ON THE COHOMOLOGY OF BRILL-NOETHER STRATA OVER QUOT SCHEMES

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Abstract. We define a Brill-Noether stratification over the Quot scheme parametrizing quotients of a trivial bundle on a curve and we compute their cohomology classes.

1. Introduction

The theory of Brill-Noether over the space of stable vector bundles or semistable bundles, has been very much studied, [BGMN], [BGN], [Te]. Let $M(r, d)$ denote the moduli space of stable vector bundles of rank r and degree d, and $\widetilde{M}(r, d)$ the moduli space of semi-stable vector bundles of rank r and degree d. For E a rank r and degree d vector bundle, the slope of E is defined as $\mu(E) = \frac{d}{r}$.

The Brill-Noether loci over the moduli space of stable bundles are defined by:

$$B_{r,d,k} = \{E \in M(r, d) |\, h^0(E) \geq k\}$$

for a fixed integer k and over the moduli space of semistable vector bundles:

$$\widetilde{B}_{r,d,k} = \{[E] \in M(r, d) | h^0(gr\,(E)) \geq k\}.$$

By the semicontinuity theorem these Brill-Noether loci are closed subschemes of the appropriate moduli spaces, and in particular it is not difficult to describe them as determinantal loci which allow us to estimate their dimension.

The main object of Brill-Noether theory is the study of these subschemes, in particular questions related to their non-emptiness, connectedness, irreducibility, dimension, topological and geometric structure. In the case of line bundles when the moduli spaces are all isomorphic to the Jacobian, these questions have been completely answered when the underlying curve is generic.

We can define in an analogous way Brill-Noether loci over the moduli of maps of fixed degree d from a curve to a projective variety, $\mathrm{Mor}_d\,(C, X)$ and the corresponding Quot schemes compactifying these spaces of morphisms. In particular we are going to consider the case in which X is the Grassmannian of m planes in $\mathbb{C}^n$.Moreover we study in detail the case in which $m = 2$, $n = 4$. In this case, this theory is connected with the geometry of ruled surfaces in $\mathbb{P}^3$. We define a Brill-Noether stratification over the moduli

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of maps and the Quot scheme and we compute the cohomology classes of these strata.

2. Brill-Noether loci

Consider the universal exact sequence over the Grassmannian $G(m,n)$:

$$0 \to \mathcal{N} \to \mathcal{O}_G^n \to \mathcal{Q} \to 0 \text{ over } G(m,n) \tag{1}$$

For every morphism $f \in M_d := \mathrm{Mor}_d\,(C, G(m,n))$ we take the pull-back of the sequence 1:

$$0 \to f^*\mathcal{N} \to f^*\mathcal{O}^n_{G(m,n)} \to f^*\mathcal{Q} \to 0. \tag{2}$$

Are we recovering all the bundles E over C when we take the pull-back of $\mathcal{Q}$ under $f \in M_d$? In other words, given $E \to C$, does there exist $f \in M_d$ such that $f^*\mathcal{Q} = E$?
Next lemma will ask to this question.

Lemma 2.1. *Given E, a degree d, rank n bundle over C, there exists a unique morphism $f \in M_d$ such that $f^*\mathcal{Q} = E$ if and only if E is generated by global sections, or equivalently is given by a quotient,*

$$\mathcal{O}_C^n \to E \to 0.$$

Proof. We suppose E is generated by global sections $s_1, \dots, s_n \in H^0(E)$, each section gives a map $s_i : \mathcal{O}_C \to E$, then E is given by a quotient

$$\mathcal{O}_C^n \to E \to 0. \tag{3}$$

We consider the Grassmannian $G(m,n)$ where $m = n - r$. By the universal property of the Grassmannian $G(m,n)$, there exists a morphism $f \in M_d$ such that $f^*\mathcal{Q} \cong E$, where $\mathcal{Q}$ is the universal quotient bundle over the Grassmannian $G(m,n)$.
Conversely, for all $f \in M_d$, $f^*\mathcal{Q}$ is generated by global sections, since $\mathcal{Q}$ is given by a quotient:

$$\mathcal{O}^n_{G(m,n)} \to \mathcal{Q} \to 0.$$

□

These quotients are parametrized by the Grothendieck's Quot schemes $Q_{d,r,n}$ of degree d, rank r quotients of $\mathcal{O}_C^n$ compactifying the spaces of morphisms M_d.

