Special 3–dimensional flips

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1. Definitions and examples

Let X be a normal complex algebraic variety. By $K = K_X$ we denote its canonical Weil divisor. A divisor of the form $K+D$ is log–canonical if

(i) all $0 \leq d_i \leq 1$ where $D = \Sigma\, d_i\, D_i \in \mathrm{Div}_{\mathbb{R}} X = \mathbb{R} \otimes \mathrm{Div}\, X$ and D_i are different prime Weil divisors.

(ii) There exists a resolution $f : \tilde{X} \longrightarrow X$ such that

$$\tilde{K} + \tilde{D} = f^*(K + D) + \Sigma\, a_i\, E_i$$

with discrepancy coefficients $a_i \geq -1$ and with non–singular normally crossing components of divisors $\tilde{D}$ and E, where $\tilde{K}$ is a canonical divisor of $\tilde{X}$, $\tilde{D}$ is the proper invers image of D and $E = \Sigma\, E_i$ is the sum of exceptional divisors. In the case when all $a_i > -1$ the divisor $K+D$ is log–terminal. These conditions are not only on singularities of X but also on that of D.

Examples. 1. If $K+D$ is log–terminal then K is also log–terminal. K is log–terminal in all non–singular points of X. Due to Kawamata a surface singular point p is log–terminal for K iff p is a quotient singularity. These singularities were classified by O. Riemenschneider. The minimal resolution of them consists of normally crossing non–singular rational curves and its graph has one of the well–known types A_n, D_n and E_6, E_7, E_8. They are types of p.

2. $K + \{y = 0\} + \frac{1}{2}\{y = x^2\}$ is log–canonical on $\mathbb{A}^2$ and log–terminal on $\mathbb{A}^2 \backslash \{(0,0)\}$.

Log–canonical $K+D$ is n–complementary if there exists a Weil divisor $\overline{D} \in |-nK - \lfloor (n+1)D \rfloor|$ such that

$$K + \lfloor (n+1)D \rfloor /n + \overline{D}/n$$

is also log–canonical. Complementary means 1–complementary.

Lemma. If $D' \geq D$ and $K+D'$ is n–complementary, then $K+D$ is also n–complementary.

Take $\overline{D} = \overline{D}' + \lfloor (n+1)D' \rfloor - \lfloor (n+1)D \rfloor$.

Proposition. Let $Z \subseteq X$ be a subvariety on which $K+D$ is negative log–terminal with $\lfloor D \rfloor = 0$. Then $K+D$ near Z is n–complementary for some natural n .

Examples. 2. Consider negative $K+D$ on $\mathbb{P}^1$ with $\lfloor D \rfloor = 0$, i.e. $D = \Sigma\, d_i\, p_i$ with

$$0 \leq d_i < 1 \quad \text{and} \quad \Sigma\, d_i < 2$$

where p_i are different points on $\mathbb{P}^1$. In addition, let $d_1 \geq d_2 \geq \dots$. Then $K+D$ is always 1–, 2–, 3–, 4– or 6– complementary. Moreover,

$K+D$ is not 1–complementary iff $d_1, d_2, d_3 \geq \frac{1}{2}$;

$K+D$ is not 1– and 2–complementary iff $d_1, d_2 \geq \frac{2}{3}$, $d_3 \geq \frac{1}{2}$ or $d_1 = \frac{2}{3}$, $d_2 = d_3 = \frac{1}{2}$ and $d_4 = \frac{1}{3}$;

$K+D$ is not 1–, 2– and 3–complementary iff $d_1 \geq \frac{3}{4}$, $d_2 \geq \frac{2}{3}$ and $d_3 \geq \frac{1}{2}$;

$K+D$ is not 1–, 2–, 3– and 4–complementary iff $d_1 \geq \frac{4}{5}$, $d_2 \geq \frac{2}{3}$ and $d_3 \geq \frac{1}{2}$.

3. Let K is log–terminal near a surface point p . Then K near p is always 1–, 2–, 3–, 4– or 6–complementary. Moreover,

K is not 1–complementary iff p has the type D_n or E_6, E_7, E_8 ;

K is not 1– and 2–complementary iff p has the type E_6, E_7 or E_8 ;

K is not 1–, 2– and 3–complementary iff p has the type E_7 or E_8 ;

K is not 1–, 2–, 3– and 4–complementary iff p has the type E_8 .

This is easy derived from the Riemanschneider classification or from the previous example.

4. (Alekseev, Reid, Shokurov). K is complementary on a Fano 3–fold with log–terminal singularities of index ≥ 1 .

5. (Mori, Reid). K is complementary near any 3–fold terminal singularity.

