of the group ring $\mathbb{Z}[H]$. By the main result of the paper [Mo], we have for $1 \le s \le p$ that

$$H \cap (1 + J_{\mathbb{Z}}[H]^s) = C_s(H).$$

If $J[H]$ is the augmentation ideal of the group ring $\mathbb{Z}_p[H]$, then $J_{\mathbf{Z}}[H]^s = J[H]^s \cap \mathbb{Z}[H]$ for $s \in \mathbb{N}$, because $\mathbb{Z}_p$ is a flat module over $\mathbb{Z}$. This gives for $1 \leq s \leq p$, that

$$H \cap (1 + J[H]^s) = C_s(H). \tag{1}$$

In particular, $H \cap (1 + J[H]^p) = 1$ and we can identify H with its image in the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[H]/J[H]^p$.

Clearly the truncated logarithm

$$\widetilde{\log}(1+x) = \sum_{1 \leqslant n < p} (-1)^{n-1} x^n / n$$

induces a one-to-one map from $(1 + J[H]) \bmod J[H]^p$ to $J[H]/J[H]^p$. The inverse map is induced by the truncated exponential

$$\widetilde{\exp}(x) = \sum_{0 \leqslant n < p} x^n / n!.$$

We note that $\widetilde{\log}$ induces for $1 \leq s < p$, an isomorphism of the multiplicative group $(1 + J[H]^s) \bmod J[H]^{s+1}$ and the additive group $J[H]^s \bmod J[H]^{s+1}$.

Consider the set $L(H) = \widetilde{\log}(H) \subset \mathbb{Z}_p[H]/J[H]^p$. Then $L(H)$ is a Lie subalgebra of the algebra $\mathbb{Z}_p[H]/J[H]^p$, i.e. the set $L(H)$ is closed under linear operations and the Lie bracket in the algebra $\mathbb{Z}_p[H]/J[H]^p$.

Indeed, if $l = \widetilde{\log} h$, where $h \in H \subset \mathbb{Z}_p[H]/J[H]^p$, then for any $t \in \mathbb{Z}_p$, we have

$$tl = t\widetilde{\log} h = \widetilde{\log}(h^t) \in L(H).$$

If $l_1 = \widetilde{\log}(h_1)$, $l_2 = \widetilde{\log}(h_2) \in L(H)$ and $t \in \mathbb{Z}_p$, then

$$\widetilde{\log}(h_1^t h_2^t) = \widetilde{\log}(\widetilde{\exp}(tl_1)\widetilde{\exp}(tl_2)) = \sum_{1 \leqslant i < p} t^i c_i(l_1, l_2) \in L(H).$$

This gives $c_1(l_1, l_2) = l_1 + l_2 \in L(H)$ and $c_2(l_1, l_2) = \frac{1}{2}[l_1, l_2] \in L(H)$.

If $L \in Lie(p)$ and $H = G(L)$, then the map

$$\widetilde{\log} : L = H \longrightarrow L(H)$$

is an isomorphism of Lie algebras. Therefore, the functor $H \mapsto L(H)$ is inverse to the functor $L \mapsto G(L)$.

Let $A(L)$ be the envelopping algebra of L. By its universal property the above embedding of L in $\mathbb{Z}_p[H]/J[H]^p$ induces the algebra morphism

$$\alpha : A(L) \longrightarrow \mathbb{Z}_p[H]/J[H]^p.$$

If $J(L)$ is the augmentation ideal of $A(L)$, then for $1 \leq s \leq p$,

$$\alpha(J(L)^s) = J[H]^s / J[H]^p,$$

(in fact, $\alpha \bmod J(L)^p$ is the isomorphism of algebras $A(L)/J(L)^p$ and $\mathbb{Z}_p[H]/J[H]^p$) and the above equality (1) implies for $1 \leq s \leq p$ that

$$L \cap J(L)^s = C_s(L).$$

This gives the following proposition:

Proposition 1. *If $L \in Lie(p)$, then*
(a) *the natural embedding $L \longrightarrow A(L)$ induces for $s \geq 1$, injective morphisms*

$$C_s(L)/C_{s+1}(L) \longrightarrow J(L)^s/J(L)^{s+1};$$

(b) *the truncated exponential $\widetilde{\exp}$ induces the injective map*

$$\widetilde{\exp} : L \longrightarrow A(L)/J(L)^p;$$

(c) *the correspondence $L \mapsto \widetilde{\exp}(L) \bmod J(L)^p$ gives a construction of the equivalence $G : Lie(p) \longrightarrow Gr(p)$ in terms of envelopping algebras of Lie algebras from $Lie(p)$.*

Remarks. 1) The parts (b) and (c) of the above proposition are formal consequences of the part (a). If all $C_s(L)$ are direct summands in the $\mathbb{Z}_p$-module L, then (a) can be proved immediately by the special choice of a system of generators of L, cf. [Kn] where the case of Lie algebras over a field was considered; the same argument was also applied in [Ab1].

2) It can be shown that in notation of the above proposition, the group $\widetilde{\exp}(L)$ is the group of "diagonal elements $\bmod \deg p$", i.e. the multiplicative group of $a \in A(L) \bmod J(L)^p$ such that

$$\Delta\hat{a} \equiv \hat{a} \otimes \hat{a} \bmod J(L \oplus L)^p,$$

where $\Delta : A(L) \longrightarrow A(L \oplus L) = A(L) \otimes A(L)$ is the diagonal morphism and $\hat{a} \in A(L)$ is such that $\hat{a} \bmod J(L)^p = a$.

We need also a slight generalization of the above construction.

Assume that R is a commutative ring with unity which is a flat $\mathbb{Z}$-module. If $L \in Lie(p)$, then $A(L_R) = A(L) \otimes R$ is the envelopping algebra of the Lie R-algebra $L_R = L \otimes R$, and $J(L_R) = J(L) \otimes R$ is its augmentation ideal. The flattness of R with the above proposition 1 gives

Proposition 2.
(a) *The natural maps $C_s(L_R)/C_{s+1}(L_R) \longrightarrow J(L_R)^s/J(L_R)^{s+1}$ are injective;*

(b) *$\widetilde{\exp}$ induces the embedding $\widetilde{\exp} : L_R \longrightarrow A(L_R)/J(L_R)^p$;*
(c) *if L_{R*} is the Lie algebra over $\mathbb{Z}_p$ obtained from L_R by restriction of scalars $\mathbb{Z}_p \longrightarrow R$, then $\widetilde{\exp}(L_R) \simeq G(L_R)$ and $\widetilde{\exp}$ induces a bijection between the set of ideals of the Lie algebra L_{R*} and the set of normal subgroups of the group $\widetilde{\exp}(L_R)$.*

1.3. *Nilpotent Artin-Schreier theory.*

Let L be a finite Lie algebra over $\mathbb{Z}_p$ of a nilpotency class $< p$. The nilpotent version of Artin-Schreier theory from [Ab2] gives the following properties:

a) If $\psi \in \mathrm{Hom}(\Gamma, G(L))$, then there exist $f \in G(L \otimes O(K_{\mathrm{sep}}))$ and $e \in G(L \otimes O(K))$ such that $\sigma f = f \circ e$ and $\psi(\tau) = (\tau f) \circ (-f)$ for any $\tau \in \Gamma$;

b) If $e_1 \in G(L \otimes O(K))$, then there exists $f_1 \in G(L \otimes O(K_{\mathrm{sep}}))$ such that $\sigma f_1 = f_1 \circ e_1$, and the correspondence $\tau \mapsto (\tau f_1) \circ (-f_1)$ determines the group homomorphism $\psi_1 : \Gamma \longrightarrow G(L)$. The conjugacy class of ψ_1 does not depend on the choice of f_1;

c) In the above notation, $\psi = \psi_1$ if and only if there exists $c \in G(L \otimes O(K))$ such that $f_1 = f \circ c$ and $e_1 = (-c) \circ e \circ (\sigma c)$.

Let $\mathbb{Z}^+(p) = \{ a \in \mathbb{N} \mid (a,p) = 1 \}$ and $\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}$. Choose $\alpha \in W(k)$ such that $\mathrm{Tr}_{W(k)/\mathbb{Z}_p}(\alpha) = 1$.

Lemma. *If $e \in G(L \otimes O(K))$, then there exists $c \in G(L \otimes O(K))$ such that*

$$(-c) \circ e \circ (\sigma c) = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0},$$

where all $D_{a0} \in L \otimes W(k)$, $\alpha^{-1} D_{00} \in L$ and the set

$$A = \{ a \in \mathbb{Z}^0(p) \mid D_{a0} \neq 0 \}$$

is finite.

Proof. We use induction on $s \geq 1$ to prove this statement modulo $C_s(L) \otimes O(K)$. If $s = 1$ there is nothing to prove.

Suppose that

$$(-c) \circ e \circ (\sigma c) \equiv \sum t^{-a} D_{a0} \bmod C_s(L),$$

where $s \geq 1$, $c \in G(L \otimes O(K))$ and the elements $D_{a0} \bmod C_s(L \otimes W(k))$ satisfy the statement of our lemma. Then

$$(-c) \circ e \circ (\sigma c) = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0} + \sum_{b > -\infty} t^b l_b \bmod C_{s+1}(L \otimes W(k)),$$

where all $l_b \in C_s(L \otimes W(k))$.

If $l_+ = \sum_{b>0} t^b l_b$, then $l_+ = \sigma l'_+ - l'_+$, where

$$l'_+ = \sum_{n \geq 0} \sigma^n l_+.$$

Consider $l_- = \sum_{b<0} t^b l_b$. This sum is finite. If $b < 0$, then there exist the unique $a_b \in \mathbb{Z}^+(p)$ and $n_b \in \mathbb{Z}_{\geq 0} := \{n \in \mathbb{Z} \mid n \geq 0\}$ such that $-b = a_b p^{n_b}$. Let

$$l_-^{(b)} = \sum_{0 \leq n < n_b} t^{-a_b p^n} \sigma^{n - n_b} l_b.$$

Then $t^b l_b = \sigma(l_-^{(b)}) - l_-^{(b)} + t^{-a_b}(\sigma^{-n_b} l_b)$. So, if $l'_- = \sum_{b<0} l_-^{(b)}$, then

$$l_- = \sigma l'_- - l'_- + \sum_{b<0} t^{-a_b}(\sigma^{-n_b} l_b).$$

If $l_0 = \sum_i w_i l_{0i}$, where all $w_i \in W(k)$ and $l_{0i} \in L$, then

$$w_i = \alpha \, \mathrm{Tr}\, w_i + \sigma w'_i - w'_i$$

for some $w_i' \in W(k)$, because $H^1(\mathrm{Gal}(k/\mathbb{F}_p), W(k)) = 0$.

So, if $l_0' = \sum_i w_i' l_{0i}$, then $l_0 = \sigma l_0' - l_0' + \alpha \sum_i \mathrm{Tr}(w_i) l_{0i}$.

Thus, for $c' = c + l_+' + l_0' + l_-'$, we have

$$(-c') \circ e \circ (\sigma c') \equiv \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0}' \bmod C_{s+1}(L),$$

where $\alpha^{-1} D_{00}' = \alpha^{-1} D_{00} + \sum_i (\mathrm{Tr}\, w_i) l_{0i} \in L$ and for $a \in \mathbb{Z}^+(p)$,

$$D_{a0}' = D_{a0} + \sum_{a_b = a} \sigma^{-n_b} l_b.$$

The lemma is proved.

In the notation of the above lemma we set $D_0 = \alpha^{-1} D_{00}$. For any $\psi \in \mathrm{Hom}(\Gamma, G(L))$, the above lemma implies the existence of $f \in G(L \otimes O(K_{\mathrm{sep}}))$ such that

$$\sigma f = f \circ (\sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0})$$

and $\psi(\tau) = (\tau f) \circ (-f)$ for any $\tau \in \Gamma$. Choose a basis $\alpha_1, \ldots, \alpha_{N_0}$ of $W(k) \simeq W(\mathbb{F}_{p^{N_0}})$ over $W(\mathbb{F}_p)$. If for $1 \leq i \leq N_0$ and $a \in \mathbb{Z}^+(p)$, the elements $D_a^{(i)} \in L$ are such that $D_{a0} = \sum_{1 \leqslant i \leqslant N_0} \alpha_i D_a^{(i)}$, then the group $\mathrm{Im}\, \psi \subset G(L)$ is generated by D_0 and all $D_a^{(i)}$, where $1 \leq i \leq N_0$ and $a \in \mathbb{Z}^+(p)$. Thus, ψ is a group epimorphism iff the above elements D_0 and $D_a^{(i)}$ generate the Lie algebra L (or equivalently, the elements D_0 and $\sigma^n D_{a0}$, where $a \in \mathbb{Z}^+(p)$ and $0 \leq n < N_0$, generate the Lie $W(k)$-algebra $L \otimes W(k)$).

1.4. *The structural constants $\eta(n_1, \ldots, n_s)$.*

If $1 \leq s < p$ and $n_1, \ldots, n_s \in \mathbb{Z}$, define constants $\eta(n_1, \ldots, n_s) \in \mathbb{Z}_p$ as follows.

If there exist $1 \leq s_1 < s_2 < \cdots < s_l = s$ such that $n_1 = \cdots = n_{s_1} > n_{s_1+1} = \cdots = n_{s_2} > \cdots > n_{s_{l-1}+1} = \cdots = n_{s_l} (= n_s)$, then we set

$$\eta(n_1, \ldots, n_s) = \frac{1}{s_1!(s_2 - s_1)! \ldots (s_l - s_{l-1})!},$$

otherwise we set $\eta(n_1, \ldots, n_s) = 0$. We extend this definition by setting for $s = 0$, $\eta(n_1, \ldots, n_s) = \eta(\emptyset) = 1$.

Lemma.

(a) *If $0 \leq s_1 \leq s < p$, then for any $n_1, \ldots, n_s \in \mathbb{Z}$, we have the identity*

$$\eta(n_1, \ldots, n_{s_1}) \eta(n_{s_1+1}, \ldots, n_s) = \sum_{\pi \in I_{s_1 s}} \eta(n_{\pi(1)}, \ldots, n_{\pi(s)}),$$

where $I_{s_1 s}$ is the subset of substitutions π of order s such that $\pi^{-1}(1), \ldots, \pi^{-1}(s_1)$ and $\pi^{-1}(s_1+1), \ldots, \pi^{-1}(s)$ are increasing sequences in $[1, s]$ (i.e. $I_{s_1 s}$ is the set of all "insertions" of the set $\{1, \ldots, s_1\}$ into the set $\{s_1+1, \ldots, s\}$);

(b) *If* $0 \leq s < p$ *and* $n_1, \dots, n_s \in \mathbb{Z}$, *then*

$$\sum_{0 \leq t \leq s} (-1)^t \eta(n_t, \dots, n_1)\eta(n_{t+1}, \dots, n_s) = \delta_{0s},$$

where δ *is the Kronecker symbol.*

Proof. Assume that $a \leq n_1, \dots, n_s \leq b$ for some $a, b \in \mathbb{Z}$. Consider the free Lie algebra L over $\mathbb{Z}_p$ with generators D_n, where $a \leq n \leq b$. Denote by A the envelopping algebra of L and by J its augmentation ideal. Any element of $A \bmod J^p$ can be uniquely presented as a linear combination over $\mathbb{Z}_p$ of the products $D_{i_1} \dots D_{i_s}$, where $0 \leq s < p$ and $i_1, \dots, i_s \in [a, b]$. Similarly, any element of $A \otimes A \bmod (1 \otimes J + J \otimes 1)^p$ can be uniquely presented as a $\mathbb{Z}_p$-linear combination of $D_{i_1} \dots D_{i_{s_1}} \otimes D_{j_1} \dots D_{j_{s_2}}$, where $s_1, s_2 \geq 0$, $s_1 + s_2 < p$ and $i_1, \dots, i_{s_1}, j_1, \dots, j_{s_2} \in [a, b]$.

Consider the diagonal algebra morphism

$$\Delta : A \bmod J^p \longrightarrow A \otimes A \bmod (1 \otimes J + J \otimes 1)^p$$

given by the correspondences $D_n \mapsto 1 \otimes D_n + D_n \otimes 1$ for all $n \in [a, b]$.

Let $\varepsilon = \widetilde{\exp}(D_b)\widetilde{\exp}(D_{b-1}) \dots \widetilde{\exp}(D_a)$, where $\widetilde{\exp}(x) = \sum_{0 \leq i < p} x^i/i!$ is the truncated exponential. From the above definition of the constants $\eta(n_1, \dots, n_s)$ it follows, that

$$\varepsilon \bmod J^p = \sum \eta(n_1, \dots, n_s) D_{n_1} \dots D_{n_s},$$

where the sum is taken for $0 \leq s < p$ and $n_1, \dots, n_s \in [a, b]$.

The identity of the part (a) is implied now by the property

$$\Delta\varepsilon \equiv \varepsilon \otimes \varepsilon \bmod (1 \otimes J + J \otimes 1)^p.$$

The identity (b) follows from the expansion

$$\varepsilon^{-1} \equiv \widetilde{\exp}(-D_a) \dots \widetilde{\exp}(-D_b) \bmod J^p = \sum_{0 \leq s < p} (-1)^s \eta(n_s, \dots, n_1) D_{n_1} \dots D_{n_s}$$

and the property $\varepsilon^{-1}\varepsilon \equiv 1 \bmod J^p$.

1.5. *The field* $K(N, r^*)$.

Let $N \in \mathbb{N}$, $q = p^N$ and let $r^* > 0$ be such that $r^*(q-1) = b^* \in \mathbb{Z}^+(p)$. We use the following generalization of the Artin-Hasse exponential

$$E(\alpha, X) = \exp\left(\alpha X + (\sigma\alpha)X^p/p + \dots + (\sigma^i\alpha)X^{p^i}/p^i + \dots\right) \in W(k)[[X]],$$

where $\alpha \in W(k)$.

Proposition. *There exists an extension $K(N,r^*)$ of the field K such that*
(a) $[K(N,r^*):K]=q$;

(b) *The Herbrandt function ψ of the extension $K(N,r^*)/K$ equals*

$$\psi(x)=\begin{cases} x, & \text{for } 0\le x\le r^* \\ (x-r^*)/q+r^*, & \text{for } x\ge r^*. \end{cases}$$

(c) *There exists a uniformising element t_0' of the field $K(N,r^*)$ such that*

$$t_0=t_0'^{q}E(1,t_0'^{b^*})^{-1}.$$

Proof. Let $r^*=m/n$, where $m,n\in\mathbb{N}$ and $(m,n)=1$. Then $m,n\in\mathbb{Z}^+(p)$, $n|(q-1)$ and $m|b^*$. Take $u_0\in K_{\rm sep}$ such that $u_0^n=t_0$. Then $L=K(u_0)$ is a totally ramified extension of K and $[L:K]=n$. Take $U\in K_{\rm sep}$ such that $U^q+r^*U=u_0^{-m}$. Then $L'=L(U)$ is a totally ramified extension of L, $[L':L]=q$ and $L'=K(U)$. Set $K'=K(U^n)\subset L'$. We want to verify that the field K' can be taken as $K(N,r^*)$, i.e. it satisfies the properties (a)-(c) of our proposition.

Lemma. $[L':K']=n$.

Proof. Denote by $K_{\rm ur}$ the maximal unramified extension of K in $K_{\rm sep}$ and set $L_{\rm ur}=LK_{\rm ur}$, $K'_{\rm ur}=K'K_{\rm ur}$ and $L'_{\rm ur}=L'K_{\rm ur}$. Because L is totally ramified over K it is sufficient to prove that $[L'_{\rm ur}:K'_{\rm ur}]=n$.

Clearly, $L'_{\rm ur}$ is a Galois extension of $K_{\rm ur}$ and there exists $\tau\in{\rm Gal}(L'_{\rm ur}/K'_{\rm ur})$ such that $\tau^n={\rm id}$ and $\tau u_0=\gamma u_0$, where $\gamma\in K_{\rm ur}$ is a primitive root of unity of order n. Because $n|(q-1)$, we have $\gamma^q=\gamma$ and therefore $\tau U=\gamma^{-m}U+\alpha$, where $\alpha\in K_{\rm ur}$ is such that $\alpha^q+r^*\alpha=0$. We can assume that $n>1$. If $\beta=\alpha/(\gamma^{-m}-1)$, then $\beta^q+r^*\beta=0$ and for $U_1=U+\beta$, we have $U_1^q+r^*U_1=u_0^{-m}$ and $\tau U_1=\gamma^{-m}U_1$.

Therefore, $L'_{\rm ur}=L_{\rm ur}(U_1)=K_{\rm ur}(U_1)$ has the degree n over $K_{\rm ur}(U_1^n)$. Applying an automorphism of the group ${\rm Gal}(L'_{\rm ur}/K_{\rm ur})$ which transforms U_1 to U we obtain $[L'_{\rm ur}:K'_{\rm ur}]=n$. The lemma is proved.

The above lemma implies that K' is a totally ramified extension of K of degree q.