In the genus 0 case, by a theorem of Grothendieck, every vector bundle E over $\mathbb{P}^1$ decomposes as a direct sum of line bundles and therefore $E \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ to be generated by global sections means that $a_i \geq 0$. In the genus 1 case, the bundles generated by global sections are the indecomposable bundles of degree $d > n$, (since $h^0(E) = d > n$ for an indecomoposable bundle of positive degree, [At]), the trivial bundle $\mathcal{O}_C$ and the direct sums of both. In genus greater or equal to 2, certain restrictions on the bundle imply that it is generated by global sections. For example, we can tensorize with a line bundle of degree m such that $E(m) = E \otimes \mathcal{O}_C(m)$ is generated by global sections and $h^1(E(m)) = 0$, [DN]. Moreover, if d is sufficiently

large and E is semistable, then $h^1(E) = 0$ and E is generated by its global sections, in fact, it is sufficient to take $d > r\,(2g-1)$.
We *define the Brill-Noether loci over the spaces of morphisms* M_d as:

$$M_{d,a} = \{f \in M_d |\, h^0(f^*\mathcal{Q}) \geq a\} \tag{4}$$

for a fixed integer a. More generally, we can tensorize the bundle with a fixed line bundle L over C and consider the following Brill-Noether loci:

$$M^L_{d,a} = \{f \in M_d |\, h^0(f^*\mathcal{Q} \otimes L) \geq a\}. \tag{5}$$

Proposition 2.2. *For d sufficiently large, there is a morphism from the Quot scheme $Q_{d,r,n}$ to the Jacobian of the curve J^d.*

Proof. Let $\mathcal{U}$ be a universal bundle over $C \times M\,(r,d)$. We consider the projective bundle $\rho : P_{d,r,n} \to M(r,d)$ whose fiber over a stable bundle $[F] \in M(r,d)$ is $\mathbb{P}(H^0(C,F)^{\oplus n})$. We take the degree sufficiently large to ensure that the dimension of $\mathbb{P}(H^0(C,F)^{\oplus n})$ is constant. Globalizing,

$$P_{d,r,n} = \mathbb{P}(\rho_*\mathcal{U}^{\oplus n}).$$

Alternatively, $P_{d,r,n}$ may be thought as a fine moduli space for $n-$pairs, $(F; \phi_1, \ldots, \phi_n)$ of a stable rank r, degree d bundle F together with a non-zero $n-$tuple of holomorphic sections $\phi = (\phi_1, \ldots, \phi_n) : \mathcal{O}^n \to F$ considered projectively. When ϕ is generically surjective, it defines a point of the Quot scheme $Q_{d,r,n}$,

$$0 \to N \to \mathcal{O}^n \to E \to 0$$

taking $N = F^\vee$. The induced map $\varphi : Q_{d,r,n} \to P_{d,r,n}$ is a birational morphism, therefore $Q_{d,r,n}$ and $P_{d,r,n}$ coincide on an open subscheme and also the universal structures coincide.
Now we consider the canonical morphism to the Jacobian of the curve:

$$det : M(r,d) \to J^d$$

and the composition of the morphisms, $F = det \circ \rho \circ \varphi$ gives a morphism from $Q_{d,r,n}$ to the Jacobian J^d. □

3. A Brill-Noether stratification over the Quot scheme.

In [Mar2] we consider the space of morphisms $R^0_d := \mathrm{Mor}_d(\mathbb{P}^1, G(2,4))$ and 2 different compactifications of this space, the Quot scheme compactification and the compactification of stable maps given by Kontsevich. We consider the following Brill-Noether locus inside the space of morphisms R^0_d:

$$R^0_{d,a} = \{f \in R^0_d |\, h^0\,(f^*\mathcal{Q}^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a)) > 0,\; h^0\,(f^*\mathcal{Q}^\vee \otimes \mathcal{O}_{\mathbb{P}^1}(a-1)) = 0\}, \tag{6}$$

for a fixed integer a.