6. (Mori). K is 1– or 2–complementary near the support of negative extremal ray of fliping type on a 3–fold with terminal singularities.

7. (Mori, Morrison, ?). There exist 4–dimensional terminal quotient singularities, which are nor 1–, nor 2– complementary.

2. Adjunction of log–canonical divisors

Consider a log–canonical divisor

$$K + D_0 + D$$

where D_0 is a sum of different prime Weil divisors of X . Let

$$\nu : D_0^{\nu} \longrightarrow D_0 \subset X$$

be the normalization of D_0. Note that normally crossing components of D_0 are considered as normal. Let

$$(K + D_0 + D)\Big|_{D_0^\nu} \overset{\mathrm{df}}{=} \nu^*(K + D_0 + D)$$

where the map $\nu^* : \mathrm{Div}_{\mathbb{R}} X \dashrightarrow \mathrm{Div}_{\mathbb{R}} D_0^\nu$ is induced by the lifting of the Cartier divisors.

Adjunction Theorem. If $K + D_0 + D$ is log–canonical (resp. log–terminal) then

$$(K + D_0 + D)\Big|_{D_0^\nu} = K_{D_0^\nu} + C$$

is also log–canonical (resp. log–terminal).

The general statement is easy derived from the 2–dimensional case. Moreover, from the standard Minimal Model Conjectures follows

$IA(D_0,D)$ Conjecture. If $(K + D_0 + D)\Big|_{D_0^\nu}$ is log–canonical (resp. log–terminal) then $K + D_0 + D$ is log–canonical (resp. log–terminal) near D_0.

Example. $IA(D_0,D)$ is true and useful in the dimension two. Indeed in this case D_0 is normal, D intersects D_0 only in non–singular points p of D_0 and $(D_0.D)_p \leq 1$.

Proposition. If $\dim X = 3$ and D is integer near D_0 then $IA(D_0,D)$ is true.

This follows from the existence of relative minimal models due to Tsunoda, Shokurov, Mori and Kawamata [SH].

Lemma. *If* $K + D_0 + D$ *is log–canonical then for any natural* n

$$(nK + nD_0 + \llcorner(n+1)D\lrcorner)\Big|_{D_0^\nu} \le nK_{D_0^\nu} + \llcorner(n+1)C\lrcorner \;.$$

The proof uses the following 2–dimensional facts

(1) If $K + D_0$ is log–terminal near p and D_0 paths through p then D_0 is a non–singular curve near p and $(K + D_0)|_{D_0} = K_{D_0} + cp$ where $c = \frac{m-1}{m}$ and m is natural. This number m is the index of $K + D_0$ in p.

(2) In addition every integer divisor near p has the index which divides m.

Epi–restriction Theorem. *Let* $Z \subseteq D_0$ *be a subvariety such that*

(i) $K + D_0 + D$ *is log–terminal;*

(ii) $K + D_0 + D$ *is negative on* Z;

(iii) $nK + nD_0 + \llcorner(n+1)D\lrcorner \ge nK + nD_0 + nD$.

Then the restriction map

$$|-nK - nD_0 - \llcorner(n+1)D\lrcorner| \xrightarrow{\;|D_0^\nu\;} |-nK_{D_0^\nu} - \llcorner(n+1)C\lrcorner| + A$$

is epi near Z, *where*

$$A = nK_{D_0^\nu} + \llcorner(n+1)C\lrcorner - (nK + nD_0 + \llcorner(n+1)D\lrcorner)\Big|_{D_0^\nu}$$

is an effective divisor according to the previous lemma.

The proof uses the Kawamata–Viehweg vanishing theorem on a desingularization of X.

Corollary. If $K + D_0 + D$ is log–terminal and $\lfloor D \rfloor = 0$ then D_0 is normal. Use locally the theorem in the case $n = 0$.

3. Classification of log–terminal surface divisors

Theorem. Let $K+D$ is log–terminal near a surface point p. Then $K+D$ is 1–, 2–, 3–, 4– or 6–complementary near p.

