REFLEXIVE MODULES OVER RATIONAL DOUBLE POINTS

by

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§0. Introduction

(0.0) This note is an appendix to the article of the same title by M. Artin and J.-L. Verdier (Math.Ann. 270 (1985), 79-82), whose main content is the following.

Let $(X,0)$ be a germ of a rational double point over an algebraically closed field k, let $\pi : \tilde{X} \longrightarrow X$ be the minimal desingularization and let E be the exceptional locus. There is a one-to-one correspondence $[M_i] \longleftrightarrow E_i$ between isomorphism classes of non trivial indecomposable reflexive modules M_i on $(X,0)$ and the irreducible components E_i of E: For such a module M_i let $\mathcal{M}_i$ be the locally free sheaf on $\tilde{X}$ defined by $\mathcal{M}_i := \pi^*(M_i)/\text{torsion}$. If r_i is the rank of M_i one has $(\Lambda^{r_i}\mathcal{M}_i) . E_j = \delta_{ij}$ and $r_i = (\Lambda^{r_i}\mathcal{M}_i) \cdot Z$, where Z denotes the fundamental cycle of $\tilde{X}$.

(0.1) We assume moreover that $(X,0)$ is a germ of a <u>quotient singularity</u>. This means that there exists a cover $q : (\mathbb{A}^2_k, 0) \longrightarrow (X,0)$ with Galois group $G \subset SL(2,k)$ whose order is prime to the characteristic of k. This is always the case in characteristic 0.

(0.2) For each M_i as in (0.0) define N_i to be the reflexive hull of $M_i \otimes \Omega^1_X$, where Ω^1_X is the module of Kähler differential forms and set $\mathcal{N}_i = \pi^*(N_i)/\text{torsion}$.

(0.3) The aim of this note is to prove the <u>multiplication formula</u>

<u>Theorem:</u> $\Lambda^{2r_i} N_i \cong (\Lambda^{r_i} M_i)^{\otimes 2} \otimes \mathcal{O}_{\tilde{X}}(E_i)$

(0.4) Originally this multiplication formula was proven case by case by G.Gonzalez-Sprinberg and J.-L. Verdier [Construction géométrique de la correspondance de Mc Kay, Ann.Sc.Éc. Norm.Sup., 16 (1983), 409-449]. Recently they computed examples of rational double points in characteristic 2,3,5 where this multiplication formula does not hold.

§1. Proof of theorem (0.3)

(1.1) Let α be the Euler differential on $(\mathbb{A}_k^2, 0)$:

$$\alpha = x \cdot dy - y \cdot dx \ .$$

As α is invariant under SL(2,k) one can think of α as being a differential one-form on X and on $\tilde{X}$ as well. One has the exact sequence

$$(1.1.1) \quad 0 \longrightarrow \mathcal{O}_{(\mathbb{A}^2,0)} \xrightarrow{\otimes\alpha} \Omega^1_{(\mathbb{A}^2,0)} \xrightarrow{\wedge\alpha} m_{(\mathbb{A}^2,0)} \otimes \omega_{(\mathbb{A}^2,0)} \longrightarrow 0$$

where m denotes the maximal ideal and ω the dualizing module. Applying q_* and taking the G-invariants one obtains the exact sequence

$$(1.1.2) \quad 0 \longrightarrow \mathcal{O}_{(X,0)} \xrightarrow{\otimes\alpha} \Omega^1_{(X,0)} \xrightarrow{\wedge\alpha} m_{(X,0)} \otimes \omega_{(X,0)} \longrightarrow 0 \ .$$

(1.2) Let F_i be the non trivial indecomposable representation of G such that $M_i \cong (\mathcal{O}_{(\mathbb{A}^2,0)} \otimes_k F_i)^G$. Tensorize (1.1.1) by F_i:

(1.2.1)

$$0 \longrightarrow (\mathcal{O}_{(\mathbb{A}^2,0)} \otimes_k F_i) \longrightarrow (\Omega^1_{(\mathbb{A}^2,0)} \otimes_k F_i) \longrightarrow (m_{(\mathbb{A}^2,0)} \otimes \omega_{(\mathbb{A}^2,0)} \otimes F_i) \longrightarrow 0 .$$

Apply q_* and take the G-invariant parts. Since the representation F_i is not trivial there are no invariants of degree 0. Therefore one obtains the exact sequence on $(X,0)$:

(1.2.2) $$0 \longrightarrow M_i \xrightarrow{\otimes \alpha} N_i \xrightarrow{\wedge \alpha} M_i \longrightarrow 0$$

where N_i is as in (0.2) and we identify $\omega_{(X,0)}$ with $\mathcal{O}_{(X,0)}$.

(1.3) Pulling back (1.1.2) on $\tilde{X}$ one obtains the complex

(1.3.1) $$0 \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\otimes \alpha} \tilde{\Omega} \xrightarrow{\wedge \alpha} \mathcal{O}_{\tilde{X}}(-Z) \longrightarrow 0$$

where $\tilde{\Omega} := \pi^*\Omega^1_{(X,0)}/\text{torsion}$. This complex is exact away from E, and also left and right exact . Let K be the kernel of $\wedge\alpha$ in (1.3.1).

<u>Claim:</u> (i) <u>K is locally free.</u>

(ii) $K \cong \mathcal{O}_{\tilde{X}}(R)$ <u>for an effective divisor</u> R <u>supported on</u> E .

Proof: Since $\mathcal{O}_{\tilde{X}}(-Z)$ is locally free one has the exact sequence

$$0 \longleftarrow K^{\vee} \longleftarrow \tilde{\Omega}^{\vee} \longleftarrow \mathcal{O}_{\tilde{X}}(Z) \longleftarrow 0$$

By definition $K^{\vee}$ is reflexive on $\tilde{X}$, and therefore $K^{\vee}$ is locally free. Dualizing once again one obtains the exact sequence

$$0 \longrightarrow K^{\vee\vee} \longrightarrow \tilde{\Omega} \longrightarrow \mathcal{O}_{\tilde{X}}(-Z) \longrightarrow 0 ,$$

and therefore $K = K^{\vee\vee}$ is locally free. Since the inclusion $\mathcal{O}_{\tilde{X}} \hookrightarrow K$ is an isomorphism outside of E one has $K \cong \mathcal{O}_{\tilde{X}}(R)$ for an effective R supported on E .

(1.4) Claim: The complex (1.3.1) is exact.

Proof: From (1.3) one obtains

$$\overset{2}{\Lambda}\,\tilde{\Omega} \cong \mathcal{O}_{\tilde{X}}(-Z+R)$$

Since $(\overset{2}{\Lambda}\Omega)\cdot E_{\ell}$ is non-negative for all ℓ and (-Z) is the largest vertical divisor intersecting each E_{ℓ} non negatively one has R = 0 , or R = Z . Assume R = Z . Then by the theorem of Artin-Verdier (0.0) , one has $\Omega^1_X = \mathcal{O}_X \oplus \mathcal{O}_X$. Therefore $m_{(X,0)}$ has two generators by (1.1.2), and (X,0) has to be smooth.

(1.5) We now pull (1.2.2) back to $\tilde{X}$. One obtains the complex

$$(1.5.1)\qquad 0 \longrightarrow M_i \xrightarrow{\otimes\alpha} N_i \xrightarrow{\wedge\alpha} M_i \longrightarrow 0 \quad .$$

This complex is left exact since M_i is torsion free, and it is right exact as π^* is. In addition the complex is exact away from E. Let K_i be the kernel of $(\wedge\alpha)$ in (1.5.1).

<u>Claim</u>: K_i <u>is locally free</u>.

The proof is the same as for (1.3).