Note that we are considering here rank two bundles, but the definition can be generalized easily to bundles of arbitrary rank r.

It is easy to see that this set can be defined alternatively as the $f \in R^0_d$ with $f^*\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a)$, for $a \leq \frac{d}{2}$ and the parameter a gives a stratification of the space R^0_d.

Geometric interpretation. The image of a curve C by f, is a geometric curve in the corresponding Grassmannian or equivalently a rational ruled surface in $\mathbb{P}^3$ for the Grassmannian of lines. Fixing the parameter a we are fixing the degree of a minimal directrix in the ruled surface. The spaces $R^0_{d,a}$ are locally closed again by the Semicontinuity Theorem and they can be shown as the degeneration locus of a morphism of bundles by means of the universal exact sequence over the corresponding Quot scheme R_d and we find that the expected dimension of $R^0_{d,a}$ as determinantal variety is $3d+2a+5$. These spaces are considered in [Mar2] as parameter spaces for rational ruled surfaces in order to solve the following enumerative problems:

(1) The problem of enumerating rational ruled surfaces through $4d+1$ points, or equivalently computing the degree of R^0_d inside the projective space of surfaces of fixed degree d,
$$R^0_d \to \mathbb{P}^{\binom{d+3}{3}-1}.$$
(2) Enumerating rational ruled surfaces with fixed splitting type. This problem raises the question of defining Gromov-Witten invariants for bundles with a fixed splitting type.

When the underlying curve C is of genus greater or equal to 1, for $f \in R^0_{C,d}$, we consider the Segre invariant s of the bundle $f^*(\mathcal{Q}^\vee)$, that is, the maximal degree of a twist $f^*\mathcal{Q}^\vee \otimes L$, having a non-zero section, or equivalently, the integer s such that the minimal degree of a line quotient $E \to L \to 0$ is $\frac{d+s}{2}$, ([LN], [CS]). If T is any algebraic variety over k and $\mathcal{A}$ is a vector bundle of rank Z on $C \times T$, then the function $s : T \to \mathbb{Z}$ defined by $s(t) = s(\mathcal{A}|_{C\times t})$ is lower semicontinuous. We recall that the bundle E is stable if $s > 0$ and semistable if $s \geq 0$. We *define the corresponding Brill-Noether loci over* $R^0_{C,d}$ as the subsets:

$$R^0_{C,d,s} = \{f \in R^0_{C,d} |\, h^0(f^*\mathcal{Q}^\vee \otimes L) \geq 1, \quad deg\, L = \frac{d+s}{2}\}. \tag{7}$$

We note that the definition is independent of the chosen line bundle L of minimal degree.

We consider the Zariski closure $R_{C,d,s}$ of the sets $R^0_{C,d,s}$ inside the Quot scheme compactification of the space of morphisms:

$$R_{C,d,s} = \{q \in R_{C,d} |\, h^0\,(C, E_{q,s}) \geq 1, E_{q,s} := \mathcal{E}^\vee \otimes \pi_2^* L|_{\{q\}\times C}\}.$$

The next theorem will exhibit $R_{C,d,s}$ as determinantal varieties which allow us to estimate their dimensions.

Let us consider the universal exact sequence in $R_{C,d} \times C$,

$$0 \to \mathcal{K} \to \mathcal{O}^n_{R_{C,d}\times C} \to \mathcal{E} \to 0. \tag{8}$$

Let K_C be the canonical bundle over C and π_1, π_2 be the projection maps over the first and second factors respectively. Tensorizing the sequence (9) with the linear sheaf $\pi_2^*(K_C \otimes L^{-1})$ gives the exact sequence:

(9)
$$0 \to \mathcal{K}\otimes\pi_2^*(K_C\otimes L^{-1}) \to \mathcal{O}^n_{R_{C,d}\times C}\otimes\pi_2^*(K_C\otimes L^{-1}) \to \mathcal{E}\otimes\pi_2^*(K_C\otimes L^{-1}) \to 0.$$