The extensions L/K and L'/K' are tamely ramified extensions of degree n, therefore their Herbrandt functions are equal $\psi_{L/K}(x)=\psi_{L'/K'}(x)=x/n$ for $x\ge 0$. For the extension L'/L, we have

$$\psi_{L'/L}=\begin{cases} x, & \text{for } 0\le x\le m \\ (x-m)/q+m, & \text{for } x\ge m. \end{cases}$$

By the composition property of the Herbrandt function we obtain, that

$$\psi_{L'/K}(x)=\psi_{L/K}\left(\psi_{L'/L}(x)\right)=(1/n)\psi_{L'/L}(x)$$

and

$$\psi_{L/K}(x)=\psi_{K'/K}\left(\psi_{L'/K'}(x)\right)=\psi_{K'/K}(x/n).$$

Therefore for $x \geq 0$, we have

$$\psi_{K'/K}(x) = \frac{1}{n}\psi_{L'/L}(nx) = \begin{cases} x, & \text{for } 0 \leq x \leq r^* \\ (x-r^*)/q + r^*, & \text{for } x \geq r^*. \end{cases}$$

Note that $U^n = u_1^{-m}$, where u_1 is a uniformising element of K'. This gives $U^{1-q} = u_1^{b^*}$ and

$$u_1^{-mq}\left(1 + r^* u_1^{b^*}\right)^n = t_0^{-m}.$$

Therefore

$$\alpha u_1^q \left(1 + r^* u_1^{b^*}\right)^{-1/r^*} = t_0$$

for some $\alpha \in K'$ such that $\alpha^m = 1$. Because $(m, q) = 1$, there exists $\alpha_1 \in K'$ such that $\alpha_1^q = \alpha$. If $u_2 = \alpha_1 u_1$ then u_2 is a uniformising element of K', $u_2^{b^*} = u_1^{b^*}$ (because $m|b^*$), and

$$u_2^q \left(1 + r^* u_2^{b^*}\right)^{-1/r^*} = t_0.$$

This gives

$$t_0 \equiv u_2^q(1 - u_2^{b^*}) \equiv u_2^q E(-1, u_2^{b^*}) \bmod u_2^{q+b^*}.$$

Now a suitable version of the Hensel Lemma gives the existence of $t_0' \in K'$ such that $t_0' \equiv u_2 \bmod u_2^{b^*+1}$ and $t_0 = t_0'^q E(-1, t_0'^{b^*})$.

The proposition is proved.

1.6. *A characterization of the ideal $\psi(\Gamma^{(v_0)}) = L^{(v_0)}$.*

Choose $M \in \mathbb{Z}_{\geq 0}$ such that $p^{M+1}L = 0$. Clearly, $L \otimes O(K) = L \otimes O_{M+1}(K)$ and $L \otimes O(K_{\text{sep}}) = L \otimes O_{M+1}(K_{\text{sep}})$. As earlier, suppose that $\psi : \Gamma \longrightarrow G(L)$ is given by the correspondence $\tau \mapsto \tau f \circ (-f)$, where $\tau \in \Gamma$, $f \in G(L \otimes O(K_{\text{sep}}))$, $e = \sum_{a \in Z^0(p)} t^{-a} D_{a0} \in G(L \otimes O(K))$ and $\sigma f = f \circ e$.

We can use the uniformizer $t_0^{p^M}$ of the field $\sigma^M K$ to construct the lifting

$$O_{M+1}(\sigma^M K) = W_{M+1}(\sigma^{2M} K)[t^{p^M}]$$

of the field $\sigma^M K$ and the lifting

$$O_{M+1}(\sigma^M K_{\text{sep}}) = W_{M+1}(\sigma^{2M} K_{\text{sep}})[t^{p^M}]$$

of the field $\sigma^M K_{\text{sep}}$. For any $n \in \mathbb{Z}$, we also use the notation $\sigma^n D_{a0} = D_{an}$.

With the above notation we obviously have $\sigma^M e = \sum_{a \in \mathbf{Z}^0(p)} t^{-ap^M} D_{aM} \in L \otimes O_{M+1}(\sigma^M K)$, $\sigma^M f \in L \otimes O_{M+1}(\sigma^M K_{\text{sep}})$, $\sigma(\sigma^M f) = (\sigma^M f) \circ (\sigma^M e)$ and $\psi(\tau) = \tau(\sigma^M f) \circ (-\sigma^M f)$ for any $\tau \in \Gamma$.

Let v_0 be a positive real number. For the ramification subgroup $\Gamma^{(v_0)}$ of Γ, we set

$$L^{(v_0)} = \psi(\Gamma^{(v_0)}).$$

Clearly, $L^{(v_0)}$ is an ideal of the Lie algebra L.

Choose $N^* \in \mathbb{N}$ and $0 < r^* < v_0$ such that $r^*(q-1) = b^* \in \mathbb{Z}^+(p)$, where $q = p^{N^*}$. Consider the field $K' = K(N^*, r^*) \subset K_{\text{sep}}$ and its uniformising element t_0', cf. n.1.5, to construct for any $N \in \mathbb{N}$, the liftings of K' and $K'_{\text{sep}} = K_{\text{sep}}$ modulo p^N. We use the following notation for these liftings: $O'_N(K') = W_N(\sigma^{N-1}K')[t_1]$ and $O'_N(K_{\text{sep}}) = W_N(\sigma^{N-1}K_{\text{sep}})[t_1]$, where $t_1 = [t_0'] \in W_N(K')$, $O'(K') = \varprojlim O'_N(K')$ and $O'(K_{\text{sep}}) = \varprojlim O'_N(K_{\text{sep}})$. Clearly, the embedding $K \subset K'$ and the identification $K_{\text{sep}} = K'_{\text{sep}}$ induce the embeddings

$$O_{M+1}(\sigma^M K) \subset W_{M+1}(\sigma^M K) \subset W_{M+1}(\sigma^M K') \subset O'_{M+1}(K') \subset W_{M+1}(K'),$$

$$O_{M+1}(\sigma^M K_{\text{sep}}) \subset W_{M+1}(\sigma^M K_{\text{sep}}) \subset O'_{M+1}(K_{\text{sep}}) \subset W_{M+1}(K_{\text{sep}}).$$

These embeddings allow us to relate constructions of the nilpotent version of the Artin-Schreier theory for different liftings $O(K)$ and $O'(K')$.

Lemma. *With respect to the above embedding $O_{M+1}(\sigma^M K) \subset O'_{M+1}(K')$ we have*

$$t^{p^M} = t_1^{qp^M} E(1, t_1^{b^*})^{-p^M}.$$

Proof. Denote by $V : W_{M+1}(K') \longrightarrow W_{M+1}(K')$ the "Verschiebung", i.e. the σ^{-1}-linear morphism given by the correspondence $(a_1, a_2, \dots, a_{M+1}) \mapsto (0, a_1, \dots, a_M)$. Clearly,

$$t \equiv t_1^q E(1, t_1^{b^*})^{-1} \operatorname{mod} VW_{M+1}(K'),$$

and for any $s \geq 0$, we have

$$t^{p^s} \equiv t_1^{qp^s} E(1, t_1^{b^*})^{-p^s} \operatorname{mod} V^{s+1}W_{M+1}(K').$$

The lemma is proved, because $V^{M+1}W_{M+1}(K') = 0$.

Let $e_1 = \sum_{a \in \mathbf{Z}^0(p)} t_1^{-a} D_{a,-N^*} \in G(L \otimes O'(K'))$ and choose $f_1 \in G(L \otimes O'(K_{\text{sep}}))$ such that $\sigma f_1 = f_1 \circ e_1$. Because $\sigma^M f \in G(L \otimes O(\sigma^M K_{\text{sep}})) \subset G(L \otimes O'(K_{\text{sep}})$, there exists $X \in G(L \otimes O'(K_{\text{sep}}))$ such that

$$\sigma^M f = (\sigma^{M+N^*} f_1) \circ X.$$

If J is an ideal of the Lie algebra L, denote by $K'_J(X)$ the field of definition of $X \operatorname{mod}(JO'(K_{\text{sep}}))$ over K'. By definition, $K'_J(X) = K_{\text{sep}}^{H_J(X)}$, where $H_J(X)$ is the subgroup of $\Gamma' = \operatorname{Gal}(K_{\text{sep}}/K')$, which consists of $\tau \in \Gamma'$ such that $\tau X \equiv X \operatorname{mod}(JO'(K_{\text{sep}}))$.

Denote by $v'_J(X)$ the maximal upper ramification number for the extension $K'_J(X)/K'$. By definition, $v'_J(X)$ is the number such that the ramification subgroups $\Gamma'^{(v)}$ of the group Γ' act trivially on $K'_J(X)$ if and only if $v > v'_J(X)$ (the existence of $v'_J(X)$ follows from left-continuity of the ramification filtration).

For $v > 0$, let $\mathcal{J}'_v(X)$ be the set of ideals J of the Lie algebra L such that $v'_J(X) < v$. If $J_1, J_2 \in \mathcal{J}'_v(X)$, then $v'_{J_1 \cap J_2}(X) = \max(v'_{J_1}(X), v'_{J_2}(X))$ and therefore $J_1 \cap J_2 \in \mathcal{J}'_v(X)$. Therefore, the set $\mathcal{J}'_v(X)$ has the minimal element $J'_v(X)$.

Proposition. $L^{(v_0)} = J'_{qv_0-b^*}(X)$.

Proof. If J is an ideal of L denote by $K_J(f)$ the field of definition of the element $f \operatorname{mod}(JO(K_{\text{sep}}))$ and by $v_J(f)$ the maximal upper ramification number of the extension $K_J(f)/K$. Clearly, $L^{(v_0)}$ is the minimal element in the set $\mathcal{J}_{v_0}(f)$ of ideals J of L such that $v_J(f) < v_0$.

Our proposition will be proved if we verify, that $\mathcal{J}_{v_0}(f) = \mathcal{J}'_{qv_0-b^*}(X)$.

Let J be an arbitrary ideal of L. Note, that the correspondences $\sigma^{-N^*} : k \longrightarrow k$, $t_0 \mapsto t'_0$, $t \mapsto t_1$, $e \mapsto e_1$ and $f \mapsto f_1$ determine an isomorphism of the extension $K_J(f)/K$ with the extension $K'_J(f_1)/K'$, where $K'_J(f_1)$ is the field of definition of $f_1 \operatorname{mod}(JO'(K_{\text{sep}}))$ over K'. Therefore, $v_J(f) = v'_J(f_1)$, where $v'_J(f_1)$ is the maximal upper ramification number of the extension $K'_J(f_1)/K'$. We also note, that $v'_J(f_1)$ is the last edge point of the graph of the Herbrandt function $\psi_{K'_J(f_1)/K'}$. Then the equality $\psi_{K'_J(f_1)/K} = \psi_{K'_J(f_1)/K'} \circ \psi_{K'/K}$ gives for the maximal upper ramification number $v_J(f_1)$ of the extension $K'_J(f_1)/K$, that

$$v_J(f_1) = \begin{cases} (v'_J(f_1) - r^*)/q + r^*, & \text{if } v'_J(f_1) \geq r^* \\ r^*, & \text{if } v'_J(f_1) \leq r^*. \end{cases}$$

In the both cases, we have $v_J(f_1) \leq \max\{r^*, v_J(f)\}$ and $v_J(f_1) \geq v_0$ implies $v_J(f) > v_J(f_1)$ (we remind that $r^* < v_0$).

Because, $K_J(\sigma^M f) = K_J(f)$ and $K'_J(\sigma^{M+N^*} f_1) = K'_J(f_1)$, the equality $X = (-\sigma^{M+N^*} f_1) \circ (\sigma^M f)$ gives $K'_J(X) \subset K_J(f)K'_J(f_1)$. Therefore, if $J \in \mathcal{J}_{v_0}(f)$ and $v_J(X)$ is the maximal upper ramification number for the extension $K'_J(X)/K$, then $v_J(X) \leq \max\{v_J(f), v_J(f_1)\} < v_0$. This is equivalent to the inequality $v'_J(X) < qv_0 - b^*$, i.e. $J \in \mathcal{J}'_{qv_0-b^*}(X)$.

If $J \in \mathcal{J}'_{qv_0-b^*}(X)$ and $v_J(f_1) \geq v_0$, then $v_J(f) > v_J(f_1) = \max\{v_J(f_1), v_J(X)\}$. This contradicts to the embedding $K_J(f) \subset K'_J(f_1)K'_J(X)$. Therefore, $v_J(f_1) < v_0$, $v_J(f) = \max\{v_J(f_1), v_J(X)\} < v_0$ and $J \in \mathcal{J}_{v_0}(f)$. The proposition is proved.

2. The Lie algebra $\mathcal{L}^0(A)$ and its filtration $\{\mathcal{L}^0(A, v)\}_{v>0}$.

2.1. Let $A \subset \mathbb{Z}^0(p)$ be a finite subset and $A^+ = A \setminus \{0\}$. Denote by $\mathcal{L}(A)$ the free Lie algebra over $W(k)$ with the set of generators

$$\{\, \mathcal{D}_a^{(m)} \mid a \in A^+,\ m \in \mathbb{Z}/N_0\mathbb{Z}\} \cup \{\, \mathcal{D}_0 \mid \text{if } 0 \in A\}.$$

Let σ be the Frobenius automorphism of $W(k)$. Define σ-linear automorphism $\sigma_{\mathcal{L}}$ of $\mathcal{L}(A)$ by correspondences $\sigma_{\mathcal{L}} : \mathcal{D}_a^{(m)} \mapsto \mathcal{D}_a^{(m+1)}$ if $a \in \mathbb{Z}^+(p)$ and $m \in \mathbb{Z}/N_0\mathbb{Z}$, and $\sigma_{\mathcal{L}} : \mathcal{D}_0 \mapsto \mathcal{D}_0$ if $0 \in A$. We shall use later the simpler notation $\sigma_{\mathcal{L}} = \sigma$. Fix $\alpha \in W(k)$ such that $\operatorname{Tr}_{W(k)/\mathbb{Z}_p} \alpha = 1$.

If $n \in \mathbb{Z}$, we set $\mathcal{D}_{an} = \mathcal{D}_a^{(n \bmod N_0)}$ if $a \in A^+$, and $\mathcal{D}_{0n} = (\sigma^n \alpha)\mathcal{D}_0$, if $0 \in A$. Clearly, $\sigma\mathcal{D}_{an} = \mathcal{D}_{a,n+1}$ for any $a \in \mathbb{Z}^0(p)$ and $n \in \mathbb{Z}$.

It is easy to see that

$$\mathcal{L}^0(A) = \{\, l \in \mathcal{L}(A) \mid \sigma l = l \,\}$$

is a free Lie algebra over $\mathbb{Z}_p$ and $\mathcal{L}^0(A)\otimes W(k)=\mathcal{L}(A)$.

In this section the set A will be fixed, so we can use the simpler notation: $\mathcal{L}(A)=\mathcal{L}$ and $\mathcal{L}^0(A)=\mathcal{L}^0$.

We use the following notation:

$(\bar{a},\bar{n})=(a_1,n_1,\dots,a_s,n_s)$, where $1\le s<p$, $a_1,\dots,a_s\in A$ and $n_1,\dots,n_s\in\mathbb{Z}$;

if $N\in\mathbb{Z}$, then $\bar{n}\ge N$ means $n_1,\dots,n_s\ge N$ (in the same way we use the notation $\bar{n}\le N$, $\bar{n}>N$ etc.);

$\gamma(\bar{a},\bar{n})=a_1p^{n_1}+\dots+a_sp^{n_s}$;

if $n\in\mathbb{Z}$, then $n^*=n$ for $n\ge 0$, and $n^*=-\infty$ for $n<0$ (in this case $p^{n^*}=0$).

For $\gamma\ge 0$ and $N\in\mathbb{Z}$, consider the element

$$\mathcal{F}_{\gamma,-N}=\sum(-1)^s\eta(n_1,\dots,n_s)a_1p^{n_1^*}[\dots[\mathcal{D}_{a_1n_1},\mathcal{D}_{a_2,n_2}],\dots,\mathcal{D}_{a_sn_s}]$$

of the Lie algebra $\mathcal{L}$, where the sum is taken for

$$(\bar{a},\bar{n})\in M_{\gamma,-N}(A):=\{\ (\bar{a},\bar{n})\ |\gamma(\bar{a},\bar{n})=\gamma,n_1\ge\dots\ge n_s\ge -N\ \}$$

(it is easy to see that this sum has only a finite number of nonzero summands).

For $v>0$, denote by $\mathcal{L}_N(A,v)=\mathcal{L}_N(v)$ the minimal $\sigma_{\mathcal{L}}$-invariant ideal of $\mathcal{L}$, which contains all $\mathcal{F}_{\gamma,-N}$ with $\gamma\ge v$, and the ideal $C_p(\mathcal{L})$ of commutators of order $\ge p$. For any $N\in\mathbb{Z}$, $\mathcal{L}_N^0(v):=\mathcal{L}_N(v)\cap\mathcal{L}^0(A)$ is an ideal of the Lie algebra $\mathcal{L}^0$ and we have $\mathcal{L}_N^0(v)W(k)=\mathcal{L}_N^0(v)\otimes W(k)=\mathcal{L}_N(v)$.

2.2. For $\gamma>0$, consider the set

$$M_\gamma(A^+)=\{\ (\bar{a},\bar{n})\ |a_1,\dots,a_s\in A^+,n_1\ge\dots\ge n_s,\gamma(\bar{a},\bar{n})=\gamma\ \}.$$

It is easy to see that $M_\gamma(A^+)$ is a finite set. Therefore, we can define

$$N(\gamma,A^+)=\min\{\ N\in\mathbb{Z}\ |\bar{n}\ge -N\quad\forall(\bar{a},\bar{n})\in M_\gamma(A^+)\ \}.$$

Definition. Let $\gamma\ge v_0>0$. Then $(\bar{a},\bar{n})\in M_\gamma(A^+)$ is (v_0,A^+)-bad, if for any $0\le t<s$ and $\gamma_t^-=\gamma_t^-(\bar{a},\bar{n}):=a_1p^{n_1}+\dots+a_tp^{n_t}$ we have either $\gamma_t^-\ge v_0$ and $n_{t+1}\ge -N(\gamma_t^-,A^+)$, or $\gamma_t^-<v_0$.

The following properties are immediate consequences of the above definition:

a) if $(\bar{a},\bar{n})=(a_1,n_1)$ and $a_1p^{n_1}\ge v_0$, then $(\bar{a},\bar{n})$ is (v_0,A^+)-bad;

b) if $(\bar{a},\bar{n})\in M_\gamma(A^+)$, $\gamma\ge v_0$ and $\gamma-a_sp^{n_s}<v_0$, then $(\bar{a},\bar{n})$ is (v_0,A^+)-bad;

c) if $(\bar{a},\bar{n})$ is (v_0,A^+)-bad and $\gamma_{s-1}^-=a_1p^{n_1}+\dots+a_{s-1}p^{n_{s-1}}\ge v_0$, then $(a_1,n_1,\dots,a_{s-1},n_{s-1})$ is (v_0,A^+)-bad;

d) if $(\bar{a},\bar{n})$ is not (v_0,A^+)-bad and $\gamma(\bar{a},\bar{n})\ge v_0$, then there exists the unique indice $s_1<s$ such that $(a_1,n_1,\dots,a_{s'},n_{s'})$ is not (v_0,A^+)-bad for $s_1<s'\le s$, and is (v_0,A^+)-bad for $s'=s_1$. In particular, $\gamma_{s_1}^-=a_1p^{n_1}+\dots+a_{s_1}p^{n_{s_1}}\ge v_0$ and $-N(\gamma_{s_1}^-,A^+)>n_{s_1+1}\ge\dots\ge n_s$.

2.3. For $M\in\mathbb{Z}$, denote by $B_M=B_M(v_0,A^+)$ the set of (v_0,A^+)-bad collections $(\bar{a},\bar{n})$ such that $\bar{n}\le M$. Clearly, $B_M=\emptyset$ for sufficiently small M.

Proposition. *The set B_M is finite.*

Proof. If $\pi = (\bar{a}, \bar{n}) \in M_\gamma(A^+)$, where $\gamma \geq v_0$, set

$$m_0(\pi) = \max\{\ 0 \leq t < s \mid \gamma_{s-t}^- = a_1 p^{n_1} + \cdots + a_{s-t} p^{n_{s-t}} \geq v_0\ \}.$$

We want to show that there exists only finitely many $\pi \in B_M$ with a given value of $m_0(\pi)$.