Scatch proof. Firstly we find such contraction $f : \tilde{X} \longrightarrow X$ that

(i) $\tilde{K} + E + \tilde{D}$ is log–terminal near E,

(ii) $\tilde{K} + E + \tilde{D}$ is numerically negative on E and

(iii) $E = \mathbb{P}^1$,

where E is the exceptional locus over p. Then we combine Example 2 from Sec. 1 and Epi–restriction theorem to choose n and $\overline{D}$ such that

(iv) $(\tilde{K} + E + \tilde{D})|_E = K_E + C$ is n–complementary where $n = 1,2,3,4$ or 6 and

(v) $\overline{D}|_E = \overline{C} + A$

where $\overline{D} \in |-n\tilde{K} - nE - \lfloor (n+1)\tilde{D} \rfloor|$, $\overline{C} \in |-nK_E - \lfloor (n+1)C \rfloor|$ and

$$K_E + \lfloor (n+1)C \rfloor /n + \overline{C}/n$$

is log–canonical. Due to Lemma from Sec. 1 decreasing D we may satisfy the condition

(iii) from Epi–restriction theorem. The divisor

$$\tilde{K} + E + \lfloor (n+1)\tilde{D} \rfloor /n + \bar{D}/n$$

is numerically trivial on E and log–canonical near E by Example of Sec. 2. So K+D is n–complementary near p with a completion $f_*\bar{D} \in |-nK - \lfloor (n+1)D \rfloor|$. By the way we obtain

Proposition. Let K+D is log–terminal near a surface point p and $d_i \geq \frac{5}{6}$ for some curve D_i through p. Then K+D is 1– or 2–complementary.

In this case $C = \Sigma c_i p_i$ with some $c_i \geq \frac{5}{6}$ and hence with $\sum_{i \neq i} c_j < 2 - \frac{5}{6} = \frac{7}{6} = \frac{1}{2} + \frac{2}{3}$. By Example 2 of Sec. 1 then $K_E + C$ is 1– or 2–complementary.

4. Special log–terminal flips

Theorem. Let dim X = 3, K+D is log–terminal near p, all $d_i \geq \frac{2}{3}$ and $d_{i_0} = 1$, $d_{i_1} \geq \frac{5}{6}$ for some different D_{i_0}, D_{i_1}. Then K+D is 1– or 2–complementary near p.

Follows from Proposition of Sec. 3 and restriction arguments. Inspite of Example of Sec. 2 use Proposition of the Sec. in the 2–complementary case on the 2–cover.

Let $f : X \longrightarrow C$ be such family of surfaces over a curve that

(i) X is non–singular,

(ii) all fibres $f^{-1}(c)$ consist of non–singular surface with normal crossing and

(iii) the general fibre is a minimal surface of the general type.

Then any divisor K+D is log–terminal if $D = \Sigma d_i D_i$ with $0 \leq d_i \leq 1$ and all D_i lie in fibres. A <u>relative minimal model</u> for K+D is a birationally transformed family $\tilde{f} : \tilde{X} \longrightarrow C$ for which

(iv) all fibres $\tilde{f}^{-1}(c)$ are proper transforms of fibres $f^{-1}(C)$,

(v) $\tilde{K} + \tilde{D}$ is log–canonical,

(vi) $\tilde{K} + \tilde{D}$ is relatively ample for $\tilde{f}$ and

(vii) discrepancy coefficients $a_i \geq -d_i$ for contracted divisors D_i .

Easy to check that such model is unique if exists. Fix a special fibre $\bigcup_{i=1}^{N} D_i = f^{-1}(c_0)$ and identify divisors $D = \Sigma d_i D_i$ with points of the cube $[0,1]^N$. From Kawamata results [Ka] follow that near $f^{-1}(c_0)$ the relative minimal models exist for a subcube $[1-\epsilon,1]^N$ where $\epsilon > 0$ depends of the family f near $f^{-1}(c_0)$.

<u>Relative Model Theorem</u>. <u>There exists</u> $0 \leq d_0 \leq \frac{5}{6}$ <u>that near</u> $f^{-1}(c_0)$ <u>the relative minimal models exist in a subcube</u> $(d_0,1]^N$ <u>and they give a locally finite convex polyhedral decomposition of</u> $(d,1]^N$.

Remain that conjecturely any extremal ray R negative for a log–terminal divisor K+D and of flipping type (i.e. in the dimension three contracting only curves) has an adjoint diagram or a flip. This should be a commutative diagram

$$\begin{array}{ccc} X & \overset{tr_R}{- - - - \rightarrow} & X^+ \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi^+} & \\ & Z & \end{array}$$

consisting of

(a) a birational map $tr_R : X - - - - \rightarrow X^+$ which is an isomorphism except for loci of codimension ≥ 2 ,

(b) a contraction φ^+ such that the divisor $K + D^+$ is log–terminal and relatively ample for φ^+ where D^+ is the proper transform of D .

Corollary 1. Let dim X = 3 , Sing X $\subseteq$ Supp D , all $d_i \geq d_0$ and D has the fibre type, i.e. $(R, \Sigma\delta_i D_i) = 0$ for some $\delta_i > 0$. Then a flip exists for R .

Corollary 2. If dim X = 3 and all $d_i \geq d_0$ then $IA(D_0,D)$ is true.

References

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