The π_{1*} direct image of the above sequence yields the following long exact sequence on $R_{C,d}$:

$$\begin{gathered} 0 \to \pi_{1*}(\mathcal{K}\otimes\pi_2^*(K_C\otimes L^{-1})) \to \pi_{1*}(\mathcal{O}^n_{R_{C,d}}\otimes\pi_2^*(K_C\otimes L^{-1})) \to \\ \to \pi_{1*}(\mathcal{E}\otimes\pi_2^*(K_C\otimes L^{-1})) \to R^1\pi_{1*}(\mathcal{K}\otimes\pi_2^*(K_C\otimes L^{-1})) \to \\ \to R^1\pi_{1*}(\mathcal{O}^n_{R_{C,d}}\otimes\pi_2^*(K_C\otimes L^{-1})) \to R^1\pi_{1*}(\mathcal{E}\otimes\pi_2^*(K_C\otimes L^{-1})) \to 0. \end{gathered}$$

Theorem 3.1. *For d sufficiently large, $R_{C,d,s}$ is the locus where the map*

$$R^1\pi_{1*}(\mathcal{K}\otimes\pi_2^*(L^{-1}\otimes K_C) \to R^1\pi_{1*}(\mathcal{O}^n_{R_{C,d}\times C}\otimes\pi_2^*(L^{-1}\otimes K_C)) \tag{10}$$

is not surjective, it is irreducible, non-empty and has expected codimension $2g-s-1$ as a determinantal variety.

Proof. The map (10) is not surjective in the support of the sheaf
$$R^1\pi_{1*}(\mathcal{E}\otimes\pi_2^*(K_C\otimes L^{-1})),$$
that is, in the points $q\in R_{C,d}$ such that $h^1\left(\mathcal{E}\otimes\pi_2^*(K_C\otimes L^{-1})|_{\{q\}\times C}\right)\geq 1$, or equivalently in $R_{C,d,s}$ by Serre duality. In other words, by semicontinuity there is an open set $P_s\subset\mathbb{P}(R^1\pi_{1*}(\mathcal{E}\otimes\pi_2^*(K_C\otimes L^{-1})))$ parametrizing points $p\in R_{C,d}$ such that $s(\mathcal{E}|_{\{p\}\times C})=s$ and by the universal property of $R_{C,d}$, there is a morphism $f: P_s\to R_{C,d}$ such that, $f(P_s)=R_{C,d,s}$. This proves that the subschemes $R_{C,d,s}$, being the image of an irreducible variety by a morphism, are irreducible. The non-emptiness follows easily from definition (7).
By Serre Duality it follows that
$$R^1\pi_{1*}\left(\mathcal{K}\otimes\pi_2^*(L^{-1}\otimes K_C)\right)\cong\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L)$$
and by the Base Change Theorem, their fibers are isomorphic to
$$H^0(C,\mathcal{K}^\vee\otimes\pi_2^*L|_{C\times\{p\}}),\quad p\in R_{C,r,d}$$
and have dimension $2d+s+m\,(1-g)$. It is enough to take $d+s>2m\,(g-1)$ to ensure the vanishing of $h^1(C,\mathcal{K}^\vee\otimes\pi_2^*L|_{C\times\{p\}})$. As a consequence, $\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L)$ is a bundle of rank $2d+s+2\,(1-g)$, (note that we are assuming that $\operatorname{rank}(\mathcal{K}^\vee)=m=2$ and $n=4$). Again by Serre Duality we see that $R^1\pi_{1*}(\mathcal{O}^n_{R_{C,d}\times C}\otimes\pi_2^*(L^{-1}\otimes K_C))\cong\pi_{1*}(\mathcal{O}^n_{R_{C,d}}\otimes\pi_2^*L)$ and it is a bundle with fiber isomorphic to
$$H^0(C,\mathcal{O}^n_{R_{C,d}}\otimes\pi_2^*L|_{C\times\{p\}})$$
of dimension $2d+2s-4g+4$. Therefore we have the following morphism of bundles:
$$\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L)\xrightarrow{\phi}\mathcal{O}^{2d+2s-4g+4}_{R_{C,d}}.$$
The expected codimension of $R_{C,d,s}$ as determinantal variety is
$$\begin{gathered}((2d-2g+s+2)-(2d+2s-4g+3)) \\ \cdot((2d+2s-4g+4)-(2d+2s-4g+3))=2g-s-1.\end{gathered}$$
$\square$