Lemma. *There exists*

$$\delta(v_0, A^+) = \min\{\ v_0 - \gamma_1 \mid \gamma_1 < v_0, M_{\gamma_1}(A^+) \neq \emptyset\ \}.$$

Proof of lemma. For $1 \leq s_0 < p$, denote by $M_{\gamma_1 s_0}(A^+)$ the subset of $M_{\gamma_1}(A^+)$, which consists of collections $(\bar{a}, \bar{n})$ of length $s \leq s_0$. We use induction on $s_0 \geq 1$ to prove the existence of

$$\delta_{s_0} = \delta_{s_0}(v_0, A^+) := \min\{\ v_0 - \gamma_1 \mid \gamma_1 < v_0, M_{\gamma_1 s_0}(A^+) \neq \emptyset\ \}.$$

The existence of δ_1 is obvious. Assume that $s_0 \geq 2$ and δ_{s_0-1} exists. Clearly, there exists $\gamma_0 \in (v_0 - \delta_{s_0-1}, v_0)$ such that $M_{\gamma_0 s_0}(A^+) \neq \emptyset$. It is sufficient to prove the existence of only finitely many $\gamma' \in (\gamma_0, v_0)$ such that $M_{\gamma' s_0}(A^+) \neq \emptyset$.

Let $(\bar{a}, \bar{n}) \in M_{\gamma' s_0}(A^+)$. Then $a_{s_0} p^{n_{s_0}} \geq \gamma_0 - (v_0 - \delta_{s_0-1}) = \delta' > 0$. This gives the existence of $N^* = N^*(A^+, \delta')$ such that $n_{s_0} \geq N^*$. Therefore, $(\bar{a}, \bar{n})$ belongs to the finite set

$$\{\ (\bar{a}, \bar{n}) \mid 1 \leq s \leq s_0, n_1, \ldots, n_s \geq N^*, \gamma(\bar{a}, \bar{n}) < v_0\ \}.$$

The lemma is proved, because $\delta(v_0, A^+) = \delta_{p-1}$.

Continue the proof of proposition by induction on $m_0(\pi)$.

Let $m_0(\pi) = 0$, i.e. $\gamma - a_s p^{n_s} < v_0$. Then $a_s p^{n_s} \geq \delta(v_0, A^+)$ and all $\pi \in B_M$ with $m_0(\pi) = 0$ belong to the finite set

$$\{\ (\bar{a}, \bar{n}) \mid N^*(A^+, \delta(v_0, A^+)) \leq n_1, \ldots, n_s \leq M\ \}.$$

Assume finiteness of $\pi' \in B_M$ with $m_0(\pi') < m^*$, where $m^* \geq 1$. Let

$$N(m^*, v_0, A^+) = \min\{-N(\gamma', A^+) \mid \exists \pi' \in B_M \cap M_{\gamma'}(A^+) \text{ such that } m_0(\pi') < m^*\ \}.$$

If $\pi = (\bar{a}, \bar{n}) \in B_M$ and $m_0(\pi) = m^*$, then $\pi_1 = (a_1, n_1, \ldots, a_{s-1}, n_{s-1}) \in B_M$ and $m_0(\pi_1) = m^* - 1$. Therefore, $n_s \geq -N(\gamma(\pi_1), A^+) \geq N(m^*, v_0, A^+)$, and π belongs to the finite set $\{\ (\bar{a}, \bar{n}) \mid N(m^*, v_0, A^+) \leq n_1, \ldots, n_s \leq M\ \}$.

The proposition is proved.

2.4. Let $\Gamma_B = \{\ \gamma(\bar{a}, \bar{n}) \mid (\bar{a}, \bar{n}) \in B_M \text{ for some } M \in \mathbb{Z}\ \}$.

Proposition. *There exists* $\max\{ N(\gamma, A^+) \mid \gamma \in \Gamma_B \}$.

Proof. Consider $M_0 = M_0(A^+, v_0) \in \mathbb{N}$ such that $ap^{M_0} \geq v_0$ for all $a \in A^+$. Let $M > M_0$ and $(\bar{a}, \bar{n}) \in B_M \setminus B_{M-1}$. It follows easily from the definition of a (v_0, A^+)-bad collection that $(a_1, M_0 - M + n_1, \dots, a_s, M_0 - M + n_s) \in B_{M_0}$, and therefore

$$\max\{ N(\gamma, A^+) \mid \gamma \in \Gamma_B \} = \max\{ N(\gamma, A^+) \mid \gamma = \gamma(\bar{a}, \bar{n}), (\bar{a}, \bar{n}) \in B_{M_0} \}$$

exists. The proposition is proved.

2.5. Let $N_B(v_0, A^+) = \max\{ N(\gamma, A^+) \mid \gamma \in \Gamma_B \}$.

Proposition. *If* $N \geq N_B(v_0, A^+)$, *then* $\mathcal{L}_N(A, v_0) = \mathcal{L}_{N_B(v_0,A^+)}(A, v_0)$.

Proof. For $\gamma \in \Gamma_B$, consider the set $B_\gamma \subset M_\gamma(A^+)$ of all (v_0, A^+)-bad collections $(\bar{a}, \bar{n})$ such that $\gamma(\bar{a}, \bar{n}) = \gamma$.

Define the subset $M(B_\gamma) \subset M_{\gamma, N(\gamma, A^+)}(A)$ consisting of collections $(\bar{a}, \bar{n})$ such that if we remove from $(\bar{a}, \bar{n})$ all $\{a_i, n_i \mid 1 \leq i \leq s, a_i = 0 \}$, then we obtain a collection $(\bar{a}', \bar{n}') \in B_\gamma$. So, for all $\gamma \in \Gamma_B$, we can define the elements

$$\mathcal{F}_{B_\gamma} = \sum (-1)^{s-1} \eta(n_1, \dots, n_s) a_1 p^{n_1^*} [\dots [\mathcal{D}_{a_1 n_1}, \mathcal{D}_{a_2 n_2}], \dots, \mathcal{D}_{a_s n_s}]$$

of the Lie algebra $\mathcal{L}$, where the sum is taken for $(\bar{a}, \bar{n}) \in M(B_\gamma)$.

Denote by $\mathcal{L}_B$ the minimal $\sigma_{\mathcal{L}}$-invariant ideal of $\mathcal{L}$, which contains all $\mathcal{F}_{B_\gamma}$, where $\gamma \in \Gamma_B$, and $C_p(\mathcal{L})$. We want to prove, that $\mathcal{L}_N(A, v_0) = \mathcal{L}_B$ for $N \geq N_B(v_0, A^+)$.

Lemma 1. *For any* $\gamma \geq v_0$, *we have* $\mathcal{F}_{\gamma, -N(\gamma, A^+)} \in \mathcal{L}_N(A, v_0)$.

Proof of lemma. If $(\bar{a}, \bar{n}) \in M_{\gamma N}(A)$, then there exists the unique indice $s_1 \leq s$ such that $(a_1, n_1, \dots, a_{s_1}, n_{s_1}) \in M_{\gamma, -N(\gamma, A^+)}(A)$, $a_{s_1+1} = \dots = a_s = 0$ and $-N(\gamma, A^+) > n_{s_1+1} \geq \dots \geq n_s \geq -N$. This gives

$$\mathcal{F}_{\gamma, -N} = \left(\mathrm{id} + \sum (-1)^t \eta(m_1, \dots, m_t) \operatorname{ad} \mathcal{D}_{0 m_1} \circ \dots \circ \operatorname{ad} \mathcal{D}_{0, m_t} \right) \mathcal{F}_{\gamma, -N(\gamma, A^+)},$$

where the sum is taken for $t \geq 1$ and all $-N \leq m_t \leq \dots \leq m_1 < -N(\gamma, A^+)$. Now the lemma follows, because the operator in brackets of the above equality is invertible.

The following two lemmas can be proved similarly by applying the property d) of (v_0, A^+)-bad collections from n.2.2.

Lemma 2. *For any* $\gamma \geq v_0$, *we have* $\mathcal{F}_{\gamma, -N} \in \mathcal{L}_B$.

From the above nn.2.3-2.4 it follows, that $\Gamma_B = \{ \gamma_1, \gamma_2, \dots \}$, where $v_0 \leq \gamma_1 \leq \gamma_2 \leq \dots$ (in the archimedian topology $\gamma_n \to +\infty$, in the p-adic topology $\gamma_n \to 0$). For $n \in \mathbb{N}$, denote by $\mathcal{L}_B^{(m)}$ the ideal of $\mathcal{L}$ generated by $\{ \mathcal{F}_{B_{\gamma_i}} \mid 1 \leq i < m \}$ and by $C_p(\mathcal{L})$.

Lemma 3. *For any $m \in \mathbb{N}$ we have*

$$\mathcal{F}_{\gamma_m,-N(\gamma_m,A^+)} \equiv \mathcal{F}_{B_{\gamma_m}} \operatorname{mod} \mathcal{L}_B^{(m)}.$$

The lemma 2 gives that $\mathcal{L}_N(A,v_0) \subset \mathcal{L}_B$.

By the lemma 3, the ideal $\mathcal{L}_B$ is the minimal σ-invariant ideal of $\mathcal{L}$ such that $\mathcal{F}_{\gamma_m,-N(\gamma_m,A^+)} \in \mathcal{L}_B$ for all $m \in \mathbb{N}$. So, the lemma 1 implies that $\mathcal{L}_B \subset \mathcal{L}_N(A,v_0)$. The proposition is proved.

2.6. Now we can set $\mathcal{L}(A,v_0) = \mathcal{L}_N(A,v_0)$ and $\mathcal{L}^0(A,v_0) = \mathcal{L}_N^0(A,v_0) = \mathcal{L}_N(A,v_0) \cap \mathcal{L}^0$, where $N \geq N_B(v_0,A^+)$. We also define the integer

$$\widetilde{N}(v_0,A^+) = \min\{\ N \in \mathbb{Z} \mid \mathcal{L}_N^0(A,v_0) = \mathcal{L}^0(A,v_0)\ \}.$$

The filtration $\{\mathcal{L}(A,v)\}_{v>0}$ is a decreasing filtration of σ-invariant ideals of the Lie algebra $\mathcal{L}$. The filtration $\{\mathcal{L}^0(A,v)\}_{v>0}$ is a decreasing filtration of ideals of the Lie algebra $\mathcal{L}^0$. The set of jumps of this filtration is contained in the set $\{\ \gamma \mid M_\gamma(A^+) \neq \emptyset\ \}$. The lemma of n.2.3 gives, that for any $v_0 > 0$, there exists $\delta = \delta(v_0,A^+) > 0$ such that $\mathcal{L}(A,v) = \mathcal{L}(A,v_0)$ and $\mathcal{L}^0(A,v) = \mathcal{L}^0(A,v_0)$ for any $v \in (v_0-\delta, v_0)$. In particular, the filtration $\{\mathcal{L}^0(A,v)\}_{v>0}$ is left-continuous.

3. Estimations in the envelopping algebra $\mathcal{A}(\mathcal{L}) \otimes O(K')$.

In this section A is a fixed finite subset of $\mathbb{Z}^0(p)$. We use the notation from n.2.1 with omitted indication to the set A. The main result of this section is the proposition of n.3.10. We use this proposition for the study of the ramification filtration of the group Γ in the section 4.

3.1. *Notation and agreements.*

Fix a positive real number v_0.

If $\delta(v_0,A^+)$ is the rational number from the lemma of n.2.3, choose $0 < \delta < \min\{\delta(v_0,A^+), v_0/3\}$ such that $v_0 - \delta \in \mathbb{Z}[1/p]$.

Choose $0 < \varepsilon < v_0/3$ such that $v_0 + \varepsilon \in \mathbb{Z}[1/p]$.

We choose $\widetilde{N} \in \mathbb{N}$ and $r^* \in \mathbb{Q}$ such that if $N^* = \widetilde{N} + 1$ and $q = p^{N^*}$, then

$\widetilde{N} \geq \max\{\ \widetilde{N}(v_0,A^+), \widetilde{N}(v_0+\varepsilon,A^+)\ \}$, cf. notation of n.2.6;

$a^* := q(v_0-\delta) \in p\mathbb{N}$ and $q(v_0+\varepsilon) \in \mathbb{N}$;

$b^* := r^*(q-1) \in \mathbb{Z}^+(p)$;

$(v_0-\delta)(q+p)/(q-1) < r^* < v_0$.

We note that $\mathcal{L}^0(v_0) = \mathcal{L}^0_{\widetilde{N}}(v_0)$, $\mathcal{L}^0(v_0+\varepsilon) = \mathcal{L}^0_{\widetilde{N}}(v_0+\varepsilon)$, and the above inequality for r^* gives $q(b^*-a^*) > pa^*$, in particular, $b^* > a^*$.

Consider the field $K' = K(r^*,N^*)$ and the element $t_1 = [t_0'] \in O'(K')$, where t_0' is the uniformising element of K', cf. n.1.5. For $N \leq \widetilde{N}$ and arbitrary γ, we have either $\mathcal{F}_{\gamma,-N} = 0$, or $q\gamma \in p\mathbb{N}$. In particular, the expression $\mathcal{F}_{\gamma,-N}t_1^{-q\gamma}$ is well-defined in the Lie algebra $\mathcal{L}^0 \otimes O'(K')$. We note also that $t_1^{-qa^*p^{-1}}, t_1^{-q(v_0+\varepsilon)} \in O'(K')$.

We use the following abbreviated notation:

$O_1 = O'(K') = W(k)((t_1)), O_0 = W(k)[[t_1]]$;

$\mathcal{A} = \mathcal{A}(\mathcal{L}^0)$ is the envelopping algebra of the Lie algebra $\mathcal{L}^0 = \mathcal{L}^0(A)$, and $J = J(\mathcal{L}^0)$ is its augmentation ideal;

In the algebra $\mathcal{A}_1 := \mathcal{A} \otimes O_1$ we set $J_1 = JO_1$ and $J_O = JO_0$;

$\mathcal{A}_1(v_0)$ and $\mathcal{A}_1(v_0 + \varepsilon)$ will denote the minimal 2-sided ideals in the O_1-algebra $\mathcal{A}_1$, which contain $\mathcal{L}^0(v_0)$ and $\mathcal{L}^0(v_0 + \varepsilon)$, respectively;

For $s \in \mathbb{N}$, we set $C_s(\mathcal{A}_1(v_0)) = \sum_{0 \leq s_1 < s} J_1^{s_1} \mathcal{A}_1(v_0) J_1^{s-1-s_1}$.

$\mathcal{A}_1^+(v_0) = C_2(\mathcal{A}_1(v_0)) + \mathcal{A}_1(v_0 + \varepsilon) + p\mathcal{A}_1(v_0)$;

As in n.2, $(\bar{a}, \bar{n}) = (a_1, n_1, \dots, a_s, n_s)$, where $0 \leq s < p$, $a_1, \dots, a_s \in A$ and $n_1, \dots, n_s \in \mathbb{Z}$;

For $a \in A$, $n \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$, we define the power series $\mathrm{ex}_l(a, n) \in \mathbb{Z}_p[[X]]$ as follows:

$$\mathrm{ex}_l(a, n) = \mathrm{ex}_l(a, n)(X) = \begin{cases} \exp(ap^{n-l} X^{p^l}), & \text{for } n > l \\ E(a, X^{p^l}), & \text{for } n = l \\ 1, & \text{for } n < l, \end{cases}$$

where E is the function from n.1.5.

If $0 \leq t \leq s$ and $(\bar{a}, \bar{n}) = (a_1, n_1, \dots, a_s, n_s)$, we set

$$\mathrm{ex}_{lt}(\bar{a}, \bar{n}) = \mathrm{ex}_l(a_{t+1}, n_{t+1}) \dots \mathrm{ex}_l(a_s, n_s)$$

(by definition, $\mathrm{ex}_{ls}(\bar{a}, \bar{n}) = 1$). If $t = 0$ we use the simpler notation $\mathrm{ex}_l(\bar{a}, \bar{n}) = \mathrm{ex}_{l0}(\bar{a}, \bar{n})$.

The results of the substitution $X \mapsto t_1^{b^*}$ in the above power series we denote by $\mathcal{E}_l(\bar{a}, \bar{n}) = \mathrm{ex}_l(\bar{a}, \bar{n})(t_1^{b^*}) \in O_0$ and $\mathcal{E}_{lt}(\bar{a}, \bar{n}) = \mathrm{ex}_{lt}(\bar{a}, \bar{n})(t_1^{b^*}) \in O_0$.

$\mathcal{D}_{\bar{a}\bar{n}} = \mathcal{D}_{a_1 n_1} \dots \mathcal{D}_{a_s n_s}$, where $\mathcal{D}_{a_i n_i}$ for $1 \leq i \leq s$, are the generators of the Lie algebra $\mathcal{L}$ from n.2.

Proposition. *For any γ, we have*
(a) $\mathcal{F}_{\gamma, -\tilde{N}} t_1^{-q\gamma} \in \mathcal{A}_1(v_0) + t_1^{-a^*} J_O$;

(b) $\mathcal{F}_{\gamma, -\tilde{N}} t_1^{-q\gamma} \in \mathcal{A}_1(v_0 + \varepsilon) + t_1^{-q(v_0+\varepsilon)} J_O$.

Proof. If $\gamma \geq v_0$, then $\mathcal{F}_{\gamma, -\tilde{N}} t_1^{-q\gamma} \in \mathcal{A}_1(v_0)$. If $\gamma < v_0$, then $\gamma \leq v_0 - \delta$ and $\mathcal{F}_{\gamma, -\tilde{N}} t_1^{-q\gamma} \in t_1^{-a^*} J_O$, because $a^* = q(v_0 - \delta)$. The part (a) is proved. Similar arguments prove the part (b).

3.2. Let $\gamma_0(\bar{a}, \bar{n}) = a_1 p^{n_1^*} + \dots + a_s p^{n_s^*} = \sum_{n_i \geqslant 0} a_i p^{n_i}$, and for $0 \leq t \leq s$, set $\gamma_{0t}(\bar{a}, \bar{n}) = a_{t+1} p^{n_{t+1}^*} + \dots + a_s p^{n_s^*}$ (cf. n.2.1 for the definition of n^*, where $n \in \mathbb{Z}$).

Proposition. *For any $\gamma \geq 0$ and $N \in \mathbb{Z}$,*

$$\mathcal{F}_{\gamma, -N} = \sum (-1)^{s+t} \eta(n_t, \dots, n_1) \eta(n_{t+1}, \dots, n_s) \gamma_{0t}(\bar{a}, \bar{n}) \mathcal{D}_{\bar{a}\bar{n}},$$

where the sum is taken for $1 \leq s < p$, $0 \leq t \leq s$ *and all* $(\bar{a}, \bar{n})$ *such that* $\gamma(\bar{a}, \bar{n}) = \gamma$ *and* $\bar{n} \geq -N$.

Remark. If $0 \in A$ then the right-hand sum contains infinitely many summands. In this and similar cases we set

$$\mathcal{F}_{\gamma,-N} = \lim_{l \to +\infty} \mathcal{F}_{\gamma,-N,l},$$

where $\mathcal{F}_{\gamma,-N,l}$ is the part of the right-hand sum which contains terms with $\bar{n} < l$. In our case it is easy to see (by applying the identity of the part (b) of the lemma from n.1.4, cf. also the proof of lemma from n.3.3 below), that if $l_0 = l_0(\gamma, A) \in \mathbb{N}$ is such that $p^{l_0} a > \gamma$ for any $a \in A^+$, then $\mathcal{F}_{\gamma,-N,l_1} = \mathcal{F}_{\gamma,-N,l_2}$ if $l_1, l_2 \geq l_0(A, \gamma)$.

Proof. For $1 \leq s < p$ and $1 \leq t \leq s$ consider the subset Φ_{st} of substitutions π of order s such that $\pi(1) = t$ and for any $1 \leq l \leq s$ the subset $\{\pi(1), \ldots, \pi(l)\}$ of $[1, s]$ is "connected", i.e. there exists $n(l) \in \mathbb{N}$ such that

$$\{\pi(1), \ldots, \pi(l)\} = \{n(l), n(l) + 1, \ldots, n(l) + l - 1\}.$$

By definition, we set $\Phi_{s0} = \Phi_{s,s+1} = \emptyset$.

Set $B_t(\bar{n}) = \sum_{\pi \in \Phi_{st}} \eta(n_{\pi(1)}, \ldots, n_{\pi(s)})$ for $0 \leq t \leq s$. We note that $B_0(\bar{n}) = B_{s+1}(\bar{n}) = 0$, $B_1(\bar{n}) = \eta(n_1, \ldots, n_s)$ and $B_s(\bar{n}) = \eta(n_s, \ldots, n_1)$.

Lemma 1. *For any* $0 \leq t \leq s$, *the set* $\Phi_{st} \cup \Phi_{s,t+1}$ *is the set of all insertions of the set* $\{t, \ldots, 1\}$ *into the set* $\{t+1, \ldots, s\}$, *i.e.* $\pi \in \Phi_{st} \cup \Phi_{s,t+1}$ *if and only if the sequences* $\{\pi^{-1}(t), \ldots, \pi^{-1}(1)\}$ *and* $\{\pi^{-1}(t+1), \ldots, \pi^{-1}(s)\}$ *are increasing.*

The proof of this lemma follows easily from the above definition of the set Φ_{st}.

The following lemma is implied by the property a) of lemma of n.1.4 of the structural constants $\eta(\bar{n})$.