(1.6) <u>Claim</u>: $(\overset{2r_i}{\Lambda} N_i) \cong (\overset{r_i}{\Lambda} M_i) \otimes (\overset{r_i}{\Lambda} M_i) \otimes \mathcal{O}_{\tilde{X}}(R_i)$ <u>for an effective divisor</u> R_i <u>supported on</u> E.

<u>Proof</u>: One has

$$(\overset{2r_i}{\Lambda} N_i) \cong (\overset{r_i}{\Lambda} M_i) \otimes (\overset{r_i}{\Lambda} K_i)$$

and the inclusion $M_i \hookrightarrow K_i$ is an isomorphism outside of E.

(1.7) <u>Claim</u>: <u>One has</u> $R_i = E_i + R_i'$ <u>for an effevtive divisor</u> R_i' <u>supported on</u> E.

<u>Proof</u>: Assume that R_i does not contain E_i. Then $R_i \cdot E_i \geq 0$. Since $(\overset{r_i}{\Lambda} M_i) \cdot E_j = 0$ for $j \neq i$ one has $R_i \cdot E_j = (\overset{2r_i}{\Lambda} N_i) \cdot E_j \geq 0$ for $i \neq j$. Therefore $R_i^2 \geq 0$. As the intersection matrix of E is negative definite one

gets $R_i = 0$. Therefore $(\overset{2r_i}{\Lambda} N_i) \cong (\overset{r_i}{\Lambda} M_i)^{\otimes 2}$, and by the theorem of Artin-Verdier (0.0) we obtain

$$N_i \cong M_i \oplus M_i$$

Restrict this isomorphism to $U := X-\{0\} = \tilde{X} - E$ and tensor with $M_i^{\vee}\big|_U$. One obtains

$$(1.7.1) \qquad (\operatorname{End} M_i)\big|_U \otimes \Omega_U^1 \cong \operatorname{End}(M_i)\big|_U \otimes (\mathcal{O}_U \oplus \mathcal{O}_U)$$

The trace map $\operatorname{End}(M_i)\big|_U \longrightarrow \mathcal{O}_U$ defines a natural splitting since the characteristic of k is prime to the order of G and hence to the rank r_i of M_i . Taking traces on both sides of (1.7.1) one gets

$$\Omega_U^1 \cong \mathcal{O}_U \oplus \mathcal{O}_U$$

which contradicts (1.4)

(1.8) By the normal basis theorem one has

$$q_*\mathcal{O}_{(\mathbb{A}^2,0)} \cong \mathcal{O}_{(X,0)} \oplus \bigoplus_i r_i M_i \qquad \text{and}$$
$$q_*\Omega^1_{(\mathbb{A}^2,0)} \cong \Omega^1_{(X,0)} \oplus \bigoplus_i r_i N_i$$

As $\Omega^1_{(\mathbb{A}^2,0)}$ is isomorphic to $\mathcal{O}_{(\mathbb{A}^2,0)} \oplus \mathcal{O}_{(\mathbb{A}^2,0)}$ as an $\mathcal{O}_{(\mathbb{A}^2,0)}$ - module one has

(1.8.1) $$\Omega^1_{(X,0)} \oplus \bigoplus_i r_i N_i \cong 2\mathcal{O}_{(X,0)} \oplus 2 \bigoplus_i r_i M_i$$

(1.9) Pull (1.8.1) back to $\tilde{X}$ and apply (1.7) and (1.4). This gives

$$\mathcal{O}_{\tilde{X}}(-Z+\Sigma r_i(E_i+R_i')) \otimes \bigotimes(\overset{r_i}{\Lambda} M_i)^{\otimes 2r_i} \cong \bigotimes_i(\overset{r_i}{\Lambda} M_i)^{\otimes 2r_i}$$

Therefore $\mathcal{O}_{\tilde{X}}(\Sigma r_i R_i') \cong \mathcal{O}_{\tilde{X}}$. As each R_i' is effective we have $R_i' = 0$ for all i , and the theorem (0.3) is proven.