Remark 3.2. *Note that as s increases, the bundle E becomes more general, so the codimension decreases. In particular when $s = g-1$, the codimension is g when fix L or 0 when we allow L to vary.*

4. Cohomology of the varieties $R_{C,d,s}$.

Let $\{1, \delta_k, 1 \leq k \leq 2g, \eta\}$ be a basis for the cohomology of C, η represents the class of a point. We will also denote $\{1, \delta_k, 1 \leq k \leq 2g, \eta\}$ the pull-backs to $R_{C,d}$ by the projection morphism. Let Θ be the theta divisor of the Jacobian variety of the curve, and Θ_r be the pull-back of Θ under the morphism F constructed in 2.2.
Let

$$c_i(\mathcal{K}^\vee) = t_i + \sum_{j=1}^{2g} s_i^j \delta_j + u_{i-1}\,\eta,\ \ t_i \in H^{2i}\,(R_{C,d}), s_i^j \in H^{2i-1}(R_{C,d}),$$

$$u_{i-1} \in H^{2i-2}\,(R_{C,d}).$$

be the Künneth decompostion of the Chern clases of $\mathcal{K}^\vee$.
Every class $z \in A(R_{C,r,d})$ can be written in the form

$$z = a + \sum_{j=1}^{2g} b^j \delta_j + f\,\eta$$

where $a = \pi_*(\eta\, z)$ and $f = \pi_*(z) \in A(R_{C,d})$. In particular $t_i = \pi_*(\eta\, c_i(\mathcal{K}^\vee))$ and $u_{i-1} = \pi_*(c_i(\mathcal{K}^\vee))$, $u_0 = \pi_*(c_i(\mathcal{K}^\vee)) = -d$.

Conjecture 4.1. *$t_1, u_1, \delta_1, \dots, \delta_{2g}$ generate $H^2(R_{C,d})$.*

For L a line bundle over C of degree $a = \frac{d+s}{2}$ its first Chern class is given by

$$c_1(\pi_2^* L) = a\,\eta.$$

Lemma 4.2. *$R_{C,d,s}$ does not have components at infinity.*

Proof. Let $q \in R_{C,d,s}$ a boundary point, that is, $E_{q,s} := \mathcal{E}^\vee \otimes \pi_2^* L|_{\{q\}}$ has a non-zero torsion T. The contribution of T to the total degree is given by the formula,

$$deg\,(E_{q,s}) = deg\,(F_{q,s}) + h^0(T),$$

where F is the locally free part of $E_{q,s}$:

$$h^0(C, E_{q,s}) = h^1(C, E_{q,s}) + deg\,(F_{q,s}) + h^0(T) + (1-g)\ \mathrm{rank}(E_{q,s}).$$

L is of maximal degree $\frac{d+s}{2}$. If $E_{q,s}$ has a non-zero torsion of degree 1, and we tensorize with a line bundle L_2 of degree $\frac{d+s}{2} - 1$, we still have $h^0(E_{q,s} \otimes L_2) \geq 1$, but this contradicts that $E_{q,s}$ has Segree invariant s, therefore these points give a component but are not in $R_{C,d,s}$.

□

Theorem 4.3. *If C is any smooth curve of genus g, and $R_{C,d,s}$ is either empty or has the expected codimension $2g-s-1$ it has fundamental class:*

$$[R_{C,d,s}] = -c_{2g-s-1}(\pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^* L)).$$

Proof. By assumption $R_{C,d,s}$ has the expected codimension as a determinantal variety and by Lemma 4.2, the Porteous formula gives the fundamental class of the varieties $R_{C,d,s}$ in terms of the Chern classes of the bundles given by Theorem 3.1.