Lemma 2. *For* $0 \leq t \leq s$, *we have*

$$B_t(\bar{n}) + B_{t+1}(\bar{n}) = \eta(n_t, \ldots, n_1)\eta(n_{t+1}, \ldots, n_s).$$

Lemma 3. *For indeterminates* $X_1, \ldots, X_s$ *we have the identity*

$$\sum_{\substack{1 \leqslant t \leqslant s \\ \pi \in \Phi_{st}}} (-1)^{t-1} X_{\pi^{-1}(1)} \ldots X_{\pi^{-1}(s)} = [\ldots[X_1, X_2], \ldots, X_s].$$

We omit the proof, which can be obtained by simple combinatorial arguments.

Now we can rewrite the right-hand side of the equality of our proposition in the form

$$\sum_{\substack{1 \leqslant t \leqslant s < p \\ \gamma(\bar{a}, \bar{n}) = \gamma, \bar{n} \geqslant -N}} (-1)^{s+t+1} B_t(\bar{n})(\gamma_{0,t-1}(\bar{a}, \bar{n}) - \gamma_{0t}(\bar{a}, \bar{n}))\mathcal{D}_{\bar{a}\bar{n}} =$$

$$\sum_{\substack{1 \leqslant s < p \\ \gamma(\bar{a}, \bar{n}) = \gamma, \bar{n} \geqslant -N}} (-1)^s \sum_{1 \leqslant t \leqslant s} (-1)^{t+1} B_t(\bar{n}) a_t p^{n_t^*} \mathcal{D}_{\bar{a}\bar{n}} =$$

$$\sum_{\substack{1\leqslant s<p\\ \gamma(\bar a,\bar n)=\gamma,\bar n\geqslant -N}} (-1)^s \sum_{\substack{1\leqslant t\leqslant s\\ \pi\in\Phi_{st}}} (-1)^{t+1}\eta(n_{\pi(1)},\dots,n_{\pi(s)})a_t p^{n_t^*}\mathcal{D}_{\bar a\bar n} =$$

$$\sum_{\substack{1\leqslant s<p\\ \gamma(\bar a,\bar n)=\gamma\\ \bar n\geqslant -N}} (-1)^s\eta(n_1,\dots,n_s)a_1p^{n_1^*}\sum_{\substack{1\leqslant t\leqslant s\\ \pi\in\Phi_{st}}}(-1)^{t+1}\mathcal{D}_{a_{\pi^{-1}(1)},n_{\pi^{-1}(1)}}\dots\mathcal{D}_{a_{\pi^{-1}(s)},n_{\pi^{-1}(s)}}.$$

By the above lemma 3 the last expression equals $\mathcal{F}_{\gamma,-N}$. The proposition is proved.

3.3. For $\gamma\geq 0$ and $N,l\in\mathbb{Z}$ such that $-N<l$, we set

$$C'_{\gamma,-N,l}=\delta_{\gamma 0}+\sum\eta(n_s,\dots,n_1)\mathcal{D}_{\bar a\bar n},\quad C''_{\gamma,-N,l}=\delta_{\gamma 0}+\sum(-1)^s\eta(n_1,\dots,n_s)\mathcal{D}_{\bar a\bar n},$$

$$\mathcal{F}_{\gamma,-N,l}=\sum(-1)^s\sum_{0\leqslant t\leqslant s}(-1)^t\eta(n_t,\dots,n_1)\eta(n_{t+1},\dots,n_s)\gamma_{0t}(\bar a,\bar n)\mathcal{D}_{\bar a\bar n},$$

where δ is the Kronecker symbol, all three sums are taken for $1\leq s<p$ and collections $(\bar a,\bar n)$ such that $\gamma(\bar a,\bar n)=\gamma$ and $-N\leq\bar n<l$.

Lemma. *For $\gamma\geq 0$ and $N,l\in\mathbb{Z}$ such that $-N<l$, we have*

$$\mathcal{F}_{\gamma,-N}=\mathcal{F}_{\gamma,-N,l}+\sum_{\gamma_1+\gamma^*+\gamma_2=\gamma}C'_{\gamma_1,-N,l}\mathcal{F}_{\gamma^*,l}C''_{\gamma_2,-N,l}\bmod J_1^p.$$

Proof. We have

$$\mathcal{F}_{\gamma,-N}-\mathcal{F}_{\gamma,-N,l}=\sum_{\substack{\gamma(\bar a,\bar n)=\gamma\\ \max\bar n\geqslant l}}(-1)^s\alpha(\bar a,\bar n)\mathcal{D}_{\bar a\bar n},$$

where $\alpha(\bar a,\bar n)=\sum_{0\leqslant t\leqslant s}(-1)^t\eta(n_t,\dots,n_1)\eta(n_{t+1},\dots,n_s)\gamma_{0t}(\bar a,\bar n)$.

Let $t_1=t_1(\bar n)=\min\{\,t\mid n_t\geq l\,\}$ and $t_2=t_2(\bar n)=\max\{\,t\mid n_t\geq l\,\}$. Then from the part (b) of the lemma from n.1.4, it follows that

$$\alpha(\bar a,\bar n)=$$

$$\eta(n_{t_1-1},\dots,n_1)\eta(n_{t_2+1},\dots,n_s)\sum(-1)^t\eta(n_t,\dots,n_{t_1})\eta(n_{t+1},\dots,n_{t_2})\gamma_{(t,t_2]}(\bar a,\bar n),$$

where $\gamma_{(t,t_2]}(\bar a,\bar n)=a_{t+1}p^{n^*_{t+1}}+\dots+a_{t_2}p^{n^*_{t_2}}$ and the sum is taken for $t_1-1\leq t\leq t_2$. From the definition of the structural constants $\eta(\bar n)$ it follows that $\alpha(\bar a,\bar n)\neq 0$ implies $n_t\geq l$ for all $t_1\leq t\leq t_2$.

It is easy to see now that the above expression for $\alpha(\bar a,\bar n)$ gives the statement of our lemma.

Proposition.
(a) *If $l \in \mathbb{Z}_{\geq 0}$ and $-\widetilde{N} \leq \bar{n} < l$, then*

$$\mathcal{D}_{\bar{a}\bar{n}} t_1^{-q\gamma(\bar{a},\bar{n})} \in \mathcal{A}_1(v_0) + \sum_{i\geq 1} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^p;$$

(b) *If $l \in \mathbb{N}$, then $\mathcal{F}_{\gamma,l} t_1^{-q\gamma} \in \mathcal{A}_1(v_0) + \sum_{i\geq 1} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^p$.*

Proof. We use induction on $2 \leq s_0 \leq p$ to prove the formulae (a) and (b) modulo $J_1^{s_0}$.

Let $s_0 = 2$.

If $s \geq 2$, then $\mathcal{D}_{\bar{a}\bar{n}} \in C_2(\mathcal{L})$. If $s = 1$, i.e. $(\bar{a},\bar{n}) = (a,n)$ with $a \in \mathbb{Z}^+(p)$ and $-\widetilde{N} \leq n < l$, then $\mathcal{D}_{an} \equiv (1/a)\sigma^n(\mathcal{F}_{a,-N}) \bmod C_2(\mathcal{L})$, with arbitrary $N \geq -n$. If we take $N = \widetilde{N}$, then for $a \geq v_0$, we have $\mathcal{D}_{an} \in \mathcal{L}(v_0) + C_2(\mathcal{L})$ and $\mathcal{D}_{an} t_1^{-q\gamma(a,n)} \in \mathcal{A}_1(v_0) + J_1^2$. If $a < v_0$, then $a \leq v_0 - \delta$ and $\mathcal{D}_{an} t_1^{-q\gamma(a,n)} \in t_1^{-a^* p^{l-1}} J_O$, because $n < l$. So, the formula (a) is proved modulo J_1^2.

Let $l \geq 1$. If $\gamma \notin p^l\mathbb{N}$, then $\mathcal{F}_{\gamma,l} = 0$. Otherwise, $\mathcal{F}_{\gamma,l} \equiv \mathcal{F}_{\gamma,-N} \bmod C_2(\mathcal{L})$, with arbitrary $-N \leq l$. Therefore, if $\gamma \geq v_0$ then $\mathcal{F}_{\gamma,l} t_1^{-q\gamma} \in \mathcal{A}_1(v_0) + J_1^2$. If $\gamma < v_0$ then $\gamma \leq v_0 - \delta$ and $\mathcal{F}_{\gamma,l} t_1^{-q\gamma} \in t_1^{-a^*} J_O$. So, the formula (b) is proved modulo J_1^2.

Assume that the relations (a) and (b) are proved modulo $J_1^{s_0}$, where $s_0 \geq 2$.

From (a) it follows now that for $s \geq 2$ and $-\widetilde{N} \leq \bar{n} < l$, we have

$$\mathcal{D}_{\bar{a}\bar{n}} t_1^{-q\gamma(\bar{a},\bar{n})} \in C_2(\mathcal{A}_1(v_0)) + \sum_{i\geq 2} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^{s_0+1}. \tag{1}$$

We obtain also from (a) that for any $\gamma \geq 0$ and $l \in \mathbb{Z}_{\geq 0}$,

$$(C'_{\gamma,-\widetilde{N},l} - \delta_{\gamma 0}) t_1^{-q\gamma}, (C''_{\gamma,-\widetilde{N},l} - \delta_{\gamma 0}) t_1^{-q\gamma} \in \mathcal{A}_1(v_0) + \sum_{i\geq 1} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^{s_0}. \tag{2}$$

Now the above lemma for $l \in \mathbb{N}$, and the inductive assumption (b) modulo $J_1^{s_0}$ give

$$\mathcal{F}_{\gamma,-\widetilde{N}} t_1^{-q\gamma} \equiv \mathcal{F}_{\gamma,-\widetilde{N},l} t_1^{-q\gamma} + \mathcal{F}_{\gamma,l} t_1^{-q\gamma} \tag{3}$$

$$\bmod C_2(\mathcal{A}_1(v_0)) + \sum_{i\geq 2} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^{s_0+1}.$$

If $\gamma \in p^l\mathbb{N}$, then $\mathcal{F}_{\gamma,-\widetilde{N},l}$ is a linear combination of $\mathcal{D}_{\bar{a}\bar{n}}$ with $s \geq 2$, $-\widetilde{N} \leq \bar{n} < l$ and $\gamma(\bar{a},\bar{n}) = \gamma$. So, we can apply the property (1) to estimate $\mathcal{F}_{\gamma,-\widetilde{N},l}$. Because $l \geq 1$, we also have

$$\mathcal{F}_{\gamma,-\widetilde{N}} t_1^{-q\gamma} \in \mathcal{A}_1(v_0) + t_1^{-a^*} J_O + J_1^p \subset \mathcal{A}_1(v_0) + t_1^{-a^* p^{l-1}} J_O + J_1^p.$$

Now the relation (3) gives the formula (b) modulo $J_1^{s_0+1}$ in the case $\gamma \in p^l\mathbb{N}$. If $\gamma \notin p^l\mathbb{N}$, then $\mathcal{F}_{\gamma,l} = 0$, and the formula (b) is valid by trivial reasons. We can use also this argument in the case $l = 1$ and $\gamma = a \in \mathbb{Z}^+(p)$ to obtain from the above relations (1) and (3) that

$$\mathcal{D}_{a0} t_1^{-qa} \in \mathcal{A}_1(v_0) + \sum_{i \geq 1} \left(t_1^{-a^*} J_O\right)^i + J_1^{s_0+1}.$$

If $l \in \mathbb{Z}_{\geq 0}$ and $n < l$, then

$$\mathcal{D}_{an} t_1^{-q\gamma(a,n)} = \sigma^n \left(\mathcal{D}_{a0} t_1^{-qa}\right) \in \sigma^n(\mathcal{A}_1(v_0) + \sum_{i \geq 1} \left(t_1^{-a^*} J_O\right)^i + J_1^{s_0+1}) \subset$$

$$\sigma^{l-1}(\mathcal{A}_1(v_0) + \sum_{i \geq 1} \left(t_1^{-a^*} J_O\right)^i + J_1^{s_0+1}) \subset \mathcal{A}_1(v_0) + \sum_{i \geq 1} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^{s_0+1}.$$

Together with the property (1) this gives obviously the formula (a) modulo $J_1^{s_0+1}$.

The proposition is proved.

Remark. If we apply the part (a) of the above proposition in the case $s_0 = 1$ and $(\bar{a}, \bar{n}) = (a, 0)$, then the proposition 2 of n.1.2 gives, that

if $a \geq sv_0$, *where* $1 \leq s < p$, *then* $\mathcal{D}_{an} \in \mathcal{L}(v_0) + C_{s+1}(\mathcal{L})$.

3.4. We also need the following modification of the above proposition.

Proposition.

(a) *If* $l \geq 0$ *and* $-\widetilde{N} \leq \bar{n} < l$, *then*

$$\mathcal{D}_{\bar{a}\bar{n}} t_1^{-q\gamma(\bar{a},\bar{n})} \in \mathcal{A}_1^+(v_0) + t_1^{-q(v_0+\varepsilon)} J_O + \sum_{i \geq 2} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^p;$$

(b) *If* $l \geq 1$, *then*

$$\mathcal{F}_{\gamma,l} t_1^{-q\gamma} \in \mathcal{A}_1^+(v_0) + t_1^{-q(v_0+\varepsilon)} J_O + \sum_{i \geq 2} \left(t_1^{-a^* p^{l-1}} J_O\right)^i + J_1^p.$$

Proof. We use the part (b) of the proposition of n.3.1 and that $\mathcal{A}_1^+(v_0)$ contains $C_2(\mathcal{A}_1(v_0)) + \mathcal{A}_1(v_0 + \varepsilon)$. If $s \geq 2$, then (a) follows from the formula (1) of n.3.3 for $s_0 = p - 1$. In the case $s = 1$, we can use arguments from the end of n.3.3. For proving the property (b), we can assume that $\gamma \in p^l\mathbb{N}$. Then (b) follows from the formula (3) of n.3.3, where $s_0 = p - 1$.

3.5. For $\gamma \geq 0$, $N \in \mathbb{Z}$ and $i \in \mathbb{N}$ set (cf. remark of n.3.2)

$$\mathcal{F}_{\gamma,-N}(i) = \sum (-1)^{s+t} \eta(n_t, \dots, n_1) \eta(n_{t+1}, \dots, n_s) \gamma_{0t}(\bar{a}, \bar{n})^i \mathcal{D}_{\bar{a}\bar{n}},$$

$$\mathcal{F}_{\gamma,-N}(0,i) = \sum (-1)^{s+t} \eta(n_t, \dots, n_1) \eta(n_{t+1}, \dots, n_s) \gamma_0(\bar{a}, \bar{n}) \gamma_{0t}(\bar{a}, \bar{n})^i \mathcal{D}_{\bar{a}\bar{n}},$$

where the both sums are taken for $1 \leq s < p$, $0 \leq t \leq s$ and $(\bar{a}, \bar{n})$ such that $\gamma(\bar{a}, \bar{n}) = \gamma$ and $\bar{n} \geq -N$.

We note that $\mathcal{F}_{\gamma,-N}(1) = \mathcal{F}_{\gamma,-N}$; if $l \geq 0$, then $\mathcal{F}_{\gamma,l}(0,i) = \gamma \mathcal{F}_{\gamma,l}(i)$; and if $\gamma \notin \mathbb{N}$, then $\mathcal{F}_{\gamma,0}(i) = 0$.

Lemma 1. *For $\gamma \geq 0$ and $N, i \in \mathbb{N}$, we have*

(a) $$\mathcal{F}_{\gamma,-N}(i) = \sum_{\gamma_1+\gamma^*+\gamma_2=\gamma} C'_{\gamma_1,-N,0}\mathcal{F}_{\gamma^*,0}(i)C''_{\gamma_2,-N,0} \bmod J_1^p;$$

(b) $$\mathcal{F}_{\gamma,-N}(0,i) = \sum_{\gamma_1+\gamma^*+\gamma_2=\gamma} C'_{\gamma_1,-N,0}\gamma^*\mathcal{F}_{\gamma^*,0}(i)C''_{\gamma_2,-N,0} \bmod J_1^p.$$

The proof is analogous to the proof of the lemma from n.3.3.

Lemma 2. *For $N \in \mathbb{Z}$, $i \in \mathbb{N}$ and $\gamma \geq 0$, we have*

$$\sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}_{\gamma_1,-N}(1)\mathcal{F}_{\gamma_2,-N}(i) = \mathcal{F}_{\gamma,-N}(i+1) - \mathcal{F}_{\gamma,-N}(0,i) \bmod J_1^p.$$

Proof. From the part (b) of the lemma of n.1.4 it follows that for any $(\bar{a},\bar{n})$

$$\sum_{0\leq t\leq s} (-1)^{s+t}\eta(n_t,\ldots,n_1)\eta(n_{t+1},\ldots,n_s)\gamma_0(\bar{a},\bar{n})\mathcal{D}_{\bar{a}\bar{n}} = 0.$$

So, if for $0 \leq t \leq s$, $\gamma_{0t}^-(\bar{a},\bar{n}) = \gamma_0(\bar{a},\bar{n}) - \gamma_{0t}(\bar{a},\bar{n}) = a_1p^{n_1^*} + \cdots + a_tp^{n_t^*}$, then

$$\mathcal{F}_{\gamma_1,-N}(1) = -\sum_{\substack{0\leqslant t\leqslant s<p\\ \gamma(\bar{a},\bar{n})=\gamma_1}} (-1)^{s+t}\eta(n_t,\ldots,n_1)\eta(n_{t+1},\ldots,n_s)\gamma_{0t}^-(\bar{a},\bar{n})\mathcal{D}_{\bar{a}\bar{n}}.$$

For computing the left-hand side of the identity of our lemma, we can use the indices (s',t'), where $1 \leq s' < p$ and $0 \leq t' \leq s'$, in the above expression for $\mathcal{F}_{\gamma_1,-N}(1)$, and the indices (s,t''), where $s' \leq s < p$ and $s' \leq t'' \leq s$, in the expression for $\mathcal{F}_{\gamma_2,-N}(i)$. Because $\gamma_{00}^-(\bar{a},\bar{n}) = \gamma_{0s}(\bar{a},\bar{n}) = 0$, we can assume also that $1 \leq t' \leq s'$ and $s' \leq t'' < s$. Now we obtain the following congruence modulo J_1^p

(1) $$\sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}_{\gamma_1,-N}(1)\mathcal{F}_{\gamma_2,-N}(i) \equiv$$

$$-\sum \eta(n_{t'},\ldots,n_1)\gamma_{0t'}^-(\bar{a},\bar{n})R_{t't''}(-1)^{s+t''}\eta(n_{t''+1},\ldots,n_s)\gamma_{0t''}(\bar{a},\bar{n})^i\mathcal{D}_{\bar{a}\bar{n}},$$

where the sum is taken for $2 \leq s < p, 1 \leq t' \leq t'' < s$ and $(\bar{a},\bar{n})$ such that $\bar{n} \geq -N$ and $\gamma(\bar{a},\bar{n}) = \gamma$, and

$$R_{t't''} = \sum_{t'\leq s_1\leq t''} (-1)^{s_1-t'}\eta(n_{t'+1},\ldots,n_{s_1})\eta(n_{t''},\ldots,n_{s_1+1}).$$

By the part (b) of the lemma from n.1.4, we have $R_{t't''} = \delta_{t't''}$, where δ is the Kronecker symbol. So, we can take $t' = t'' = t$ and after applying the identity

$$\gamma_{0t}^-(\bar{a},\bar{n})\gamma_{0t}(\bar{a},\bar{n})^i = \gamma_0(\bar{a},\bar{n})\gamma_{0t}(\bar{a},\bar{n})^i - \gamma_{0t}(\bar{a},\bar{n})^{i+1},$$

we can rewrite the right-hand side of (1) in the form

$$-\sum(-1)^{s+t}\eta(n_t,\dots,n_1)\eta(n_{t+1},\dots,n_s)\gamma_0(\bar a,\bar n)\gamma_{0t}(\bar a,\bar n)^i\mathcal{D}_{\bar a\bar n}+$$

$$\sum(-1)^{s+t}\eta(n_t,\dots,n_1)\eta(n_{t+1},\dots,n_s)\gamma_{0t}(\bar a,\bar n)^{i+1}\mathcal{D}_{\bar a\bar n},$$

where the both sums are taken for $2\le s<p$, $1\le t<s$ and $(\bar a,\bar n)$ such that $\bar n\ge -N$ and $\gamma(\bar a,\bar n)=\gamma$. Now we note, that in the above expression we can use summation for the indices (s,t) such that $1\le s<p$ and $0\le t\le s$. Indeed, for any $1\le s<p$, we have $\gamma_{0s}(\bar a,\bar n)=0$, and for $t=0$ we have $\gamma_0(\bar a,\bar n)\gamma_{0t}(\bar a,\bar n)^i=\gamma_{0t}(\bar a,\bar n)^{i+1}$.

So, the above expression is equal to $-\mathcal{F}_{\gamma,-N}(0,i)+\mathcal{F}_{\gamma,-N}(i+1)$, and the lemma is proved.