$$[R_{C,d}] = \triangle_{2g-s-1,1}\left(c_t(-\pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^* L)\right),$$

where $\triangle_{p,q}(a) = det \begin{pmatrix} a_p & \dots & a_{p+q-1} \\ \vdots & & \vdots \\ a_{p-q+1} & \dots & a_p \end{pmatrix}$, for any formal series $a(t) = \sum_{k=-\infty}^{k=+\infty} a_k\, t^k$. □

4.0.1. *Computations of Chern classes.* Applying Grothendieck-Riemann-Roch theorem to the morphism π_{1*}, it follows that

$$\operatorname{ch}(\pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^* L))) = \pi_{1*}(\operatorname{Td}(R_{C,d} \times C)/R_{C,d}) \cdot \operatorname{ch}(\mathcal{K}^\vee \otimes \pi_2^* L)). \tag{11}$$

First we compute the Chern classes of $\mathcal{K}^\vee \otimes \pi_2^* L$:

$$c_1(\mathcal{K}^\vee \otimes \pi_2^* L)) = t_1 + (\sum_{j=1}^{2g} s_1^j \delta_j) + 2\,(d+am)\,\eta,$$

$$c_2\,(\mathcal{K}^\vee \otimes \pi_2^* L) = t_2 + (\sum_{j=1}^{2g} s_2^j \delta_j) + u_1\eta + a\eta t_1 + a(\sum_{j=1}^{2g} s_1^j \delta_j)\eta,$$

$$c_3\,(\mathcal{K}^\vee \otimes \pi_2^* L) = a\eta\, t_2 + a\eta\alpha_2.$$

We will call α_1 and α_2 to the classes $\sum_{j=1}^{2g} s_1^j \delta_j$ and $\sum_{j=1}^{2g} s_2^j \delta_j$ respectively. The intersection numbers for the δ_i imply the following relations:

$$\alpha_1^2 = -2A\eta, \quad A = \sum_{j=1}^{g} s_1^j s_1^{j+g} \in H^2\,(R_{C,d} \times C), \quad \alpha_1^3 = 0,$$

$$\alpha_2^2 = -2\gamma\eta, \quad \gamma = \sum_{i=1}^{g} s_2^j s_2^{j+g} \ \in H^6\,(R_{C,d} \times C), \ \alpha_2^3 = 0,$$

$$\alpha_1\alpha_2 = B\eta, \quad B = (\sum_{i=1}^{g} -s_1^i s_1^{i+g} + s_1^{i+g} s_2^i) \in H^4(R_{C,d} \times C),$$

By proposition 2.2 and [ACGH], A coincides with the divisor Θ_r.
Let us denote by ch_i, the $i-$homogeneous part of the Chern character of a bundle.

$$\operatorname{ch}_0\,(\mathcal{K}^\vee \otimes \pi_2^* L) = m,$$

$$\operatorname{ch}_1\,(\mathcal{K}^\vee \otimes \pi_2^* L) = t_1 + \alpha_1 + \eta\,(d+am),$$

$$\begin{aligned}\operatorname{ch}_2\,(\mathcal{K}^\vee \otimes \pi_2^* L) = \frac{1}{2}[&t_1^2 + \alpha_1^2 + 2\,t_1\,\alpha_1 + 2t_1\eta\,(d+am) + 2\alpha_1\eta\,(d+am) \\ &- 2t_2 - 2\alpha_2 - 2u_1\eta - 2a\eta t_1 - 2a\eta\alpha_1,]\end{aligned}$$

$$\begin{aligned}\mathrm{ch}_3\,(\mathcal{K}^\vee\otimes\pi_2^*L) = \frac{1}{6}[&t_1^3+3\,(d+am)\eta\,t_1^2+3\,(d+am)\eta\alpha_1^2+6(d+am)\eta\alpha_1t_1\\ &+3t_1\alpha_1^2+3\alpha_1t_1^2-3t_1t_2-3\alpha_2t_1-3u_1t_1\eta-3a\eta t_1^2-3\eta\alpha_1^2\\ &-3\alpha_1t_2+3t_1\alpha_1^2+3\alpha_1t_1^2-3t_1t_2-3\alpha_2t_1-3u_1t_1\eta-3a\eta t_1^2\\ &-3\eta\alpha_1^2-3\alpha_1t_23t_3+3\alpha_3+3u_2\eta+3a\eta t_2+3\eta\alpha_2]\end{aligned}$$

$$\mathrm{ch}_n\,(\mathcal{K}^\vee\otimes\pi_2^*L)=\mathrm{coeff}_{(t^n)}\left(\sum_n\frac{(-1)^{n-1}}{n}\sum_j c_j(\mathcal{K}^\vee\otimes\pi_2^*L)t^j\right)^n$$