Proposition 1. *If $i\in\mathbb{N}$ and $\gamma\ge 0$ then*

(a) $$\mathcal{F}_{\gamma,-\tilde N}(i)t_1^{-q\gamma}\in\mathcal{A}_1(v_0)+\sum_{1\leqslant j\leqslant i}\left(t_1^{-a^*}J_O\right)^j+J_1^p;$$

(b) $$\mathcal{F}_{\gamma,-\tilde N}(i)t_1^{-q\gamma}\in\mathcal{A}_1^+(v_0)+t_1^{-q(v_0+\varepsilon)}J_O+\sum_{2\leqslant j\leqslant i}\left(t_1^{-a^*}J_O\right)^j+J_1^p.$$

(c) $$\mathcal{F}_{\gamma,0}(i)t_1^{-q\gamma}\in\mathcal{A}_1(v_0)+\sum_{1\leqslant j\leqslant i}\left(t_1^{-a^*}J_O\right)^j\sum_{j_1\geqslant 0}\left(t_1^{-a^*p^{-1}}J_O\right)^{j_1}+J_1^p.$$

Proof. If $i=1$ the formulae (a) and (b) are proved, cf. proposition of n.3.1.

Assume by induction that (a) and (b) are proved for $i=i_0\ge 1$.

Now we assume that (c) is proved for $i=i_0$ modulo $J_1^{s_0}$, where $1\le s_0<p$. Then the property (2) of n.3.3 with $l=0$, and the part (a) of lemma 1 imply that

$$\mathcal{F}_{\gamma,-\tilde N}(i_0)t_1^{-q\gamma}\equiv\mathcal{F}_{\gamma,0}(i_0)t_1^{-q\gamma}$$

$$\operatorname{mod}C_2(\mathcal{A}_1(v_0))+\sum_{1\leqslant j\leqslant i_0}\left(t_1^{-a^*}J_O\right)^j\sum_{j_1\geqslant 1}\left(t_1^{-a^*p^{-1}}J_O\right)^{j_1}+J_1^{s_0+1}.$$

By the inductive assumption (a) for $i=i_0$, this gives the formula (c) modulo $J_1^{s_0+1}$. So, the part (c) is proved by induction on $1\le s_0\le p$ for $i=i_0$.

We apply it with the property (2) of n.3.3 and the part (b) of lemma 1 to obtain

(1) $$\mathcal{F}_{\gamma,-\tilde N}(0,i_0)t_1^{-q\gamma}\in\mathcal{A}_1(v_0)+\sum_{1\leqslant j\leqslant i_0}\left(t_1^{-a^*}J_O\right)^j\sum_{j_1\geqslant 0}\left(t_1^{-a^*p^{-1}}J_O\right)^{j_1}+J_1^p.$$

The inductive assumption (a) for $i=i_0$ gives

(2) $$\sum_{\gamma_1+\gamma_2=\gamma}\mathcal{F}_{\gamma_1,-\tilde N}(1)\mathcal{F}_{\gamma_2,-\tilde N}(i_0)t_1^{-q\gamma}\in C_2(\mathcal{A}_1(v_0))+\sum_{2\leqslant i\leqslant i_0+1}\left(t_1^{-a^*}J_O\right)^j+J_1^p.$$

Now the above relations (1) and (2) with lemma 2 give the formula (a) for $i = i_0 + 1$ (we use that $\left(t_1^{-a^*p^{-1}} J_O\right)^{j_1} \subset t_1^{-a^*} J_O$ for $1 \le j_1 < p$).

Proceeding in the same way we obtain

$$\mathcal{F}_{\gamma,-\tilde{N}}(i_0+1)t_1^{-q\gamma} \equiv \mathcal{F}_{\gamma,-\tilde{N}}(0,i_0)t_1^{-q\gamma} \equiv \gamma\mathcal{F}_{\gamma,0}(0,i_0)t_1^{-q\gamma} \equiv \gamma\mathcal{F}_{\gamma,-\tilde{N}}(i_0)t_1^{-q\gamma}$$

$$\bmod C_2(\mathcal{A}_1(v_0)) + \sum_{2\leqslant i\leqslant i_0+1} \left(t_1^{-a^*} J_O\right)^j + J_1^p$$

(we use the lemma 2 for the first congruence and the lemma 1 for the second and the third congruences). The above congruences give the part (b) for $i = i_0 + 1$.

The proposition is proved.

Proposition 2. *If $l, i \in \mathbb{N}$, then*

$$\text{(a)} \qquad \mathcal{F}_{\gamma,l}(i)t_1^{-q\gamma} \in \sum_{j=1}^{i} p^{(i-j)l} C_j(\mathcal{A}_1(v_0)) + \sum_{\substack{1\leqslant j\leqslant i\\ j_1\geqslant j}} p^{(i-j)l}\left(t_1^{-a^*p^{l-1}} J_O\right)^{j_1} + J_1^p;$$

$$\text{(b)} \qquad \mathcal{F}_{\gamma,l}(i)t_1^{-q\gamma} \in p^{(i-1)l}\mathcal{A}^+(v_0) + \sum_{j=2}^{i} p^{(i-j)l} C_j(\mathcal{A}_1(v_0)) +$$

$$p^{(i-1)l} t_1^{-q(v_0+\varepsilon)} J_O + \sum_{\substack{2\leqslant j\leqslant i\\ j_1\geqslant j}} p^{(i-j)l}\left(t_1^{-a^*p^{l-1}} J_O\right)^{j_1} + J_1^p;$$

Proof. We use induction on $i \ge 1$.

If $i = 1$, then

$$\mathcal{F}_{\gamma,l}(1)t_1^{-q\gamma} = \mathcal{F}_{\gamma,l}t_1^{-q\gamma} \in \mathcal{A}_1(v_0) + \sum_{j_1\geqslant 1}\left(t_1^{-a^*p^{l-1}} J_O\right)^{j_1} + J_1^p$$

by the part (b) of the proposition of n.3.3.

Assume that our proposition is proved for $i = i_0 \ge 1$. Then

$$\text{(1)} \qquad \sum_{\gamma_1+\gamma_2=\gamma} \mathcal{F}_{\gamma_1,l}(1)\mathcal{F}_{\gamma_2,l}(i_0)t_1^{-q\gamma} \in$$

$$\sum_{2\leqslant j\leqslant i_0+1} p^{(i_0+1-j)l} C_j(\mathcal{A}_1(v_0)) + \sum_{\substack{2\leqslant j\leqslant i_0+1\\ j_1\geqslant j}} p^{(i_0+1-j)l}\left(t_1^{-a^*p^{l-1}} J_O\right)^{j_1} + J_1^p.$$

Clearly, $\mathcal{F}_{\gamma,l}(0,i_0) = \gamma\mathcal{F}_{\gamma,l}(i_0)$ and we have either $\gamma \in p^l\mathbb{N}$, or $\mathcal{F}_{\gamma,l}(i_0) = 0$. Therefore,

$$\mathcal{F}_{\gamma,l}(0,i_0)t_1^{-q\gamma} \in \sum_{1\leqslant j\leqslant i_0} p^{(i_0+1-j)l} C_j(\mathcal{A}_1(v_0)) + \sum_{\substack{1\leqslant j\leqslant i_0\\ j_1\geqslant j}} p^{(i_0+1-j)l}\left(t_1^{-a^*p^{l-1}} J_O\right)^{j_1} + J_1^p,$$

and we obtain the part (a) for $i = i_0 + 1$ from the above lemma 2 with $-N = l$ and $i = i_0$.

The part (b) for $i = i_0 + 1$, can be now obtained by the use of the above formula (1), the lemma 2, and the inductive assumption (b) for $i = i_0$.

Corollary. *If $i, l \geq 1$, then*

$$\text{(a)} \qquad (\mathcal{F}_{\gamma,l}(i)/i!)t_1^{-q\gamma} \in \mathcal{A}_1(v_0) + \sum_{1 \leqslant j < p} \left(t_1^{-a^* p^{l-1}} J_O\right)^j + J_1^p;$$

$$\text{(b)} \qquad (\mathcal{F}_{\gamma,l}(i)/i!)t_1^{-q\gamma} \in \mathcal{A}_1^+(v_0) + t_1^{-q(v_0+e)} J_O + \sum_{2 \leqslant j < p} \left(t_1^{-a^* p^{l-1}} J_O\right)^j + J_1^p.$$

Proof. Clearly, if $\bar{n} \geq l$, then $\gamma_{0t}(\bar{a}, \bar{n}) \in p^l \mathbb{N}$ for $0 \leq t \leq s$. Therefore,

$$\mathcal{F}_{\gamma,l}(i)/i! \in \left(p^{li}/i!\right) \mathcal{A}_1 \subset \mathcal{A}_1.$$

If $i \in \mathbb{N}$, then by the proposition 2, $\mathcal{F}_{\gamma,l}(i)t_1^{-q\gamma}$ belongs to

$$\sum_{1 \leqslant j \leqslant \min\{i, p-1\}} p^{(i-j)l} C_j(\mathcal{A}_1(v_0)) + \sum_{1 \leqslant j \leqslant j_1 \leqslant \min\{i, p-1\}} p^{(i-j)l} \left(t_1^{-a^* p^{l-1}} J_O\right)^{j_1} + J_1^p$$

$$\subset p^{\tilde{i}} \mathcal{A}_1(v_0) + p^{\tilde{i}} \sum_{1 \leqslant j_1 < p} \left(t_1^{-a^* p^{l-1}} J_O\right)^{j_1} + J_1^p,$$

where $\tilde{i} = \max\{0, i - (p-1)\}$.

One can easily verify that $v_p(i!) \leq \tilde{i}$ for $i \in \mathbb{N}$. Therefore, the "J_1^p-part" of $\mathcal{F}_{\gamma,l}(i)t_1^{-q\gamma}$ is also divisible by $i!$ and we obtain the part (a) of the corollary.

The part (b) can be proved similarly.

3.6. We use the notation from n.3.1 to define for $N \leq \tilde{N}$ and $l \geq 0$ the following elements of the algebra $\mathcal{A} \otimes O_1[[X]]$:

$$\Psi_{-N}^{(l)} =$$

$$\sum (-1)^{s+t} \eta(n_t, \ldots, n_1) \eta(n_{t+1}, \ldots, n_s) \operatorname{ex}_{0t}(\bar{a}, \bar{n}) t_1^{-q\gamma(\bar{a}, \bar{n})} \mathcal{D}_{\bar{a}\bar{n}} \prod_{l_1=1}^{l} \operatorname{ex}_{l_1}(\bar{a}, \bar{n}),$$

where the sum is taken for $0 \leq s < p$, $0 \leq t \leq s$ and $(\bar{a}, \bar{n})$ such that $\bar{n} \geq -N$ (the part of this sum which corresponds to $s = 0$ is assumed to be equal to 1).

Remark. The meaning of the above right-hand sum can be clarified as follows. If $l_1 \in \mathbb{N}$ and $\Psi_{-N,l_1}^{(l)}$ is the part of this sum consisting of terms with $\bar{n} < l_1$, then $\Psi_{-N,l_1}^{(l)} \in \mathcal{A} \otimes O_0[t_1^{-1}][[X]]$. It is easy to see that, if $l_1 \geq l_2 > l$, then

$$\Psi_{-N,l_1}^{(l)} \equiv \Psi_{-N,l_2}^{(l)} \bmod p^{l_2 - l} \mathcal{A} \otimes O_0[t_1^{-1}][[X]]$$

and therefore,

$$\Psi_{-N}^{(l)} := \lim_{l_1 \to +\infty} \Psi_{-N,l_1}^{(l)} \in \mathcal{A} \otimes O_1[[X]] = \mathcal{A}_1 \otimes_{O_1} O_1[[X]].$$

For $l, N \in \mathbb{Z}$ such that $-N < l$, set

$$C'_{-N,l} = 1 + \sum \eta(n_s, \ldots, n_1) \mathcal{D}_{\bar{a}\bar{n}} t_1^{-q\gamma(\bar{a},\bar{n})} \prod_{1 \leq l_1 < l} \operatorname{ex}_{l_1}(\bar{a}, \bar{n}),$$

$$C''_{-N,l} = 1 + \sum (-1)^s \eta(n_1, \ldots, n_s) \mathcal{D}_{\bar{a}\bar{n}} t_1^{-q\gamma(\bar{a},\bar{n})} \prod_{0 \leq l_1 < l} \operatorname{ex}_{l_1}(\bar{a}, \bar{n}),$$

where the both sums are taken for $1 \leq s < p$ and $(\bar{a}, \bar{n})$ such that $-N \leq \bar{n} < l$.

Lemma 1. *For $l_1, N \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$, we have*

$$\Psi^{(l)}_{-N} = C'_{-N,l_1}\Psi^{(l)}_{l_1}C''_{-N,l_1}.$$

The proof is similar to the proof of the lemma of the n.3.3. We only note that $\mathrm{ex}_{l_1}(\bar a, \bar n) = 1$ if $\bar n < l_1$.

For $l \in \mathbb{Z}_{\geq 0}$ and $0 \leq t \leq s$, consider the expansions in powers of X:

$$\mathrm{ex}_l(\bar a, \bar n) = \sum_{j \geq 0} c_j^{(l)}(\bar a, \bar n) X^{jp^l}, \qquad \mathrm{ex}_{0t}(\bar a, \bar n) = \sum_{j\geq 0} c_j^{(0)}(t, \bar a, \bar n) X^j.$$

If $l \in \mathbb{Z}_{\geq 0}$, we set $\gamma_l(\bar a, \bar n) = a_1 p^{(n_1 - l)^*} + \dots + a_s p^{(n_s - l)^*}$, and note that this definition coincides with the definition from the n.3.2 in the case $l = 0$, and for $\bar n \geq l$, we have $\gamma_l(\bar a, \bar n) = p^{-l}\gamma_0(\bar a, \bar n)$.

Lemma 2.
(a) *For $1 \leq j < p$ and $l \in \mathbb{Z}_{\geq 0}$, we have*

$$c_j^{(l)}(\bar a, \bar n) = \gamma_l(\bar a, \bar n)^j / j!, \qquad c_j^{(0)}(t, \bar a, \bar n) = \gamma_{0t}(\bar a, \bar n)^j / j!;$$

(b) *For $l \in \mathbb{Z}_{\geq 0}$ and $\bar n > l$, we have $\mathrm{ex}_l(\bar a, \bar n) = \exp(\gamma_l(\bar a, \bar n) X^{p^l})$.*

Proof. The part (a) follows, because the Artin-Hasse exponential $E(1, X)$ coincides with the standard exponential in degrees $< p$. The part (b) follows from the definition of $\mathrm{ex}_l(\bar a, \bar n)$, cf. n.3.1.

For any $N \leq \widetilde{N}$ and $l \in \mathbb{Z}_{\geq 0}$, we use the above expansions of $\mathrm{ex}_{0t}(\bar a, \bar n)$ and $\mathrm{ex}_{l_1}(\bar a, \bar n)$, where $1 \leq l_1 \leq l$, to obtain the expansion of $\Psi^{(l)}_{-N}$:

$$\Psi^{(l)}_{-N} = \sum_{i \geqslant 0} \Psi^{(l)}_{-N}(i) X^i.$$

By the part (b) of the lemma of n.1.4, $\Psi^{(l)}_{-N}(0) = 1$. If $0 \leq l_1 < l$ and $1 \leq i < p^{l_1 + 1}$, then

$$\Psi^{(l)}_{-N}(i) = \Psi^{(l_1)}_{-N}(i),$$

because $\mathrm{ex}_{l_1+1}(\bar a, \bar n) \equiv \dots \equiv \mathrm{ex}_l(\bar a, \bar n) \equiv 1 \bmod(\deg p^{l_1+1})$.

Lemma 3. *If $1 \leq i < p$, then*

(a)
$$\Psi^{(0)}_{-\widetilde{N}}(i) \in \mathcal{A}_1(v_0) + \sum_{1 \leqslant j \leqslant i} \left(t_1^{-a^*} J_O\right)^j + J_1^p;$$

(b)
$$\Psi^{(0)}_{-\widetilde{N}}(i) \in \mathcal{A}_1^+(v_0) + t_1^{-q(v_0+\varepsilon)} J_O + \sum_{2 \leqslant j \leqslant i} \left(t_1^{-a^*} J_O\right)^j + J_1^p.$$

Proof. Indeed, by the part (a) of the above lemma 2 for $1 \leq i < p$, we have

$$\Psi^{(0)}_{-\widetilde{N}}(i) = \sum_\gamma (\mathcal{F}_{\gamma, -\widetilde{N}}(i)/i!) t_1^{-q\gamma},$$

and our lemma follows from the proposition 1 of n.3.5.

Lemma 4. *If $l \in \mathbb{N}$ and $p^l \leq i < p^{l+1}$, then*

(a)
$$\Psi^{(l)}_{-\tilde{N}}(i) \in \mathcal{A}_1(v_0) + \sum_{1\leqslant j<p} \left(t_1^{-a^*p^{l-1}} J_O\right)^j + J_1^p;$$

(b)
$$\Psi^{(l)}_{-\tilde{N}}(i) \in \mathcal{A}_1^+(v_0) + t_1^{-q(v_0+\varepsilon)} J_O + \sum_{2\leqslant j<p} \left(t_1^{-a^*p^{l-1}} J_O\right)^j + J_1^p;$$

Proof. Consider the decomposition $\Psi^{(l)}_{-\tilde{N}} = C'_{-\tilde{N},l}\Psi_l^{(l)}C''_{-\tilde{N},l}$ from the lemma 1 and let $C'_{-\tilde{N},l} = \sum_{i\geqslant 0} C'_{-\tilde{N},l}(i)X^i$ and $C''_{-\tilde{N},l} = \sum_{i\geqslant 0} C''_{-\tilde{N},l}(i)X^i$. By the proposition of n.3.3, we have for $i \geq 1$,

$$C'_{-\tilde{N},l}(i), C''_{-\tilde{N},l}(i) \in \mathcal{A}_1(v_0) + \sum_{j\geqslant 1} \left(t_1^{-a^*p^{l-1}} J_O\right)^j + J_1^p.$$

From lemma 2, it follows that for $1 \leq i < p^{l+1}$, $\Psi_l^{(l)}(i)$ coincides with the coefficient for X^i of the expression

$$\sum (-1)^{s+t} \eta(n_t, \dots, n_1)\eta(n_{t+1}, \dots, n_s) \exp(\gamma_{0t}(\bar{a}, \bar{n})X)\times$$
$$\times \mathcal{D}_{\bar{a}\bar{n}} t_1^{-q\gamma(\bar{a},\bar{n})} \prod_{1\leqslant u\leqslant l} \exp\left(\gamma(\bar{a},\bar{n})X^{p^u}/p^u\right)$$
$$= \sum_{\substack{\gamma\in p^l\mathbb{N}\\ j\geqslant 1}} \exp\left[\gamma\left(\frac{X^{p^l}}{p^l} + \dots + \frac{X^p}{p}\right)\right] (\mathcal{F}_{\gamma,l}(j)/j!) t_1^{-q\gamma} X^j,$$

where the first sum is taken for $1 \leq s < p$, $0 \leq t \leq s$ and $(\bar{a}, \bar{n})$ such that $\bar{n} \geq l$. Therefore, $\Psi_l^{(l)}(i)$ is a linear combination with p-integral coefficients of $\sum_\gamma (\mathcal{F}_{\gamma,l}(j)/j!) t_1^{-q\gamma}$, where $j \in \mathbb{N}$, and therefore belongs (cf. corollary of n.3.5) to

$$\mathcal{A}_1(v_0) + \sum_{1\leqslant j_1<p} \left(t_1^{-a^*p^{l-1}} J_O\right)^{j_1} + J_1^p.$$

Now the part (a) of the lemma follows from the relation

$$\Psi^{(l)}_{-\tilde{N}}(i) = \sum_{i_1+i^*+i_2=i} C'_{-\tilde{N},l}(i_1)\Psi_l^{(l)}(i^*)C''_{-\tilde{N},l}(i_2).$$

The part (b) can be obtained similarly by the use of the estimations

$$C'_{-\tilde{N},l}(i), C''_{-\tilde{N},l}(i), \mathcal{F}_{\gamma,l}(i)/i! \in \mathcal{A}_1^+(v_0) + t_1^{-q(v_0+\varepsilon)} J_O + \sum_{j\geqslant 2} \left(t_1^{-a^*p^{l-1}} J_O\right)^j + J_1^p,$$

where $i \geq 1$.