Now applying formula (11), we have

$$\mathrm{ch}\,(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))=\pi_{1*}\left((1+(1-g)\eta)\,(\mathrm{ch}\,(\mathcal{K}^\vee\otimes\pi_2^*L)\right).$$

The $i-$homogeneous term of $\mathrm{ch}(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))$ is,

$$\begin{aligned}&\mathrm{ch}_0=\mathrm{rank}\,(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))=d+am+(1-g)\,m\\ &\mathrm{ch}_1=t_1\,(d+am)+\alpha_1\,(d+am)-at_1-a\alpha_1-u_1+(1-g)\alpha_1\\ &\mathrm{ch}_2=(1-g)[\frac{1}{2}\,t_1^2+\frac{1}{2}\,\alpha_1^2+t_1\alpha_1-t_2-\alpha_2+3(d+am)t_1^2+3(d+am)\alpha_1^2]\\ &+6(d+am)\alpha_1t_1-3u_1t_1-3at_1^2-3\alpha_1^2-6a\alpha_1t_1-3u_1\alpha_1-3(d+am)t_2\\ &-3(d+am)\alpha_2+3u_2+3at_2+3\alpha_2\\ &\mathrm{ch}_3=(1-g)\,[t_1^3+\alpha_1^3+3t_1\alpha_1^2+3t_1^2\alpha_1-3t_1t_2-3\alpha_1\alpha_2-3\alpha_1t_2+3t_3+3\alpha_3]\end{aligned}$$

$$\vdots$$

$$\mathrm{ch}_i(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))=(1-g)\;\mathrm{ch}(\mathcal{K}^\vee\otimes\pi_2^*L)+\mathrm{coeff}_\eta\left(\mathrm{ch}_i(\mathcal{K}^\vee\otimes\pi_2^*L)\right).$$

Finally, we get that the Chern classes of $\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L)$ are,

$$c_1\,(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))=(d+am-a+(1-g))t_1+(d+am+(1-g)-a)\alpha_1-u_1.$$

$$\begin{aligned}c_2\,(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))=\frac{1}{2}\,c_1^2-\frac{1}{2}&(1-g)(t_1^2+\alpha_1^2+t_1\alpha_1-t_2-\alpha_2)\\ &+3\,(d+am)(t_1^2+\alpha_1^2+2\alpha_1t_1-t_2-\alpha_2)\\ &+3(\alpha_1^2+at_1^2+u_1t_1+2a\alpha_1t_1-u_2-at_2-\alpha_2).\end{aligned}$$

$$\vdots$$

$$c_n(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))=-\sum_{r=1}^n\frac{(-1)^{r-1}}{n}r!\,\mathrm{ch}_r(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))\,c_{n-r}(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L)).$$

Corollary 4.4. *If C is a curve of genus 1, then*

$$[R_{C,d,0}]=-c_1(\pi_{1*}(\mathcal{K}^\vee\otimes\pi_2^*L))=-(d+am-a)t_1-(d+am-a)\alpha_1+u_1.$$

$$\begin{aligned}[R_{C,d,-1}] = -c_2(\pi_{1*}(\mathcal{K}^\vee \otimes \pi_2^* L)) = -\frac{1}{2}\,&((d+am-a)t_1 + (d+am-a)\alpha_1 - u_1)^2\\ &- 3\,(d+am)(t_1^2+\alpha_1^2+2\alpha_1 t_1 - t_2 - \alpha_2)\\ &- 3(\alpha_1^2 + a t_1^2 + u_1 t_1 - 2a\alpha_1 t_1 - u_2 - a t_2 - \alpha_2).\end{aligned}$$

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