Lemma 5. *If $l \geq 0$ and $i \geq p^{l+1}$, then*

(a)
$$\Psi^{(l)}_{-\tilde{N}}(i) \in \mathcal{A}_1(v_0) + \sum_{1 \leqslant j < p} \left(t_1^{-a^* p^l} J_O\right)^j + J_1^p;$$

(b)
$$\Psi^{(l)}_{-\tilde{N}}(i) \in \mathcal{A}_1^+(v_0) + t_1^{-q(v_0+\varepsilon)} J_O + \sum_{2 \leqslant j < p} \left(t_1^{-a^* p^l} J_O\right)^j + J_1^p.$$

Proof. Consider the decomposition $\Psi^{(l)}_{-\tilde{N}} = C'_{-\tilde{N},l+1} \Psi^{(l)}_{l+1} C''_{-\tilde{N},l+1}$. Proceeding as in the proof of the above lemma 4, we obtain the part (a) of our lemma, because here we have

$$\Psi^{(l)}_{l+1} = \sum_{\substack{\gamma \in p^{l+1}\mathbb{N} \\ j \geqslant 1}} \exp\left[\gamma\left(\frac{X^{p^l}}{p^l} + \cdots + \frac{X^p}{p}\right)\right] (\mathcal{F}_{\gamma,l+1}(j)/j!) t_1^{-q\gamma} X^j.$$

The part (b) of the lemma can be obtained similarly to the proof of the part (b) of the above lemma 4.

3.7. For any $\gamma \geq 0$, $l \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{Z}$, set

$$\Psi^{(l)}_{\gamma,-N} = \sum (-1)^{s+t} \eta(n_t, \ldots, n_1) \eta(n_{t+1}, \ldots, n_s) \operatorname{ex}_{0t}(\bar{a}, \bar{n}) \mathcal{D}_{\bar{a}\bar{n}} \prod_{l_1=1}^{l} \operatorname{ex}_{l_1}(\bar{a}, \bar{n}),$$

where the sum is taken for $0 \leq t \leq s < p$ and $(\bar{a}, \bar{n})$ such that $\bar{n} \geq -N$ and $\gamma(\bar{a}, \bar{n}) = \gamma$. Then $\Psi^{(l)}_{\gamma,-N} \in \mathcal{L} \otimes_{W(k)} W(k)[[X]]$ and it is easy to see that if $\gamma \to +\infty$, then $\Psi^{(l)}_{\gamma,-N} \to 0$ in the p-adic topology. Therefore,

$$\Psi^{(l)}_{-N} = \sum_{\gamma} \Psi^{(l)}_{\gamma,-N} t_1^{-q\gamma}$$

and $\Psi^{(l)}_{-N}$ converges for $X = t_1^{b^*}$. We set $\Theta^{(l)}_{-N} = \Psi^{(l)}_{-N}(t_1^{b^*}) \in \mathcal{L} \otimes O_1$.

Proposition.

(a) *For any $l \in \mathbb{Z}_{\geq 0}$, $\Theta^{(l)}_{-\tilde{N}} \in 1 + \mathcal{A}_1(v_0) + t_1^{b^*-a^*} J_O + J_1^p$;*

(b)
$$\Theta^{(0)}_{-\tilde{N}} \equiv 1 + \sum_{\gamma} \mathcal{F}_{\gamma,-\tilde{N}} t_1^{-q\gamma+b^*} \bmod \left(\mathcal{A}_1^+(v_0) + t_1^{b^*-a^*} J_O + J_1^p\right);$$

(c) *For any $m \in \mathbb{N}$, $\Theta^{(m)}_{-\tilde{N}} \equiv \Theta^{(m-1)}_{-\tilde{N}} \bmod \left(\mathcal{A}_1^+(v_0) + t_1^{p^m(b^*-a^*)} J_O + J_1^p\right)$.*

Proof. As it was noted in n.3.6, if $0 \leq l_1 < l$, then for $1 \leq i < p^{l_1+1}$, we have $\Psi^{(l_1)}_{-\tilde{N}}(i) = \Psi^{(l)}_{-\tilde{N}}(i)$. Therefore,

$$\Theta^{(l)}_{-\tilde{N}} = \sum_{i=0}^{p-1} \Psi^{(0)}_{-\tilde{N}}(i) t_1^{ib^*} + \cdots + \sum_{i=p^l}^{p^{l+1}-1} \Psi^{(l)}_{-\tilde{N}}(i) t_1^{ib^*} + \sum_{i \geqslant p^{l+1}} \Psi^{(l)}_{-\tilde{N}}(i) t_1^{ib^*},$$

and the statement (a) follows from the parts (a) of the above lemmas 3-5 because for any $l \in \mathbb{Z}_{\geq 0}$, we have $p^{l+1}b^* - a^*p^l(p-1) \geq b^* - a^*$.

Consider the equality

$$\Theta^{(0)}_{-\widetilde{N}} = 1 + \Psi^{(0)}_{-\widetilde{N}}(1)t_1^{b^*} + \sum_{i\geqslant 2} \Psi^{(0)}_{-\widetilde{N}}(i)t_1^{ib^*}.$$

If $i \geq 2$, then $-q(v_0 + \varepsilon) + ib^* \geq b^* - a^*$.

Indeed, $-q(v_0+\varepsilon) + ib^* \geq -q(v_0+\varepsilon) + 2b^* = b^* - a^* - q(\delta+\varepsilon) + r^*(q-1)$. But, cf. n.3.1.1, $\varepsilon, \delta < v_0/3$ and $r^* > (q+p)(v_0-\delta)/(q-1)$ give $-q(\delta+\varepsilon) + r^*(q-1) > q(v_0 - \varepsilon - 2\delta) + pa^*/q > 0$.

Thus, by the parts (b) of the lemmas 3-5 from n.3.6, we have

$$\sum_{i\geqslant 2} \Psi^{(0)}_{-\widetilde{N}}(i)t_1^{ib^*} \in \mathcal{A}_1^+(v_0) + t_1^{b^*-a^*} J_O + J_1^p.$$

Because $\Psi^{(0)}_{-\widetilde{N}}(1)t_1^{b^*} = \sum_\gamma \mathcal{F}_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*}$, the part (b) of the proposition is proved. Similarly we can prove the part (c) because

$$\Theta^{(m)}_{-\widetilde{N}} - \Theta^{(m-1)}_{-\widetilde{N}} = \sum_{i\geqslant p^m} \Psi^{(m)}_{-\widetilde{N}}(i)t_1^{ib^*} - \sum_{i\geqslant p^m} \Psi^{(m-1)}_{-\widetilde{N}}(i)t_1^{ib^*}.$$

The proposition is proved.

3.8. For $M \in \mathbb{Z}_{\geq 0}$, consider the following three subgroups $\mathcal{H}_1 \supset \mathcal{H}_1^0 \supset \mathcal{H}_1^+$ of the group of invertible elements of the algebra $\mathcal{A}_1$:

$$\mathcal{H}_1 = 1 + \mathcal{A}_1(v_0) + \sum_{j\geqslant 1} \left(t_1^{-a^*p^M} J_O\right)^j + J_1^p;$$

$$\mathcal{H}_1^0 = 1 + \mathcal{A}_1(v_0) + \left(t_1^{(b^*-a^*)qp^M} J_O\right) \sum_{j\geqslant 0} \left(t_1^{-a^*p^M} J_O\right)^j + J_1^p.$$

$$\mathcal{H}_1^+ = 1 + \mathcal{A}_1^+(v_0) + \left(t_1^{(b^*-a^*)qp^M} J_O\right) \sum_{j\geqslant 0} \left(t_1^{-a^*p^M} J_O\right)^j + J_1^p.$$

It is easy to see that $\mathcal{H}_1^0$ and $\mathcal{H}_1^+$ are normal subgroups in the group $\mathcal{H}_1$ and $\mathcal{H}_1^0/\mathcal{H}_1^+$ is a central subgroup of $\mathcal{H}_1/\mathcal{H}_1^+$ of exponent p.

Proposition. *If $\widehat{N} = \widetilde{N} + M$ then*

(a) *For $m, l, n \in \mathbb{Z}_{\geq 0}$ such that $n + m - 1 \leq \widehat{N}$, we have $\sigma^n \Theta^{(l)}_{-\widetilde{N}+m} \in \mathcal{H}_1$;*

(b) *If m, l, n satisfy the above assumptions from (a) and $n + l \geq \widehat{N} + 1$, then*

$$\sigma^n \Theta^{(l)}_{-\widetilde{N}+m} \equiv \sigma^n \Theta^{(l-1)}_{-\widetilde{N}+m} \bmod \mathcal{H}_1^+;$$

(c)
$$\sigma^{\widehat{N}+1} \Theta^{(0)}_{-\widetilde{N}} \equiv \widetilde{\exp}\left(\sigma^{\widehat{N}+1} \sum_\gamma \mathcal{F}_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*}\right) \bmod \mathcal{H}_1^+.$$

Proof. From the part (a) of the proposition of n.3.7, it follows that for any $l, n \in \mathbb{Z}_{\geq 0}$, we have $\sigma^n \Theta^{(l)}_{-\widetilde{N}} \in \mathcal{H}_1$.

Now we note that the power series $C'_{-N,l}$ and $C''_{-N,l}$ from the beginning of the n.3.6 converge for $X = t_1^{b^*}$. Then $\sigma^n C'_{-\widetilde{N},-\widetilde{N}+m}(t_1^{b^*}) - 1$ and $\sigma^n C''_{-\widetilde{N},-\widetilde{N}+m}(t_1^{b^*}) - 1$ are linear combinations with coefficients from O_0 of $\mathcal{D}_{\bar{a}\bar{n}} t_1^{-q\gamma(\bar{a},\bar{n})}$, where

$$\bar{n} \leq n - \widetilde{N} + m - 1 \leq M$$

by assumption (a) of our proposition. Therefore by the proposition of n.3.3, we have

$$\sigma^n C'_{-\widetilde{N},-\widetilde{N}+m}(t_1^{b^*}), \sigma^n C''_{-\widetilde{N},-\widetilde{N}+m}(t_1^{b^*}) \in \mathcal{H}_1.$$

So, the part (a) follows from the identity, cf. n.3.6,

$$(1) \quad \sigma^n \Theta^{(l)}_{-\widetilde{N}+m} = \left(\sigma^n C'_{-\widetilde{N},-\widetilde{N}+m}(t_1^{b^*})\right)^{-1} \left(\sigma^n \Theta^{(l)}_{-\widetilde{N}}\right) \left(\sigma^n C''_{-\widetilde{N},-\widetilde{N}+m}(t_1^{b^*})\right)^{-1}.$$

From the parts (a) and (c) of the proposition of n.3.7, it follows that

$$\Theta^{(l)}_{-\widetilde{N}} \left(\Theta^{(l-1)}_{-\widetilde{N}}\right)^{-1} \in 1 + \mathcal{A}_1^+(v_0) + t_1^{p^l(b^*-a^*)} J_O + J_1^p,$$

and therefore, if $n + l \geq \widehat{N} + 1 = N^* + M$, we have

$$\sigma^n \Theta^{(l)}_{-\widetilde{N}} \equiv \sigma^n \Theta^{(l-1)}_{-\widetilde{N}} \bmod \mathcal{H}_1^+.$$

Now the above identities (1) give that $\sigma^n \Theta^{(l)}_{-\widetilde{N}+m}$ and $\sigma^n \Theta^{(l-1)}_{-\widetilde{N}+m}$ have the same image in the group $\mathcal{H}_1/\mathcal{H}_1^+$ and the part (b) is proved.

The part (c) follows easily from the part (b) of the proposition of n.3.7.

3.9. For $N \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$, consider (cf. remark of n.3.6)

$$\Phi_m^{(N)} = \sum (-1)^{s+t} \eta(n_t, \dots, n_1) \eta(n_{t+1}, \dots, n_s) t_1^{-q\gamma(\bar{a},\bar{n})} \mathcal{D}_{\bar{a}\bar{n}} \prod_{0 \leqslant i \leqslant N} \mathcal{E}_{it}(\bar{a}, \bar{n}),$$

where the sum is taken for $0 \leq t \leq s < p$ and $(\bar{a}, \bar{n})$ such that $\bar{n} \geq m$.

We note, that $\Phi_m^{(0)} = \Theta_m^{(0)}$.

For $M \in \mathbb{Z}_{\geq 0}$, consider

$$E_M = \sum \eta(n_s, \dots, n_1) \mathcal{E}_M(\bar{a}, \bar{n}) \dots \mathcal{E}_0(\bar{a}, \bar{n}) t_1^{-q\gamma(\bar{a},\bar{n})} \mathcal{D}_{\bar{a}\bar{n}}$$

and

$$E'_M = \sum \eta(n_s, \dots, n_1) t_1^{-q\gamma(\bar{a},\bar{n})} \mathcal{D}_{\bar{a}\bar{n}},$$

where the both sums are taken for $0 \leq s < p$ and $(\bar{a}, \bar{n})$ such that $n_1 = \dots = n_s = M$. By the proposition of n.3.3, we have $E_M, E'_M \in \mathcal{H}_1$.

Proposition.
(a) $\Phi_M^{(\widehat{N})}, \sigma\Phi_M^{(\widehat{N})} \in \mathcal{H}_1$;

(b) $\Phi_M^{(\widehat{N})} E_M \equiv E'_M \left(\sigma\Phi_M^{(\widehat{N})}\right) \operatorname{mod} \mathcal{H}_1^0(1+p^{M+1}\mathcal{A}_1)$;

(c) $\Phi_M^{(\widehat{N})} E_M \equiv E'_M(\sigma\Phi_M^{(\widehat{N})})\widetilde{\exp}(-\sigma^{\widehat{N}+1}\sum_{\gamma}\mathcal{F}_{\gamma,-\widetilde{N}}t_1^{-q\gamma+b^*}) \operatorname{mod} \mathcal{H}_1^+(1+p^{M+1}\mathcal{A}_1)$.

Proof.

Lemma 1. $\Phi_M^{(\widehat{N})} E_M \equiv E'_M \Phi_{M+1}^{(\widehat{N})} \operatorname{mod} J_1^p$.

Proof. In the notation from the beginning of n.3.6 it is easy to see that $E_M^{-1} = C''_{M,M+1}(t_1^{b^*})$. The identity $\Phi_M^{(\widehat{N})} = E'_M \widetilde{\Phi}_{M+1}^{(\widehat{N})} C''_{M,M+1}(t_1^{b^*}) \operatorname{mod} J_1^p$ is quite analogous to the identity of the lemma 1 from n.3.6, and can be obtained by similar arguments. The lemma is proved.

Lemma 2. *If $N \in \mathbb{N}$ and $m \in \mathbb{Z}$, then* $\left(\sigma\Phi_{m-1}^{(N-1)}\right)\Theta_m^{(N)} = \Phi_m^{(N)}$.

Proof. For $0 \le t \le s < p$ and $l \in \mathbb{N}$, set $(\bar{a},\bar{n})_{(0,t]} = (a_1, n_1, \dots, a_t, n_t)$ and $\mathcal{E}_{lt}^-(\bar{a},\bar{n}) = \mathcal{E}_l((\bar{a},\bar{n})_{(0,t]})$. Then

$$\sigma\Phi_{m-1}^{(N-1)} =$$

$$\sum(-1)^{s+t}\eta(n_t,\dots,n_1)\left[\eta(n_{t+1},\dots,n_s)\mathcal{E}_{Nt}(\bar{a},\bar{n})\dots\mathcal{E}_{1t}(\bar{a},\bar{n})\right]t_1^{-q\gamma(\bar{a},\bar{n})}\mathcal{D}_{\bar{a}\bar{n}},$$

$$\Theta_m^{(N)} = \sum(-1)^s\left[(-1)^t\eta(n_t,\dots,n_1)\mathcal{E}_{Nt}^-(\bar{a},\bar{n})\dots\mathcal{E}_{1t}^-(\bar{a},\bar{n})\right]\times$$

$$\eta(n_{t+1},\dots,n_s)\mathcal{E}_{Nt}(\bar{a},\bar{n})\dots\mathcal{E}_{0t}(\bar{a},\bar{n})t_1^{-q\gamma(\bar{a},\bar{n})}\mathcal{D}_{\bar{a}\bar{n}},$$

where the both sums are taken for $0 \le t \le s < p$ and $(\bar{a},\bar{n})$ such that $\bar{n} \ge m$. When multiplying these expressions we can use the part b) of the lemma from n.1.4 to simplify the product of square brackets. This gives the formula of our lemma.

From this lemma and the part (a) of the proposition of n.3.8, we obtain

$$\Phi_M^{(\widehat{N})} = \left(\sigma^{\widehat{N}}\Theta_{-\widetilde{N}}^{(0)}\right)\left(\sigma^{\widehat{N}-1}\Theta_{-\widetilde{N}+1}^{(1)}\right)\dots\left(\sigma\Theta_{M-1}^{(\widehat{N}-1)}\right)\Theta_M^{(\widehat{N})} \in \mathcal{H}_1,$$

$$\sigma\Phi_M^{(\widehat{N})} = \left(\sigma^{\widehat{N}+1}\Theta_{-\widetilde{N}}^{(0)}\right)\prod_{l=1}^{\widehat{N}}\sigma^{\widehat{N}+1-l}\Theta_{-\widetilde{N}+l}^{(l)} \in \mathcal{H}_1.$$

The part (a) is proved.

Similarly, we have

$$\Phi_{M+1}^{(\widehat{N})} = \prod_{l=1}^{\widehat{N}}\left(\sigma^{\widehat{N}+1-l}\Theta_{-\widetilde{N}+l}^{(l-1)}\right)\Theta_{M+1}^{(\widehat{N})}.$$

By the part (b) of the proposition of n.3.8, we have

$$\sigma^{\widehat{N}+1-l}\Theta^{(l)}_{-\widetilde{N}+l} \equiv \sigma^{\widehat{N}+1-l}\Theta^{(l-1)}_{-\widetilde{N}+l} \operatorname{mod} \mathcal{H}_1^+$$

for all $1 \le l \le \widehat{N}$. If $\bar{n} \ge M+1$, then $\mathcal{E}_{0t}(\bar{a},\bar{n}) \in 1 + p^{M+1}O_0$ and the lemma from n.1.4 gives that $\Theta^{(\widehat{N})}_{M+1} \in 1 + p^{M+1}\mathcal{A}_1$. So,

$$\Phi^{(\widehat{N})}_{M+1} \equiv \left(\sigma^{\widehat{N}+1}\Theta^{(0)}_{-\widetilde{N}}\right)^{-1}\left(\sigma\Phi^{(\widehat{N})}_M\right) \operatorname{mod} \mathcal{H}_1^+(1+p^{M+1}\mathcal{A}_1),$$

and the statements (b) and (c) follow from the above lemma 1 and the parts (a) and (c) of the proposition of n.3.8.

3.10. Let $\hat{t} = t_1^q E(-1, t_1^{b^*}) \in O_1$. We note, cf. lemma of n.1.6, that

$$\hat{t}^{p^M} \operatorname{mod} p^{M+1}O_1 = t^{p^M} \in O_{M+1}(\sigma^M K).$$

Let

$$e'_{\mathcal{L}} = \sum_{a \in A} t_1^{-a}\mathcal{D}_{a,-N^*}, \quad e^{(M)}_{\mathcal{L}} = \sum_{a \in A} \hat{t}^{-ap^M}\mathcal{D}_{aM}.$$

Lemma 1. $E'_M = \sigma^{\widehat{N}+1}\widetilde{\exp}(e'_{\mathcal{L}})$, $E_M = \widetilde{\exp}(e^{(M)}_{\mathcal{L}})$.

Proof. The lemma follows, because E'_M and E_M could be written in the following form:

$$E'_M = \sigma^{\widehat{N}+1} \sum_{\substack{0 \leqslant s < p \\ a_1,\dots,a_s \in A}} (1/s!) \prod_{i=1}^{s} \left(t_1^{-a_i}\mathcal{D}_{a_i,-N^*}\right),$$

$$E_M = \sum_{\substack{0 \leqslant s < p \\ a_1,\dots,a_s \in A}} (1/s!) \prod_{i=1}^{s} \left(\hat{t}^{-a_i p^M}\mathcal{D}_{a_i M}\right).$$

Lemma 2. *There exists* $\phi^{(\widehat{N})}_M \in \mathcal{L}^0 \otimes O_1 = \mathcal{L}_1$ *such that*

$$\Phi^{(\widehat{N})}_M \equiv \widetilde{\exp}(\phi^{(\widehat{N})}_M) \operatorname{mod} J_1^p.$$

Proof. If $m \in \mathbb{N}$, let $\Phi^{(\widehat{N})}_{M,m}$ be given by the same expression as $\Phi^{(\widehat{N})}_M$, but the sum is taken with the additional restriction $\bar{n} < \widehat{N} + m$. Then we have

$$\Phi^{(\widehat{N})}_M \equiv \Phi^{(\widehat{N})}_{M,m} \operatorname{mod} p^m \mathcal{A}_1,$$

cf. remark of n.3.6.

It is easy to see that $\Phi^{(\widehat{N})}_{M,m} \equiv \Phi_{1m}\Phi_{2m} \operatorname{mod} J_1^p$, where

$$\Phi_{1m} = \sum \eta(n_s,\dots,n_1)t_1^{-q\gamma(\bar{a},\bar{n})}\mathcal{D}_{\bar{a}\bar{n}},$$

$$\Phi_{2m} = \sum (-1)^s \eta(n_1, \dots, n_s) \mathcal{E}_{0-\widehat{N}}(\bar{a}, \bar{n}) t_1^{-q\gamma(\bar{a},\bar{n})} \mathcal{D}_{\bar{a}\bar{n}},$$

where the both sums are taken for $0 \le s < p$, $\bar{a} \in A^s$ and $M \le \bar{n} < \widehat{N} + m$, and we use the notation $\mathcal{E}_{0-\widehat{N}}(\bar{a}, \bar{n}) = \mathcal{E}_0(\bar{a}, \bar{n}) \dots \mathcal{E}_{\widehat{N}}(\bar{a}, \bar{n})$.

From the definition of the structural constants, cf. n.1.4, it follows now that

$$\Phi_{1m} = \prod_{n=\widehat{N}+m}^{M} \widetilde{\exp} \left(\sigma^n \sum_{a \in A} t_1^{-qa} \mathcal{D}_{a0} \right) \bmod J_1^p,$$

$$\Phi_{2m} = \prod_{n=M}^{\widehat{N}+m} \widetilde{\exp} \left(- \sum_{a \in A} \mathcal{E}_{0-\widehat{N}}(a, n) \sigma^n (t_1^{-qa} \mathcal{D}_{a0}) \right) \bmod J_1^p.$$

This gives the existence of $\phi_{Mm}^{(\widehat{N})} \in \mathcal{L}_1$ such that $\Phi_{M,m}^{(\widehat{N})} = \widetilde{\exp}(\phi_{Mm}^{(\widehat{N})})$. Clearly, there exists $\phi_M^{(\widehat{N})} = \lim_{m \to \infty} \phi_{Mm}^{(\widehat{N})} \in \mathcal{L}_1$ and $\Phi_M^{(\widehat{N})} = \widetilde{\exp}(\phi_M^{(\widehat{N})})$.

The lemma is proved.

Let $\mathcal{L}_1(v_0) = \mathcal{L}^0(v_0) O_1$. Define the ideal

$$\mathcal{L}^{0+}(v_0) = [\mathcal{L}^0, \mathcal{L}^0(v_0)] + p\mathcal{L}^0(v_0) + \mathcal{L}^0(v_0 + \varepsilon)$$

of the Lie algebra $\mathcal{L}^0$. We also set $\mathcal{L}_1^+(v_0) = \mathcal{L}^{0+}(v_0) O_1$ and $\mathcal{L}_O = \mathcal{L}^0 O_0$.

We note that

$$\mathcal{L}_1(v_0) = \mathcal{L}_1 \cap (\mathcal{A}_1(v_0) + J_1^p), \quad \mathcal{L}_1^+(v_0) = \mathcal{L}_1 \cap (\mathcal{A}_1^+(v_0) + J_1^p).$$

Consider the following O_0-submodules of $\mathcal{L}_1$:

$$\mathcal{LH}_1 = \mathcal{L}_1(v_0) + \sum_{1 \leqslant j < p} t_1^{-ja^* p^M} C_j(\mathcal{L}_O) + p^{M+1} \mathcal{L}_1,$$

$$\mathcal{LH}_1^0 = \mathcal{L}_1(v_0) + t_1^{qp^M(b^* - a^*)} \sum_{1 \leqslant j < p} t_1^{-(j-1)a^* p^M} C_j(\mathcal{L}_O) + p^{M+1} \mathcal{L}_1,$$

$$\mathcal{LH}_1^+ = \mathcal{L}_1^+(v_0) + t_1^{qp^M(b^* - a^*)} \sum_{1 \leqslant j < p} t_1^{-(j-1)a^* p^M} C_j(\mathcal{L}_O) + p^{M+1} \mathcal{L}_1.$$

With respect to induced Lie brackets, $\mathcal{LH}_1$ is a Lie algebra over O_0, $\mathcal{LH}_1^0$ and $\mathcal{LH}_1^+$ are its ideals and $\mathcal{LH}_1^0 / \mathcal{LH}_1^+$ is an abelian ideal of $\mathcal{LH}_1 / \mathcal{LH}_1^+$ annihilated by p.

Lemma 3. *If $l \in \mathcal{L}_1$, then*

(a) $l \in \mathcal{LH}_1 \iff \widetilde{\exp}(l) \in \mathcal{H}_1(1 + p^{M+1}\mathcal{A}_1)$;

(b) $l \in \mathcal{LH}_1^0 \iff \widetilde{\exp}(l) \in \mathcal{H}_1^0(1 + p^{M+1}\mathcal{A}_1)$;

(c) $l \in \mathcal{LH}_1^+ \iff \widetilde{\exp}(l) \in \mathcal{H}_1^+(1 + p^{M+1}\mathcal{A}_1)$;

Proof. Consider the Lie $\mathbb{Z}_p$-algebra

$$\widetilde{L}^0 = \mathcal{L}^0(A) / (\mathcal{L}^0(A, v_0) + p^{M+1} \mathcal{L}^0(A)).$$

Then $\widetilde{L}^0$ is a finite Lie algebra of nilpotency class $< p$.

Let $\widetilde{L} = \widetilde{L}^0 \otimes W(k)$, $\widetilde{L}_O = \widetilde{L}^0 \otimes O_0$, $\widetilde{L}_1 = \widetilde{L}^0 \otimes O_1$. If $\widetilde{J}^0$ is the augmentation ideal of the envelopping algebra of the Lie algebra $\widetilde{L}^0$, we set $\widetilde{J} = \widetilde{J}^0 \otimes W(k)$, $\widetilde{J}_O = \widetilde{J}^0 \otimes O_0$ and $\widetilde{J}_1 = \widetilde{J}^0 \otimes O_1$. With this notation the part (a) of our lemma is equivalent to the following statement:

$$(a') \qquad \sum_{1\leqslant j<p} t_1^{-ja^*p^M} C_j(\widetilde{L}_O) = \widetilde{L}_1 \cap (\sum_{j\geqslant 1} t_1^{-ja^*p^M} \widetilde{J}_O^j + J_1^p).$$

Clearly, the left-hand side of (a') is contained in its right-hand side. Further we note that any element $\tilde{j} \in \widetilde{J}_1$ can be uniquely expressed in the form

$$\tilde{j} = \sum_{a>>-\infty} t_1^a j_a,$$

where $j_a \in \widetilde{J}$ for all $a \in \mathbb{Z}$ and $j_a = 0$ for sufficiently small a. In this notation, $\tilde{j} \in \widetilde{L}_1$ iff $j_a \in \widetilde{L}_1$ for all $a \in \mathbb{Z}$, and

$$\tilde{j} \in \sum_{j\geqslant 1} t_1^{-ja^*p^M} \widetilde{J}_O^j + J_1^p$$

iff for $1 \leq s < p$ we have: $j_a \in \widetilde{J}^{s+1}$ for $a < -sa^*p^M$.

Therefore, if $\tilde{j}$ belongs to the right-hand side of (a'), then for $1 \leq s < p$ and $a < -sa^*p^M$, we have $j_a \in \widetilde{L} \cap \widetilde{J}^{s+1}$. By the proposition 2 (a) of n.1.2, $\widetilde{L} \cap \widetilde{J}^{s+1} = C_{s+1}(\widetilde{L})$, and therefore $\tilde{j}$ belong to the left-hand side of (a').

The parts (b) and (c) of our lemma can be proved similarly.

Now the proposition of n.3.9 can be stated in the following form.

Proposition.

$$\text{(a)} \qquad \phi_M^{(\widehat{N})}, \sigma\phi_M^{(\widehat{N})} \in G(\mathcal{LH}_1);$$

$$\text{(b)} \qquad \phi_M^{(\widehat{N})} \circ \hat{e}_{\mathcal{L}}^{(M)} \equiv \left(\sigma^{\widehat{N}+1} e'_{\mathcal{L}}\right) \circ \left(\sigma\phi_M^{(\widehat{N})}\right) \operatorname{mod} G(\mathcal{LH}_1^0);$$

(c)

$$\phi_M^{(\widehat{N})} \circ \hat{e}_{\mathcal{L}}^{(M)} \equiv \left(\sigma^{\widehat{N}+1} e'_{\mathcal{L}}\right) \circ \left(\sigma\phi_M^{(\widehat{N})}\right) \circ \left(-\sigma^{\widehat{N}+1} \sum_{\gamma} \mathcal{F}_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*}\right) \operatorname{mod} G(\mathcal{LH}_1^+).$$

4. The main theorem.

In this section we consider a group epimorphism $\psi : \Gamma \longrightarrow G(L)$, where L is a finite Lie algebra over $\mathbb{Z}_p$ of a nilpotency class $< p$.

By the n.1, for any $\tau \in \Gamma$, we have $\psi(\tau) = \tau f \circ (-f)$, where $f \in G(L \otimes O(K_{\mathrm{sep}}))$ is such that $\sigma f = f \circ e$, $e = \sum_{a \in A} t^{-a} D_{a0} \in G(L \otimes O(K))$ and $A \subset \mathbb{Z}^0(p)$ is a finite subset.

Consider the Lie algebras $\mathcal{L}^0 = \mathcal{L}^0(A)$ and $\mathcal{L} = \mathcal{L}(A)$ from n.2. Then the correspondences $\mathcal{D}_{a0} \mapsto D_{a0}$ where $a \in A$, define the unique σ-invariant morphism of Lie algebras

$$\pi : \mathcal{L} \longrightarrow L \otimes W(k).$$

For any $n \in \mathbb{Z}$ and $a \in \mathbb{Z}^0(p)$, we set $D_{an} = \sigma^n D_{a0} = \pi(\mathcal{D}_{an})$. Clearly, π induces the epimorphic morphism of Lie algebras over $\mathbb{Z}_p$

$$\pi^0 : \mathcal{L}^0 \longrightarrow L$$

and we have the induced decreasing filtration $\{L(v)\}_{v>0}$ of the ideals $L(v) = \pi^0(\mathcal{L}^0(A, v))$ in the Lie algebra $\mathcal{L}^0$.

For any $\gamma \geq 0$ and $N \in \mathbb{Z}$, set $F_{\gamma,-N} = \pi(\mathcal{F}_{\gamma,-N})$. If $v_0 > 0$ and $N \geq \widetilde{N}(v_0, A^+)$, then $L(v)W(k)$ is the minimal σ-invariant ideal of $L \otimes W(k)$ which contains the set $\{F_{\gamma,-N} \mid \gamma \geq v_0\}$.

Theorem. *If $v > 0$ and $\Gamma^{(v)}$ is the ramification subgroup of Γ in upper numbering, then*

$$\psi(\Gamma^{(v)}) = G(L(v)) \subset G(L).$$

Proof.

4.1. *Inductive assumption.*

Let $M \in \mathbb{Z}_{\geq 0}$ be such that $p^{M+1}L = 0$ and let $1 \leq s_0 < p$ be such that $C_{s_0+1}(L) = 0$. By induction we can assume that the theorem is proved for the compositions of the morphism ψ with the natural projections $G(L) \longrightarrow G(L/p^M L)$ and $G(L) \longrightarrow G(L/C_{s_0}(L))$.

This assumption gives for any $v > 0$, that

$$L^{(v)} \equiv L(v) \bmod C_{s_0}(L), \quad L^{(v)} \equiv L(v) \bmod p^M L, \tag{1}$$

where $L^{(v)}$ is the ideal of L such that $\psi(\Gamma^{(v)}) = G(L^{(v)})$.

Consider the set $\mathcal{R} = \{v \in \mathbb{R}_{>0} \mid L^{(v)} \neq L(v)\}$.

It is easy to see that for a sufficiently large v we have $L^{(v)} = L(v) = 0$. This implies that either $\mathcal{R} = \emptyset$ (and the theorem is proved), or there exists $v_0 = \sup \mathcal{R} > 0$. In this case $L^{(v_0)} \neq L(v_0)$. Indeed, the both filtrations $\{L(v)\}_{v>0}$ and $\{L^{(v)}\}_{v>0}$ are finite and left-continuous. Therefore, there exists $\delta > 0$ such that for any $v \in (v_0 - \delta, v_0]$, we have $L^{(v)} = L^{(v_0)}$ and $L(v) = L(v_0)$, and the equality $L(v_0) = L^{(v_0)}$ gives the contradiction $v_0 = \sup \mathcal{R} \leq v_0 - \delta$.

So, the theorem will be proved, if we take an arbitrary $v_0 > 0$, assume that

$$L(v) = L^{(v)} \quad \forall v > v_0 \tag{2}$$

and show that $L^{(v_0)} = L(v_0)$.

4.2. For the above $v_0 > 0$ and $A \subset \mathbb{Z}^0(p)$, we use the choice of ε, δ, $\widetilde{N}$, N^*, q, a^*, b^* and notation of n.3.

We set $L_1 = L\otimes O_1$, $L_O = L\otimes O_0$, $L_1(v_0) = L(v_0)O_1$, $L_1(v_0+\varepsilon) = L(v_0+\varepsilon)O_1$, $L^+(v_0) = [L, L(v_0)] + pL(v_0) + L(v_0+\varepsilon)$ and $L_1^+(v_0) = L^+(v_0)O_1$.

Let $\pi_1 = \pi^0\otimes O_1 : \mathcal{L}_1 \longrightarrow L_1$. Then

$$\pi_1(e'_{\mathcal{L}}) = e_1 = \sum_{a\in A} t_1^{-a} D_{a,-N^*};$$

$$\pi_1(\hat{e}_{\mathcal{L}}^{(M)}) = \sigma^M e = \sum_{a\in A} t^{-ap^M} D_{aM} \in L\otimes O_{M+1}(\sigma^M K) \subset L_1;$$

$$\pi_1(\mathcal{LH}_1) = L_1(v_0) + \sum_{j\geqslant 1} t_1^{-ja^*p^M} C_j(L_O) := LH_1;$$

$$\pi_1(\mathcal{LH}_1^0) = L_1(v_0) + t_1^{qp^M(b^*-a^*)} \sum_{j\geqslant 1} t_1^{-(j-1)a^*p^M} C_j(L_O) := LH_1^0;$$

$$\pi_1(\mathcal{LH}_1^+) = L_1^+(v_0) + t_1^{qp^M(b^*-a^*)} \sum_{j\geqslant 1} t_1^{-(j-1)a^*p^M} C_j(L_O) := LH_1^+.$$

Clearly, LH_1 is a Lie algebra over O_0, LH_1^0 and LH_1^+ are its ideals and the quotient LH_1^0/LH_1^+ is an abelian ideal in LH_1/LH_1^+ annihilated by p.

With the above notation the proposition of n.3.10 gives the following

Proposition. *If* $\phi^* = \pi_1(\phi_M^{(\widehat{N})})$, *then*

(a) $\phi^*, \sigma\phi^* \in G(LH_1)$;

(b) $\phi^*\circ(\sigma^M e) \equiv (\sigma^{\widehat{N}+1}e_1)\circ(\sigma\phi^*) \bmod G(LH_1^0)$;

(c) $\quad \phi^*\circ(\sigma^M e) \equiv (\sigma^{\widehat{N}+1}e_1)\circ(\sigma\phi^*)\circ(-\sigma^{\widehat{N}+1}\sum_{\gamma} F_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*}) \bmod G(LH_1^+)$.

4.3. We set

$$L_{\mathrm{sep}} = L\otimes O'(K_{\mathrm{sep}}),\quad L(v_0)_{\mathrm{sep}} = L(v_0)O'(K_{\mathrm{sep}}),\quad L(v_0)_{\mathrm{sep}}^+ = L^+(v_0)O'(K_{\mathrm{sep}}).$$

Similarly to n.4.2, we also set

$$LH_{\mathrm{sep}} = L(v_0)_{\mathrm{sep}} + \sum_{j\geqslant 1} t_1^{-ja^*p^M} C_j(L_O),$$

$$LH_{\mathrm{sep}}^0 = L(v_0)_{\mathrm{sep}} + t_1^{qp^M(b^*-a^*)} \sum_{j\geqslant 1} t_1^{-(j-1)a^*p^M} C_j(L_O),$$

$$LH_{\mathrm{sep}}^+ = L(v_0)_{\mathrm{sep}}^+ + t_1^{qp^M(b^*-a^*)} \sum_{j\geqslant 1} t_1^{-(j-1)a^*p^M} C_j(L_O).$$

As in the n.4.2 we have that LH_{sep} is a Lie algebra over O_0, LH_{sep}^0 and LH_{sep}^+ are its ideals and $LH_{\mathrm{sep}}^0/LH_{\mathrm{sep}}^+$ is an abelian ideal in $LH_{\mathrm{sep}}/LH_{\mathrm{sep}}^+$ annihilated by p. We have also the natural inclusions:

$$LH_1 \subset LH_{\mathrm{sep}},\ LH_1^0 \subset LH_{\mathrm{sep}}^0,\ LH_1^+ \subset LH_{\mathrm{sep}}^+.$$

Proposition. *There exists* $f_1^0 \in \{f_1 \in G(L_{\mathrm{sep}}) \mid \sigma f_1 = f_1 \circ e\}$ *such that for* $X^0 = (-\sigma^{\widehat{N}+1} f_1^0) \circ (\sigma^M f)$, *we have* $X^0 \equiv \phi^* \operatorname{mod} G(LH^0_{\mathrm{sep}})$.

Remark. In particular, we obtain that $X^0 \in G(LH_{\mathrm{sep}})$. In fact, one can show that $\sigma^M f, \sigma^{\widehat{N}+1} f_1 \in G(LH_{\mathrm{sep}})$.

Proof. By induction we can assume the existence of $f_1' \in G(L_{\mathrm{sep}})$ such that $\sigma f_1' = f_1' \circ e_1$ and for $X' = (-\sigma^{\widehat{N}+1} f_1') \circ (\sigma^M f)$ we have

$$X' \equiv \phi^* \operatorname{mod} G(LH_{\mathrm{sep}} + C_{s_0}(L_{\mathrm{sep}})). \tag{1}$$

The equalities $\sigma f_1' = f_1' \circ e_1$ and $\sigma f = f \circ e$ give

$$X' \circ (\sigma^M e) = (\sigma^{\widehat{N}+1} e_1) \circ (\sigma X').$$

The congruence (1) gives $X' = \phi^* \circ U$, where

$$U \in G(LH_{\mathrm{sep}} + C_{s_0}(L_{\mathrm{sep}})).$$

We note that the quotient $G(LH^0_{\mathrm{sep}} + C_{s_0}(L_{\mathrm{sep}}))/G(LH^0_{\mathrm{sep}})$ is a central subgroup in $G(LH_{\mathrm{sep}})/G(LH^0_{\mathrm{sep}})$. From the congruence, cf. proposition of n.4.2,

$$\phi^* \circ (\sigma^M e) \equiv (\sigma^{\widehat{N}+1} e_1) \circ (\sigma\phi^*) \operatorname{mod} G(LH^0_{\mathrm{sep}}),$$

we obtain that $\sigma U - U \in G(LH^0_{\mathrm{sep}})$.

Therefore, $U \in LH^0_{\mathrm{sep}} + L$ because $G(L_{\mathrm{sep}})|_{\sigma=\mathrm{id}} = G(L)$ and for any

$$l \in t_1^{qp^M(b^*-a^*)} \sum_{j \geqslant 1} t_1^{-(j-1)a^* p^M} C_j(L_O),$$

we have

$$l_1 = -\sum_{n \geqslant 0} \sigma^n l \in t_1^{qp^M(b^*-a^*)} \sum_{j \geqslant 1} t_1^{-(j-1)a^* p^M} C_j(L_O)$$

and this l_1 satisfies the identity $\sigma l_1 - l_1 = l$.

So, we have $U \equiv u \operatorname{mod} G(LH_{\mathrm{sep}})$, where $u \in C_{s_0}(L)$.

Let $f_1^0 = f_1' \circ u$, then $\sigma f_1^0 = f_1^0 \circ e_1$ and

$$X^0 = (-\sigma^{\widehat{N}+1} f_1^0) \circ (\sigma^M f) = X' \circ (-u) \equiv \phi^* \operatorname{mod} G(LH_{\mathrm{sep}}).$$

The proposition is proved.

Corollary. $L^{(v_0)} \subset L(v_0)$.

Proof. In notation of n.1.6 the above proposition implies that for the field of definition of $X \operatorname{mod} L(v_0)_{\mathrm{sep}}$ over K', we have

$$K'(X \operatorname{mod} L(v_0)_{\mathrm{sep}}) = K',$$

i.e. $L(v_0) \in \mathcal{J}'_{qv_0-b^*}(X)$, and by the proposition of n.1.6, we have $L(v_0) \supset L^{(v_0)}$.

4.4. By the unductive assumption (1) of n.4.1 and the corollary of n.4.3,

$$L^+(v_0) \subset L^{(v_0)} \subset L(v_0).$$

Therefore for $\widetilde{L} = L(v_0)/L^{(v_0)}$, we have $p\widetilde{L} = C_2(\widetilde{L}) = 0$.

Let $\widetilde{L}_{\text{sep}} = \widetilde{L} \otimes K_{\text{sep}}$, $\widetilde{L}_1 = \widetilde{L} \otimes K'$ and $\widetilde{L}_k = \widetilde{L} \otimes k$.

For $\gamma \geq v_0$, denote by $\widetilde{F}_{\gamma,-\widetilde{N}}$ the image of $F_{\gamma,-\widetilde{N}}$ under the natural projection $L(v_0) \longrightarrow \widetilde{L}_k$ and consider $\widetilde{U} \in \widetilde{L}_{\text{sep}}$ such that

$$\sigma\widetilde{U} - \widetilde{U} = \sum_{v_0 \leq \gamma < v_0+\varepsilon} \widetilde{F}_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*}.$$

Proposition. *If $K'(\widetilde{U})$ is the field of definition of $\widetilde{U}$ over K', then for the maximal upper ramification number $v'(\widetilde{U})$ of the extension $K'(\widetilde{U})/K'$, we have*

$$v'(\widetilde{U}) < qv_0 - b^*.$$

Proof. By the proposition of n.4.3, we have $X^0 = \phi^* \circ U$, where $U \in G(LH^0_{\text{sep}})$. Now we use the equality

$$X^0 \circ (\sigma^M e) = (\sigma^{\widehat{N}+1} e_1) \circ (\sigma X^0),$$

the part (c) of the proposition of n.4.2, and that $G(LH^0_{\text{sep}})/G(LH^+_{\text{sep}})$ is a central subgroup of $G(LH_{\text{sep}})/G(LH^+_{\text{sep}})$, to obtain the following congruence:

$$\sigma U - U \equiv \sigma^{\widehat{N}+1} \sum_{\gamma} F_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*} \operatorname{mod}(L^+(v_0)_{\text{sep}} + t_1 L_O). \tag{1}$$

In the right-hand sum all summands with $\gamma < v_0$ can be omitted, because in this case $-q\gamma + b^* > -a^* + b^* > 0$ and $F_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*} \in t_1 L_O$. We can also omit all terms with $\gamma \geq v_0 + \varepsilon$, because here $F_{\gamma,-\widetilde{N}} t_1^{-q\gamma+b^*} \in L_1(v_0+\varepsilon) \subset L^+(v_0)_{\text{sep}}$.

If U_1, U_2 are any two solutions of the above congruence (1), then

$$U_1 - U_2 \in L^+(v_0)_{\text{sep}} + t_1 L_O + L.$$

Therefore, the fields of definition of these elements modulo $L^{(v_0)}O'(K_{\text{sep}})$ coincide. We denote this field by $\widetilde{K}$.

Clearly, $\widetilde{K} = K'(X^0 \operatorname{mod} L^{(v_0)}O'(K_{\text{sep}}))$. By the proposition of n.1.6, the maximal upper ramification number of the extension $\widetilde{K}/K'$ is less that $qv_0 - b^*$. It is easy to see, that there exists a solution $U_1 \in G(L_{\text{sep}})$ of the congruence (1) such that $U_1 \operatorname{mod} L^{(v_0)}O'(K_{\text{sep}}) = \sigma^{\widehat{N}+1}\widetilde{U}$ with respect to the natural embedding $\widetilde{L}_{\text{sep}} \subset L_{\text{sep}}/L^{(v_0)}O'(K_{\text{sep}})$, and therefore $\widetilde{K} = K'(\widetilde{U})$.

4.5. Assume that $\widetilde{L} = L(v_0)/L^{(v_0)} \neq 0$, and $l_1, \dots, l_n$ is a basis of $\widetilde{L}$ over $\mathbb{F}_p$. We note that the set $\{\sigma^n \widetilde{F}_{\gamma,-\widetilde{N}} \mid v_0 \leq \gamma < v_0 + \varepsilon, n \in \mathbb{Z}/N_0\mathbb{Z}\}$ generates $\widetilde{L}_k$ over k. Therefore the set of coefficients

$$\{ a_{\gamma i} \in k \mid v_0 \leq \gamma < v_0 + \varepsilon\}$$

of the decompositions

$$\widetilde{F}_{\gamma,-\widetilde{N}} = \sum_{i=1}^{n} a_{\gamma i} l_i,$$

contains at least one non-zero element, i.e. there exist $\gamma_0 \geq v_0$ and $1 \leq i_0 \leq n$ such that $a_{\gamma_0, i_0} \neq 0$.

Consider the decomposition

$$\widetilde{U} = \sum_{i=1}^{n} U_i l_i,$$

where for $i = 1, \dots, n$, $U_i \in K_{\text{sep}}$ and satisfy the relations

$$U_i^p - U_i = \sum_{v_0 \leq \gamma < v_0 + \varepsilon} a_{\gamma i} t_0^{\prime - q\gamma + b^*}.$$

Clearly, $K'(\widetilde{U})$ is the composite of the fields $K(U_i)$, $i = 1, \dots, n$. Therefore,

$$v'(\widetilde{U}) \geq v'_{i_0},$$

where v'_{i_0} is the maximal upper ramification number of the extension $K'(U_{i_0})/K'$. By the choice of $\widetilde{N}$, q and b^* from n.3.1, we have either $F_{\gamma,-\widetilde{N}} = 0$, or $q\gamma - b^* \in \mathbb{Z}$ and $(q\gamma - b^*, p) = 1$. We also note that $\gamma \geq v_0$ implies $q\gamma - b^* > 0$. For this reason, $a_{\gamma_0, i_0} \neq 0$ implies that $K'(U_{i_0})$ has degree p over K' and $v'_{i_0} = q\gamma_{i_0} - b^*$, where $\gamma_{i_0} = \max\{ \gamma \mid a_{\gamma i_0} \neq 0\} \geq \gamma_0 \geq v_0$.

Therefore, for the maximal upper ramification number $v'(\widetilde{U})$ of the extension $\widetilde{K}/K'$ we have $v'(\widetilde{U}) \geq q\gamma_0 - b^* \geq v_0$. This contradicts to the proposition of the above n.4.4. So, $L(v_0) = L^{(v_0)}$, and the theorem is proved.

5. Description of the ramification filtration modulo p^{th} commutators.

5.1. $\mathbb{Z}/p^M\mathbb{Z}$*-module* K^*/K^{*p^M}.

As earlier K is a complete discrete valuation field of characteristic $p > 0$ with finite residue field $k \simeq \mathbb{F}_{p^{N_0}}$ and a fixed uniformising element t_0. For $M \in \mathbb{N}$, consider the lifting $O_M(K)$ of the field K modulo p^M from n.1.1. Let $\mathrm{Col}_M : K^* \longrightarrow O_M(K)^*$ be Coleman's multiplicative section of the projection $O_M(K) \longrightarrow K$. This homomorphism is uniquely defined by conditions: $t_0 \mapsto t$ and $E(\alpha, t_0^a) \mapsto E(\alpha, t^a)$, where $\alpha \in W(k)$, $a \in \mathbb{Z}^+(p)$ and $E(\alpha, X)$ is the power series from n.1.5.

Consider the Witt pairing

$$(\ ,\ ) : O_M(K) \times K^* \longrightarrow \mathbb{Z}/p^M\mathbb{Z}$$

explicitly given by the Witt reciprocity law, cf. [Fo],

$$(f,g) = (\mathrm{Res}\circ\mathrm{Tr})(f\, \mathrm{d}_{\log} \mathrm{Col}_M(g)),$$

where $f \in O_M(K), g \in K^*$ and Tr is induced by the trace of the quotient field of $W(k)$ over $\mathbb{Q}_p$. We have the induced identification:

$$K^*/K^{*p^M} = \mathrm{Hom}\left(O_M(K)/(\sigma - \mathrm{id})O_M(K), \mathbb{Z}/p^M\mathbb{Z}\right).$$

As in n.1.3 fix $\alpha \in W(k)$ such that $\mathrm{Tr}\,\alpha = 1$. Then we have the decomposition

$$O_M(K) = (\sigma - \mathrm{id})O_M(K) \oplus \left(\oplus_{a\in\mathbb{Z}^+(p)} W_M(k)t^{-a} \oplus (\mathbb{Z}/p^M\mathbb{Z})\alpha\right).$$

Therefore,

$$K^*/K^{*p^M} \otimes W_M(k) = \prod_{a\in\mathbb{Z}^+(p)} \mathrm{Hom}(W_M(k)t^{-a}, W_M(k)) \times \mathrm{Hom}((\mathbb{Z}/p^M\mathbb{Z})\alpha, W_M(k)).$$

For $a \in \mathbb{Z}^+(p)$ (resp., $a = 0$) and $n \in \mathbb{Z}$ denote by $D_{an}^{(M)}$ the element of $K^*/K^{*p^M} \otimes W_M(k)$ with the only non-zero component in $\mathrm{Hom}(W_M(k)t^{-a}, W_M(k))$ (resp., $\mathrm{Hom}((\mathbb{Z}/p^M\mathbb{Z})\alpha, W_M(k)))$ given by the correspondence $wt^{-a} \mapsto \sigma^n w$ (resp., $\alpha \mapsto \sigma^n\alpha$). Clearly, for any $a \in \mathbb{Z}^0(p)$, we have $D_{an}^{(M)} = D_{a,n+N_0}^{(M)}$ and the set

$$\{\ D_{an}^{(M)} \mid a \in \mathbb{Z}^0(p), 0 \le n < N_0\ \}$$

generates the $W_M(k)$-module $K^*/K^{*p^M} \otimes W_M(k)$.

Let A be a finite subset of $\mathbb{Z}^0(p)$. Then the set

$$\{\ D_{an}^{(M)} \mid a \in A, 0 \le n < N_0\ \}$$

generates the free $W_M(k)$-module $\mathcal{M}(A,M) \otimes W_M(k)$, where $\mathcal{M}(A,M)$ is the image in K^*/K^{*p^M} of the subgroup of K^* generated by the set

$$\{\ E(\alpha, t^a) \mid \alpha \in W(k), a \in A \cap \mathbb{Z}^+(p)\ \}$$

and by t_0 if $0 \in A$ (this follows easily from the Witt explicit reciprocity law). If $A_1 \subset A$, then we have a natural epimorphism of modules

$$\mathcal{M}(A,M) \longrightarrow \mathcal{M}(A_1,M)$$

induced by the correspondences $D_{an}^{(M)} \mapsto D_{an}^{(M)}$ if $a \in A_1$, and $D_{an}^{(M)} \mapsto 0$ if $a \in A \setminus A_1$. With respect to these epimorphisms we have obviously that

$$\varprojlim_A \mathcal{M}(A,M) = K^*/K^{*p^M}.$$

The above considerations give also

$$K^*/K^{*p^M}\otimes O_M(K)=(K^*/K^{*p^M}\otimes W_M(k))\otimes_{W_M(k)}O_M(K)=$$

$$\mathrm{Hom}\left(\oplus_{a\in\mathbf{Z}^+(p)}W_M(k)t^{-a}\oplus(\mathbb{Z}/p^M\mathbb{Z})\alpha,O_M(K)\right).$$

Denote by $e^{(M)}\in K^*/K^{*p^M}\otimes O_M(K)$ the element which corresponds to the natural inclusion of $\oplus_{a\in\mathbf{Z}^+(p)}W_M(k)t^{-a}\oplus(\mathbb{Z}/p^M\mathbb{Z})\alpha$ in $O_M(K)$. It is easy to see that

$$e^{(M)}=\sum_{a\in\mathbf{Z}^0(p)}t^{-a}D_{a0}^{(M)}$$

and the image of $e^{(M)}$ under the natural projection of K^*/K^{*p^M} to $\mathcal{M}(A,M)$ equals

$$e^{(A,M)}=\sum_{a\in A}t^{-a}D_{a0}^{(M)}.$$

Finally, we remark that under the modulo p^M reduction map

$$K^*/K^{*p^{M+1}}\longrightarrow K^*/K^{*p^M},$$

we have $D_{an}^{(M+1)}\mapsto D_{an}^{(M)}$ and $e^{(M+1)}\mapsto e^{(M)}$.

5.2. *The Lie algebra $\mathcal{L}^0$ and the identification $\bar{\psi}$.*

Let $K^*(p)=\varprojlim_M K^*/K^{*p^M}$ with respect to morphisms of reduction modulo p^M from $K^*/K^{*p^{M+1}}$ to K^*/K^{*p^M}, $M\in\mathbb{N}$. Denote by $\mathcal{L}^0$ (resp., $\mathcal{L}^0(M)$) the free Lie algebra with topological module of generators $K^*(p)$ (resp., K^*/K^{*p^M}) over $\mathbb{Z}_p$ (resp., $\mathbb{Z}/p^M\mathbb{Z}$). If $A\subset\mathbb{Z}^0(p)$ is a finite subset, let $\mathcal{L}^0(A,M)$ be a free Lie algebra over $\mathbb{Z}/p^M\mathbb{Z}$ with the generating module $\mathcal{M}(A,M)$, cf. n.5.1.

Clearly, the projective system of $\mathbb{Z}_p$-modules $\{\mathcal{M}(A,M)\}_{A,M}$ defines the projective system of Lie algebras $\{\mathcal{L}^0(A,M)\}_{A,M}$ and

$$\varprojlim_A\mathcal{L}^0(A,M)=\mathcal{L}^0(M),\qquad\varprojlim_{A,M}\mathcal{L}^0(A,M)=\varprojlim_A\mathcal{L}^0(A)=\mathcal{L}^0,$$

where $\mathcal{L}^0(A)=\varprojlim_M\mathcal{L}^0(A,M)$.

Set $\mathcal{L}=\mathcal{L}^0\otimes W(k)$, $\mathcal{L}(M)=\mathcal{L}^0(M)\otimes W_M(k)$ and $\mathcal{L}(A,M)=\mathcal{L}^0(A,M)\otimes W_M(k)$. We note that the Lie algebras $\mathcal{L}^0(A)$ and $\mathcal{L}(A)=\mathcal{L}^0(A)\otimes W(k)$ are naturally identified with the Lie algebras from n.2 denoted by the same symbols. Under this identification for all $a\in A$ and $n\in\mathbb{Z}$, we have $\varprojlim_M D_{an}^{(M)}=\mathcal{D}_{an}$, where the elements $\mathcal{D}_{an}\in\mathcal{L}(A)$ were introduced in n.2.1. The algebra $\mathcal{L}$ is a profree Lie algebra over $W(k)$, the set

$$\{\mathcal{D}_{an}\mid a\in\mathbb{Z}^0(p),n\in\mathbb{Z}\}$$

generates $\mathcal{L}$ and $\sigma\mathcal{D}_{an} = \mathcal{D}_{a,n+1}$ for any $a \in \mathbb{Z}^0(p)$ and $n \in \mathbb{Z}$. We shall use a tilde in notation of any of the above Lie algebras for its quotient by the ideal of commutators of order $\geq p$.

Consider the elements $e^{(A,M)} \in G(\widetilde{\mathcal{L}}^0(A,M) \otimes O_M(K))$ from n.5.1. These elements are compatible in the projective system $\{G(\widetilde{\mathcal{L}}^0(A,M) \otimes O_M(K))\}_{A,M}$. If

$$\mathcal{F}(A,M) = \left\{ f \in G(\widetilde{\mathcal{L}}^0(A,M) \otimes O(K_{\text{sep}})) \mid \sigma f = f \circ e^{(A,M)} \right\},$$

then $\{\mathcal{F}(A,M)\}_{A,M}$ is a projective system of non-empty finite sets and therefore, its projective limit is not empty (in fact, all connecting morphisms of this projective system are epimorphisms). Choose $f \in \varprojlim_{A,M} \mathcal{F}(A,M)$ and denote by $f^{(A,M)}$ its projection to $\mathcal{F}(A,M)$. Then the correspondences

$$\tau \mapsto \tau f^{(A,M)} \circ (-f^{(A,M)})$$

define the compatible system of group homomorphisms

$$\psi^{(A,M)} : \Gamma \longrightarrow G(\widetilde{\mathcal{L}}^0(A,M)).$$

It is easy to see that $\psi^{(M)} = \varprojlim_{A} \psi^{(A,M)}$ induces the group isomorphism

$$\bar{\psi}^{(M)} : \Gamma/\Gamma^{p^M} C_p(\Gamma) \longrightarrow G(\widetilde{\mathcal{L}}^0(M))$$

and $\psi = \varprojlim_{A,M} \psi^{(A,M)}$ induces the group isomorphism

$$\bar{\psi} : \Gamma(p)/C_p(\Gamma(p)) \longrightarrow G(\widetilde{\mathcal{L}}^0),$$

where $\Gamma(p) = \varprojlim_{M} \Gamma/\Gamma^{p^M}$ is the Galois group of the maximal p-extension of K in K_{sep}. We note that $\bar{\psi} \bmod C_2(\Gamma(p)) : \Gamma(p)^{\text{ab}} \simeq K^*(p)$ is induced by the reciprocity map of local class field theory.

5.3. For any finite subset $A \subset \mathbb{Z}^0(p)$ and $v > 0$, consider the ideal $\mathcal{L}^0(A,v)$ of the Lie algebra $\mathcal{L}^0(A)$ from n.2.6. It is easy to see that $\{\mathcal{L}^0(A,v)\}_A$ is a projective subsystem in the projective system $\{\mathcal{L}^0(A)\}_A$. Therefore, $\varprojlim_{A} \mathcal{L}^0(A,v) = \mathcal{L}^0(v)$ is an ideal of $\mathcal{L}^0$. If $\widetilde{\mathcal{L}}^0(v) = \mathcal{L}^0(v)/C_p(\mathcal{L}^0)$, then the main theorem of n.4 gives

Theorem A. *For any $v > 0$, we have*

$$\bar{\psi}(\Gamma(p)^{(v)} \bmod C_p(\Gamma(p))) = G(\widetilde{\mathcal{L}}^0(v)).$$

The above ideals $\widetilde{\mathcal{L}}^0(v)$ can be described as follows.

For any $a \in \mathbb{Z}^0(p)$ and $n \in \mathbb{Z}$, denote by $\widetilde{\mathcal{D}}_{an}$ the image of $\mathcal{D}_{an} \in \mathcal{L}$ in $\widetilde{\mathcal{L}} = \mathcal{L}/C_p(\mathcal{L})$. If $v > 0$ and $\widetilde{\mathcal{L}}(v) = \widetilde{\mathcal{L}}^0(v) \otimes W(k) \subset \widetilde{\mathcal{L}}$, then by the remark from n.3.3, we have:

if $a \geq sv$, where $1 \leq s < p$, then for any $n \in \mathbb{Z}$, $\widetilde{\mathcal{D}}_{an} \in \widetilde{\mathcal{L}}(v) + C_{s+1}(\widetilde{\mathcal{L}})$.

Let $A(v) = \mathbb{Z}^0(p) \cap [1, (p-1)v)$ and in notation of n.2.6 let $N(v) = \widetilde{N}(v, A(v)^+)$. We use the constants $\eta(n_1, \dots, n_s)$ from n.1.4 to define for any $\gamma > 0$, the following elements $\widetilde{\mathcal{F}}_\gamma(v)$ of the Lie algebra $\widetilde{\mathcal{L}}$:

$$\widetilde{\mathcal{F}}_\gamma(v) = \sum (-1)^s \eta(n_1, \dots, n_s) a_1 p^{n_1} \left[\dots \left[\widetilde{\mathcal{D}}_{a_1 n_1}, \widetilde{\mathcal{D}}_{a_2 n_2} \right], \dots, \widetilde{\mathcal{D}}_{a_s n_s} \right],$$

where the above sum is taken for $1 \leq s < p$, $a_1, \dots, a_s \in A(v)$ and $n_1, \dots, n_s \in \mathbb{Z}$ such that $n_1 \geq 0$, $n_1 \geq \dots \geq n_s \geq -N(v)$ and $a_1 p^{n_1} + \dots + a_s p^{n_s} = \gamma$.

It is easy to see that: 1) for any $\gamma > 0$, the above expression for $\widetilde{\mathcal{F}}_\gamma(v)$ contains only finitely many terms; 2) the set $\mathcal{S}(v) = \{\gamma > 0 \mid \widetilde{\mathcal{F}}_\gamma(v) \neq 0\}$ is discrete in the archimedean topology, and therefore, $\mathcal{S}(v) = \{\gamma_1, \dots, \gamma_m, \dots\}$, where $0 < \gamma_1 < \dots < \gamma_m < \dots$; 3) in the p-adic topology we have $\lim_{m \to \infty} \widetilde{\mathcal{F}}_{\gamma_m}(v) = 0$. So, we have the following description of the ideals $\widetilde{\mathcal{L}}^0(v)$:

Theorem B. *For any $v > 0$, $\widetilde{\mathcal{L}}^0(v)$ is the minimal closed ideal of the Lie algebra $\widetilde{\mathcal{L}}^0$ such that $\widetilde{\mathcal{L}}(v) = \widetilde{\mathcal{L}}^0(v) \otimes W(k)$ contains the set*

$$\left\{ \widetilde{\mathcal{F}}_\gamma(v) \mid \gamma \geq v \right\} \cup \left\{ \widetilde{\mathcal{D}}_{a0} \mid a \geq (p-1)v \right\}.$$

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